A NEW STRONG INVARIANCE PRINCIPLE FOR SUMS OF INDEPENDENT RANDOM VECTORS

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ABSTRACT. We provide a strong invariance principle for sums of independent, identically distributed random vectors which need not have finite second absolute moments. Various applications are indicated. In particular, we show how one can re-obtain some recent LIL type results from this invariance principle.

1. INTRODUCTION

Let $X, X_1, X_2, \ldots$ be independent, identically distributed (i.i.d.) random vectors in $\mathbb{R}^d$ and set $S_n = \sum_{i=1}^n X_i, n \geq 1, S_0 := 0$. If the random vectors have mean zero and a finite covariance matrix Σ it follows from the multidimensional central limit theorem that

$$S_n/\sqrt{n} \xrightarrow{d} Y \sim \text{normal}(0, \Sigma), \tag{1.1}$$

where $\xrightarrow{d}$ stands for convergence in distribution.
There is also a much more general weak convergence result available, namely Donsker's Theorem. To formulate this result we first have to recall the definition of the partial sum process sequence $S_{(n)} : \Omega \to C_d[0,1]$:

$$S_{(n)}(t) = \begin{cases} S_k & \text{if } t = k/n, 0 \leq k \leq n, \\ \text{linearly interpolated} & \text{elsewhere.} \end{cases}$$

Let $\{W(t), t \geq 0\}$ be a standard d-dimensional Brownian motion and denote the Euclidean norm on $\mathbb{R}^d$ by $|\cdot|$. Then the d-dimensional version of Donsker's theorem can be formulated as follows,

Theorem 1.1 (Donsker). *Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors such that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$. Let Γ be the positive definite, symmetric matrix satisfying $\Gamma^2 = \text{cov}(X) =: \Sigma$. Then we have,*

$$S_{(n)}/\sqrt{n} \xrightarrow{d} \Gamma \cdot \overline{W},$$

where $\overline{W}(t), 0 \leq t \leq 1$ is the restriction of W to $[0,1]$.

In order to prove this result one can use a coupling argument, that is one can construct the random variables $X_1, X_2, \ldots$ and a d-dimensional Brownian motion $\{W(t) : t \geq 0\}$ on a suitable p-space so that one has

$$\|S_{(n)} - \Gamma \cdot W_{(n)}\|/\sqrt{n} \xrightarrow{\mathbb{P}} 0, \tag{1.2}$$

where $W_{(n)})(t) = W(nt), 0 \leq t \leq 1$, $\xrightarrow{\mathbb{P}}$ stands for convergence in probability, and $\|\cdot\|$ is the sup-norm on $C_d[0,1]$.

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Relation (1.2) clearly implies Donsker's theorem since we have $W_{(n)}/\sqrt{n} \stackrel{d}{=} \overline{W}$.

It is natural now to ask whether one can replace convergence in probability by almost sure convergence. This is not only a formal improvement of the above coupling result, but it also makes it possible to infer almost sure convergence results for partial sum processes from the corresponding results for Brownian motion. This was pointed out in the classical paper by Strassen [15] who obtained a functional law of the iterated logarithm for general partial sum processes along these lines. So one can pose the following

Question 1.2. Given a monotone sequence c_n, when is a construction possible such that with probability one,

$$\|S_{(n)} - \Gamma \cdot W_{(n)}\| = O(c_n) \text{ as } n \to \infty?$$

If such a construction is possible, one speaks of a strong invariance principle with rate $O(c_n)$.

We first look at the 1-dimensional case. (Then Γ is simply the standard deviation σ of X.) Though it was already known at an early stage that no better convergence rate than $O(\log n)$ is feasible unless of course the variables $X, X_1, X_2, \ldots$ are normally distributed, it had been an open question for a long time whether a strong invariance principle with such a rate is actually attainable. Very surprisingly, Komlós, Major and Tusnády [10] eventually were able to show that such a construction is possible in dimension 1 if and only if the moment generating function of X is finite and if X has mean zero. More generally, they proved that a strong invariance principle with rate $O(c_n)$ is possible for any sequence c_n of positive real numbers such that c_n/n^α is decreasing for some $\alpha < 1/3$ and $c_n/\log n$ is non-decreasing, if and only if

$$\sum_{n=1}^{\infty} \mathbb{P}\{|X| \geq c_n\} < \infty \text{ and } \mathbb{E}X = 0. \tag{1.3}$$

Major [11] obtained analogous results for sequences c_n satisfying c_n/n^α is non-increasing for some $\alpha < 1/2$ and $c_n/n^{1/3}$ is non-decreasing. This includes especially the sequences $c_n = n^\gamma, 1/3 \leq \gamma < 1/2$. For sequences c_n in this range one can also get a strong invariance principle with rate $o(c_n)$ rather than $O(c_n)$. Moreover, it is well known that it is impossible to obtain an analogous result for the sequence $c_n = \sqrt{n}$. Note that in this case condition (1.3) is equivalent with the classical condition $\mathbb{E}X^2 < \infty$ and $\mathbb{E}X = 0$. In this case the best possible strong invariance principle is of order $o(\sqrt{n \log\log n})$. The remaining gap, namely the determination of the optimal convergence rates for "big" sequences c_n of order $o(\sqrt{n})$ where no $\alpha < 1/2$ exists such that c_n/n^α is non-decreasing, was closed by Einmahl [3]. (Note that this includes all sequences of the form $\sqrt{n}/h(n)$ where $h : [1,\infty[\to]0,\infty[$ is slowly varying at infinity and $h(x) \to \infty$ as $x \to \infty$.) We next mention the work of Major [12] who showed that under the classical condition $\mathbb{E}X^2 < \infty$ and $\mathbb{E}X = 0$ a strong approximation with rate $o(\sqrt{n})$ is possible if one replaces the Brownian motion by a slightly different Gaussian process.

Following up the ideas from [12, 3], Einmahl and Mason [9] finally obtained the following strong invariance principle.

Theorem 1.3. *Let $X, X_1, X_2, \dots$ be i.i.d. random variables satisfying condition (1.3) for a non-decreasing sequence c_n of positive real numbers such that $c_n/n^{1/3}$ is eventually non-decreasing and $c_n/\sqrt{n}$ is eventually non-increasing. If the underlying p-space is rich enough, one can construct a 1-dimensional Brownian motion such that with probability one,*

$$\|S_{(n)} - \sigma_n W_{(n)}\| = o(c_n) \text{ as } n \to \infty,$$

where $\sigma_n^2 = \mathbb{E}\left[X^2 I\{|X| \le c_n\}\right]$.

Using this result, one can easily determine the optimal convergence rate for the strong invariance principle in its classical formulation for all sequences c_n in this range. (See the subsequent Corollary 2.2 for more details.) Note that Theorem 1.3 only applies if $\mathbb{E}X^2 < \infty$. This follows from the fact that $c_n = O(\sqrt{n})$ under the above assumptions and the second moment is finite if condition (1.3) holds for such a sequence. Very recently, Einmahl [6] showed that Theorem 1.3 has also a version in the infinite variance case and he used this one to prove new functional LIL type results in this setting.

We return to the multidimensional case. Most of the results for (1-dimensional) random variables have been extended to random vectors by now. We mention the work of Philipp [13] who extended Strassen's strong invariance principle with rate $o(\sqrt{n \log\log n})$ to the d-dimensional case (actually also to Banach spaced valued random elements) and that of Berger [1] who generalized Major's result from [11] to the d-dimensional case. This led to the best possible rate of $o(n^{1/3})$ in the multidimensional invariance principle at that time. This rate was further improved in [3] to $o(n^{\alpha})$, for $\alpha > 1/4$. The next major step was taken by Einmahl [5] who was able to extend all the results of Komlós, Major and Tusnády [10] up to order $O((\log n)^2)$ to the multivariate case. Moreover, it was shown in this article that under an extra smoothness assumption on the distribution of X strong approximations with even better rates, especially with rate $O(\log n)$ are possible in higher dimensions as well. Zaitsev [16] finally showed that such constructions are also possible for random vectors which do not satisfy the extra smoothness condition so that we now know that all the results of [10] have versions in higher dimensions.

Given all this work, one has now a fairly complete picture for the strong invariance principle for sums of i.i.d. random vectors. In the present paper we shall close one of the remaining gaps. We shall show that it is also possible to extend Theorem 1.3 to the d-dimensional case. Actually, this is not too difficult if one proves it as the original result is stated above, but as we have indicated, there is also a version of this result in the infinite variance case. The purpose of this paper is to establish a general multidimensional version of Theorem 1.3 which also applies if $\mathbb{E}|X|^2 = \infty$. In this case the problem becomes more delicate since one has to use truncation arguments which lead to random vectors with possibly very irregular covariance matrices. Most of the existing strong approximation techniques for sums of independent random vectors require some conditions on the ratio of the largest and smallest eigenvalues of the covariance matrices (see, for instance, [4, 16]) and, consequently, they cannot be applied in this case. Here a new strong approximation method which is due to Sakhanenko [14] will come in handy.

2. The main result and some corollaries.

We first state our new strong invariance principle where we only assume that $\mathbb{E}|X| < \infty$. (This follows from the subsequent assumption (2.1) since all sequences c_n considered are of order $O(n)$. If condition (2.1) is satisfied for such a sequence, we have $\mathbb{E}|X| < \infty$.)

Theorem 2.1. *Let $X, X_1, X_2, \dots$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$. Assume that*

$$\sum_{n=1}^{\infty} \mathbb{P}\{|X| \geq c_n\} < \infty, \tag{2.1}$$

where c_n is a non-decreasing sequence of positive real numbers such that

$$\exists\, \alpha \in]1/3, 1[: c_n/n^{\alpha} \text{ is eventually non-decreasing, and} \tag{2.2}$$

$$\forall\, \epsilon > 0 \,\exists\, m_\epsilon \geq 1 : c_n/c_m \leq (1+\epsilon)(n/m), m_\epsilon \leq m < n. \tag{2.3}$$

If the underlying p-space is rich enough, one can construct a d-dimensional standard Brownian motion $\{W(t), t \geq 0\}$ such that with probability 1,

$$\|S_{(n)} - \Gamma_n \cdot W_{(n)}\| = o(c_n) \text{ as } n \to \infty, \tag{2.4}$$

where Γ_n is the sequence of positive semidefinite, symmetric matrices determined by

$$\Gamma_n^2 = \left(\mathbb{E}\left[X^{(i)} X^{(j)} I\{|X| \leq c_n\}\right]\right)_{1 \leq i,j \leq d}. \tag{2.5}$$

As a first application of our above strong invariance principle, we show how one can re-obtain the main results of [3] from it. Here we are assuming that $\mathbb{E}|X|^2 < \infty$ so that $\mathrm{cov}(X)$ (= covariance matrix of X) exists.

Corollary 2.2. *Let $X, X_1, X_2, \dots$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$ and assume that $\mathbb{E}|X|^2 < \infty$. Let Γ be the positive semidefinite, symmetric matrix satisfying $\Gamma^2 = \mathrm{cov}(X)$. Assume that condition (2.1) is satisfied for a sequence c_n such that $c_n/\sqrt{n}$ is eventually non-increasing and (2.2) holds. Then a construction is possible such that we have with probability one:*

$$\|S_{(n)} - \Gamma \cdot W_{(n)}\| = o(c_n \vee c_n^2 \sqrt{\log\log n/n}). \tag{2.6}$$

Furthermore, we have,

$$\|S_{(n)} - \Gamma \cdot W_{(n)}\|/c_n \xrightarrow{\mathbb{P}} 0 \tag{2.7}$$

Remark 2.3. We get the following results due to [3] from (2.6):

(1) If c_n satisfies additionally $c_n = O(\sqrt{n/\log\log n})$, then we also have the almost sure rate $o(c_n)$ for the "standard" approximation by $\Gamma \cdot W_{(n)}$.
(2) Set $\rho_n = \frac{c_n}{\sqrt{n/\log\log n}}$. If $\liminf_{n\to\infty} \rho_n > 0$, we get the rate $o(c_n \rho_n)$, where the extra factor ρ_n is sharp (see [3]).

Note also that (2.7) (with $c_n = \sqrt{n}$) immediately implies Donsker's theorem.

To formulate the following corollary we need somewhat more notation: For any (d,d)-matrix A we set $\|A\| := \sup\{|A \cdot v| : |v| \leq 1\}$. We recall that $\|A\|^2$ is equal to the largest eigenvalue of the symmetric matrix $A^t A$. This is due to the well known fact that the largest eigenvalue $\Lambda(C)$ of a positive semidefinite, symmetric (d,d)-matrix C satisfies $\Lambda(C) = \sup\{\langle v, Cv\rangle : |v| \le 1\}$, where $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^d$. Furthermore, let for any $t \ge 0$,

$$H(t) := \sup\{\mathbb{E}[\langle v, X\rangle^2 I\{|X| \le t\}] : |v| \le 1\}.$$

If we look at the matrices Γ_n we see that $\|\Gamma_n\|^2 = H(c_n)$. Similarly as in [8] we set for any sequence c_n as in Theorem 2.1,

$$\alpha_0 = \sup\left\{\alpha \ge 0 : \sum_{n=1}^{\infty} n^{-1}\exp\left(-\frac{\alpha^2 c_n^2}{2nH(c_n)}\right) = \infty\right\}.$$

Using Theorem 2.1 we now can give a very short proof of Theorem 3 [8] in the finite-dimensional case. This result is the basis for all the LIL type results in [7, 8] and, consequently, we can prove all these results in the finite-dimensional case via Theorem 2.1.

Corollary 2.4. *Let $X, X_1, X_2, \dots$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$. Assume that condition (2.1) holds for a non-decreasing sequence c_n of positive real numbers such that $c_n/\sqrt{n}$ is eventually non-decreasing and condition (2.3) is satisfied. Then we have with probability one,*

$$\limsup_{n\to\infty} \frac{|S_n|}{c_n} = \alpha_0. \tag{2.8}$$

We finally show how the general law of the iterated logarithm (see Corollary 2.5), follows directly from Corollary 2.4. (In [7, 8] we had obtained this result as a corollary to another more general result, the law of a very slowly varying function which also follows from Corollary 2.4, but requires a more delicate proof.)
As usual, we set $Lt = \log(t \vee e)$ and $LLt = L(Lt), t \ge 0$.

Corollary 2.5 (General LIL)**.** *Let $X, X_1, X_2, \dots$ be i.i.d. random vectors in $\mathbb{R}^d$. Let $p \ge 1$ and $\lambda \ge 0$. Then the following are equivalent:*

(a) *We have with probability one,* $\limsup_{n\to\infty} |S_n|/\sqrt{2n(LLn)^p} = \lambda$
(b) $\limsup_{t\to\infty} H(t)/(LLt)^{p-1} = \lambda^2$ *and* $\mathbb{E}[X] = 0$.

Note that we do not explicitely assume that

$$\sum_{n=1}^{\infty} \mathbb{P}\{|X| \ge \sqrt{n(LLn)^p}\} < \infty,$$

or, equivalently, $\mathbb{E}[|X|^2/(LL|X|)^p] < \infty$. In the finite dimensional case this condition follows from (b). This was already pointed out in the 1-dimensional case (see, for instance, [6]), and we shall give here a detailed proof of this fact in arbitrary finite dimension. We mention that this implication does not hold in the infinite dimensional setting so that one has an extra condition in this case (see [8]).

The remaining part of this paper is organized as follows: The proof of Theorem 2.1 will be given in Section 3 and then we shall show in Section 4 how the corollaries can be obtained.

3. Proof of the strong invariance principle.

3.1. **Some auxiliary results.** Our proof is based on the following strong approximation result which follows from the work of Sakhanenko [14]. (See his Corollary 3.2.)

Theorem 3.1. *Let $X_j^*, 1 \le j \le n$ be independent mean zero random vectors on $\mathbb{R}^d$ such that $\mathbb{E}|X_j^*|^3 < \infty, 1 \le j \le n$. Let $x > 0$ be fixed. If the underlying p-space is rich enough, one can construct independent normal$(0, I)$-distributed random vectors $Y_j, 1 \le j \le n$ such that*

$$\mathbb{P}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^k (X_j^* - A_j \cdot Y_j)\right| \ge x\right\} \le C\sum_{j=1}^d \mathbb{E}|X_j^*|^3/x^3, \tag{3.1}$$

where A_j is the positive semidefinite, symmetric matrix satisfying $A_j^2 = \mathrm{cov}(X_j^*), 1 \le j \le n$ and C is a positive constant depending on d only.

Proof. From Corollary 3.2. in [14] we get independent random vectors $Y_1, \dots, Y_n$ so that the probability in (3.1) is

$$\le C' x^{-3} \sum_{j=1}^n (\mathbb{E}|X_j^*|^3 + \mathbb{E}|Y_j^*|^3),$$

where $Y_j^* := A_j Y_j, 1 \le j \le n$ and C' is a positive constant depending on d only. Writing $Y_j^* = (Y_{j,1}^*, \dots, Y_{j,d}^*)^t$ and using the inequality $|v|^3 \le d^{1/2}\sum_{i=1}^d |v_i|^3, v \in \mathbb{R}^d$ (which follows from the Hölder inequality), we get for $1 \le j \le n$,

$$\begin{aligned}\mathbb{E}|Y_j^*|^3 &\le d^{1/2}\sum_{i=1}^d \mathbb{E}|Y_{j,i}^*|^3 = d^{1/2}\mathbb{E}|Z|^3\sum_{i=1}^d (\mathbb{E}|Y_{j,i}^*|^2)^{3/2} \\ &= d^{1/2}\mathbb{E}|Z|^3\sum_{i=1}^d (\mathbb{E}|X_{j,i}^*|^2)^{3/2} \le d^{1/2}\mathbb{E}|Z|^3\sum_{i=1}^d \mathbb{E}|X_{j,i}^*|^3 \le d^{3/2}\mathbb{E}|Z|^3\mathbb{E}|X_j^*|^3,\end{aligned}$$

where $Z : \Omega \to \mathbb{R}$ is standard normal. Thus we have,

$$\mathbb{E}|Y_j^*|^3 \le C''\mathbb{E}|X_j^*|^3, 1 \le j \le n, \tag{3.2}$$

where C'' is a positive constant depending on d only and Theorem 3.1 has been proved. □

Corollary 3.2. *Let $X_n^*, n \ge 1$ be a sequence of independent mean zero random vectors on $\mathbb{R}^d$ such that we have for a non-decreasing sequence c_n of positive real numbers which converges to infinity,*

$$\sum_{n=1}^\infty \mathbb{E}|X_n^*|^3/c_n^3 < \infty.$$

If the underlying p-space is rich enough, one can construct a sequence of independent normal$(0, I)$-distributed random vectors such that with probability one,

$$\sum_{j=1}^n (X_j^* - A_j \cdot Y_j) = o(c_n) \text{ as } n \to \infty,$$

where A_n is the sequence of positive semidefinite, symmetric matrices satifying $A_n^2 = \mathrm{cov}(X_n^*), n \ge 1$.

Proof. We employ a similar argument as on p.95, [4]. It is easy to see that one can find another non-decreasing sequence $\tilde{c}_n$ so that $\tilde{c}_n \to \infty$, $\tilde{c}_n = o(c_n)$ as $n \to \infty$ and still

$$\sum_{n=1}^{\infty} \mathbb{E}|X_n^*|^3/\tilde{c}_n^3 < \infty. \tag{3.3}$$

Set

$$m_0 := 1, m_n := \min\{k : \tilde{c}_k \geq 2\tilde{c}_{m_{n-1}}\}, n \geq 1.$$

By the definition of the subsequence m_n we have

$$\tilde{c}_{m_n-1}/\tilde{c}_{m_{n-1}} \leq 2 \leq \tilde{c}_{m_n}/\tilde{c}_{m_{n-1}}, n \geq 1.$$

Theorem 3.1 enables us to define independent normal$(0, I)$-distributed random vectors $\{Y_j : m_{n-1} \leq j < m_n\}$ in terms of the random vectors $\{X_j^* : m_{n-1} \leq j < m_n\}$ (for any $n \geq 1$) such that

(3.4)
$$\begin{aligned}\mathbb{P}\left\{\max_{m_{n-1}\leq k<m_n}\left|\sum_{j=m_{n-1}}^{k}(X_j^* - A_j \cdot Y_j)\right| \geq \tilde{c}_{m_{n-1}}\right\} &\leq C\sum_{j=m_{n-1}}^{m_n-1}\mathbb{E}|X_j^*|^3/\tilde{c}_{m_{n-1}}^3 \\ &\leq 8C\sum_{j=m_{n-1}}^{m_n-1}\mathbb{E}|X_j^*|^3/\tilde{c}_j^3.\end{aligned}$$

The resulting sequence $\{Y_n : n \geq 1\}$ consists of independent random vectors since the "blocks" $\{X_j^* : m_{n-1} \leq j < m_n\}$ are independent.
Recalling (3.3) and using the Borel-Cantelli Lemma we see that we have with probability one,

$$\max_{m_{n-1}\leq k<m_n}\left|\sum_{j=m_{n-1}}^{k}(X_j^* - A_j \cdot Y_j)\right| \leq \tilde{c}_{m_{n-1}} \text{ eventually}$$

Employing the triangular inequality and adding up the above inequalities we get with probability one,

$$\left|\sum_{j=1}^{k}(X_j^*(\omega) - A_j \cdot Y_j(\omega))\right| \leq K(\omega) + \sum_{i=1}^{n-1}\tilde{c}_{m_i} \leq K(\omega) + 2\tilde{c}_{m_{n-1}}, m_{n-1} \leq k < m_n$$

and we see that our corollary holds. □

The following lemma collects some more or less known facts.

Lemma 3.3. *Let $X : \Omega \to \mathbb{R}^d$ be a random vector such that (2.1) holds for a non-decreasing sequence c_n of positive real numbers.*

(a) *If c_n satisfies condition (2.2), we have:*

$$\sum_{n=1}^{\infty}\mathbb{E}[|X|^3 I\{|X| \leq c_n\}]/c_n^3 < \infty.$$

(b) *If c_n satisfies condition (2.3), we have*

$$\mathbb{E}[|X|I\{|X| > c_n\}] = o(c_n/n) \text{ as } n \to \infty.$$

(c) *If* $\mathbb{E}[X] = 0$*, and both conditions (2.2), (2.3) are satisfied, we have:*

$$\sum_{k=1}^{n} \mathbb{E}[XI\{|X| \le c_k\}] = o(c_n) \text{ as } n \to \infty.$$

Proof. First observe that setting $p_j = \mathbb{P}\{c_{j-1} < |X| \le c_j\}, j \ge 1$, where $c_0 = 0$, we have by our assumption (2.1),

$$\sum_{j=1}^{\infty} jp_j < \infty \tag{3.5}$$

To prove (a) we note that we have on account of (2.2):

$$c_j/j^\alpha \le c_n/n^\alpha \text{ for } n \ge j \ge j_0 \text{ (say)},$$

which in turn implies that $c_j/j^\alpha \le K_1 c_n/n^\alpha, 1 \le j \le n, n \ge 1$, where $K_1 > 0$ is a suitable constant. It follows that

$$c_j/c_n \le K_1(j/n)^\alpha, 1 \le j \le n, n \ge 1. \tag{3.6}$$

We now see that

$$\sum_{n=1}^{\infty} \mathbb{E}[|X|^3 I\{|X| \le c_n\}]/c_n^3 \le \sum_{n=1}^{\infty}\sum_{j=1}^{n} c_j^3 p_j/c_n^3 = \sum_{j=1}^{\infty}\left(\sum_{n=j}^{\infty}(c_j/c_n)^3\right)p_j$$
$$\le K_1^3 \sum_{j=1}^{\infty}\left(\sum_{n=j}^{\infty} n^{-3\alpha}\right) j^{3\alpha}p_j \le K_2 \sum_{j=1}^{\infty} jp_j < \infty.$$

Here we have used the fact that $\sum_{n=j}^{\infty} n^{-3\alpha} = O(j^{1-3\alpha})$ as $j \to \infty$ which follows easily by comparing this series with the integral $\int_j^\infty x^{-3\alpha}dx < \infty$. (Recall that $\alpha > 1/3$.)
To prove (b) we observe that

$$n\mathbb{E}[|X|I\{|X| > c_n\}]/c_n \le \sum_{j=n+1}^{\infty} n(c_j/c_n)p_j \le K_3 \sum_{j=n+1}^{\infty} jp_j, \tag{3.7}$$

where we have used the fact that $c_j/c_n \le K_3 j/n, j \ge n$ for some positive constant K_3. (This easily follows from condition (2.3).) Recalling (3.5) we readily obtain (b).
We turn to the proof of (c). Let $\delta > 0$ be fixed and choose an $m_\delta \ge 1$ so that $m\mathbb{E}[|X|I\{|X| > c_m\}]/c_m \le \delta$ for $m \ge m_\delta$, which is possible due to (b).
Since $\mathbb{E}X = 0$, we trivially have $\mathbb{E}[XI\{|X| \le c_m\}] = -\mathbb{E}[XI\{|X| > c_m\}]$ and we can conclude that

$$\left|\sum_{k=1}^{n} \mathbb{E}[XI\{|X| \le c_k\}]/c_n\right| \le m_\delta \mathbb{E}|X|/c_n + \delta \sum_{k=m_\delta+1}^{n} c_k/(kc_n).$$

Due to (3.6) we further have,

$$\sum_{k=m_\delta+1}^{n} c_k/(kc_n) \le K_1 \sum_{k=m_\delta+1}^{n} k^{\alpha-1}/n^\alpha \le K_1/\alpha.$$

Consequently, we have,

$$\limsup_{n\to\infty}\left|\sum_{k=1}^{n}\mathbb{E}[XI\{|X|\le c_k\}]/c_n\right|\le K_1\delta/\alpha.$$

This implies (c) since we can choose δ arbitrarily small. □

The next lemma gives us more information on the matrices Γ_n.

Lemma 3.4. *Let the sequence Γ_n be defined as in Theorem 2.1. Then we have for $n\ge m\ge 1$,*

(a) $\Gamma_n-\Gamma_m$ *is positive semidefinite.*
(b) $\|\Gamma_n-\Gamma_m\|^2\le\mathbb{E}[|X|^2I\{c_m<|X|\le c_n\}]$.

Proof. By definition we have

$$\langle v,(\Gamma_n^2-\Gamma_m^2)v\rangle=\mathbb{E}[\langle X,v\rangle^2I\{c_m<|X|\le c_n\}]\ge 0, v\in\mathbb{R}^d,$$

which clearly shows that $\Gamma_n^2-\Gamma_m^2$ is positive semidefinite. This in turn implies that this also holds for $\Gamma_n-\Gamma_m$ since $f(t)=\sqrt{t}, t\ge 0$ is an operator monotone function (see Proposition V.1.8, [2]). We thus have proved (a).
Furthermore, we can conclude from the above formula that

$$\|\Gamma_n^2-\Gamma_m^2\|\le\mathbb{E}[|X|^2I\{c_m<|X|\le c_n\}].$$

Here we have used the fact that if A is a positive semidefinite, symmetric (d,d)-matrix, we have $\|A\|=\sup\{\langle v,Av\rangle:|v|\le 1\}$.
Finally, noting that by Theorem X.1.1, [2]

$$\|\Gamma_n-\Gamma_m\|^2\le\|\Gamma_n^2-\Gamma_m^2\|,$$

we see that (b) also holds. □

3.2. **Conclusion of the proof.**
(i) Set $X_n'=X_nI\{|X_n|\le c_n\}, X_n^*=X_n'-\mathbb{E}X_n', n\ge 1$. Then we clearly have by assumption (2.1),

$$\sum_{n=1}^{\infty}\mathbb{P}\{X_n\ne X_n'\}<\infty, \tag{3.8}$$

which via the Borel-Cantelli lemma trivially implies that with probability one, $\sum_{j=1}^{n}(X_j-X_j')=o(c_n)$ as $n\to\infty$. Recalling Lemma 3.3(c), we see that with probability one,

$$S_n-\sum_{j=1}^{n}X_j^*=o(c_n)\text{ as }n\to\infty. \tag{3.9}$$

(ii) Noting that $\mathbb{E}|X_n^*|^3\le 8\mathbb{E}[|X|^3I\{|X|\le c_n\}], n\ge 1$, we get from Lemma 3.3(a) that

$$\sum_{n=1}^{\infty}\mathbb{E}|X_n^*|^3/c_n^3<\infty. \tag{3.10}$$

In view of Corollary 3.2 we now can find a sequence $\{Y_n\}$ of independent normal$(0, I)$-distributed random vectors such that with probability one,

$$\sum_{j=1}^{n}(X_j^* - A_j \cdot Y_j) = o(c_n) \text{ as } n \to \infty, \tag{3.11}$$

where A_n are the positive semidefinite symmetric matrices satisfying $A_n^2 = \mathrm{cov}(X_n^*) = \mathrm{cov}(X_n')$.
(iii) We next claim that with probability one,

$$\sum_{j=1}^{n}(\Gamma_j - A_j) \cdot Y_j = o(c_n) \text{ as } n \to \infty. \tag{3.12}$$

In order to prove that it is sufficient to show that

$$\sum_{j=1}^{\infty} \frac{\mathbb{E}[|(\Gamma_j - A_j) \cdot Y_j|^2]}{c_j^2} < \infty. \tag{3.13}$$

To see that we argue as follows:
Using a standard 1-dimensional result on random series componentwise, we then can conclude that the random series $\sum_{j=1}^{\infty}(\Gamma_j - A_j) \cdot Y_j / c_j$ is convergent in $\mathbb{R}^d$ with probability one, which in turn via Kronecker's lemma (applied componentwise) implies (3.12).
Next observe that $\mathbb{E}[|(\Gamma_j - A_j) \cdot Y_j|^2] \leq d\|\Gamma_j - A_j\|^2, j \geq 1$ so that (3.13) follows once we have shown that

$$\sum_{j=1}^{\infty} \frac{\|\Gamma_j - A_j\|^2}{c_j^2} < \infty. \tag{3.14}$$

From the definition of these matrices we immediately see that for any $v \in \mathbb{R}^d$,

$$\langle v, (\Gamma_j^2 - A_j^2)v \rangle = (\mathbb{E}[\langle X, v \rangle I\{|X| \leq c_j\}])^2$$

which on account of $\mathbb{E}[\langle X, v \rangle] = 0$ implies,

$$\|\Gamma_j^2 - A_j^2\| = \sup_{|v| \leq 1} (\mathbb{E}[\langle X, v \rangle I\{|X| > c_j\}])^2 \leq \mathbb{E}[|X| I\{|X| > c_j\}]^2.$$

Using once more Theorem X.1.1. in [2] and recalling Lemma 3.3(b), we find that

$$\|\Gamma_j - A_j\|^2 \leq \|\Gamma_j^2 - A_j^2\| \leq \epsilon_j c_j^2 / j^2, j \geq 1$$

where $\epsilon_j \to 0$ as $j \to \infty$. This trivially implies (3.14).
(iv) Combining relations (3.9), (3.11) and (3.12), we see that with probability one,

$$S_n - \sum_{j=1}^{n} \Gamma_j \cdot Y_j = o(c_n) \text{ as } n \to \infty.$$

This of course implies that with probability one,

$$\max_{1 \leq k \leq n} |S_k - \sum_{j=1}^{k} \Gamma_j \cdot Y_j| = o(c_n) \text{ as } n \to \infty. \tag{3.15}$$

Set

$$\Delta_n := \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} (\Gamma_n - \Gamma_j) Y_j \right|, n \geq 1.$$

We claim that with probability one,

$$\Delta_n/c_n \to 0 \text{ as } n \to \infty. \tag{3.16}$$

We first show that with probability one,

$$\Delta_{2^\ell}/c_{2^\ell} \to 0 \text{ as } \ell \to \infty. \tag{3.17}$$

To that end we note that by combining Lévy's inequality and the Markov inequality, we get for any $\epsilon > 0$,

$$\mathbb{P}\{\Delta_{2^\ell} \geq \epsilon c_{2^\ell}\} \leq 2\mathbb{P}\left\{\left|\sum_{j=1}^{2^\ell} |(\Gamma_{2^\ell} - \Gamma_j)Y_j\right| \geq \epsilon c_{2^\ell}\right\} \leq 2\epsilon^{-2} c_{2^\ell}^{-2} \sum_{j=1}^{2^\ell} \mathbb{E}[|(\Gamma_{2^\ell} - \Gamma_j)Y_j|^2].$$

As we have $\mathbb{E}[|(\Gamma_{2^\ell} - \Gamma_j)Y_j|^2] \leq d\|\Gamma_{2^\ell} - \Gamma_j\|^2$, it suffices to show,

$$\sum_{\ell=1}^{\infty} \sum_{j=1}^{2^\ell} \|\Gamma_{2^\ell} - \Gamma_j\|^2 / c_{2^\ell}^2 < \infty. \tag{3.18}$$

Using the inequality $\|\Gamma_{2^\ell} - \Gamma_j\|^2 \leq \mathbb{E}[|X|^2 I\{c_j < |X| \leq c_{2^\ell}\}]$ (see Lemma 3.4(b)), we can prove this by essentially the same argument as on page 908 in [6]. (Note that we now have $c_j^2/c_{2^\ell}^2 \leq (j/2^\ell)^{2\alpha}$ so that one has to modify the last two bounds on this page slightly.)
(v) Let $2^\ell < n < 2^{\ell+1}$. Then we have by the triangular inequality,

$$\Delta_n \leq \max_{1 \leq k \leq n} \left|\sum_{j=1}^{k} (\Gamma_{2^{\ell+1}} - \Gamma_j)Y_j\right| + \max_{1 \leq k \leq n} \left|(\Gamma_{2^{\ell+1}} - \Gamma_n) \sum_{j=1}^{k} Y_j\right|$$

which in turn is

$$\leq \Delta_{2^{\ell+1}} + \|\Gamma_{2^{\ell+1}} - \Gamma_{2^\ell}\| \max_{1 \leq k \leq 2^{\ell+1}} \left|\sum_{j=1}^{k} Y_j\right|.$$

Here we have used the fact that $\|\Gamma_{2^{\ell+1}} - \Gamma_n\| \leq \|\Gamma_{2^{\ell+1}} - \Gamma_{2^\ell}\|, 2^\ell \leq n \leq 2^{\ell+1}$ which follows from Lemma 3.4(a).
Using obvious modifications of the proof of relation (3.11) in [6], we can conclude that with probabilty one,

$$\|\Gamma_{2^{\ell+1}} - \Gamma_{2^\ell}\| \max_{1 \leq k \leq 2^{\ell+1}} \left|\sum_{j=1}^{k} Y_j\right| = o(c_{2^\ell}) \text{ as } \ell \to \infty. \tag{3.19}$$

Combining relations (3.17) and (3.19), we see that (3.16) holds.
(vi) In view of (3.15) and (3.16) we have with probability one,

$$\max_{1 \leq k \leq n} \left|S_k - \Gamma_n \sum_{j=1}^{k} Y_j\right| = o(c_n) \text{ as } n \to \infty.$$

Letting $T_{(n)} : \Omega \to C_d[0,1]$ be the partial sum process sequence based on $\sum_{j=1}^n Y_j, n \geq 1$, we see that with probability one

$$\|S_{(n)} - \Gamma_n \cdot T_{(n)}\| = o(c_n) \text{ as } n \to \infty. \tag{3.20}$$

If the underlying p-space is rich enough, we can find a d-dimensional Brownian motion $\{W(t) : t \geq 0\}$ such that $W(n) = \sum_{j=1}^{n} Y_j, n \geq 1$. Using the corresponding result in the 1-dimensional case (see [9]) componentwise, we find that with probability one,

$$\|T_{(n)} - W_{(n)}\| = O(\sqrt{\log n}) \text{ as } n \to \infty$$

and consequently we have with probability one,

$$\|\Gamma_n \cdot T_{(n)} - \Gamma_n \cdot W_{(n)}\| \leq \|\Gamma_n\| \|T_{(n)} - W_{(n)}\| = O(\|\Gamma_n\|\sqrt{\log n}) = o(c_n), \tag{3.21}$$

where we have used the fact that $\|\Gamma_n\|^2 \leq \mathbb{E}[|X|^2 I\{|X| \leq c_n\}] \leq c_n \mathbb{E}|X|$ and (2.2). Combining (3.20) and (3.21), we obtain the assertion and the theorem has been proved.

4. Proofs of the corollaries

4.1. **Proof of Corollary 2.2.** We need the following lemma.

Lemma 4.1. *Let $X : \Omega \to \mathbb{R}^d$ be a mean zero random vector with $\mathbb{E}|X|^2 < \infty$. Assume that (2.1) holds, where c_n is a non-decreasing sequence of positive real numbers such that $c_n/\sqrt{n}$ is eventually non-increasing. Then we have for Γ_n defined as in Theorem 2.1,*

$$\|\Gamma_n^2 - \text{cov}(X)\| = o(c_n^2/n) \text{ as } n \to \infty.$$

Proof. We have,

$$\begin{aligned}\|\Gamma_n^2 - \text{cov}(X)\| &= \sup_{|v|\leq 1} \langle v, (\text{cov}(X) - \Gamma_n^2)v\rangle \\ &= \sup_{|v|\leq 1} \mathbb{E}[\langle v, X\rangle^2 I\{|X| > c_n\}] \\ &\leq \mathbb{E}[|X|^2 I\{|X| > c_n\}].\end{aligned} \tag{4.1}$$

Furthermore, using the fact that c_m^2/m is eventually non-increasing, we get for large n,

$$\begin{aligned}\mathbb{E}[|X|^2 I\{|X| > c_n\}] &\leq \sum_{k=n}^{\infty} c_{k+1}^2 \mathbb{P}\{c_k < |X| \leq c_{k+1}\} \\ &\leq \frac{c_n^2}{n} \sum_{k=n}^{\infty} (k+1)\mathbb{P}\{c_k < |X| \leq c_{k+1}\},\end{aligned}$$

which is of order $o(c_n^2/n)$ since the series $\sum_{k=1}^{\infty} k\mathbb{P}\{c_k < |X| \leq c_{k+1}\}$ converges by (2.1). □

We next show that $\|\Gamma_n - \Gamma\|$ is of the same order. This is trivial in dimension 1, but in higher dimensions one needs some extra arguments.

Lemma 4.2. *Let Γ be the positive semidefinite symmetric matrix satisfying $\Gamma^2 =$ cov(X). Under the assumptions of Lemma 4.1 we have:*

$$\|\Gamma_n - \Gamma\| = o(c_n^2/n) \text{ as } n \to \infty.$$

Proof. We first look at the case where cov(X) is not positive definite.
Set d_1 =rank(cov(X)) and choose an orthonormal basis $\{v_1, \ldots, v_d\}$ of $\mathbb{R}^d$ consisting of eigenvectors of cov(X), where the vectors $v_i, i > d_1$ correspond to the eigenvalue

0. Let S be the orthogonal matrix with column vectors $v_1, \dots, v_d$. Then we clearly have,

$$S^t \mathrm{cov}(X) S = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array} \right),$$

where C is a positive definite symmetric (d_1, d_1)-matrix. (C is actually a diagonal matrix). Choosing the unique positive definite symmetric (d_1, d_1)-matrix $\bar{C}$ such that $\bar{C}^2 = C$, we readily obtain (by unicity of the square root matrix) that

$$\Gamma = S \left(\begin{array}{c|c} \bar{C} & 0 \\ \hline 0 & 0 \end{array} \right) S^t.$$

Noticing that $\mathbb{E}[\langle X, v_j \rangle^2] = 0, j > d_1$, we see that we have also for the matrices Γ_n^2,

$$S^t \Gamma_n^2 S = \left(\begin{array}{c|c} C_n & 0 \\ \hline 0 & 0 \end{array} \right),$$

where C_n are positive semidefinite symmetric (d_1, d_1)-matrices (not necessarily diagonal). This implies that

$$\Gamma_n = S \left(\begin{array}{c|c} \bar{C}_n & 0 \\ \hline 0 & 0 \end{array} \right) S^t,$$

where $\bar{C}_n$ are the positive semidefinite symmetric matrices satisfying $\bar{C}_n^2 = C_n$.
If $\mathrm{cov}(X)$ is positive definite, we set $\bar{C} = \Gamma, C = \mathrm{cov}(X), \bar{C}_n = \Gamma_n, C_n = \Gamma_n^2, n \geq 1$.
Using Theorem X.1.1 in [2] we can conclude from Lemma 4.1 that

$$\|\bar{C}_n - \bar{C}\| = \|\Gamma_n - \Gamma\| \leq \|\Gamma_n^2 - \mathrm{cov}(X)\|^{1/2} \to 0 \text{ as } n \to \infty.$$

This implies that $\bar{C}_n$ is positive definite for large n. Moreover, we have that the smallest eigenvalue λ_n of $\bar{C}_n$ converges to that one of $\bar{C}$ which is equal to the smallest positive eigenvalue of Γ. If we denote this eigenvalue by λ we find that $\lambda_n \geq \lambda/2 > 0$ for large n.
Applying Theorem X.3.7. in [2] (with $A = C_n, B = C$ and $\alpha = \lambda^2/4$) we see that for large n,

$$\|\Gamma_n - \Gamma\| = \|\bar{C}_n - \bar{C}\| \leq \lambda^{-1} \|C_n - C\| = \lambda^{-1} \|\Gamma_n^2 - \mathrm{cov}(X)\|,$$

which in conjunction with Lemma 4.1 implies the above assertion. □

Now we can conclude the proof of Corollary 2.2 by a simple application of the triangular inequality. Just observe that by Theorem 2.1, with probability one

$$\begin{aligned} \|S_{(n)} - \Gamma \cdot W_{(n)}\| &\leq \|S_{(n)} - \Gamma_n \cdot W_{(n)}\| + \|(\Gamma_n - \Gamma) \cdot W_{(n)}\| \\ &\leq o(c_n) + \|\Gamma_n - \Gamma\| \|W_{(n)}\| \end{aligned}$$

Note that we can apply Theorem 2.1 since we are assuming that $c_n/\sqrt{n}$ is eventually non-increasing and we thus have for some $m_0 \geq 1$, $c_n/\sqrt{n} \leq c_m/\sqrt{m}, m_0 \leq m \leq n$ which implies that condition (2.3) holds.
By the law of the iterated logarithm for Brownian motion we have with probability one,

$$\|\Gamma_n - \Gamma\| \|W_{(n)}\| = O(\|\Gamma_n - \Gamma\| \sqrt{n \log \log n})$$

which is in view of Lemma 4.2 of order $o(c_n^2/\sqrt{n/\log \log n})$.
Since $W_{(n)}/\sqrt{n} \stackrel{d}{=} \overline{W}$, where $\overline{W}(t), 0 \leq t \leq 1$ is the Brownian motion on the

compact interval $[0,1]$, we also have,

$$\|\Gamma_n - \Gamma\| \|W_{(n)}\| = O_{\mathbb{P}}(\|\Gamma_n - \Gamma\|\sqrt{n}) = o_{\mathbb{P}}(c_n^2/\sqrt{n}) = o_{\mathbb{P}}(c_n).$$

Corollary 2.2 has been proved.

4.2. **Proof of Corollary 2.4.** We shall use the following d-dimensional version of Lemma 3 of [6]. The proof is almost the same as in dimension 1 and it is omitted. Recall that $H(c_n) = \sup\{\mathbb{E}[\langle v, X\rangle^2 I\{|X| \le c_n\}] : |v| \le 1\} = \|\Gamma_n\|^2, n \ge 1$.

Lemma 4.3. *Let $X : \Omega \to \mathbb{R}^d$ be a mean zero random vector and assume that condition (2.1) holds for a sequence c_n of positive real numbers such that $c_n/\sqrt{n}$ is non-decreasing. Whenever $n_k \nearrow \infty$ is a subsequence satisfying for large enough k,*

$$1 < a_1 < n_{k+1}/n_k \le a_2 < \infty,$$

we have:

$$\sum_{k=1}^{\infty} \exp\left(-\frac{\alpha^2 c_{n_k}^2}{2n_k \|\Gamma_{n_k}\|^2}\right) \begin{cases} = \infty & \text{if } \alpha < \alpha_0, \\ < \infty & \text{if } \alpha > \alpha_0. \end{cases} \tag{4.2}$$

4.2.1. *The upper bound part.* W.l.o.g. we can assume that $\alpha_0 < \infty$.
We first show that under the assumptions of the corollary we have with probability one,

$$\limsup_{n\to\infty} |S_n|/c_n \le \alpha_0. \tag{4.3}$$

To that end it is sufficient to show that we have for any $\delta > 0$ and $n_k = n_k(\delta) = [(1+\delta)^k], k \ge 1$ with probability one,

$$\limsup_{k\to\infty} \max_{1\le n\le n_k} |S_n|/c_{n_k} \le \alpha_0 \tag{4.4}$$

Note that we trivially have,

$$\max_{n_{k-1}\le n\le n_k} |S_n|/c_n \le (c_{n_k}/c_{n_{k-1}}) \max_{1\le n\le n_k} |S_n|/c_{n_k}.$$

Moreover, it follows from condition (2.3) and the definition of n_k that

$$\limsup_{k\to\infty} c_{n_k}/c_{n_{k-1}} \le \limsup_{k\to\infty} n_k/n_{k-1} = 1+\delta.$$

Combining these two observations with (4.4) we get for any $\delta > 0$ with probability one,

$$\limsup_{n\to\infty} |S_n|/c_n \le \alpha_0(1+\delta),$$

which clearly implies (4.3).
In view of our strong invariance principle, (4.4) follows if we can show that with probability one,

$$\limsup_{k\to\infty} \|\Gamma_{n_k} \cdot W_{(n_k)}\|/c_{n_k} \le \alpha_0. \tag{4.5}$$

In order to prove the last relation, we need a deviation inequality for $\max_{0\le t\le 1} |W(t)|$. The following simple (suboptimal) inequality will be sufficient for our purposes.

Lemma 4.4. *Let $\{W(t) : t \ge 0\}$ be a standard d-dimensional Brownian motion and let δ be a positive constant. Then there exists a constant $C_\delta = C_\delta(d) > 0$ which depends only on δ and d such that*

$$\mathbb{P}\left\{\max_{0\le t\le 1} |W(t)| \ge u\right\} \le C_\delta \exp(-u^2/(2+2\delta)), u \ge 0 \tag{4.6}$$

Proof. Since $W \stackrel{d}{=} -W$, we can infer from the Lévy inequality that for $u \geq 0$,

$$\mathbb{P}\left\{\max_{0\leq t\leq 1} |W(t)| \geq u\right\} \leq 2\mathbb{P}\left\{|W(1)| \geq u\right\}$$

The random variable $|W(1)|^2$ has a chi-square distribution with d degrees of freedom and thus we have

$$\begin{aligned}\mathbb{P}\left\{|W(1)| \geq u\right\} &= 2^{-d/2}\Gamma(d/2)^{-1}\int_{u^2}^{\infty} x^{d/2-1}\exp(-x/2)dx \\ &\leq Ku^{d-2}\exp(-u^2/2), u \geq 1,\end{aligned}$$

where $K > 0$ is a constant depending on d only.
Obviously we can find a positive constant C'_δ so that the last term is bounded above by

$$\leq C'_\delta \exp(-u^2/(2+2\delta)).$$

Setting $C_\delta = 2C'_\delta \vee e^{1/(2+2\delta)}$, we see that inequality (4.6) holds for any $u \geq 0$ and the lemma has been proved. □

We are ready to prove (4.5). Let $\delta > 0$ be fixed and set $\alpha_\delta = (1+\delta)(\alpha_0+\delta)$. Recall that $(W_{(n)}(t)/\sqrt{n})_{0\leq t\leq 1} \stackrel{d}{=} (W(t))_{0\leq t\leq 1}$. Then we can infer from (4.2) that

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{P}\{\|\Gamma_{n_k}\cdot W_{(n_k)}\| \geq \alpha_\delta c_{n_k}\} &\leq \sum_{k=1}^{\infty}\mathbb{P}\{\|\Gamma_{n_k}\|\|W_{(n_k)}\| \geq \alpha_\delta c_{n_k}\} \\ &\leq C_\delta \sum_{k=1}^{\infty}\exp\left(-\frac{(1+\delta)(\alpha_0+\delta)^2 c_{n_k}^2}{2n_k\|\Gamma_{n_k}\|^2}\right) < \infty.\end{aligned}$$

This implies via the Borel-Cantelli lemma that with probability one,

$$\limsup_{k\to\infty} \|\Gamma_{n_k}\cdot W_{(n_k)}\|/c_{n_k} \leq (1+\delta)(\alpha_0+\delta).$$

Since this holds for any $\delta > 0$ we get (4.5) and consequently (4.3).

4.2.2. *The lower bound part.* We assume that $\alpha_0 > 0$. Otherwise, there is nothing to prove.
Furthermore, we can assume that $c_n/\sqrt{n} \to \infty$. If $c_n = O(\sqrt{n})$, then we have $\alpha_0 = \infty$ unless of course $X = 0$ with probability one. Applying Corollary 2.4 with $c_n = \sqrt{n}(\log\log n)^{1/4}$, it follows that even $\limsup_{n\to\infty}|S_n|/(\sqrt{n}(\log\log n)^{1/4}) = \infty$ if X is non-degenerate. This trivially implies Corollary 2.4 for any sequence c_n of order $O(\sqrt{n})$.
We need the following lemma. Since the proof is almost identical with that one in the 1-dimensional case (see Lemma 1, [7]) it is omitted. An inspection of this proof also reveals that one needs not assume that X has a finite mean and thus we have,

Lemma 4.5. *Let $X : \Omega \to \mathbb{R}^d$ be a random vector satisfying condition (2.1) for a sequence c_n of positive real numbers such that $c_n/\sqrt{n}$ is non-decreasing and converges to infinity. Then we have,*

$$\mathbb{E}[|X|^2 I\{|X| \leq c_n\}] = o(c_n^2/n) \text{ as } n \to \infty. \tag{4.7}$$

Let $\delta \in]0,1[$ be fixed and $m \geq 1+\delta^{-1}$ a natural number. Consider the subsequence $n_k = m^k, k \geq 1$. We first show that if $0 < \alpha(1+\delta) < \alpha_0$ we have with probability one,

$$\limsup_{k\to\infty} |S_{n_{k+1}} - S_{n_k}|/c_{n_{k+1}} \geq \alpha. \tag{4.8}$$

Rewriting $S_{n_{k+1}} - S_{n_k}$ as $S_{(n_{k+1})}(1) - S_{(n_{k+1})}(1/m)$, we see that Theorem 2.1 implies that (4.8) holds if and only if one has with probability one,

$$\limsup_{k\to\infty} |\Gamma_{n_{k+1}} \cdot (W(n_{k+1}) - W(n_k))|/c_{n_{k+1}} \geq \alpha. \tag{4.9}$$

Consider the independent events

$$A_k := \{|\Gamma_{n_{k+1}} \cdot (W(n_{k+1}) - W(n_k))| \geq \alpha c_{n_{k+1}}\}, k \geq 1.$$

As $\|\Gamma_{n_{k+1}}\|$ is the largest eigenvalue of $\Gamma_{n_{k+1}}$, we can find an orthonormal vector $v_{k+1} \in \mathbb{R}^d$ so that $\Gamma_{n_{k+1}} v_{k+1} = \|\Gamma_{n_{k+1}}\| v_{k+1}$ and we can conclude that

$$\begin{aligned}\mathbb{P}(A_k) &\geq \mathbb{P}\{|\langle v_{k+1}, \Gamma_{n_{k+1}} \cdot (W(n_{k+1}) - W(n_k))\rangle| \geq \alpha c_{n_{k+1}}\} \\ &= \mathbb{P}\{\|\Gamma_{n_{k+1}}\| \sqrt{n_{k+1} - n_k} |Z| \geq \alpha c_{n_{k+1}}\},\end{aligned}$$

where $Z : \Omega \to \mathbb{R}$ is standard normal.
Employing the trivial inequality $\mathbb{P}\{|Z| \geq t\} \geq \exp(-t^2(1+\delta)/2), t \geq t_\delta$, where t_δ is a positive constant depending on δ only, we see that for large k,

$$\mathbb{P}(A_k) \geq \exp\left(-\frac{\alpha^2(1+\delta)c^2_{n_{k+1}}}{2(n_{k+1} - n_k)\|\Gamma_{n_{k+1}}\|^2}\right).$$

We can apply the above inequality for large k since by Lemma 4.5

$$\|\Gamma_n\|^2 \leq \mathbb{E}[|X|^2 I\{|X| \leq c_n\}] = o(c_n^2/n) \text{ as } n \to \infty$$

and, consequently,

$$c_{n_{k+1}}/(\sqrt{n_{k+1} - n_k}\|\Gamma_{n_{k+1}}\|) \to \infty \text{ as } k \to \infty.$$

Since we have chosen $m \geq 1+\delta^{-1}$, it follows that $n_{k+1} - n_k = n_{k+1}(1-1/m) \geq n_{k+1}(1+\delta)^{-1}$. We can conclude that for large enough k,

$$\mathbb{P}(A_k) \geq \exp\left(-\frac{\alpha^2(1+\delta)^2 c^2_{n_{k+1}}}{2n_{k+1}\|\Gamma_{n_{k+1}}\|^2}\right),$$

and, consequently, we have on account of (4.2),

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty.$$

Using the Borel-Cantelli lemma, we see that (4.9) holds which in turn implies (4.8).
If $\boxed{\alpha_0 = \infty}$, we use the trivial inequality

$$\limsup_{k\to\infty} |S_{n_{k+1}} - S_{n_k}|/c_{n_{k+1}} \leq 2 \limsup_{k\to\infty} |S_{n_k}|/c_{n_k},$$

which in conjunction with (4.8) (where we set $\delta = 1/2$ and $m = 3$) implies that we have for any $\alpha > 0$ with probability one,

$$\limsup_{k\to\infty} |S_{n_k}|/c_{n_k} \geq \alpha/2.$$

It is now obvious that $\limsup_{n\to\infty} |S_n|/c_n = \alpha_0 = \infty$ with probability one.

If $\boxed{\alpha_0 < \infty}$ we get from the upper bound part and the definition of n_k with probability one,

$$\limsup_{k\to\infty} |S_{n_k}|/c_{n_{k+1}} \le \alpha_0 \limsup_{k\to\infty} c_{n_k}/c_{n_{k+1}} \le 2\alpha_0/\sqrt{m}.$$

Combining this with (4.8) we see that we have if $\alpha(1+\delta) < \alpha_0$ for any $m \ge 1+\delta^{-1}$ with probability one,

$$\limsup_{n\to\infty} |S_n|/c_n \ge \alpha - 2\alpha_0/\sqrt{m}.$$

Since we can make δ arbitrarily small, we see that $\limsup_{n\to\infty} |S_n|/c_n \ge \alpha_0$ with probability one and Corollary 2.4 has been proved.

4.3. **Proof of Corollary 2.5.** We only show how (b) implies (a) and we do this if $p > 1$. For the implication "(a) $\Rightarrow$ (b)" we refer to [7]. We need another lemma.

Lemma 4.6. *Let $X : \Omega \to \mathbb{R}^d$ be a random vector and set*

$$\tilde{H}(t) = \mathbb{E}[|X|^2 I\{|X| \le t\}] \vee 1, t \ge 0$$

Then we have for any $\delta > 0 : \mathbb{E}[|X|^2/(\tilde{H}(|X|))^{1+\delta}] < \infty$.

Proof. Without loss of generality we can assume that $\mathbb{E}|X|^2 = \infty$ and consequently that $\bar{H}(t) \to \infty$ as $t \to \infty$, where $\bar{H}(t) = \mathbb{E}[|X|^2 I\{|X| \le t\}], t \ge 0$. Obviously, $\bar{H}$ is right continuous and non-decreasing. Therefore there exists a unique Lebesgue-Stieltjes measure μ on the Borel subsets of $\mathbb{R}_+$ satisfying,

$$\mu(]a,b]) = \bar{H}(b) - \bar{H}(a), 0 \le a < b < \infty.$$

Let G be the generalized inverse function of $\bar{H}$, i.e.

$$G(u) = \inf\{x \ge 0 : \bar{H}(x) \ge u\}, 0 < u < \infty.$$

As $\bar{H}$ is right continuous, the above infimum is actually a minimum. In particular we have $\bar{H}(G(u)) \ge u, u > 0$. Moreover:

$$G(u) \le x \iff u \le \bar{H}(x). \tag{4.10}$$

Let $\bar{\lambda}$ the Lebesgue measure on the Borel subsets of $\mathbb{R}_+$. From (4.10) it easily follows that μ is equal to the image measure $\bar{\lambda}_G$.
Next set $\alpha = G(1)$ so that $\tilde{H}(x) = 1, x < \alpha$ and $\bar{H}(x) = \tilde{H}(x), x \ge \alpha$. It trivially follows that

$$\mathbb{E}\left[|X|^2/\tilde{H}(|X|)^{1+\delta}\right] \le \mathbb{E}\left[|X|^2 I\{|X| \le \alpha\}\right] + \int_{]\alpha,\infty[} \bar{H}(x)^{-1-\delta}\mu(dx).$$

The first term is obviously finite. As for the second term we have

$$\begin{aligned}\int_{]\alpha,\infty[} \bar{H}(x)^{-1-\delta}\mu(dx) &= \int_{]\alpha,\infty[} \bar{H}(x)^{-1-\delta}\bar{\lambda}_G(dx)\\ = \int_{\bar{H}(\alpha)}^{\infty} \bar{H}(G(u))^{-1-\delta}du &\le \int_1^{\infty} u^{-1-\delta}du < \infty\end{aligned}$$

and the lemma has been proved. □

As we trivially have $\bar{H}(t) \le dH(t), t \ge 0$, we get from (b) that $\tilde{H}(t) = O((LLt)^{p-1})$ as $t \to \infty$ and we readily obtain that for some positive constant C,

$$\mathbb{E}[|X|^2/(LL|X|)^p] \le C\mathbb{E}[|X|^2/(\tilde{H}(|X|))^{p/(p-1)}]$$

which is finite in view of Lemma 4.6.
Consequently, we have,

$$\sum_{n=1}^{\infty} \mathbb{P}\{|X| \geq \sqrt{n(LLn)^p}\} < \infty.$$

We can apply Corollary 2.4 with $c_n = \sqrt{n(LLn)^p}$ and we see that with probability one,

$$\limsup_{n\to\infty} |S_n|/\sqrt{2n(LLn)^p} = \alpha_0/\sqrt{2},$$

where

$$\alpha_0 = \sup\left\{\alpha \geq 0 : \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-\frac{\alpha^2 (LLn)^p}{2H(\sqrt{n(LLn)^p})}\right) = \infty\right\}.$$

It remains to show that $\alpha_0 = \lambda\sqrt{2}$.
Consider $\alpha = \lambda_2\sqrt{2}$, where $\lambda_2 > \lambda$. If $\lambda_1 \in]\lambda, \lambda_2[$, we clearly have by (b) for large n,

$$H(\sqrt{n(LLn)^p}) \leq \lambda_1^2 (LLn)^{p-1}$$

and it follows that

$$\frac{1}{n} \exp\left(-\frac{\alpha^2 (LLn)^p}{2H(\sqrt{n(LLn)^p})}\right) \leq \frac{1}{n(Ln)^{(\lambda_2/\lambda_1)^2}}.$$

which leads to a convergent series. Thus, we have $\alpha_0 \leq \lambda\sqrt{2}$.
As for the opposite inequality, we can and do assume that $\lambda > 0$.
Consider $\alpha = \beta_1\sqrt{2}$, where $0 < \beta_1 < \lambda$. Let further β_2, β_3 be positive numbers such that $\beta_1 < \beta_2 < \beta_3 < \lambda$.
Choose a sequence $t_k \uparrow \infty$ such that

$$H(t_k) \geq \lambda^2 (1 - 1/k)(LLt_k)^{p-1}.$$

Set

$$m_k = \min\{m : t_k \leq \sqrt{m(LLm)^p}\}.$$

It is easy to see that $t_k \sim \sqrt{m_k(LLm_k)^p}$ as $k \to \infty$ and we thus have for large k,

$$H(\sqrt{m_k(LLm_k)^p}) \geq \beta_3^2 (LLm_k)^{p-1},$$

Here we have used the fact that $LLt \sim LL(t^2)$ as $t \to \infty$, from which we can also infer that for large k,

$$(LLn)^p \leq (\beta_2/\beta_1)^2 (LLm_k)^p, m_k \leq n \leq m_k^2 =: n_k.$$

Recalling that $\alpha = \sqrt{2}\beta_1$ we get for large k,

$$\sum_{n=m_k}^{n_k} \frac{1}{n} \exp\left(-\frac{\alpha^2 (LLn)^p}{2H(\sqrt{n(LLn)^p})}\right) \geq \sum_{n=m_k}^{n_k} \frac{1}{n(Lm_k)^{(\beta_2/\beta_3)^2}} \geq (Lm_k)^{1-(\beta_2/\beta_3)^2}.$$

The last term converges to infinity and thus the series in the definition of α_0 diverges, which means that $\alpha_0 \geq \beta_1\sqrt{2}$ for any $\beta_1 < \lambda$. Thus we have $\alpha_0 \geq \lambda\sqrt{2}$ and the corollary has been proved.

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