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Two-boundary problems for semi-Markov walk with a linear drift

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Abstract. In this article we determine the joint distribution of the first exit time from an interval and the value of the overshoot and the joint distribution of the supremum, infimum and the value of the semi-Markov walk with a linear drift. The results obtained are applied for the case when the jumps of the process are exponential. The corresponding distributions is found in terms of the resolvent of the process. We established the weak convergence of the distributions two-boundary functionals of the semi-Markov walk with a linear drift to the corresponding functionals of the Wiener process.

Key words. Key words; the semi-Markov walk with a linear drift, exit from an interval, the joint distribution of the supremum, infimum and the value of the process.

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1. Introduction

One of the most important two-boundary functionals of the process is the joint distribution of the first exit time from an interval and the value of the overshoot by the process at the instant of the first exit. This distribution was determined for Lévy processes in [1]. The authors suggested to use one-boundary functionals of the process, i.e. the joint distributions of the first passage time of the positive (negative) level and the value of the overshoot by the process. Their approach is based on employing the probability methods (total probability formula, homogeneity and strong Markov property of the process). In order to determine the Laplace transforms of the joint distribution of the first exit time and the value of the overshoot, the authors set up and solve a system of linear integral equations (see for instance [1]).

As a result the Laplace transforms of the first exit time and the value of the overshoot are given only in terms of the joint distributions of the one-boundary functionals of the process. This fact allows us to solve another two-boundary problems for Lévy a linear component to the processes mentioned we obtain the processes which are homogeneous with respect tot the first component [4]. In this paper we will solve another two-boundary problem for the semi-Markov walk with a linear drift, i.e. we will determine the joint distribution of the infimum, the supremum and the value of the process. The results obtained can be applied in queueing theory and financial mathematics.

2. Definitions and auxiliary results

Let $\eta, \delta \in (0,\infty)$, be independent random variables and $F(x) = \mathbf{P}[\eta \le x]$, $x \in \mathbb{R}_+ = [0,\infty)$ be a continuous distribution function of η. We will suppose that $\mathbf{E}\eta,\ \mathbf{E}\delta < \infty$. Introduce a random variable $\xi \doteq \eta - \delta \in \mathbb{R}$, and define the random sequences $\{\eta, \eta'_n\}$, $\{\delta, \delta'_n\}$, $\{\xi, \xi'_n\}$, $n \in \mathbb{N} = \{1, 2, \dots\}$ of independent identically distributed random variables. The symbol $\doteq$ means the equity of the corresponding distributions. Define a random walk $\{\delta_n\}_{n\in\mathbb{N}\cup 0} : \delta_0 = 0$, $\delta_n = \delta'_1 + \cdots + \delta'_n$ and random sequences

$$\eta_0(x) = 0, \quad \eta_1(x) = \eta_x, \quad \eta_{n+1}(x) = \eta_x + \eta'_1 + \cdots + \eta'_n, \qquad n \in \mathbb{N},$$
$$\xi_0(x) = 0, \quad \xi_1(x) = \eta_x - \delta, \quad \xi_{n+1}(x) = \xi_1(x) + \xi'_1 + \cdots + \xi'_n, \qquad n \in \mathbb{N},$$

where $\eta_x \in (0,\infty)$ is a random variable with the distribution function

$$F_x(u) = \mathbf{P}[\eta_x \le u] = (F(x+u) - F(x))(1 - F(x))^{-1}, \quad u \in \mathbb{R}_+.$$

For all $t \ge 0$ define a renewal process generated by a sequence $\{\eta_n(x)\}_{n\in\mathbb{N}\cup 0}$:

$$N_x(t) = \max\{\, n \in \mathbb{N}\cup 0 :\ \eta_n(x) \le t \,\} \in \mathbb{N}\cup 0, \qquad x \in \mathbb{R}_+. \tag{2.1}$$

For $x \in \mathbb{R}_+$ introduce a right-continuous random process

$$\{Z_x(t)\}_{t\ge 0} = \left\{t - \delta_{N_x(t)}\right\}_{t\ge 0} \in \mathbb{R}, \qquad Z_x(0) = 0. \tag{2.2}$$

Observe that the sample paths of this process increase linearly on the intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{N}\cup 0$ and negative jumps of the process (2.2) occur at the time instants $\eta_{n+1}(x)$. The value of the jumps is identically distributed with δ'_{n+1}. We will call the process $\{Z_x(t)\}_{t\ge 0}$ defined by (2.2) a semi-Markov random walk with a linear drift [5]. Note, that the process $\{Z_x(t)\}_{t\ge 0}$ is not a Markov process in general (besides the case when η is exponentially distributed). Therefore, we define a linear component is the time elapsed since the last jump instant of the process $\{Z_x(t)\}_{t\ge 0}$ up to time a t_0. Adding a linear component (2.3) to the process (2.2), we obtain a right-continuous Markov process

$$\{X_t\}_{t\ge 0} = \left\{Z_x(t),\, \eta_x^+(t)\right\}_{t\ge 0} \in \mathbb{R}\times\mathbb{R}_+, \qquad X_0 = \{0, x, \}, \quad x\in\mathbb{R}_+, \quad (2.4)$$

which governs the process $\{Z_x(t)\}_{t\ge 0}$. The process (2.4) is a Markov process homogeneity with respect to the first component [4]. In other words, if $X_{t_0} = \{k, u\}$, $k \in \mathbb{R}$, $u \in \mathbb{R}_+$ for some t_0, then the further evolution of the process a $\{X_t\}_{t\ge t_0}$ does not depend on the value k of the first component. After alapse of a t_0 the first component of the process $\{X_t\}_{t\ge t_0}$ increases linearly and the first jump of the process (which is identically distributed with δ) will occur after elapsing of a random time η_u. Observe, that the random points $\{\eta_n(x)\}_{n\in\mathbb{N}}$ are the switching points of the renewal process $\{N_x(t)\}_{t\ge 0}$ and they are the times at which $\{\eta_x^+(t)\}_{t\ge 0}$ is killed to zero. Moreover, $\{\eta_n(x)\}_{n\in\mathbb{N}}$ are the Markov times of the processes (2.1)-(2.4). We would like to emphasize that the property of homogeneity with respect to the first component of $\{X_t\}_{t\ge 0}$, and the Markov property of the random times $\{\eta_n(x)\}_{n\in\mathbb{N}\cup 0}$ will be used throughout the paper. We now introduce an auxiliary process which will be used in the sequel. Define

$$\{S_x(t)\}_{t\ge 0} = \left\{\xi_{N_x(t)}(x)\right\}_{t\ge 0} \in \mathbb{R}, \qquad S_x(0) = 0, \quad x\in\mathbb{R}_+. \quad (2.5)$$

The process (2.5) is a right-continuous step process. The sample paths of this process are the constants on the intervals $[\eta_n(x), \eta_{n+1}(x))$, $n\in\mathbb{N}\cup 0$ and the jumps $\xi \doteq \eta - \delta$, $n\in\{2,3,\dots\}$ (or $\xi_1(x) \doteq \eta_x - \delta$ for $n = 1$) of the process (2.5) occur at the instants $\eta_n(x)$, $n\in\mathbb{N}$. We will call the process $\{S_x(t)\}_{t\ge 0}$ a semi-Markov random walk [5], generated by the random sequence $\{\eta_n(x)\}$, $\{\xi_n(x)\}$, $n\in\mathbb{N}\cup 0$.

Proposition 2.1. *Let $N_x(t)$, $Z_x(t)$, $\eta_x^+(t)$, $S_x(t)$, $t\ge 0$, be random processes defined by the formulae (2.1)-(2.3), (2.5), and $\nu_s \sim \exp(s)$ be an exponential random variable independent of these processes, $\tilde f_x(s) = \mathbf{E}\exp\{-s\eta_x\}$, $\tilde f(s) = \tilde f_0(s) = \mathbf{E}\exp\{-s\eta\}$. Then for all $x\in\mathbb{R}_+$, $s>0$, $|\theta|\le 1$, $\Re(a) = \Re(c) = 0$, $\Re(b)\ge 0$ the following equality holds*

$$\mathbf{E}\theta^{N_x(\nu_s)} e^{-aZ_x(\nu_s) - b\eta_x^+(\nu_s) - cS_x(\nu_s)} = \frac{s}{s+a+b}(1-\tilde f_x(s+a+b))e^{-bx}$$

Proof. It is not difficult to establish that the mathematical expectation

$$E_x^s(\theta,a,b,c)=\mathbf{E}\theta^{N_x(\nu_s)}\exp\{-aZ_x(\nu_s)-b\eta_x^+(\nu_s)-cS_x(\nu_s)\}$$

obeys the following equation for $x\in\mathbb{R}_+$, $s>0$, $|\theta|\leq 1$, $\Re(a)=\Re(c)=0$, $\Re(b)\geq 0$

$$E_x^s(\theta,a,b,c)=s\frac{1-\tilde{f}_x(s+a+b)}{s+a+b}e^{-bx}+\theta\tilde{f}_x(s+a+c)\mathbf{E}e^{(a+c)\delta}E_0^s(\theta,a,b,c).$$

In order to write this equation we used the total probability formula, independence of the random variables δ and $\eta_1(x)$, homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component and the fact that the random time $\eta_1(x)$ is a Markov time. Setting $x=0$ in this equations we get

$$E_0^s(\theta,a,b,c)=\frac{s}{s+a+b}\,\frac{1-\tilde{f}(s+a+b)}{1-\theta\tilde{f}(s+a+c)\mathbf{E}e^{(a+c)\delta}}.$$

Substituting the expression for $E_0^s(\theta,a,b,c)$ into the previous equality we get (2.6). The formula (2.7) follows from (2.6), for $\theta=1$, $a=b=0$. □

We now introduce one-boundary functionals of the semi-Markov walk $\{S_x(t)\}_{t\geq 0}$ (2.5). For $x,k\in\mathbb{R}_+$ denote by

$$\tilde{\tau}^k(x)=\inf\{t:S_x(t)>k\},\qquad \tilde{T}^k(x)=S_x(\tilde{\tau}^k(x))-k,$$

$\tilde{\tau}^k=\tilde{\tau}^k(0)$, $\tilde{T}^k=\tilde{T}^k(0)$, the first overshoot time of the positive level k by the random walk $\{S_x(t)\}_{t\geq 0}$ and the value of the overshoot at this instant;

$$\tilde{\tau}_k(x)=\inf\{t:S_x(t)<-k\},\qquad \tilde{T}_k(x)=-S_x(\tilde{\tau}_k(x))-k,$$

$\tilde{\tau}_k=\tilde{\tau}_k(0)$, $T_k=\tilde{T}_k(0)$, the first overshoot time of the negative level $-k$ by the random walk $\{S_x(t)\}_{t\geq 0}$, and the value of the overshoot at this instant. Random variables $\tilde{\tau}^k(x)$, $\tilde{\tau}_k(x)$ take their values from a countable set $\{\eta_n(x),\,n\in\mathbb{N}\}$ and they are Markov times of the process $\{S_x(t)\}_{t\geq 0}$. It is worth of mentioning that there is no principal difference between a semi-Markov walk and a 'usual' random walk with respect to studying one-boundary characteristics. In order to illustrate this, we now derive for $\{S_x(t)\}_{t\geq 0}$ analogues of the well-known results of [18], [17], [16] for random

(i) *the semi-Markov walk* $\{S_0(t)\}_{t\geq 0}$ *admits the Spitzer-Rogozin factorization [18],[17]*

$$\mathbf{E}e^{-pS_0(\nu_s)} = \frac{1-\tilde{f}(s)}{1-\tilde{f}(s+p)\,\mathbf{E}e^{p\delta}} = \mathbf{E}e^{-pS^+_{\nu_s}}\,\mathbf{E}e^{-pS^-_{\nu_s}}, \qquad \Re(p)=0, \quad (2.8)$$

where the variables $-S^-_{\nu_s}, S^+_{\nu_s} \in \mathbb{R}_+$ *are infinitely divisible with the Laplace transforms of the form*

$$\mathbf{E}e^{-pS^{\pm}_{\nu_s}} = \exp\left\{\sum_{n\in\mathbb{N}} \frac{1}{n}\,\mathbf{E}\left[e^{-s\eta_n}\left(e^{-p\xi_n}-1\right);\,\pm\xi_n>0\right]\right\}, \qquad \pm\Re(p)\geq 0;$$

(ii) *the integral transforms of the joint distributions of* $\{\tilde{\tau}^k, \tilde{T}^k\}$, $\{\tilde{\tau}_k, \tilde{T}_k\}$, $k\in\mathbb{R}_+$ *of the semi-Markov walk* $\{S_0(t)\}_{t\geq 0}$ *obey the formulae [16] for* $\Re(p)\geq 0$

$$\mathbf{E}\left[e^{-s\tilde{\tau}^k-p\tilde{T}^k};\tilde{\tau}^k<\infty\right] = \left(\mathbf{E}\,e^{-pS^+_{\nu_s}}\right)^{-1}\mathbf{E}\left[e^{-p(S^+_{\nu_s}-k)};S^+_{\nu_s}>k\right],$$
$$\mathbf{E}\left[e^{-s\tilde{\tau}_k-p\tilde{T}_k};\,\tilde{\tau}_k<\infty\right] = \left(\mathbf{E}\,e^{pS^-_{\nu_s}}\right)^{-1}\mathbf{E}\left[e^{p(S^-_{\nu_s}+k)};\,S^-_{\nu_s}<-k\right]; \qquad (2.9)$$

(iii) *the integral transforms of the joint distributions of* $\{S_0(\nu_s),\, S^{\pm}_{\nu_s}\}$ *satisfy the following equalities for* $k\in\mathbb{R}_+$

$$\mathbf{E}\left[e^{-pS_0(\nu_s)};S^+_{\nu_s}\leq k\right] = \mathbf{E}e^{-pS^-_{\nu_s}}\mathbf{E}\left[e^{-pS^+_{\nu_s}};S^+_{\nu_s}\leq k\right], \qquad \Re(p)\leq 0,$$
$$\mathbf{E}\left[e^{-pS_0(\nu_s)};S^-_{\nu_s}\geq -k\right] = \mathbf{E}e^{-pS^+_{\nu_s}}\mathbf{E}\left[e^{-pS^-_{\nu_s}};S^-_{\nu_s}\geq -k\right], \ \Re(p)\geq 0. \quad (2.10)$$

A simple proof of this lemma is given in [6]. It is based on applying factorization-probability methods.

3. One-boundary characteristics of the process $\{Z_x(t)\}_{t\geq 0}$.

In this section we will determine the one-boundary functionals generated by the first overshoot time of a fixed level by the process $\{Z_x(t)\}_{t\geq 0}$. Let $X_0=\{0,x\}$, $x,k\in\mathbb{R}_+$. Define

$$\tau_k(x)=\inf\{t: Z_x(t)<-k\}, \qquad T_k(x)=-Z_x(\tau_k(x))-k, \qquad \inf\{\emptyset\}=\infty,$$

of the joint distribution of $\{\tau_k(x), T_k(x)\}$ *satisfies the following formulae*

$$\tilde{f}_k^0(z,s) = \left(\mathbf{E}\, e^{zS_{\nu_s}^-}\right)^{-1} \mathbf{E}\left[e^{z(S_{\nu_s}^- + k)};\ S_{\nu_s}^- < -k\right], \tag{3.1}$$

$$\tilde{f}_k^x(z,s) = \mathbf{E}\left[e^{-s\eta_x + z(\xi_1(x)+k)}; \xi_1(x) < -k\right] + \int_{-k}^{\infty} \mathbf{E}\left[e^{-s\eta_x}; \xi_1(x) \in dv\right] \tilde{f}_{k+v}^0(z,s),$$

where $\xi_1(x) \doteq \eta_x - \delta$ *and* $S_{\nu_s}^- = \inf_{t \le \nu_s} S_0(t)$ *is the infimum of the semi-Markov walk* $\{S_0(t)\}_{t \ge 0}$.

Proof. Observe, that the processes $Z_x(t)$, $S_x(t)$ $t \ge 0$ do not decrease on the intervals $[\eta_n(x), \eta_{n+1}(x))$. It follows from the definitions of these processes (2.2), (2.5) that the negative jumps of $Z_x(t),\ S_x(t), t \ge 0$ can only occur at time points $\{\eta_n(x),\ n \in \mathbb{N}\}$. It also follows from (2.2), (2.5) that

$$Z_x(\eta_n(x)) = S_x(\eta_n(x)) = \eta_n(x) - \delta_n, \qquad n \in \mathbb{N} \cup 0.$$

Thus, the first overshoot time $\tilde{\tau}_k(x)$ of the negative level $-k$ and the value of the overshoot $\tilde{T}_k(x)$ through this level by the semi-Markov walk $\{S_x(t)\}_{t \ge 0}$ coincide in distribution with the first overshoot time $\tau_k(x)$ and the value of the overshoot $T_k(x)$ by the process $\{Z_x(t)\}_{t \ge 0}$. The first equality of (3.1) follows straightforwardly from the second formula of (2.9). In order to get the second formula of (3.1) we used the total probability formula, the Markov property of $\eta_1(x) \doteq \eta_x$ and homogeneity of the process $\{X_t\}_{t \ge 0}$ with respect to the first component. □

We now consider a more interesting problem, i.e. we will determine the integral transforms of the joint distributions of the upper one-boundary functionals of the process $\{Z_x(t)\}_{t \ge 0}$. Let $X_0 = \{0, x\}$, $x, k \in \mathbb{R}_+$. Introduce a random variables

$$\tau^k(x) = \inf\{t : Z_x(t) > k\}, \qquad \eta^k(x) = \eta_x^+(\tau^k(x)),$$

i.e. the first overshoot time of the positive level k by the process $\{Z_x(t)\}_{t \ge 0}$ and the value of the linear component at the instant of the first overshoot of the positive level. We will suppose that $\inf\{\emptyset\} = \infty$ and on the event $\{\tau^k(x) = \infty\}$ we set per definition $\eta^k(x) = \infty$. For $\Re(p) \ge 0$, $s > 0$, $\mathfrak{B}^k(x) = \{\tau^k < \infty\}$ denote

$$C_x^p(dl,s) = \int_0^\infty e^{-pk} \mathbf{E}\left[e^{-s\tau^k(x)}; \eta^k(x) \in dl, \mathfrak{B}^k(x)\right] dk, \quad C_x^p(s) = \int_0^\infty C_x^p(dl,s).$$

Theorem 3.2 ([6]). *Let* $\{Z_x(t)\}_{t>0}$, *be a semi-Markov process with a linear drift de-*

(i) *the integral transform of the joint distribution of* $\{\tau^k(x), \eta^k(x)\}$ *satisfies the following formulae*

$$C_x^p(dl, s) = \frac{1 - F(l)}{1 - F(x)} e^{-(l-x)(s+p)} \mathbf{I}_{\{l \geq x\}} dl$$
$$+[1 - F(l)]e^{-sl} \frac{\tilde{f}_x(s+p)}{1 - \tilde{f}(s)} \mathbf{E} e^{-pS^+_{\nu_s}} \mathbf{E}\left[e^{-p(l-\delta+S^-_{\nu_s})}; l - \delta + S^-_{\nu_s} \geq 0\right] dl,$$
$$C_x^p(s) = \frac{1}{s+p}\left[1 - \frac{\tilde{f}_x(s+p)}{1 - \tilde{f}(s)} \mathbf{E} e^{-pS^+_{\nu_s}} \left(\mathbf{E} e^{sS^-_{\nu_s}} - \mathbf{E} e^{s(S^-_{\nu_s} - \delta)}\right)\right]; \tag{3.2}$$

(ii) *the Laplace transform of the distribution of the supremum* $Z_x^+(\nu_s)$ *of the process obeys the formula*

$$\mathbf{E} e^{-pZ_x^+(\nu_s)} = \frac{s}{s+p} + \frac{p}{s+p} \frac{\tilde{f}_x(s+p)}{1 - \tilde{f}(s)} \mathbf{E} e^{-pS^+_{\nu_s}} \left[\mathbf{E} e^{sS^-_{\nu_s}} - \mathbf{E} e^{s(S^-_{\nu_s} - \delta)}\right] \tag{3.3}$$

and for the infimum $Z_x^-(\nu_s)$ *of the process the following equalities are valid*

$$\mathbf{P}[Z_0^-(\nu_s) \geq -k] = \mathbf{P}[S^-_{\nu_s} \geq -k], \tag{3.4}$$
$$\mathbf{P}[Z_x^-(\nu_s) \geq -k] = 1 - \mathbf{E}\left[e^{-s\eta_x}; \xi_1(x) < -k\right]$$
$$- \int_{-k}^{\infty} \mathbf{E}\left[e^{-s\eta_x}; \xi_1(x) \in dv\right] \mathbf{P}[-S^-_{\nu_s} > k + v].$$

The proof of this theorem in a more general case (i.e. when $\delta \in \mathbb{R}$) is given in [6]. It is worth of mentioning that the distribution of $\tau^k(0)$ have been studied by Gusak [8]. The analytical expressions for the integral transform of $\tau^k(0)$ contain projectors which are not calculated. Therefore, in order to solve the two-sided exit problem we will require a more general one-boundary functional of the process $\{Z_x(t)\}_{t\geq 0}$, i.e. the joint distribution of $\{\tau^k(x), \eta^k(x)\}$, $x \in \mathbb{R}_+$. Hence, we will not use the results given in [8] but rather apply the results of Theorem 3.2.

Theorem 3.3. *Let* $\{Z_x(t)\}_{t\geq 0}$, *be a semi-Markov process with a linear drift defined by (2.2).*

$\pm\Re(p) \geq 0$

$$\mathbf{E}e^{-aZ_x(\nu_s)-pZ_x^+(\nu_s)} = \frac{s}{s+a+p} + \tag{3.5}$$

$$+\frac{s}{s+a}\frac{\tilde{f}_x(s+a+p)}{1-\tilde{f}(s)}\left[\frac{p}{s+a+p}B_s(-s) - B_s(a)\right]\mathbf{E}e^{-(a+p)S^+_{\nu_s}},$$

$$\mathbf{E}\left[e^{-aZ_0(\nu_s)}, Z_0^-(\nu_s) \geq -k\right] = \frac{s}{s+a}\frac{1-\tilde{f}(s+a)}{1-\tilde{f}(s)}\mathbf{E}\left[e^{-aS^-_{\nu_s}}; S^-_{\nu_s} \geq -k\right]\mathbf{E}e^{-aS^+_{\nu_s}},$$

where $S^+_{\nu_s}$, $S^-_{\nu_s}$ are the supremum and the infimum (2.8) of the semi-Markov walk $\{S_0(t)\}_{t\geq 0}$ on the time interval $[0,\nu_s]$ and $B_s(a) = \left(1-\mathbf{E}e^{a\delta}\right)\mathbf{E}e^{-aS^-_{\nu_s}}$.

Corollary 3.4. *The Laplace transforms of the distribution of the increments of the process*

$$Z_x^+(\nu_s) - Z_x(\nu_s) \in \mathbb{R}_+, \qquad Z_0(\nu_s) - Z_0^-(\nu_s) \in \mathbb{R}_+,$$

on the time interval $[0,\nu_s]$ obey the following relations for $\Re(p) \geq 0$

$$\mathbf{E}e^{-p(Z_x^+(\nu_s)-Z_x(\nu_s))} = 1 + \frac{1}{s-p}\frac{\tilde{f}_x(s)}{1-\tilde{f}(s)}\left[pB_s(-s) - sB_s(-p)\right],$$

$$\mathbf{E}e^{-p(Z_0(\nu_s)-Z_0^-(\nu_s))} = \frac{s}{s+p}\frac{1-\tilde{f}(s+p)}{1-\tilde{f}(s)}\mathbf{E}e^{-pS^+_{\nu_s}}.$$

Proof. Let us verify the first equality of (3.5). One can write the following equation for $k \in \mathbb{R}_+$, $\Re(a) = 0$

$$\mathbf{E}e^{-aZ_x(\nu_s)} = \mathbf{E}[e^{-aZ_x(\nu_s)}; Z_x^+(\nu_s) \leq k] + \int_0^\infty \mathbf{E}[e^{-s\tau^k(x)}; \eta^k(x) \in dl]e^{-ak}\mathbf{E}e^{-aZ_l(\nu_s)}.$$

The validity of this equation is based on the total probability formula, the Markov property of the random time $\tau^k(x)$, homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect of the first component, properties of the exponential variable ν_s and the following reasoning. The increments of the process $\{Z_x(t)\}_{t\geq 0}$ on the time interval $[0,\nu_s]$ (the expression in the left-hand side of the equation) can be realized either on sample paths of the process which do not intersect the upper boundary $k \in \mathbb{R}_+$ (the first term in the right-hand side) or on the sample paths which do intersect the upper boundary and

(3.2). Computing the integral in the right-hand side of the latter equality we get the first formula of (3.5). Let us give another approach to verify this equality. Using the total probability formula, the Markov property of the random variable $\eta_1(x) \doteq \eta_x$, homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component and properties of the exponential variable ν_s we establish that the mathematical expectation

$$U_x^k(a,s) = \mathbf{E}[e^{-aZ_x(\nu_s)}; Z_x^+(\nu_s) \leq k], \qquad x, k \in \mathbb{R}_+, \quad \Re(a) = 0$$

obeys the following equation

$$U_x^k(a,s) = s\int_0^k e^{-u(s+a)}\mathbf{P}[\eta_x > u]\,du + \int_0^k e^{-u(s+a)}\mathbf{P}[\eta_x \in du]\int_0^\infty e^{av}\mathbf{P}[\delta \in dv]\,U_0^{k-u+v}(a,s).$$

Define the following projectors [11] for the Laplace transforms (Fourier transform) $\tilde{F}(p) = \int_{-\infty}^{\infty} e^{-pu} f(u)\,du$, $\Re(p) = 0$ such that $\int_{-\infty}^{\infty} e^{-pu}|f(u)|\,du < \infty$

$$\mathfrak{P}_{\mathbb{R}_+}[\tilde{F}(p)] = \int_{0-}^{\infty} e^{-pu} f(u)\,du, \qquad \mathfrak{P}_{(-\infty,0)}[\tilde{F}(p)] = \int_{-\infty}^{0-} e^{-pu} f(u)\,du.$$

Multiplying the latter equation by e^{-pk} and integrating it over $k \in \mathbb{R}_+$ for $\Re(p) \geq 0$, $p \neq 0$ we find

$$\tilde{U}_x^p(a,s) = \frac{s}{p}\,\frac{1-\tilde{f}_x(s+a+p)}{s+a+p} + \tilde{f}_x(s+a+p)\mathfrak{P}_{\mathbb{R}_+}\left[\mathbf{E}e^{(a+p)\delta}\tilde{U}_0^p(a,s)\right], \quad (3.6)$$

where $\tilde{U}_x^p(a,s) = \int_0^\infty e^{-pk} U_x^k(a,s)\,dk$. It follows from this equality for $x = 0$ that the function $\tilde{U}_0^p(a,s)$ obeys the following equation

$$\tilde{U}_0^p(a,s) = \frac{s}{p}\,\frac{1-\tilde{f}(s+a+p)}{s+a+p} + \tilde{f}(s+a+p)\mathfrak{P}_{\mathbb{R}_+}\left[\mathbf{E}e^{(a+p)\delta}\tilde{U}_0^p(a,s)\right]. \quad (3.7)$$

In order to solve it we will apply the Wiener-Hopf factorization. First, we will determine an auxiliary function $I_p^+(a,s) = \mathfrak{P}_{\mathbb{R}_+}\left[\mathbf{E}e^{(a+p)\delta}\tilde{U}_0^p(a,s)\right]$. Then by means of (3.6) we will obtain the function $\tilde{U}_x^p(a,s)$. Multiplying the equation (3.7) by $\mathbf{E}e^{(a+p)\delta}$, we get for $\Re(a) = \Re(p) = 0$

$$I_p^+(a,s)(1-\tilde{f}(s+a+p)\mathbf{E}e^{(a+p)\delta}) = \frac{s}{p}\,\frac{1-\tilde{f}(s+a+p)}{s+a+p}\mathbf{E}e^{(a+p)\delta} - I_p^-(a,s),$$

where $I_p^-(a,s) = \mathfrak{P}_{(-\infty,0)}\left[\mathbf{E}e^{(a+p)\delta}\tilde{U}_0^p(a,s)\right]$. Employing the Spitzer-Rogozin facwhere $B_s(a) = (1 - \mathbf{E}e^{a\delta})\mathbf{E}e^{-aS^-_{\nu_s}}$. In order to calculate the projectors which enter (3.8) we need the following proposition.

Proposition 3.5. *Let* $\varepsilon \in \mathbb{C}$, $\Re(\varepsilon) \geq 0$, $A(p)$ *be a bounded analytical function in the semi-plane* $\Re(p) < 0$, *and continuous on* $\Re(p) = 0$. *Then for* $\Re(p) \in [-\Re(\varepsilon), 0]$, $p \neq -\varepsilon$ *it admits the following additive factorization expansion*

$$\frac{A(p)}{\varepsilon + p} = \frac{A(-\varepsilon)}{\varepsilon + p} + \frac{A(p) - A(-\varepsilon)}{\varepsilon + p},$$

where $A(-\varepsilon)(\varepsilon + p)^{-1}$ *is a bounded analytical function in* $\Re(p) > -\Re(\varepsilon)$, $[A(p) - A(-\varepsilon)](\varepsilon+p)^{-1}$ *is a bounded analytical function in* $\Re(p) < 0$, *continuous on* $\Re(p) = 0$ *and*

$$\mathfrak{P}_{\mathbb{R}_+}\left[\frac{A(p)}{\varepsilon + p}\right] = \frac{A(-\varepsilon)}{\varepsilon + p}, \qquad \mathfrak{P}_{(-\infty,0)}\left[\frac{A(p)}{\varepsilon + p}\right] = \frac{A(p) - A(-\varepsilon)}{\varepsilon + p}.$$

The statements of Proposition 3.5 are obvious, so we will not concentrate on the proof. It is clear, that the function $B_s(a + p)$ is analytical and bounded in $\Re(p) < 0$, also continuous on the $\Re(p) = 0$. Employing the equality of Proposition 3.5 to calculate the projectors (3.8), we find that

$$\mathfrak{P}_{\mathbb{R}_+}\left[\frac{1}{p}\,\frac{B_s(a + p)}{s + a + p}\right] = \frac{1}{s + a}\,\frac{1}{p}\left[B_s(a) - \frac{p}{s + a + p}B_s(-s)\right].$$

It is clear that the functions $I_p^{\pm}(a, s)$ are bounded and analytical in $\pm\Re(p) \geq 0$, and are continuous on $\Re(p) = 0$. It follows from the Wiener-Lévy lemma [10] that $\left(\mathbf{E}e^{-pS^+_{\nu_s}}\right)^{-1}$ is a bounded analytical function in $\Re(p) > 0$, continuous on the boundary. Let us turn now to the equality (3.8) and apply the factorization reasoning [7]. The function which enters the left-hand side of this equality is bounded analytical in the semi-plane $\Re(p) > 0$, and continuous on $\Re(p) = 0$. Hence, by means of this equality it can be extended to the semi-plane $\Re(p) < 0$, remaining bounded and analytical. Then, in view of the Liouville theorem it is equal to a constant $C(s)$ in the whole complex plane. In order to determine this constant, we calculate the limit in the left-hand side of (3.8) as $p \to \infty$, which yields that $C(s) = 0$. Equating the left-hand side of (3.8) to zero we get

In order to obtain this equation we have used properties of the process $\{X_t\}_{t\geq 0}$, the variables τ_k, ν_s and the following reasoning.

The increments of the process $\{Z_0(t)\}_{t\geq 0}$ on the time interval $[0,\nu_s]$ (the left-hand side) can be realized either on without intersecting the lower boundary $-k$ (the first term in the left-hand side) or with intersection of the lower boundary $-k$ and the further realization of the process on $[0,\nu_s]$. Substituting the expressions for the mathematical expectations $\mathbf{E}e^{-aZ_0(\nu_s)}$ (2.6), $\mathbf{E}[e^{-aZ_0(\nu_s)}; Z_0^-(\nu_s) \geq -k]$ (2.10) into this equation we obtain the second formula of (3.5). Formulae of Corollary 3.4 follow straightforwardly from the statements of Theorem 3.3. This completes the proof. □

4. Two-sided boundary problems for process $\{Z_x(t)\}_{t\geq 0}$.

Let $B \in \mathbb{R}_+$ be fixed, $k \in [0,B]$, $r = B-k$, $X_0 = \{0,x\}$, $x \in \mathbb{R}_+$ and define a random variable

$$\chi = \inf\{\, t: \ Z_x(t) \notin [-r,k]\,\}$$

i.e. the first exit time from the interval $[-r,k]$ by the process $\{Z_x(t)\}_{t\geq 0}$. Denote by $\mathfrak{A}^k = \{Z_x(\chi) > k\}$ the event that the first exit from the interval $[-r,k]$ takes place through the upper boundary by means of the linear increase of the process $\{Z_x(t)\}_{t\geq 0}$; $\mathfrak{A}_r = \{\, Z_x(\chi) < -r\,\}$ the event that the first exit from the interval occurs through the lower boundary $-r$. Observe, that χ is a Markov time of the process $\{X_t\}_{t\geq 0}$ and $\mathbf{P}\left[\mathfrak{A}^k + \mathfrak{A}_r\right] = 1$. Define random variables

$$L = \eta_x^+(\chi)\,\mathbf{I}_{\mathfrak{A}^k} + 0\cdot \mathbf{I}_{\mathfrak{A}_r}, \qquad T = 0\cdot \mathbf{I}_{\mathfrak{A}^k} + (-Z_x(\chi) - r)\,\mathbf{I}_{\mathfrak{A}_r},$$

i.e. the value of the linear component $\eta_x^+(t)$ at the instant of the first exit and the value of the overshoot through the boundary at this instant. Here $\mathbf{I}_{\mathfrak{A}} = \mathbf{I}_{\mathfrak{A}}(\omega)$ is the indicator of a set $\mathfrak{A}$. Denote for $k \in [0,B]$, $r = B-k$

$$F_x^k(dl,s) = f_x^k(dl,s) - \int_0^\infty f_r^x(dv,s)\, f_0^{v+B}(dl,s),$$
$$F_r^x(du,s) = f_r^x(du,s) - \int_0^\infty f_x^k(dl,s)\, f_B^l(du,s),$$

of the joint distribution of $\{\chi, L, T\}$ *satisfy the following formulae for* $s > 0$

$$V_x^k(dl,s) = F_x^k(dl,s) + \int_0^\infty F_x^k(dl_1,s)\,\mathfrak{K}_{l_1}^+(dl,s),$$
$$V_r^x(du,s) = F_r^x(du,s) + \int_0^\infty F_r^x(dv,s)\,\mathfrak{K}_v^-(du,s), \tag{4.1}$$

where

$$\mathfrak{K}_{l_1}^+(dl,s) = \sum_{n\in\mathbb{N}} K_{l_1}^+(dl,s)^{*(n)}, \quad \mathfrak{K}_v^-(du,s) = \sum_{n\in\mathbb{N}} K_v^-(du,s)^{*(n)}, \tag{4.2}$$

are the uniformly convergent series of the successive iterations; and

$$K_{l_1}^+(dl,s)^{*(1)} \stackrel{\text{def}}{=} K_{l_1}^+(dl,s), \qquad K_v^-(du,s)^{*(1)} \stackrel{\text{def}}{=} K_v^-(du,s),$$
$$K_{l_1}^+(dl,s)^{*(n+1)} = \int_0^\infty K_{l_1}^+(dl_2,s)\,K_{l_2}^+(dl,s)^{*(n)}, \quad n\in\mathbb{N},$$
$$K_v^-(du,s)^{*(n+1)} = \int_0^\infty K_v^-(du_1,s)\,K_{u_1}^-(du,s)^{*(n)}, \quad n\in\mathbb{N}, \tag{4.3}$$

are the iterations of the kernels $K_{l_1}^+(dl,s)$, $K_v^-(du,s)$, *which are defined by the formulae*

$$K_{l_1}^+(dl,s) = \int_0^\infty f_B^{l_1}(du,s)\,f_0^{u+B}(dl,s),$$
$$K_v^-(du,s) = \int_0^\infty f_0^{v+B}(dl,s)\,f_B^{l}(du,s). \tag{4.4}$$

Proof. Let us mention that the joint distribution of the first exit time and the value of the overshoot for Lévy processes has been determined in [1]. This distribution was obtained in terms of the joint distributions of the one-boundary characteristics of the process. The authors solved a system of the integral equations for the Laplace transforms of the two-boundary characteristics studied. This idea was applied for solving the two-sided problem for another classes of stochastic processes, e.g. the difference of two renewal processes [3]. The processes mentioned and $\{Z_x(t), \eta_x^+(t)\}_{t\geq 0}$ belong to the class of Markov processes homogeneous with respect to the first component [4]. Now, following the approach from [1], we write a system of integral equations to de-

This system was obtained by using the total probability formula, homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component, Markov property of the random variables $\tau^k(x)$, $\tau_k(x)$, χ and the following probabilistic reasoning. The first equation of the system represents the fact that the crossing of the upper boundary k by the process $\{Z_x(t)\}_{t\geq 0}$ (the left-hand side of the equation) can occur either on the sample paths of the process which do not intersect the lower boundary $-r$ (the first term in the right-hand side of the equation) or on the sample paths which do intersect the lower boundary $-r$ and then cross the upper boundary k. The second equation is written analogously. This system of equations is similar to a system of linear equations with two unknowns, and its solution is analogous to the solution of the system of linear equations. Thus, we proceed as follows. Substituting the expression for the function $V_x^k(dl, s)$ from the first equation into the second one we get

$$V_r^x(du, s) = f_r^x(du, s) - \int_0^\infty f_x^k(dl, s)\, f_B^l(du, s) +$$
$$+ \int_{l=0}^\infty \int_{v=0}^\infty V_r^x(dv, s)\, f_0^{v+B}(dl, s)\, f_B^l(du, s).$$

Changing the order of integration in the third term of the right-hand side of the latter equation we obtain a linear integral equation for the function $V_r^x(du, s)$

$$V_r^x(du, s) = F_r^x(du, s) + \int_0^\infty V_r^x(dv, s)\, K_v^-(du, s), \tag{4.6}$$

where

$$K_v^-(du, s) = \int_0^\infty f_0^{v+B}(dl, s)\, f_B^l(du, s),$$

is the kernel of this equation. Since $\tau_k(x) \in \{\eta_n(x),\, n \in \mathbb{N}\}$, then $\mathbf{E}[e^{-s\tau_k(x)}] \leq \mathbf{E}[e^{-s\eta_x}]$. It is clear that $\mathbf{P}[\tau^k(x) > t] = \mathbf{P}[Z_x^+(t) \leq k]$ for all $v \in \mathbb{R}_+$. Thus, we may write

$$\mathbf{E}\left[e^{-s\tau^B(x)}; \mathfrak{B}^B(x)\right] - \mathbf{E}\left[e^{-s\tau^{v+B}(x)}; \mathfrak{B}^{v+B}(x)\right] = \mathbf{P}\left[Z_x^+(\nu_s) \in (B, v+B]\right] \geq 0.$$

Hence, the kernel of the equation (4.6) enjoys the following property for a fixed $s_0 > 0$ for all $s > s_0$, $u, v \in \mathbb{R}_+$

$$K_v^-(du, s) \leq \int_0^\infty f_0^{v+B}(dl, s)\, \mathbf{E}e^{-s\eta_l} \leq \int_0^\infty f_0^{v+B}(dl, s)$$

converges uniformly for all $s > s_0$, $v, u \in \mathbb{R}_+$. Applying the method of successive iterations [15] to solve the linear integral equation (4.6), we obtain the second equality of Theorem 4.1 for $s > s_0$. Letting $s_0 \to 0$, we verify this equality for all $s > 0$. In order to verify the first equality of the theorem, we first substitute the expression for the function $V_r^x(du, s)$ from the second equation of the system (4.5) into the first one. This yields

$$V_x^k(dl, s) = f_x^k(dl, s) - \int_0^\infty f_r^x(dv, s)\, f_0^{v+B}(dl, s) + \\ + \int_{v=0}^\infty \int_{l_1=0}^\infty V_x^k(dl_1, s)\, f_B^{l_1}(dv, s)\, f_0^{v+B}(dl, s).$$

Changing the order of integration in the third term of the right-hand side of this equation we get a linear integral equation for the function $V_x^k(dl, s)$

$$V_x^k(dl, s) = F_x^k(dl, s) + \int_0^\infty V_x^k(dl_1, s)\, K_{l_1}^+(dl, s). \tag{4.7}$$

The kernel of this equation

$$K_{l_1}^+(dl, s) = \int_0^\infty f_B^{l_1}(dv, s)\, f_0^{v+B}(dl, s)$$

enjoys the following property for all $l, l_1 \in \mathbb{R}_+$, $s > s_0$

$$K_{l_1}^+(dl, s) \le \mathbf{E}\left[e^{-s\tau^B}; \mathfrak{B}^B\right] \int_0^\infty f_B^{l_1}(dv, s) \le \mathbf{E}\left[e^{-s\tau^B}; \mathfrak{B}^B\right] \mathbf{E} e^{-s\eta_{l_1}} < a < 1.$$

Hence, the series of the successive iterations

$$\mathfrak{K}_{l_1}^+(dl, s) = \sum_{n \in \mathbb{N}} K_{l_1}^+(dl, s)^{*(n)} < a(1-a)^{-1} < \infty$$

converges uniformly for all $l, l_1 \in \mathbb{R}_+$. Applying the method of successive iterations [15] to solve (4.7) we get the first equality of Theorem 4.1 for $s > s_0$. Letting $s_0 \to 0$, we establish its validity for all $s > 0$. □

Theorem 4.2. *Let $\{Z_x(t)\}_{t \ge 0}$ be a semi-Markov walk with a linear drift given by (2.2), $k, r \in \mathbb{R}_+$, $B = k + r$, and*

$$\tilde{Q}_x^s(a) = \mathbf{E}\left[e^{-aZ_x(\nu_s)}; Z_x^-(\nu_s) \ge -r, Z_x^+(\nu_s) \le k\right] = \mathbf{E}\left[e^{-aZ_x(\nu_s)}; \chi > \nu_s\right]$$

where the mathematical expectations

$$U_x^k(a,s) = \mathbf{E}\left[e^{-aZ_x(\nu_s)}; Z_x^+(\nu_s) \le k\right], \qquad U_r^x(a,s) = \mathbf{E}\left[e^{-aZ_x(\nu_s)}; Z_x^-(\nu_s) \ge -r\right]$$

are given by (3.5). In particular, the following identity is true

$$\mathbf{P}[\chi > \nu_s] = \mathbf{P}[Z_x^+(\nu_s) \le k] - \int_0^\infty \mathbf{E}[e^{-s\chi}; T \in dv, \mathfrak{A}_r]\mathbf{P}[Z_0^+(\nu_s) \le v + B]$$
$$= \mathbf{P}[Z_x^-(\nu_s) \ge -r] - \int_0^\infty \mathbf{E}[e^{-s\chi}; L \in dl, \mathfrak{A}^k]\mathbf{P}[Z_l^-(\nu_s) \ge -B].$$

Proof. According to the total probability formula, the Markov property of χ, homogeneity of the process $\{X_t\}_{t\ge 0}$ with respect to the first component and the properties of ν_s, we write

$$\mathbf{E}[e^{-aZ_x(\nu_s)}; Z_x^+(\nu_s) \le k] = \mathbf{E}\left[e^{-aZ_x(\nu_s)}; Z_x^+(\nu_s) \le k, Z_x^-(\nu_s) \ge -r\right]$$
$$+ e^{ar}\int_0^\infty e^{av}\mathbf{E}[e^{-s\chi}; T \in dv, \mathfrak{A}_r]\mathbf{E}\left[e^{-aZ_0(\nu_s)}; Z_0^+(\nu_s) \le v + B\right].$$

This equation shows that the increments of the process $\{Z_x(t)\}_{t\ge 0}$ on the time interval $[0, \nu_s]$ given that there is no intersection of the upper level $k \in \mathbb{R}_+$ (the left-hand side of the equation) are realized either on the samples paths which do not intersect the lower level $-r \le 0$, (the first term in the right-hand side) or on the sample paths which do intersect the lower level $-r$ and further do not cross the upper level k (the second term in the right-hand side. A more detailed derivation of this equation for Lévy processes and random walks is given [1]. One can notice, that the first formula of (4.8) follows immediately from the latter equation. The second formula of (4.8) follows from the equation

$$\mathbf{E}[e^{-aZ_x(\nu_s)}; Z_x^-(\nu_s) \ge -r] = \mathbf{E}\left[e^{-aZ_x(\nu_s)}; Z_x^-(\nu_s) \ge -r, Z_x^+(\nu_s) \le k\right]$$
$$+ e^{-ak}\int_0^\infty \mathbf{E}[e^{-s\chi}; L \in dl, \mathfrak{A}^k]\mathbf{E}\left[e^{-aZ_l(\nu_s)}; Z_l^-(\nu_s) \ge -B\right].$$

The identity of Theorem 4.1 can be obtained from (4.8) by letting $a = 0$. This completes the proof. □

5. Possible applications.

Lemma 5.1. *Let* $\tilde{f}(s) = \mathbf{E}[e^{-s\eta}]$, $\rho = \lambda\mathbf{E}\eta$. *Then*

(i) *for* $s > 0$ *the equation*

$$\lambda\tilde{f}(s+p) + p - \lambda = 0 \tag{5.1}$$

has a unique root $c(s)$ *in the semi-plane* $\Re(p) > 0$. *This root is positive and* $c(s) \in (0, \lambda)$;

(ii) *if* $\rho \leq 1$, *then* $\lim_{s\to 0} c(s) = 0$; *and if* $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (0, \lambda)$.

Proof. Let $s > 0$, $p \geq 0$. One can establish the existence of the root $c(s) \in (0, \lambda)$ of the equation (5.1) by means of analysis of behavior of the functions $y_1(p) = \lambda - p$, $y_2(s,p) = \lambda\tilde{f}(s+p)$ on the interval $[0, \lambda]$. Analyzing behavior of the derivative $\frac{d}{dp}\, y_2(0,p)\Big|_{p\downarrow 0}$ we get the first statement of Lemma 5.1. Let us prove the uniqueness of the root of (5.1) in $\Re(p) > 0$. In order to do this we will use the Rouche theorem. Let $\varepsilon, R > 0$ be such that for a fixed $s_0 > 0$ the following inequalities hold

$$\varepsilon < \lambda(1 - \tilde{f}(s_0)), \qquad R > \lambda(1 + \tilde{f}(s_0)). \tag{5.2}$$

Note, that these inequalities are valid for any $s > s_0$. Define a closed contour $\Gamma_\varepsilon(R) = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{p : \Re(p) = \varepsilon,\ |p| \leq R\}$ is a vertical line segment and $\Gamma_2 = \{p : |\arg p| \leq \arccos(\varepsilon/R),\ |p| = R\}$ is an arc of the circle with center in the origin and connecting the ends of the line segment Γ_1. Denote by $G_\varepsilon(R)$ the area enclosed inside the contour $\Gamma_\varepsilon(R)$. It is clear that $\lambda \in G_\varepsilon(R)$.

Let $p \in \mathbb{C}$, and define the complex-valued functions $f(p) = \lambda - p$, $g(p) = \lambda\tilde{f}(s+p) + p - \lambda$.One can see that $f(p) + g(p) = \lambda\tilde{f}(s+p)$. It follows from the first inequality (5.2) that on the line Γ_1, the following chain of inequalities holds for $s > s_0$

$$|f(p) + g(p)| = \lambda|\tilde{f}(s+p)| \leq \lambda\tilde{f}(s) < \lambda - \varepsilon \leq |f(p)|, \qquad p \in \Gamma_1.$$

The second inequality of (5.2) implies that on Γ_2 the following relation holds for $s > s_0$,

$$|f(p) + g(p)| \leq \lambda\tilde{f}(s) < R - \lambda \leq |f(p)|, \qquad p \in \Gamma_2.$$

Hence, the following inequality is valid on the contour $\Gamma_\varepsilon(R)$, for $s > s_0$

$$|f(p) + g(p)| < |f(p)|, \qquad p \in \Gamma_\varepsilon(R).$$

Thus, in view of Rouche theorem the number of zeros of the function $g(p) = \lambda\tilde{f}(s+$

Lemma 5.2. *Let $\{S_0(t)\}_{t\geq 0}$ be a semi-Markov walk (2.5), $\delta \sim \exp(\lambda)$, $S_t^+ = \sup_{u\leq t} S_0(u)$, $S_t^- = \inf_{u\leq t} S_0(u)$, be the supremum and the infimum of the process $\{S_0(t)\}_{t\geq 0}$. Then*

(i) *the Laplace transforms of the distributions of $S_{\nu_s}^+$, $S_{\nu_s}^-$ satisfy the following formulae for $\pm\Re(p) \geq 0$, $s > 0$*

$$\mathbf{E}e^{-pS_{\nu_s}^+} = \frac{\lambda}{c(s)}(1-\tilde{f}(s))\frac{p-c(s)}{\lambda\tilde{f}(s+p)+p-\lambda}, \quad \mathbf{P}\left[S_{\nu_s}^+ = 0\right] = \frac{\lambda}{c(s)}(1-\tilde{f}(s)),$$
$$\mathbf{E}e^{-pS_{\nu_s}^-} = \frac{c(s)}{\lambda}\,\frac{\lambda-p}{c(s)-p}, \qquad \mathbf{P}\left[S_{\nu_s}^- = 0\right] = \frac{c(s)}{\lambda}, \tag{5.3}$$

where $c(s)$ is the unique positive root of the equation (5.1);

(ii) *for the joint distributions of $\{\tilde{\tau}^k, \tilde{T}^k\}$, $\{\tilde{\tau}_k, \tilde{T}_k\}$ the following equalities are valid for $k \in \mathbb{R}_+$, $\Re(p) \geq 0$, $\Re(z) \geq 0$*

$$\mathbf{E}\left[e^{-s\tilde{\tau}_k}; \tilde{T}_k \in du, \tilde{\tau}_k < \infty\right] = \left(1 - \frac{c(s)}{\lambda}\right) e^{-kc(s)}\lambda e^{-\lambda u}\,du,$$
$$\int_0^\infty e^{-kp}\mathbf{E}\left[e^{-s\tilde{\tau}^k - z\tilde{T}^k}; \tilde{\tau}^k < \infty\right] dk = \frac{1}{p-z}\left(1 - \mathbf{E}e^{-pS_{\nu_s}^+}/\mathbf{E}e^{-zS_{\nu_s}^+}\right). \tag{5.4}$$

Corollary 5.3. *Let $\{Z_x(t)\}_{t\geq 0}$ be a semi-Markov walk with a linear drift (2.2), $x \in \mathbb{R}_+$, $\delta \sim \exp(\lambda)$. Then*

(i) *the Laplace transforms of the joint distribution of $\{\tau_k(x), T_k(x)\}$, $x, k \in \mathbb{R}_+$ satisfy the following equality for $s > 0$*

$$f_k^x(du, s) = \mathbf{E}\left[e^{-s\tau_k(x)}; T_k(x) \in du, \tau_k(x) < \infty\right] = \lambda\tilde{f}_x(s+c(s))e^{-kc(s)-u\lambda}du, \tag{5.5}$$

where $c(s) \in (0, \lambda)$ is the unique root of the equation (5.1) in the semi-plane $\Re(p) > 0$;

(ii) *if $\rho > 1$, then $\mathbf{P}[\tau_k(x) < \infty] = \tilde{f}_x(c)\,e^{-kc} < 1$ and $\tau_k(x)$ is a defective random variable for all $x, k \in \mathbb{R}_+$; if $\rho \leq 1$, then $\mathbf{P}[\tau_k(x) < \infty] = 1$ and $\tau_k(x)$ is a proper variable for all $x, k \in \mathbb{R}_+$.*

is analytical in $\Re(p) > c(s)$ and $\lim\limits_{|p|\to\infty} \mathbb{R}_p^s(x) = 0$. Thus, it can be represented by an absolutely convergent Laplace integral for all $s, x \in \mathbb{R}_+$ [9]

$$\mathbb{R}_p^s(x) = \frac{\tilde{f}_x(s+p)}{\lambda \tilde{f}(s+p) + p - \lambda} = \int_0^\infty e^{-pu} R_u^s(x)\, du, \qquad \Re(p) > c(s). \tag{5.6}$$

This formula can be considered as a definition of the function $R_u^s(x) : \mathbb{R}_+ \to \mathbb{R}_+$ (with respect to u). A similar function was first introduced by Takács [19] for semi-continuous processes with independent increments and semi-continuous random walks. Let us establish basic properties of this function. The first formula of (5.3) implies that

$$\mathbb{R}_p^s(x) = \frac{\lambda}{c(s)}(1-\tilde{f}(s))^{-1} \frac{\tilde{f}_x(s+p)}{p - c(s)}\, \mathbf{E} e^{-pS_{\nu_s}^+}, \qquad \Re(p) > c(s).$$

The functions which enter the right-hand side of this equation are the Laplace transforms for $\Re(p) > c(s)$. Thus, their original functions should coincide with the original function of those in the left-hand side. Hence, for $u \in \mathbb{R}_+$

$$R_u^s(x) = \frac{\lambda e^{uc(s)}}{c(s)(1-\tilde{f}(s))} \int_0^u \mathbf{P}[S_{\nu_s}^+ \in dv] \mathbf{E}\left[e^{-(s+c(s))\eta_x}; \eta_x < u - v\right]. \tag{5.7}$$

The formula (5.7) implies that for fixed $s, x \in \mathbb{R}_+$ the function $R_u^s(x)$, $u \geq 0$ is a positive monotonically increasing continuous function. Moreover, $R_u^s(x) < A(s)e^{uc(s)}$, $A(s) < \infty$. Therefore, $\int_0^\infty R_u^s(x) e^{-\alpha u} du < \infty$ for $\alpha > c(s)$. Additionally, the function $R_u^s(x)$ has bounded variation in any neighbourhood of the point $u \geq 0$. Thus, [9, p 68] the following inversion formula is valid

$$R_u^s(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{up}\, \mathbb{R}_p^s(x)\, dp, \qquad u \in \mathbb{R}_+, \qquad \alpha > c(s). \tag{5.8}$$

The formula (5.8) defines the resolvent of the semi-Markov walk with a linear drift $\{Z_x(t)\}_{t\geq 0}$ given by (2.2), when $\delta \sim$ exp(λ). We will suppose that $R_u^s(x) = 0$ for $u < 0$ and for all $s, x \in \mathbb{R}_+$. It follows from the definition that $R_0^s(x) = \lim\limits_{p\to\infty} p\, \mathbb{R}_p^s(x) = 0$. It is worth of mentioning that the defining formula (5.8) of the resolvent allows

(i) *the Laplace transforms* $f_x^k(dl,s)$ *of the joint distributions of* $\{\tau^k(x),\, \eta^k(x)\}$ *satisfy the following equality for* $x,k\in\mathbb{R}_+$

$$f_x^k(dl,s)=e^{-s(l-x)}\,\frac{1-F(l)}{1-F(x)}\,\mathbf{I}_{\{l\ge x\}}\,\delta(l-x-k)\,dl \\ +\lambda e^{-sl}\,[1-F(l)]\left[e^{-lc(s)}R_k^s(x)-R_{k-l}^s(x)\right]dl; \tag{5.9}$$

(ii) *for the Laplace transforms of the first overshoot time of the upper boundary by the process* $\{Z_x(t)\}_{t\ge0}$ *the following formula is valid for* $s>0$, $x,k\in\mathbb{R}_+$

$$\mathbf{E}\left[e^{-s\tau^k(x)};\mathfrak{B}^k(x)\right]=e^{-sk}-\frac{sR_k^s(x)}{s+c(s)}+s\int_0^k e^{-su}R_{k-u}^s(x)\,du; \tag{5.10}$$

(iii) *if* $\rho<1$, *then* $\tau^k(x)$ *is a defective random variable for all* $x,k\in\mathbb{R}_+$, *and*

$$\mathbf{P}[\tau^k(x)<\infty]=1-(1-\rho)R_k(x)<1,\qquad R_k(x)\overset{\text{def}}{=}R_k^0(x),$$

if $\rho\ge1$, *then* $\tau^k(x)$ *is a proper variable for all* $x,k\in\mathbb{R}_+$.

Define for $x,k,B\in\mathbb{R}_+$ random variables

$$i^k(x)=\inf\{t>\tau^k(x):\ Z_x(t)<k-B\},\quad I^k(x)=k-B-Z_x(i^k(x)),$$

the first downward crossing time of the interval $[k-B,k]$ by the process $\{Z_x(t)\}_{t\ge0}$ and the value of the overshoot of the lower boundary $k-B$ at this instant. We will use the convention that $\inf\{\emptyset\}\overset{\text{def}}{=}\infty$, $I^k(x)=\infty$ on the event $\{i^k(x)=\infty\}$. It is clear that $\eta_x^+(i^k(x))=0$, because $i^k(x)\in\{\eta_n(x);n\in\mathbb{N}\}$.

Lemma 5.5. *Let* $\{Z_x(t)\}_{t\ge0}$ *be a semi-Markov walk with a drift (2.2),* $\delta\sim\exp(\lambda)$, $\{R_u^s(x)\}_{u\in\mathbb{R}_+}$, $s,x\in\mathbb{R}_+$ *be the resolvent (5.8) of the process* $\{Z_x(t)\}_{t\ge0}$. *Then*

(i) *the Laplace transforms of the joint distribution of* $\{i^k(x),I^k(x)\}$

$$\varphi_x^k(du,s)=\mathbf{E}[e^{-si^k(x)};\ I^k(x)\in du, i^k(x)<\infty]$$

satisfy the following equality for $x,k\in\mathbb{R}_+$

$$\varphi_x^k(du,s)=\left[c_x(s)e^{-c(s)(B-k)}-e^{-c(s)B}R_k^s(x)r(s)\right]\lambda e^{-\lambda u}du, \tag{5.11}$$

where $c_x(s)=\tilde f_x(s+c(s))$, $r(s)=1+\lambda\tilde f'(s+c(s))$;

Theorem 5.6. *Let $\{Z_x(t)\}_{t\geq 0}$ be a semi-Markov walk with a linear drift given by (2.2), $\delta \sim \exp(\lambda)$, $\{R_u^s(x)\}_{u\in\mathbb{R}_+}$, $s, x \in \mathbb{R}_+$ be a resolvent of the process $\{Z_x(t)\}_{t\geq 0}$ given by (5.8). Then*

(i) *the Laplace transforms of the joint distribution of $\{\chi, L, T\}$*

$$V_x^k(dl, s) = \mathbf{E}[e^{-s\chi};\, L \in dl, \mathfrak{A}^k], \qquad V_r^x(du, s) = \mathbf{E}\left[e^{-s\chi};\, T \in du, \mathfrak{A}_r\right]$$

satisfy the following equalities for $s > 0$, $x \in \mathbb{R}_+$

$$V_r^x(du, s) = \frac{R_k^s(x)}{\mathbf{E}\, R_{\delta+B}^s}\, \lambda e^{-\lambda u}\, du,$$

$$V_x^k(dl, s) = f_x^k(dl, s) - \frac{R_k^s(x)}{\mathbf{E}\, R_{\delta+B}^s}\, \mathbf{E} f_0^{\delta+B}(dl, s), \tag{5.12}$$

where the function $f_x^k(dl, s)$ is given by (5.9),

$$\mathbf{E}\, R_{\delta+B}^s = \int_0^\infty \lambda e^{-\lambda v} R_{v+B}^s\, dv, \quad \mathbf{E} f_0^{\delta+B}(dl, s) = \int_0^\infty \lambda e^{-\lambda v} f_0^{v+B}(dl, s)\, dv;$$

(ii) *the Laplace transforms of the first exit time χ from the interval by the process $\{Z_x(t)\}_{t\geq 0}$ admit the following representation for $s > 0$*

$$\mathbf{E}[e^{-s\chi};\, \mathfrak{A}_r] \stackrel{\text{def}}{=} V_r^x(s) = \frac{R_k^s(x)}{\mathbf{E}\, R_{\delta+B}^s}, \tag{5.13}$$

$$\mathbf{E}[e^{-s\chi};\, \mathfrak{A}^k] \stackrel{\text{def}}{=} V_x^k(s) = e^{-sk} - s\frac{R_k^s(x)}{s+\lambda} + sA_k^s(s, x) - \lambda\frac{V_r^x(s)}{s+\lambda}\left[e^{-sB} + sA_B^s(s)\right],$$

where

$$A_k^s(a, x) = \int_0^k e^{-au} R_{k-u}^s(x)\, du, \qquad A_k^s(a, 0) \stackrel{\text{def}}{=} A_k^s(a);$$

(iii) *the probabilities of the first exit from the interval by the process $\{Z_x(t)\}_{t\geq 0}$ are given as follows*

$$\mathbf{P}[\mathfrak{A}_r] = \frac{R_k(x)}{\mathbf{E}\, R_{\delta+B}}, \qquad \mathbf{P}[\mathfrak{A}^k] = 1 - \frac{R_k(x)}{\mathbf{E}\, R_{\delta+B}},$$

formula of (5.12). Using the corollaries, Lemma 5.5 and defining formulae (4.2)-(4.4), after some calculations we obtain for $s, v, u > 0$ $n \in \mathbb{N}$

$$K_v^-(du,s) = \varphi_0^{v+B}(du,s), \tag{5.14}$$

$$K_v^-(du,s)^{*(n)} = \varphi_0^{v+B}(du,s)\left(\mathbf{E}\,\varphi_0^{\delta+B}(s)\right)^{n-1},$$

$$\mathfrak{K}_v^-(du,s) = \varphi_0^{v+B}(du,s)\left(1-\mathbf{E}\,\varphi_0^{\delta+B}(s)\right)^{-1},$$

$$F_r^x(du,s) = c_x(s)\, e^{-c(s)\,r}\lambda e^{-\lambda u}du - \varphi_x^k(du,s),$$

where $c_x(s) = \tilde{f}_x(s + c(s))$ and the mathematical expectations $\varphi_x^k(du,s)$, $x,k \in \mathbb{R}_+$ are given by (5.11), $\varphi_x^k(s) = \int_0^\infty \varphi_x^k(du,s)$. Substituting (5.14) into the second formula of (4.1) we get the first equality of (5.12). We now verify the second formula of (5.12). It follows from Lemma 5.5, the formulae (4.2)-(4.4) that for all $s > 0$, $x, l, l_1 \in \mathbb{R}_+, n \in \mathbb{N}$

$$K_{l_1}^+(dl,s) = c_{l_1}(s)e^{-c(s)B}\mathbf{E}\,f_0^{\delta+B}(dl,s),$$

$$K_{l_1}^+(dl,s)^{*(n)} = c_{l_1}(s)e^{-c(s)B}\left(\mathbf{E}\,\varphi_0^{\delta+B}(s)\right)^{n-1}\mathbf{E}\,f_0^{\delta+B}(dl,s), \tag{5.15}$$

$$\mathfrak{K}_{l_1}^+(dl,s) = c_{l_1}(s)e^{-c(s)B}\left(1-\mathbf{E}\,\varphi_0^{\delta+B}(s)\right)^{-1}\mathbf{E}\,f_0^{\delta+B}(dl,s),$$

$$F_x^k(dl,s) = f_x^k(dl,s) - c_x(s)e^{-c(s)\,r}\,\mathbf{E}\,f_0^{\delta+B}(dl,s),$$

where the mathematical expectations $f_x^k(dl,s)$, $x, k \in \mathbb{R}_+$ are given by (5.11). Substituting (5.15) into the first formula of (4.1) we obtain the second formula of (5.12). It is worth of noticing that instead of the calculations (5.15) to obtain the second formula of (5.12) we could use the first equation of system (4.5), since the function $V_r^x(du,s)$ is already determined by the first equality of (5.12). Integrating the first equality of (5.12) with respect to $u \in \mathbb{R}_+$, and the second equality of (5.12) with respect to $l \in \mathbb{R}_+$, we get the formulae a (5.13). Passing $s \to 0$ in (5.13) we find the probabilities of the first exit through the upper and lower boundary by the process $\{Z_x(t)\}_{t\geq 0}$.

We now present another approach to prove the theorem. It is based on the system (4.5). This system becomes more simple for the case when $\delta \sim \exp(\lambda)$. In view of (5.5) this system is of the following form

$$f^k(dl,s) = V^k(dl,s) + \int^\infty V^x(du,s)\, f^{u+B}(dl,s) \tag{5.16}$$

because by means of (5.16) we then determine the function $V_r^x(du,s)$, and due to the first equality we will find the function $V_x^k(dl,s)$. Substituting the expression for the function $V_r^x(du,s)$ from the second equation of (5.16) into the first one, we get

$$f_x^k(dl,s) = V_x^k(dl,s) + c_x(s)e^{-c(s)\,r}\mathbf{E}\, f_0^{\delta+B}(dl,s) - \tilde{V}_x^k(s)\mathbf{E}\, f_0^{\delta+B}(dl,s).$$

Let us multiply this equation by $c_l(s)e^{-c(s)B}$ and integrate it with respect to $l \in \mathbb{R}_+$. This yields

$$\varphi_x^k(s) = \tilde{V}_x^k(s) + c_x(s)e^{-c(s)\,r}\mathbf{E}\,\varphi_0^{\delta+B}(s) - \tilde{V}_x^k(s)\mathbf{E}\,\varphi_0^{\delta+B}(s),$$

which is a linear equation for the function $\tilde{V}_x^k(s)$. In view of the formula (5.11) we find from the latter equation that

$$\tilde{V}_x^k(s) = c_x(s)e^{-c(s)\,r} - \frac{R_k^s(x)}{\mathbf{E}\,R_{\delta+B}^s}.$$

Substituting the right-hand side of this equality into the second equation of (5.16), we get

$$V_r^x(du,s) = \frac{R_k^s(x)}{\mathbf{E}\,R_{\delta+B}^s}\,\lambda e^{-\lambda u}du,$$

i.e. the first formula of (5.12). Substituting the right-hand side of the latter equality into the first equation of (5.16) we find the second formula of (5.12). This completes the proof. □

Theorem 5.7. *Let $\{Z_x(t)\}_{t\ge 0}$ be a semi-Markov walk with a linear drift given by (2.2), $\delta \sim \exp(\lambda)$, $\{R_u^s(x)\}_{u\in\mathbb{R}_+}$, $s, x \in \mathbb{R}_+$ be a resolvent of the process $\{Z_x(t)\}_{t\ge 0}$, $R_u^s \overset{\text{def}}{=} R_u^s(0)$, $r, k \in \mathbb{R}^+$, $r + k = B$,*

$$q_x^s(-r,u,k) = \mathbf{P}\left[-r \le \inf_{t\le\nu_s} Z_x(t),\; Z_x(\nu_s) \le u,\; \sup_{t\le\nu_s} Z_x(t) \le k\right]$$

be the joint distribution of the infimum, the supremum and the value of the process on the exponential time interval $[0,\nu_s]$. Then the following equalities hold for $s > 0$

$$q_x^s(-r,u,k) = \mathbf{P}\,[\nu_s \le u] + \frac{se^{-\lambda(k-u)}}{\cdots}R_k^s(x) - sA^s(s,x) + V_r^x(s)\left[\frac{\lambda e^{-s(r+u)}}{\cdots} - 1\right.$$

Proof. The formulae (3.5), (5.3), (5.6) imply for $\Re(a) = 0$, $\Re(p) > c(s)$

$$\tilde{U}_x^p(a,s) = \int_0^\infty e^{-pk}\mathbf{E}\left[e^{-aZ_x(\nu_s)}; Z_x^+(\nu_s) \le k\right] dk = \frac{1}{p}\,\frac{s}{s+a+p}$$
$$+ \frac{s}{s+c(s)}\,\frac{c(s)}{c(s)-a}\,\mathbb{R}_{a+p}^s(x) - \frac{s}{s+a}\,\mathbb{R}_{a+p}^s(x)\left[\frac{s}{s+a+p} + \frac{a}{p}\right].$$

Using the definition of the resolvent (5.18), and inverting the Laplace transforms in both sides with respect to p, we find that

$$U_x^k(a,s) = \frac{s}{s+a}\left(1 - e^{-k(s+a)}\right) + \frac{s}{s+c(s)}\,\frac{c(s)}{c(s)-a}\,e^{-ak}R_k^s(x)$$
$$- \frac{s}{s+a}\left[aA_k^s(-a,x) + sA_k^s(s,x)\right]e^{-ak}.$$

The first formula of (4.8) and the formula (5.12) yield

$$\tilde{Q}_x^s(a) = U_x^k(a,s) - e^{ar}\int_0^\infty \lambda e^{-\lambda v}e^{av}\,\frac{R_k^s(x)}{\hat{R}^s(\lambda,B)}\,\frac{1}{\lambda}e^{-\lambda B}U_0^{v+B}(a,s), dv$$
$$= U_x^k(a,s) + \frac{R_k^s(x)}{\hat{R}^s(\lambda,B)}\left[\int_0^B e^{-v(\lambda-a)}U_0^v(a,s)\,dv - \tilde{U}_0^{\lambda-a}(a,s)\right]e^{-ak},$$

where $\hat{R}^s(\lambda,B) = \int_B^\infty e^{-\lambda k}R_k^s\,dk$. Substituting the expressions for the functions $U_x^k(a,s)$, $\tilde{U}_0^{\lambda-a}(a,s)$ into the latter equation and performing some calculations we get

$$\tilde{Q}_x^s(a) = \frac{s}{s+a}\left(1 - e^{-k(s+a)}\right) + \frac{s\lambda}{s+\lambda}\frac{e^{-ak}}{\lambda-a}R_k^s(x)$$
$$- \frac{se^{-ak}}{s+a}\left[aA_k^s(-a,x) + sA_k^s(s,x)\right]$$
$$+ \frac{s\lambda}{s+a}V_r^x(s)\left[a\frac{A_k^s(-a)}{\lambda-a} + s\frac{A_k^s(s)}{s+\lambda} + \frac{e^{-sB}}{s+\lambda} - \frac{e^{aB}}{\lambda-a}\right]e^{-ak}.$$

It follows from (5.13) that

Substituting the expression for the functions $\tilde{Q}_x^s(a)$, $\mathbf{P}\left[\chi > \nu_s\right]$ from the previous equalities into the latter one, we find that

$$\int_{-r}^{k} e^{-au} q_x^s(-r,u,k)\,du = \frac{1-e^{-ak}}{a} - \frac{1-e^{-k(s+a)}}{s+a} + \frac{s}{s+\lambda}\frac{e^{-ak}}{\lambda-a} R_k^s(x)$$
$$+ \frac{se^{-ak}}{s+a}[A_k^s(s,x) - A_k^s(-a,x)] + \frac{s\lambda e^{-ak}}{s+a} V_r^x(s)\left[\frac{A_B^s(-a)}{\lambda-a} - \frac{A_B^s(s)}{\lambda-a}\right]$$
$$+ \frac{e^{-ak}}{s+a} V_r^x(s)\left[1 - \frac{\lambda e^{-sB}}{s+\lambda} + \frac{s}{a}\left(1-\frac{\lambda e^{aB}}{\lambda-a}\right)\right], \qquad \Re(a)=0.$$

Inverting the Laplace (Fourier) transforms with respect to a in both sides of this equality we derive the formula (5.17). □

We will now study the liming behaviour of the two-boundary functionals of the process $\{Z_x(t)\}_{t\geq 0}$. Denote by $w_t \in \mathbb{R}$, $t \geq 0$ a standard Wiener process, $\mathbf{E}w_1 = 0$, $\mathbf{D}w_1 = \sigma^2 > 0$, and let

$$\chi^* = \inf\{t:\ w_t \notin (-r,k)\}, \qquad k \in (0,1), \quad r = 1-k,$$

denote the first exit time from the interval $(-r,k)$ by the process w_t. It is well-known (see for instance [12]) that the Laplace transforms of χ^* satisfy the following formulae

$$\mathbf{E}\left[e^{-s\chi^*}; A^k\right] = \frac{\mathrm{sh}\left(\frac{r}{\sigma}\sqrt{2s}\right)}{\mathrm{sh}\left(\frac{1}{\sigma}\sqrt{2s}\right)}, \qquad \mathbf{E}\left[e^{-s\chi^*}; A_r\right] = \frac{\mathrm{sh}\left(\frac{k}{\sigma}\sqrt{2s}\right)}{\mathrm{sh}\left(\frac{1}{\sigma}\sqrt{2s}\right)},$$

where $A^k = \{w_{\chi^*} = k\}$, $A_r = \{w_{\chi^*} = -r\}$ are the events denoting the exit from the interval $(-r,k)$ through the upper boundary k and through the lower boundary $-r$.

Corollary 5.8. *Let $\{Z_x(t)\}_{t\geq 0}$ be a semi-Markov walk with a linear drift given by (2.2), $\delta \sim \exp(\lambda)$, $\mu = \mathbf{E}\eta$, $d = \mathbf{E}\eta^2 < \infty$,*

$$\chi(B) = \inf\{t : Z_x(t) \notin [-rB, kB]\}, \qquad k \in (0,1), \quad r = 1-k,$$

$\mathfrak{A}^k(B) = \{Z_x(\chi(B)) > kB\}$, $\mathfrak{A}_r(B) = \{Z_x(\chi(B)) < -rB\}$. Then the following limiting equalities hold for $\rho = \lambda\mu = 1$

$$\lim_{B\to\infty} \mathbf{P}\left[\frac{\chi(B)}{B^2} \in dt; \mathfrak{A}^k(B)\right] = \mathbf{P}\left[\chi^* \in dt; A^k\right]$$

where $\sigma = \sqrt{\lambda d}$. In particular, the limiting exit probabilities admit the following representations

$$\lim_{B\to\infty} \mathbf{P}\left[\mathfrak{A}^k(B)\right] = \frac{2}{\pi}\sum_{n\in\mathbb{N}} \frac{\sin(k\pi n)}{n} = r, \quad \lim_{B\to\infty} \mathbf{P}\left[\mathfrak{A}_r(B)\right] = \frac{2}{\pi}\sum_{n\in\mathbb{N}} \frac{\sin(r\pi n)}{n} = k. \tag{5.19}$$

Proof. Taking into the account the conditions of the corollary we can establish the following equality for the Laplace transforms of the random variable η_x as $s \to 0$

$$\tilde{f}_x(s) = 1 - s\mu_x + \frac{1}{2}\, s^2 d_x + o(s^2), \qquad \mu_x = \mathbf{E}\eta_x, \quad d_x = \mathbf{E}\eta_x^2, \tag{5.20}$$

where $\lim\limits_{a\to 0} \frac{o(a)}{a} = 0$. We will now verify that the following formulae valid for all $x, k \in \mathbb{R}_+$

$$\begin{aligned} \lim_{B\to\infty} \frac{1}{B^2}\, A^{s/B^2}_{kB}\left(s/B^2, x\right) &= \frac{1}{s}\left(\operatorname{ch}\left(\frac{k}{\sigma}\sqrt{2s}\right) - 1\right), \\ \lim_{B\to\infty} \frac{1}{B}\int_0^\infty \lambda e^{-\lambda v} R^{s/B^2}_{v+B}\, dv &= \frac{2}{\sigma\sqrt{2s}}\operatorname{sh}\left(\frac{1}{\sigma}\sqrt{2s}\right), \\ \lim_{B\to\infty} \frac{1}{B}\, R^{s/B^2}_{kB}(x) &= \frac{2}{\sigma\sqrt{2s}}\operatorname{sh}\left(\frac{k}{\sigma}\sqrt{2s}\right), \end{aligned} \tag{5.21}$$

where $\sigma = \sqrt{\lambda d}$. It follows from the definition of the function $A^s_k(a, x)$ (5.13), and of the resolvent $R^s_k(x)$ (5.18) that

$$\tilde{A}^s_p(s, x) = \int_0^\infty e^{-pk} A^s_k(a, x)\, dk = \frac{1}{s+p}\, \frac{\tilde{f}_x(s+p)}{\lambda\tilde{f}(s+p) + p - \lambda}.$$

The limiting equality a (5.20) and the latter formula imply the following chain of the equalities

$$\lim_{B\to\infty} \frac{1}{B^3}\, \tilde{A}^{s/B^2}_{p/B}\left(s/B^2, x\right) = \lim_{B\to\infty} \frac{1}{p + \frac{s}{B}}\, \frac{1 - \frac{p}{B}\mu_x + o(\frac{1}{B})}{\frac{1}{2}\,\lambda d p^2 - s + o(\frac{1}{B^2})} =$$

$$= \frac{1}{}\, \frac{1}{} = \int^{\infty} e^{-pk} \lim \frac{1}{}\, A^{s/B^2}_{kB}\left(s/B^2, x\right) dk$$

It follows from (5.20) that the function

$$\tilde{R}^s_p(\lambda) = \int_0^\infty e^{-pk} \int_0^\infty \lambda e^{-\lambda v} R^s_{v+k}\, dv\, dk = \frac{1}{\lambda \tilde{f}(s+p) + p - \lambda}$$

obeys the following chain of the equalities

$$\lim_{B\to\infty} \frac{1}{B^2} \tilde{R}^{s/B^2}_{p/B}(\lambda) = \frac{1}{\frac{1}{2}\sigma^2 p^2 - s} = \int_0^\infty e^{-pk} \lim_{B\to\infty} \frac{1}{B} \int_0^\infty \lambda e^{-\lambda v} R^s_{v+kB}\, dv\, dk.$$

Inverting the Laplace transforms in both sides of the latter equality for $\alpha > \sqrt{2s}$, we get

$$\lim_{B\to\infty} \frac{1}{B} \int_0^\infty \lambda e^{-\lambda v} R^s_{v+kB}\, dv = \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{pk}\, dp}{\frac{1}{2}\sigma^2 p^2 - s} = \frac{2}{\sigma\sqrt{2s}} \operatorname{sh}\left(\frac{k}{\sigma}\sqrt{2s}\right).$$

In particular, for $k = 1$ we derive the second formula of (5.21). The third formula of (5.21) can be verified analogously. It is worth of mentioning that the limit representations for the resolvent of spectrally one-sided Lévy processes have been obtained in [13]. Having derived the limiting equalities required we now verify the formulae (5.18). It follows from (5.13) and (5.21) that

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}_r(B)\right] = \lim_{B\to\infty} V^x_{rB}\left(s/B^2\right) =$$

$$= \lim_{B\to\infty} \frac{R^{s/B^2}_{kB}(x)}{\int_0^\infty \lambda e^{-\lambda v} R^{s/B^2}_{v+B}\, dv} = \frac{\operatorname{sh}\left(\frac{k}{\sigma}\sqrt{2s}\right)}{\operatorname{sh}\left(\frac{1}{\sigma}\sqrt{2s}\right)},$$

$$\lim_{B\to\infty} \mathbf{E}\left[e^{-s\frac{\chi(B)}{B^2}}; \mathfrak{A}^k(B)\right] = \lim_{B\to\infty} V^{kB}_x\left(s/B^2\right) =$$

$$= \operatorname{ch}\left(\frac{k}{\sigma}\sqrt{2s}\right) - \frac{\operatorname{sh}\left(\frac{k}{\sigma}\sqrt{2s}\right)}{\operatorname{sh}\left(\frac{1}{\sigma}\sqrt{2s}\right)} \operatorname{ch}\left(\frac{1}{\sigma}\sqrt{2s}\right) = \frac{\operatorname{sh}\left(\frac{r}{\sigma}\sqrt{2s}\right)}{\operatorname{sh}\left(\frac{1}{\sigma}\sqrt{2s}\right)}.$$

Inverting the Laplace transforms in the right-hand sided of these formulae [20], we get the equalities (5.18). Integrating these equalities with respect to $t \in \mathbb{R}_+$ we get (5.19). □

Corollary 5.9. *Let $\{Z_x(t)\}_{t\ge 0}$ be a semi-Markov walk with a linear drift given by (2.2), $\delta_\nu \sim \exp(\lambda)$, $\mu = \mathbf{E}\eta$, $d = \mathbf{E}\eta^2 < \infty$, $k \in (0,1)$, $r = 1-k$.*

of the infimum, the supremum and the value of the Wiener process with the dispersion $\sigma = \sqrt{\lambda d}$ and the following limiting equality holds

$$\lim_{B\to\infty} q_x^t(u,B) = \frac{4}{\pi}\sum_{n\in\mathbb{N}} \frac{(-1)^{n+1}}{n}\, e^{-\frac{t}{2}(\pi n\sigma)^2} \sin(k\pi n)\sin^2\left(\frac{r+u}{2}\, n\pi\right). \quad (5.22)$$

The distribution of the random variable $\chi(B)$ admits the following representation

$$\lim_{B\to\infty} \mathbf{P}\left[\frac{\chi(B)}{B^2} > t\right] = \frac{4}{\pi}\sum_{n\in\mathbb{N}\cup 0} \frac{e^{-\frac{t}{2}(\pi(2n+1)\sigma)^2}}{2n+1}\, \sin(k(2n+1)\pi).$$

Proof. In order to prove Corollary 5.8 we will need the limiting formulae (5.21) and the following equality

$$\lim_{B\to\infty} \frac{1}{B^2}\, e^{-\lambda B(k-u)} \int_0^{(k-u)B} e^{\lambda v}\, R_{B-v}^{s/B^2}\, dv = 0, \qquad u\in[-r,k]. \quad (5.23)$$

It is true, since the function R_u^s, $u\in\mathbb{R}_+$ monotonically increases for a fixed $s>0$ then

$$e^{-\lambda(k-u)B} \int_0^{(k-u)B} e^{\lambda v}\, R_{B-v}^{s/B^2}\, dv < \frac{1}{\lambda}\, R_B^{s/B^2}\left(1 - e^{-\lambda(k-u)B}\right).$$

This equality and the third formula of (5.21) for $k=1$ imply (5.23). Let us verify (5.22). It is clear that

$$s\int_0^{\infty} e^{-st} q_x^t(u,B)\, dt = q_x^{s/B^2}(-rB, uB, kB), \qquad k\in(0,1) \quad r = 1-k,$$

where the function $q_x^s(-r,u,k)$ is determined by (5.27). Thus,

$$\lim_{B\to\infty}\int_0^{\infty} e^{-st} q_x^t(uB)\, dt = \frac{1}{s}\lim_{B\to\infty} q_x^{s/B^2}(-rB,uB,kB) \stackrel{\text{def}}{=} q^*(s) =$$

$$= \frac{1}{s}\,\frac{\text{sh}\frac{k}{\sigma}\sqrt{2s}}{\text{sh}\frac{1}{\sigma}\sqrt{2s}}\left[\text{ch}\left(\frac{r+u}{\sigma}\sqrt{2s}\right) - 1\right] + \frac{1}{s}\left[1 - \text{ch}\left(\frac{u^+}{\sigma}\sqrt{2s}\right)\right],$$

where $u^+ = \max(0,u)$. In order to compute this limit we used the formulae (5.21), (5.23). Note, that the Laplace transform in the right-hand side of this equality was

Therefore, we established the weak convergence of the joint distribution $q_x^t(u, B)$ as $B \to \infty$ to the corresponding distribution of the Wiener process and also verified the formula (5.26). The second formula of Corollary 5.9 follows from the first one for $u = k$. $\square$

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References

1. V. F. Kadankov, and T. Kadankova. On the distribution of the first exit from an interval and the value of a jump over the boundary for processes with independent increments and random walks. *Ukr. Math. J.* **57**(2005), 1590–1620.
2. V. F. Kadankov, and T. Kadankova. Two-boundary problems for Poisson process with exponential component. *Ukr. Math. J.* **58(7)** (2006), 922–953.
3. V. F. Kadankov. Exit from an interval by a difference of two renewal processes. *Theory of Stochastic Processes.* **11(27)** , (2005), (3-4), 92–96.
4. I. I. Ezhov, and A.V. Skorohod. Markov processes homogeneous with respect to the second component. *Theor. Prob. and its. appl.* **14(1)** (1969), 3–14.
5. T. I. Nasirova. *Semi-Markov walk processes*. Baku, ELM. (1984).
6. V. F. Kadankov. Exit from the interval by the semi-Markov walk with linear drift. *Ukr. Math. J* (submitted).
7. A. A. Borovkov. *Stochastic processes in queueing theory.* Springer Verlag, (1976).
8. D. V. Gusak. On intersection of the level by a process defined by the sum of a random number of summands. *Ukr. Math. J.* **47**, (1995), 897-914.
9. V. A. Ditkin, and A. P. Prudnikov. *Operational Calculus,* Moscow, Vushaya shkola.(1966).
10. M. G. Krein. Integral equations on the semi-axis with a kernel dependening on the difference of the arguments. *Contributions of Math. Sciences*, **13(5)**, (1958). 3-120.
11. Kemperman. A Winer-Hopf type method for a general random walk with a two-sided boundary. *Ann. Math. Statist.* **34**, 4, (1963). 1168-1193.
12. K. Ito, and H. McKean. *Diffusion processes and their sample paths.* Springer-Verlag, Berlin · Heidelberg · NewYork. (1965).

16. E. A. Pecherskii, and B. A. Rogozin. On joint distributions of random variables assosiated with fluctuations of a process with independent increments. *Theor. Prob. and its Appl.* **14,** (1969), 410–423.

17. B. A. Rogozin. On the distribution of some functionals related to boundary problems for processes with independent increments. *Theor. Prob. and its Appl.* **11(4)**, (1966), 656-670.

18. F. Spitzer. *Principles of random walk.* Princeton New Jersy, Toronto·New York·London. (1964).

19. L. Takács. *Combinatorial methods in the theory of stochastic processes.* John Wiley & Sons, Inc. (1967).

20. V. A. Ditkin, and P. I Kuznecov. *Handbook in operational calculus.* Moskow. (1951).

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