Queueing Syst (2010) 65: 175–209
DOI 10.1007/s11134-010-9170-5

Busy period, virtual waiting time and number of customers in $G^\delta|M^\varkappa|1|B$ system

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Received: 2 October 2008 / Revised: 22 October 2009 / Published online: 10 April 2010

Abstract In this article, we determine the integral transforms of several two-boundary functionals for a difference of a compound Poisson process and a compound renewal process. Another part of the article is devoted to studying the above-mentioned process reflected at its infimum. We use the results obtained to study a $G^\delta|M^\varkappa|1|B$ system with batch arrivals and finite buffer in the case when $\delta \sim \mathrm{ge}(\lambda)$. We derive the distributions of the main characteristics of the queuing system, such as the busy period, the time of the first loss of a customer, the number of customers in the system, the virtual waiting time in transient and stationary regimes. The advantage is that these results are given in a closed form, namely, in terms of the resolvent sequences of the process.

Keywords Busy period · Time of the first loss of the customer · First exit time · Value of the overshoot · Linear component · Resolvent sequence

Mathematics Subject Classification (2000) 60G40 · 60K20

1 Introduction

The queueing models with batch arrivals and a finite buffer of the $G^\delta|M^\varkappa|1|B$ type arise in telecommunication networks, transportation systems and manufacturing systems. One of the most important performance issues of the queue with finite buffer

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is losses which are caused due to the buffer overflow, see [1, 13]. In this article, we address the problem of losses by determining the Laplace transforms of the time of the first loss and the number of lost customers. In addition, we consider other performance characteristics such as the busy period and the virtual waiting time. We propose a probabilistic approach based on the solution of the boundary problems for a two-component Markov process which governs functioning of the above-mentioned queueing system.

Experience shows that finding the aforementioned characteristics is not an easy task. So far, the embedded Markov chain approach has been widely used for their analysis (see [4] and references therein). The idea to use embedded Markov chains for studying queueing systems was introduced by Erlang in the 1930s. The main disadvantage of this approach is that the analysis of the evolution of the system is performed only on the states of the system which constitute a countable set. So far, major efforts of the researches were focused on establishing the equivalence between the stationary distribution of certain characteristics of the system and stationary distribution of the corresponding characteristics of the embedded Markov chain. It is clear that many characteristics cannot be determined based only on the states which constitute the countable set.

A two-component Markov process with continuous time appears naturally to mathematically describe the evolution of a queueing system. The first component of this process describes the phase state of the system, whereas the second component ensures the Markov property of the process.

The evolution of the number of customers in such systems is described by a process with two reflecting boundaries. In the general case, this process is a difference of two renewal processes. Reflections from the upper boundary are generated by the supremum (infimum) of the process. Reflections from the lower boundary govern the server's behavior. In general, such processes are not Markovian, but by adding a complementary linear component we obtain a Markov process which describes the functioning of the queueing system. Determining primary performance characteristics of the queueing system translates to studying boundary functionals of this governing process. We have followed this approach for the queueing systems $G^{\delta}|\mathrm{M}^{\varkappa}|1|B$ with a finite buffer in the case when $\delta \sim \mathrm{ge}(\lambda)$.

For the queueing systems of $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ and $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ type, the governing process is the difference of the compound Poisson process and the compound renewal process complemented with a linear component. The main two-boundary characteristic of this random process is the joint distribution of $\{\chi, L, T\}$, i.e., of the first exit time from the interval, the value of the overshoot and the value of the linear component at this instant. Other two-boundary characteristics, such as the number of intersections of the interval, the joint distribution of the supremum, infimum and the position of the process, the sojourn time of staying inside the interval, the first entry time into the interval and also boundary characteristics of the reflected process can be derived in terms of this functional. In order to determine the Laplace transforms of the main two-boundary characteristic for the difference of a compound Poisson process and a compound renewal process [8, 9], we employ the method suggested in [7] for general Lévy processes. The main contribution of this approach is that the joint distribution of $\{\chi, L, T\}$ is derived in terms of more simple joint distributions of the one-boundary functionals of the process which are known. For an overview of the existing results on the two-boundary problems, we refer to [5, 19]. For the state-of-the-art in queueing systems with batch arrivals and finite buffer, see [16–18, 22] and the references therein.

The rest of the article is structured as follows. In Sect. 1 we introduce the process and necessary notations. In Sect. 2 we consider the lower and upper one-boundary characteristics of the process. The main result is the solution of the two-sided exit problem for the underlying process. Section 3 deals with the process reflected at its infimum. We then consider the first passage time of the upper boundary, the distribution of the increments of the process and its asymptotic behavior. Finally, in Sect. 4 we apply the results obtained in the previous sections to study primary performance characteristics of the queueing system $G^{\delta}|M^{\varkappa}|1|B$, such as the busy period, the time of the first loss of a customer, the number of the lost customers, the number of customers in the system and the virtual waiting time in transient and stationary regimes. In addition, we derive the above-mentioned characteristics for the partial case of $G^{\delta}|M^{\varkappa}|1|B$, namely when $\mathbf{P}[\delta = 1] = 1$.

2 Preliminaries

Let $\varkappa, \delta \in \mathbb{N} = \{1, 2, \dots\}$ be positive independent integer random variables, and $\eta \in (0, \infty)$ be a positive random variable independent of $\varkappa, \delta$ with the distribution function $F(x) = \mathbf{P}[\eta \le x]$, $x \ge 0$. We will assume that $\mathbf{E}\varkappa$, $\mathbf{E}\delta$, $\mathbf{E}\eta < \infty$. Introduce the sequences $\{\eta, \eta'_n\}$, $\{\varkappa, \varkappa'_n\}$, $\{\delta, \delta'_n\}$, $n \in \mathbb{N}$, of i.i.d. (inside of each sequence) variables and define the monotone sequences

$$\begin{aligned} &\eta_0(x) = 0, \qquad \eta_1(x) = \eta_x, \\ &\eta_{n+1}(x) = \eta_x + \eta'_1 + \dots + \eta'_n, \quad n \in \mathbb{N}, \\ &\varkappa_0 = 0, \qquad \varkappa_n = \varkappa'_1 + \dots + \varkappa'_n; \\ &\delta_0 = 0, \qquad \delta_n = \delta'_1 + \dots + \delta'_n; \quad n \in \mathbb{N}, \end{aligned} \tag{1}$$

where $\eta_x \in (0, \infty)$ is a random variable with the following distribution function

$$F_x(u) = \mathbf{P}[\eta_x \le u] = \big[F(x+u) - F(x)\big]\big(1 - F(x)\big)^{-1}, \quad u \ge 0.$$

Denote by $\{\pi(t)\}_{t \ge 0} \in \mathbb{Z}^+ = \{0, 1, \dots\}$ a compound Poisson process

$$\mathbf{E}\, z^{\pi(t)} = e^{tk(z)}, \quad k(z) = \mu\big(\mathbf{E}\, z^{\varkappa} - 1\big), \quad |z| \le 1,$$

where $\mu > 0$ is the intensity of the jumps and $\varkappa$ is a jump size. For all $t \ge 0$ define a renewal process generated by the random sequence $\{\eta_n(x)\}_{n \in \mathbb{Z}^+}$:

$$N_x(t) = \max\big\{n \in \mathbb{Z}^+ : \eta_n(x) \le t\big\} \in \mathbb{Z}^+, \quad x \ge 0.$$

Introduce a right-continuous step process

$$D_x(t) = \pi(t) - \delta_{N_x(t)} \in \mathbb{Z} = \{0, \pm 1, \dots\}, \quad t \ge 0; \qquad D_x(0) = 0. \tag{2}$$

Note, that inter-arrival times of the positive jumps are exponentially distributed with parameter μ, the positive jumps themselves are of a random size $\varkappa$, and there occur negative jumps of size δ'_n at time instants $\eta_n(x)$, $n \in \mathbb{N}$. We will call the process $\{D_x(t)\}_{t\geq 0}$ a difference of the compound Poisson process and a compound renewal process. Observe, that this process is not a Markov process in general. For all $t \geq 0$, introduce a right-continuous linear component

$$\eta_x^+(t) = \begin{cases} t + x, & 0 \leq t < \eta_x, \\ t - \eta_{N_x(t)}(x), & t \geq \eta_x \end{cases} \in \mathbb{R}_+ = [0, \infty), \quad x \geq 0. \tag{3}$$

The process $\{\eta_x^+(t)\}_{t\geq 0}$ increases linearly on the intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{Z}^+$, it is killed to zero at the points $\eta_n(x)$, $n \in \mathbb{N}$, and the value of the process at the instant $t_0 \geq \eta_x$ is equal to the time elapsed from the moment of the last negative jump of Process (2) until t_0. We will call Process (3) a linear component. By adding this linear component to the process $\{D_x(t)\}_{t\geq 0}$, we obtain a right-continuous Markov process

$$\begin{aligned} &\{X_t\}_{t\geq 0} = \big\{D_x(t), \eta_x^+(t)\big\}_{t\geq 0} \in \mathbb{Z} \times \mathbb{R}_+, \\ &X_0 = \{0, x\}, \quad x \geq 0, \end{aligned} \tag{4}$$

which governs the process $\{D_x(t)\}_{t\geq 0}$. The process defined in (4) is a Markov process. Note, that it is homogeneous with respect to the first component [6]. This fact will be used constantly when setting up the equations.

For all $x \in \mathbb{R}_+$, $|z| \leq 1$, denote

$$\begin{aligned} &\tilde{f}_x(s) = \mathbf{E}e^{-s\eta_x}, \qquad \tilde{f}(s) = \tilde{f}_0(s), \\ &\tilde{f}_x(s, z) = \tilde{f}_x\big(s - k(z)\big) = \mathbf{E}\big[e^{-s\eta_x} z^{\pi(\eta_x)}\big]. \end{aligned}$$

Lemma 1 (Kadankov et al. [10]) *Let $D_x(t)$, $\eta_x^+(t)$, $t \geq 0$ be the random processes defined by (2)–(3), and $\nu_s \sim \exp(s)$ be an exponential random variable independent of these processes. Then for all $x \in \mathbb{R}_+$, $s > 0$, $|z| = 1$, $p \geq 0$ the following equality holds*

$$\begin{aligned} \mathbb{D}_x^s(z, p) &= \mathbf{E}z^{D_x(\nu_s)} e^{-p\eta_x^+(\nu_s)} \\ &= \frac{se^{-px}}{s + p - k(z)} \big(1 - \tilde{f}_x(s + p, z)\big) \\ &\quad + \frac{s}{s + p - k(z)} \tilde{f}_x(s, z) \mathbf{E}z^{-\delta} \frac{1 - \tilde{f}(s + p, z)}{1 - \tilde{f}(s, z)\mathbf{E}z^{-\delta}}. \end{aligned} \tag{5}$$

In particular, for all $x \in \mathbb{R}_+$, $s > 0$, the following formulae are valid

$$\mathbb{D}_x^s(z) = \mathbf{E}z^{D_x(\nu_s)} = \frac{s}{s - k(z)} + \frac{s\tilde{f}_x(s, z)}{s - k(z)} \frac{\mathbf{E}z^{-\delta} - 1}{1 - \tilde{f}(s, z)\mathbf{E}z^{-\delta}}, \quad |z| = 1. \tag{6}$$

Here, and in the sequel, we will assume that the random variable $\delta \in \mathbb{N}$ is geometrically distributed with parameter $\lambda \in [0, 1)$:

$$\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}, \quad n \in \mathbb{N},$$
$$\mathbf{E}z^{\delta} = z\frac{1-\lambda}{1-\lambda z}, \quad |z| \le 1.$$

This assumption means that the process $\{D_x(t)\}_{t\ge 0}$ has geometrically distributed negative jumps which occur at time instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$. Here, and in the sequel, we will use the following short notation $\delta \sim \mathrm{ge}(\lambda)$. In this case, it is possible to obtain closed-form solutions for the one- and the two-sided boundary problems. In order to determine the main two-boundary characteristic of the process, we will require the one-boundary functionals of the process. These are the Laplace transforms of the joint distributions of the upper and lower one-boundary functionals of the process $\{X_t\}_{t\ge 0}$. In the sequel we will use the following result.

Lemma 2 *Let* $\tilde{f}(s) = \mathbf{E}e^{-s\eta}$. *Then for* $s > 0$ *the equation*

$$z - \lambda = (1-\lambda)\tilde{f}\big(s - k(z)\big), \quad |z| < 1, \quad \lambda \in [0, 1) \tag{7}$$

has a unique solution $z = c(s)$ *inside the circle* $|z| < 1$. *This solution is positive and* $c(s) \in (\lambda, 1)$. *If* $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$, $\rho = \mu(1-\lambda)\mathbf{E}[\varkappa]\mathbf{E}[\eta]$, *then for* $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (\lambda, 1)$; *and for* $\rho \le 1$, $\lim_{s\to 0} c(s) = 1$.

A detailed proof of an analogous proposition for semi-continuous random walks can be found in the monograph of Spitzer [20]. The reasoning in that proof can be also applied to (7) (see also Lemma 1, [9]).

3 One- and two-sided exit problems

Let $X_0 = \{0, x\}$, $x \in \mathbb{R}_+$, $k \in \mathbb{Z}^+$. Define

$$\tau_k(x) = \inf\big\{t : D_x(t) < -k\big\},$$
$$T_k(x) = -D_x\big(\tau_k(x)\big) - k,$$
$$\inf\{\emptyset\} = \infty,$$

i.e., the first undershoot time of the negative level $-k$ by the process $\{D_x(t)\}_{t\ge 0}$. We will use the convention that on the event $\{\tau_k(x) = \infty\}$ $T_k(x) = \infty$. Denote $\mathfrak{B}_k(x) = \{\tau_k(x) < \infty\}$. We define the lower one-boundary functional of the process as follows

$$f_k(x, m, s) = \mathbf{E}\big[e^{-s\tau_k(x)};\ T_k(x) = m, \mathfrak{B}_k(x)\big], \quad m \in \mathbb{N}.$$

This functional is determined by means of the following lemma.

Lemma 3 (Kadankov and Kadankova [8]) *Let* $\{D_x(t)\}_{t\ge 0}$ *be the difference of a compound Poisson process and a compound renewal process*, $\delta \sim \mathrm{ge}(\lambda)$. *Then*

(i) *The Laplace transform of the joint distribution of* $\{\tau_k(x), T_k(x)\}$, $k \in \mathbb{Z}^+$, $x \geq 0$ *satisfies the following equality for* $s > 0$, $m \in \mathbb{N}$

$$f_k(x, m, s) = \tilde{f}_x\big(s - k\big(c(s)\big)\big)c(s)^k(1-\lambda)\lambda^{m-1}, \tag{8}$$

where $c(s) \in (\lambda, 1)$ *is the unique solution of* (7) *inside the circle* $|z| < 1$, $\tilde{f}_x(s) = \mathbf{E}\,e^{-s\eta_x}$, $\tilde{f}(s) = \mathbf{E}\,e^{-s\eta} = \tilde{f}_0(s)$.

(ii) *If* $\rho > 1$, *then* $\mathbf{P}[\tau_k(x) < \infty] = \tilde{f}_x(-k(c))\,c^k < 1$, *and* $\tau_k(x)$, *for all* $k \in \mathbb{Z}^+$, $x \geq 0$, *is a defective random variable*; *if* $\rho \leq 1$, *then* $\mathbf{P}[\tau_k(x) < \infty] = 1$, *and* $\tau_k(x)$ *is a proper variable for all* $k \in \mathbb{Z}^+$, $x \geq 0$.

Note that the value of the overshoot $T_k(x)$ does not depend on $\tau_k(x)$ for any $k \in \mathbb{Z}^+$, and it is geometrically distributed: $T_k(x) \sim \mathrm{ge}(\lambda)$.

We now introduce a sequence which will be used to obtain the results in the sequel. The idea to employ this sequence for semi-continuous random walks and semi-continuous Lévy processes is due to Takács [21]. Since the function

$$\begin{aligned}\tilde{f}_x\big(s - k(z)\big) &= \mathbf{E}\big[e^{-s\eta_x} z^{\pi(\eta_x)}\big] \\ &= \sum_{i\in\mathbb{Z}^+} z^i \int_0^\infty e^{-st}\mathbf{P}\big[\eta_x \in dt, \pi(t) = i\big], \quad |z| \leq 1\end{aligned}$$

is analytic inside the unit circle for all $s, x \geq 0$, the function

$$\mathbb{Q}_z^s(x) = \frac{(1-\lambda)\tilde{f}_x(s-k(z))}{(1-\lambda)\tilde{f}(s-k(z)) + \lambda - z}, \quad s, x \geq 0, \quad |z| < c(s) \tag{9}$$

is analytic on the open set $|z| < c(s)$. In this region, it can be represented as a power series

$$\mathbb{Q}_z^s(x) = \sum_{k\in\mathbb{Z}^+} z^k Q_k^s(x), \quad s, x \geq 0, \quad |z| < c(s).$$

The coefficients of this expansion can be calculated by means of the inversion formula

$$Q_k^s(x) = \frac{1}{2\pi i}\oint_{|z|=\alpha} \frac{1}{z^{k+1}} \frac{(1-\lambda)\tilde{f}_x(s-k(z))}{(1-\lambda)\tilde{f}(s-k(z)) + \lambda - z}\,dz, \quad \alpha \in \big(0, c(s)\big). \tag{10}$$

We will call the sequence $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$, defined by Formula (10), the resolvent sequence of the process $\{D_x(t)\}_{t\geq 0}$.

We now explain a probabilistic meaning of this sequence. Introduce a random sequence as follows (see [10]):

$$\begin{aligned} &Y_0(x) = 0, \qquad Y_1(x) = \pi(\eta_x) - \delta, \\ &Y_{n+1}(x) = Y_1(x) + \sum_{i=1}^n Y_i', \qquad Y_n = X_n(0), \end{aligned}$$

where $Y = \pi(\eta) - \delta \in \mathbb{Z}$, $\{Y, Y_n'\}$, $n \in \mathbb{N}$, is a sequence of i.i.d. random variables. Define a right-continuous step process in the following way:

$$\{R_x(t)\}_{t\geq 0} = \{Y_{N_x(t)}(x)\}_{t\geq 0} \in \mathbb{Z}, \quad R_x(0) = 0, \quad x \in \mathbb{R}_+.$$

The sample paths of the process are constant on the time intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{Z}^+$ and they jump at the instants $\eta_n(x)$, $n \in \mathbb{N}$. These jumps have the same distribution as $Y \doteq \pi(\eta) - \delta$, where $n \in \{2, 3, \dots\}$, and $Y_1(x) \doteq \pi(\eta_x) - \delta$ for $n = 1$. Here, and in the sequel, we will call the process $\{R_x(t)\}_{t\geq 0}$ a semi-Markov random walk generated by the sequences $\{\eta_n(x)\}$, $\{Y_n(x)\}$, $n \in \mathbb{Z}^+$. Let $R_t^+ = \sup_{u\leq t} R_0(u)$ be the supremum of $\{R_0(t)\}_{t\geq 0}$. The generating function of R_t^+ was found in [10]:

$$\mathbf{E} z^{R_{\nu_s}^+} = \frac{1-\lambda}{1-c(s)} \frac{(1-\tilde{f}(s))(z-c(s))}{z-\lambda-(1-\lambda)\tilde{f}(s-k(z))}, \quad |z| \leq 1,$$

where ν_s is an exponential variable with parameter $s > 0$, independent from the process $\{R_x(t)\}_{t\geq 0}$. It follows from (9) and from the latter formula that for $|z| < c(s)$

$$\mathbb{Q}_z^s(x) = \frac{1-c(s)}{1-\tilde{f}(s)} \frac{\tilde{f}_x(s-k(z))}{c(s)-z} \mathbf{E} z^{R_{\nu_s}^+}, \quad |z| < c(s).$$

Comparing the coefficients of z^k, $k \in \mathbb{Z}^+$, on both sides yields

$$Q_k^s(x) = \frac{1-c(s)}{1-\tilde{f}(s)} \sum_{i=0}^{k} c(s)^{i-k-1} \sum_{j=0}^{i} \mathbf{E}\big[e^{-s\eta_x}, \pi(\eta_x) = j\big] \mathbf{P}\big[R_{\nu_s}^+ = i-j\big].$$

Denote by $\pi^s(\eta_x) \in \mathbb{Z}^+$, $s > 0$, a random variable given by its distribution:

$$\mathbf{P}\big[\pi^s(\eta_x) = k\big] = \frac{1}{\tilde{f}_x(s)} \mathbf{E}\big[e^{-s\eta_x}, \pi(\eta_x) = k\big], \quad k \in \mathbb{Z}^+.$$

Then the previous equality implies that

$$Q_k^s(x) = \frac{\tilde{f}_x(s)}{1-\tilde{f}(s)} \frac{1-c(s)}{c(s)^{k+1}} \sum_{i=0}^{k} c(s)^i \mathbf{P}\big[\pi^s(\eta_x) + R_{\nu_s}^+ = i\big], \quad k \in \mathbb{Z}^+,$$

which explains the probabilistic meaning of the resolvent sequence. Asymptotically, one has that $Q_k^s(x) \sim c(s)^{-k}$ as $k \to \infty$.

Let $X_0 = \{0, x\}$, $x \geq 0$, $k \in \mathbb{Z}^+$, and introduce the upper one-boundary functionals of the process $\{X_t\}_{t\geq 0}$:

$$\begin{aligned}
\tau^k(x) &= \inf\{t : D_x(t) > k\}, \\
T^k(x) &= D_x(\tau^k(x)) - k, \\
\eta^k(x) &= \eta_x^+(\tau^k(x)),
\end{aligned}$$

i.e., the instant of the first crossing of the level k by the process $\{D_x(t)\}_{t\ge 0}$, the value of the overshoot across the upper level and the value of the linear component $\eta_x^+(\cdot)$ at the instant of the first crossing (time since the last renewal). Denote $\mathfrak{B}^k(x) = \{\tau^k(x) < \infty\}$,

$$f^k(x, dl, m, s) = \mathbf{E}\big[e^{-s\tau^k(x)};\ \eta^k(x) \in dl, T^k(x) = m, \mathfrak{B}^k(x)\big], \quad m \in \mathbb{N}.$$

We will now determine the Laplace transforms of the upper one-boundary functionals of the process $\{D_x(t)\}_{t\ge 0}$. Let $k \in \mathbb{Z}^+$ and $\tilde{\tau}^k = \inf\{t : \pi(t) > k\}$, $\tilde{T}^k = \pi(\tilde{\tau}^k) - k$ be the first crossing time through the upper level k by the compound Poisson process $\{\pi(t)\}_{t\ge 0}$ and the value of the overshoot at this instant. Denote by

$$\rho_k(t) = \mathbf{P}[\pi(t) = k], \qquad \sum_{k=0}^{\infty} z^k \rho_k(t) = \mathbf{E} z^{\pi(t)} = e^{tk(z)}, \quad |z| \le 1,$$

$$p_k^m(dt) = \mathbf{P}\big[\tilde{\tau}^k \in dt,\ \tilde{T}^k = m\big] = \mu \sum_{i=0}^{k} \rho_i(t) \mathbf{P}[\varkappa = k - i + m]\, dt, \quad m \in \mathbb{N}.$$

Lemma 4 (Kadankov and Kadankova [8]) *Let $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$ be the resolvent sequence of the process $\{D_x(t)\}_{t\ge 0}$ given by (10), $k \in \mathbb{Z}^+$, $s, x \ge 0$. The following equalities hold*:

$$f^k(x, dl, m, s) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m\big(d(l - x)\big) + \Phi_\lambda^s(0, dl, m)\, Q_k^s(x) - e^{-sl}\big[1 - F(l)\big] \sum_{i=0}^{k} Q_i^s(x) p_{k-i}^m(dl), \tag{11}$$

where $\Phi_\lambda^s(0, dl, m) = e^{-sl}[1 - F(l)] \sum_{k=0}^{\infty} c(s)^k p_k^m(dl)$;

$$f^k(x, s) = \mathbf{E} e^{-s\tau^k(x)} = 1 - A_x^k(s) - \frac{s}{s - k(c(s))} \frac{Q_k^s(x)}{1 - \lambda}, \tag{12}$$

and $A_x^k(s) = \sum_{i=0}^{k} \tilde{\rho}_i(s)[1 - Q_{k-i}^s(x)(1 - \lambda)^{-1}]$, $\tilde{\rho}_k(s) = s \int_0^\infty e^{-st} \rho_k(t)\, dt$.

For $\mathbf{E}[\varkappa]$, $\mathbf{E}[\eta] < \infty$ and $\rho < 1$, $\tau^k(x)$ is a defective random variable and

$$\mathbf{P}\big[\tau^k(x) < \infty\big] = 1 - (1 - \rho)(1 - \lambda)^{-1} Q_k(x) < 1,$$

where $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $x \ge 0$ is the resolvent sequence of the process $\{D_x(t)\}_{t\ge 0}$ given by (10) for $s = 0$:

$$Q_k(x) = \frac{1}{2\pi i} \oint_{|z|=\alpha} \frac{dz}{z^{k+1}} \frac{(1 - \lambda) \tilde{f}_x(-k(z))}{(1 - \lambda) \tilde{f}(-k(z)) + \lambda - z}, \quad \alpha \in \big(0, c(0)\big); \tag{13}$$

if $\rho \ge 1$ then, for all $k \in \mathbb{Z}^+$ and $x \ge 0$, $\tau^k(x)$ is a proper random variable.

Along with Expression (13) there exists another way to calculate $Q_k(x)$, which is more applicable from a practical point of view. We will now derive the recurrence formula for $Q_k(x)$. It follows from (9) for $s, z = 0$ that

$$Q_0(x) = (1-\lambda)\big(\lambda + (1-\lambda) f_0\big)^{-1} f_0(x),$$

where, for all $k \in \mathbb{Z}^+$, $f_k(x) = \mathbf{P}[\pi(\eta_x) = k] = \int_0^\infty \mathbf{P}[\eta_x \in dt, \pi(t) = k]$, $f_k = f_k(0)$. Again, it follows from (9) for $s = 0$ that

$$(1-\lambda)\tilde{f}_x\big(-k(z)\big) = (1-\lambda)\tilde{f}\big(-k(z)\big)\mathbb{Q}_z(x) + (\lambda - z)\mathbb{Q}_z(x).$$

Comparing the coefficients of z^k, $k \in \mathbb{N}$, on both sides implies that

$$(1-\lambda) f_k(x) = (1-\lambda)\sum_{i=0}^{k} Q_i(x) f_{k-i} + \lambda Q_k(x) - Q_{k-1}(x).$$

Combining similar terms yields

$$\big(\lambda + (1-\lambda) f_0\big) Q_k(x) = (1-\lambda) f_k(x) + Q_{k-1}(x) - (1-\lambda)\sum_{i=0}^{k-1} Q_i(x) f_{k-i}.$$

The latter formula is a recurrence relation which allows to calculate successively the terms $Q_k(x)$ given the previous terms $Q_0(x), \ldots, Q_{k-1}(x)$. For instance, given the expression for $Q_0(x)$ one finds that

$$Q_1(x) = \frac{1-\lambda}{\lambda + (1-\lambda) f_0}\left[f_1(x) + \frac{1-(1-\lambda) f_0}{\lambda + (1-\lambda) f_0} f_0(x)\right].$$

Denote by $D_x^-(t) = \inf_{[0,t]} D_x(\cdot)$ the running infimum of the process on $[0, t]$.

Theorem 1 *Let $r \in \mathbb{Z}^+$, $E_r^-(x, z, s) = \mathbf{E}[z^{D_x(\nu_s)}; D_x^-(\nu_s) \geq -r]$, $|z| \leq 1$ be the generating function of the joint distribution of $\{D_x(\nu_s), D_x^-(\nu_s)\}$. Then*

(i) *The generating function $E_r^-(x, z, s)$ is such that*

$$E_r^-(x, z, s) = (1-z)\mathbb{A}_x^z(s) + (1-z) f_r(x, s)\frac{1-\lambda}{\lambda - z}\mathbb{A}_0^z(s) z^{-r}, \tag{14}$$

where $f_r(x, s) = \mathbf{E}[e^{-s\tau_r(x)}; \mathfrak{B}_r(x)] = c_x(s) c(s)^r$, $c_x(s) = \tilde{f}_x(s - k(c(s)))$,

$$\mathbb{A}_x^z(s) = \sum_{k \in \mathbb{Z}+} z^k A_x^k(s) = \frac{s}{s - k(z)}\left(\frac{1}{1-z} - \frac{1}{1-\lambda}\mathbb{Q}_z^s(x)\right).$$

(ii) *The joint distribution $\mathbb{E}_r^-(x, u, s) = \mathbf{P}[D_x(\nu_s) \geq u, D_x^-(\nu_s) \geq -r]$, $u \in [-r, \infty)$ satisfies the following equality*

$$\mathbb{E}_r^-(x, u, s) = 1 - A_x^{u-1}(s) - f_r(x, s)\big(1 - \mathbf{E}A_0^{u+r+\delta-1}(s)\big), \tag{15}$$

where $\mathbf{E}A_x^{k+\delta}(s) = (1-\lambda)\sum_{i \in \mathbb{N}} \lambda^{i-1} A_x^{k+i}(s)$, $A_x^u(s) = 0$ *for* $u < 0$.

(iii) *Under Condition* (A), *namely*,

$$\begin{aligned}\rho &= (1-\lambda)\mu \mathbf{E}\eta \mathbf{E}\varkappa = 1,\\ \sigma^2 &= \mu\left[\mathbf{E}\varkappa(\varkappa-1) + \frac{\mathbf{E}\varkappa\mathbf{E}\eta^2}{(1-\lambda)(\mathbf{E}\eta)^2}\right] < \infty,\end{aligned} \tag{A}$$

the following limiting equality holds as $B \to \infty$, $r > 0$,

$$\mathbf{P}\big[D_x\big(tB^2\big) \geq [uB], D_x^-\big(tB^2\big) \geq [-rB]\big] \to \frac{1}{\sigma\sqrt{2\pi t}} \int_u^{u+2r} e^{-v^2/2\sigma^2 t}\, dv,$$

where $[a]$ *is the integer part of the number* a, $u \in [-r, \infty)$.

Proof In accordance with the total probability law and Markov property of $\tau_r(x)$ for all $r \in \mathbb{Z}^+$, we derive the following equation

$$\mathbf{E}z^{D_x(\nu_s)} = \mathbf{E}\big[z^{D_x(\nu_s)}; D_x^-(\nu_s) \geq -r\big] + f_r(x,s)\frac{1-\lambda}{z-\lambda} z^{-r}\mathbf{E}z^{D_0(\nu_s)}, \quad |z| = 1.$$

To write this equation, we used the path decomposition principle. It means that the increments of the process $D_x(\nu_s)$ on the interval $[0, \nu_s]$ are realized either on the sample paths which do not cross the negative level $-r$, or on the sample paths which do intersect the level $-r$, and further evolution of the process is its probabilistic replica on $[0, \nu_s]$. From the latter equation, we find that

$$E_r^-(x,z,s) = \mathbb{D}_x^s(z) - f_r(x,s)\frac{1-\lambda}{z-\lambda} z^{-r}\mathbb{D}_0^s(z).$$

Observe that the function which enters the left-hand side of the equation is analytic in $\{z : |z| \leq 1\}$. Therefore, the right-hand side is also an analytic function for all $|z| \leq 1$. Employing (6) and the definition of the resolvent (9), we get

$$E_r^-(x,z,s) = (1-z)\mathbb{A}_x^z(s) + (1-z) f_r(x,s)\frac{1-\lambda}{\lambda - z}\mathbb{A}_0^z(s) z^{-r}, \quad |z| \leq 1.$$

It is not difficult to establish the following equality

$$\sum_{u=-r}^{\infty} z^u \mathbb{E}_r^-(x,u,s) = \frac{1-f_r(x,s)}{1-z} z^{-r} - \frac{z}{1-z} E_r^-(x,z,s), \quad |z| \leq 1.$$

In view of the latter and the previous equality, we find that

$$\sum_{u=-r}^{\infty} z^u \mathbb{E}_r^-(x,u,s) = \frac{1-f_r(x,s)}{1-z} z^{-r} - z\mathbb{A}_x^z(s) - f_r(x,s)\frac{1-\lambda}{\lambda-z}\mathbb{A}_0^z(s) z^{-r+1}.$$

Comparing the coefficients of z^u, $u \in [-r, \infty)$, and taking into account that $\mathbb{A}_x^\lambda(s) = 0$, we obtain (15).

Denote $\tilde{e}_r^t(x,u,B) = \mathbf{P}[D_x(tB^2) \geq [uB], D_x^-(tB^2) \geq [-rB]]$, $r > 0$, $u \geq -r$. It is clear that

$$\lim_{B\to\infty} \int_0^\infty e^{-st} \tilde{e}_k^t(x,u,B)\,dt = \frac{1}{s} \lim_{B\to\infty} \mathbb{E}_{[kB]}^{s/B^2}\big(x,[uB]\big).$$

To be able to perform asymptotic analysis and verify the third statement of the theorem we will use the following equalities (see [10] where they were derived)

$$\begin{aligned}
&c\big(s/B^2\big) = 1 - B^{-1}\sqrt{2s}/\sigma + o\big(B^{-1}\big),\\
&\lim_{B\to\infty} B^{-1} Q_{[kB]}^{s/B^2}(x) = \frac{2\sinh(k\sqrt{2s}/\sigma)}{\sigma\sqrt{2s}\mathbf{E}\eta} = \lim_{B\to\infty} B^{-1}\mathbf{E}Q_{\delta+[kB]}^{s/B^2}, \qquad (16)\\
&\lim_{B\to\infty} A_x^{[kB]}\big(s/B^2\big) = 1 - \cosh(k\sqrt{2s}/\sigma) = \lim_{B\to\infty} \mathbf{E}A_0^{\delta+[kB]}\big(s/B^2\big).
\end{aligned}$$

Employing these equalities and also (15), we find that

$$\begin{aligned}
&\lim_{B\to\infty} \int_0^\infty e^{-st} \tilde{e}_k^t(x,u,B)\,dt\\
&\quad = s^{-1}\mathbf{I}_{\{u>0\}}\big(e^{-u\sqrt{2s}/\sigma}/2 - e^{-(2r+u)\sqrt{2s}/\sigma}/2\big)\\
&\qquad + s^{-1}\mathbf{I}_{\{u\in[-r,0]\}}\big(1 - e^{u\sqrt{2s}/\sigma}/2 - e^{-(2r+u)\sqrt{2s}/\sigma}/2\big), \quad u \geq -r.
\end{aligned}$$

Denote by $w_{\{t\geq 0\}}$ the symmetric Wiener process with dispersion σ. Let $\tau^a = \inf\{t : w_t \geq a\}$ be the first passage time of the level $a \in \mathbb{R}_+$. The Lévy identity $\mathbf{P}[\tau \leq t] = 2\mathbf{P}[w_t \geq a]$ implies the following relation for the Laplace transforms

$$\frac{1}{s} e^{-a\sqrt{2s}/\sigma} = 2\int_0^\infty e^{-st}\mathbf{P}[w_t \geq a]\,dt.$$

Using this formula to invert the Laplace transforms in the previous equality, we obtain the limiting equality of the theorem. □

We will now consider the two-sided exit problem for the underlying process (2). Let $B \in \mathbb{Z}^+$ be fixed, $k \in [0,B]$, $r = B - k$, $X_0 = \{0,x\}$, $x \geq 0$, and introduce the random variable

$$\chi_r^B(x) = \inf\big\{t : D_x(t) \notin [-r,k]\big\} \stackrel{\text{def}}{=} \chi$$

as the first exit time from the interval $[-r,k]$ by the process $\{D_x(t)\}_{t\geq 0}$. This random variable takes values from a countable set $\{\xi_n, n \in \mathbb{N}\} \cup \{\eta_n(x), n \in \mathbb{N}\}$, and it is a Markov time of the process $\{X_t\}_{t\geq 0}$. (Here ξ_n, $n \in \mathbb{N}$, is a sequence of the jump times of the compound Poisson process.) Observe that exit from the interval can occur either through the upper boundary k, or through the lower boundary $-r$. In view of this remark, we introduce the events: $\mathfrak{A}^k = \{D_x(\chi) > k\}$, i.e., the process $\{D_x(t)\}_{t\geq 0}$ exits the interval $[-r,k]$ through the upper boundary k; $\mathfrak{A}_r = \{D_x(\chi) < -r\}$, i.e., the process $\{D_x(t)\}_{t\geq 0}$ exits the interval $[-r,k]$ through the lower boundary $-r$.

Denote by

$$T = \big(D_x(\chi) - k\big)\mathbf{I}_{\mathfrak{A}^k} + \big(-D_x(\chi) - r\big)\mathbf{I}_{\mathfrak{A}_r},$$
$$L = \eta_x^+(\chi)\mathbf{I}_{\mathfrak{A}^k} + 0 \cdot \mathbf{I}_{\mathfrak{A}_r},$$
$$\mathbf{P}\big[\mathfrak{A}^k + \mathfrak{A}_r\big] = 1$$

the value of the overshoot through the boundaries of the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\ge 0}$ and the value of the linear component at the instant of the first exit. Here $\mathbf{I}_{\mathfrak{A}} = \mathbf{I}_{\mathfrak{A}}(\omega)$ is the indicator function of the event $\mathfrak{A}$. Denote

$$V^k(x, dl, m, s) = \mathbf{E}\big[e^{-s\chi}; L \in dl, T = m, \mathfrak{A}^k\big],$$
$$V_r(x, m, s) = \mathbf{E}\big[e^{-s\chi}; T = m, \mathfrak{A}_r\big].$$

Theorem 2 (Kadankov and Kadankova [8]) *Let* $\{D_x(t)\}_{t\ge 0}$ *be the difference of the compound Poisson process and the renewal process* (2), $\delta \sim \mathrm{ge}(\lambda)$, $Q_k^s \overset{\mathrm{def}}{=} Q_k^s(0)$. *Then*

(i) *The Laplace transforms of the joint distribution of* $\{\chi, L, T\}$ *satisfy the following equalities for all* $x, s \ge 0$, $m \in \mathbb{N}$,

$$V_r(x, m, s) = \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s}(1-\lambda)\lambda^{m-1},$$
$$V^k(x, dl, m, s) = f^k(x, dl, m, s) - \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s}\mathbf{E}f^{\delta+B}(0, dl, m, s), \tag{17}$$

where the function $f^k(x, dl, m, s)$ *is given by* (11),

$$\mathbf{E}Q_{\delta+B}^s = \sum_{k\in\mathbb{N}}(1-\lambda)\lambda^{k-1}Q_{k+B}^s,$$
$$\mathbf{E}f^{\delta+B}(0, dl, m, s) = \sum_{k\in\mathbb{N}}(1-\lambda)\lambda^{k-1}f^{k+B}(0, dl, m, s).$$

(ii) *For the Laplace transforms of the first exit time* χ, *the following formulae hold*

$$\mathbf{E}\big[e^{-s\chi}; \mathfrak{A}_r\big] = \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s},$$
$$\mathbf{E}\big[e^{-s\chi}; \mathfrak{A}^k\big] = 1 - A_x^k(s) - \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s}\big(1 - \mathbf{E}A_0^{\delta+B}(s)\big), \tag{18}$$

where $\mathbf{E}A_0^{\delta+B}(s) = \sum_{k\in\mathbb{N}}(1-\lambda)\lambda^{k-1}A_0^{k+B}(s)$.

It is worth noting that the Laplace transforms $V_r(x, m, s)$, $V^k(x, dl, m, s)$ were determined in [8] for the case when $\delta \in \mathbb{N}$ is a positive arbitrarily distributed random variable. The results of Theorem 1 were obtained as a corollary for $\delta \sim \mathrm{ge}(\lambda)$. In the following section we will employ this joint distribution to study the boundary characteristics of the process reflected from the boundaries.

4 Reflections from the boundary

Denote by $D_x^r(t) = r + D_x(t),\ t \geq 0$, the process starting from $r \in \mathbb{Z}$ when $\eta_x^+(0) = x \geq 0$. Let $r \in \mathbb{Z}^+$, and for all $t \geq 0$ we define a right-continuous process reflected at the boundary 0 as follows:

$$\begin{aligned} &\underline{D}_0^r(x,t) = D_x^r(t) - \min\Big\{0, \inf_{[0,t]} D_x^r(\cdot)\Big\} \in \mathbb{Z}^+, \\ &\underline{D}_r^0(x,0) = r. \end{aligned} \tag{19}$$

The reflection from the lower boundary 0 is generated by the infimum of the process $D_x^r(t)$. Observe that the first hitting of 0 by the process $\underline{D}_0^r(x,t)$ occurs at time $\tau_r(x)$. Subsequent time periods between the hitting times have the same distribution as $\tau_0(0)$.

Note that reflections from the boundaries generated by the infimum (supremum) were introduced by Lévy for a standard Wiener process. Applying the symmetry principle and the mirror reflection principle, Lévy determined the distributions of the boundary functionals of the reflected standard Wiener process. It appears that these distributions are the limit distributions for the reflected process after an appropriate scaling of time and space.

4.1 Passage of the upper boundary

We will now determine the one-boundary functionals for Process (19). For $B \in \mathbb{Z}^+$, $r \in [0, B]$, denote by

$$\begin{aligned} &\underline{\tau}_r^B(x) = \inf\big\{t : \underline{D}_0^r(x,t) > B\big\} \stackrel{\text{def}}{=} \underline{\tau}, \\ &\underline{T}_r^k(x) = \underline{D}_r^0(\underline{\tau}) - B, \qquad \underline{L}_r^k(x) = \eta_x^+(\underline{\tau}) \end{aligned}$$

the first crossing time of the upper level B by the process $\underline{D}_0^r(x,t)$, the value of the overshoot and the value of the linear component at this instant.

Note that these boundary functionals were studied in [14] for the reflected Lévy processes generated by the infimum (supremum). The reflected spectrally one-sided Lévy processes generated by the infimum (supremum) of the process were considered in [2, 15]. An interesting application in queueing theory for the spectrally one-sided Lévy process reflected by its infimum was given in [3].

Lemma 5 *Let $\underline{D}_0^r(x,t)_{\{t\geq 0\}}$ be the process reflected by infimum* (19). *Then*

(i) *The Laplace transform $\underline{v}_x^r(dl,m,s) = \mathbf{E}[e^{-s\underline{\tau}_r^B(x)}; \underline{L} \in dl, \underline{T} = m]$ of the joint distribution of $\{\underline{\tau}_r^B(x), \underline{L}, \underline{T}\}$ satisfies the following equality for $s > 0$*

$$\underline{v}_x^r(dl,m,s) = V^k(x,dl,m,s) + \frac{V_r(x,s)}{1 - V_0(0,s)}\, V^B(0,dl,m,s), \tag{20}$$

where $k=B-r$, $V_r(x,s)=\sum_{m\in\mathbb{N}}^{\infty} V_r(x,m,s)$, *and the functions* $V^k(x,dl,m,s)$, $V_r(x,m,s)$ *are determined by* (17); *in particular,*

$$\underline{v}_x^r(s)=\mathbf{E}e^{-s\underline{\tau}_r^B(x)}=1-A_x^k(s)+Q_k^s(x)\frac{\mathbf{E}A_0^{\delta+B}(s)-A_0^B(s)}{\mathbf{E}Q_{\delta+B}^s-Q_B^s}. \tag{21}$$

(ii) *Under Condition* (A), *the following equality is valid*

$$\lim_{B\to\infty}\mathbf{P}\big[\underline{\tau}_{rB}^B(x)/B^2>t\big]=\frac{4}{\pi}\sum_{n\in\mathbb{Z}^+}\frac{e^{-\frac{t}{2}(\pi(n+\frac{1}{2})\sigma)^2}}{2n+1}\sin\big(k(2n+1)\pi/2\big),$$

where $r\in(0,1)$, $k=1-r$.

(iii) *The random variable* $\underline{\tau}_r^B(x)$ *is proper* ($\mathbf{P}[\underline{\tau}_r^B(x)<\infty]=1$) *and*

$$\mathbf{E}\underline{\tau}_r^B(x)=A_x^k+Q_k(x)\frac{\mathbf{E}A_0^{\delta+B}-A_0^B}{\mathbf{E}Q_{\delta+B}-Q_B(0)}<\infty,$$

where $Q_k(x)=Q_k^0(x)$, $\mathbf{E}Q_{\delta+B}=\mathbf{E}Q_{\delta+B}^0$,

$$A_x^k=\sum_{i=0}^k\rho_i\big[1-(1-\lambda)^{-1}Q_{k-i}(x)\big],$$

$$\rho_i=\int_0^\infty\mathbf{P}\big[\pi(t)=i\big]dt<\infty.$$

Let us verify Formula (20). It follows from the definition of the process $\underline{D}_0^r(x,t)$ (see (19)), the total probability law and the Markov property of χ that the following equation is valid:

$$\underline{v}_x^r(dl,m,s)=V^k(x,dl,m,s)+V_r(x,s)\underline{v}_0^0(dl,m,s).$$

Letting $x=r=0$ in this equation, we find that

$$\underline{v}_0^0(dl,m,s)=V^B(0,dl,m,s)\big(1-V_0(0,s)\big)^{-1}.$$

Substituting the expression for the function $\overline{v}_0^0(dl,du,s)$ into the previous equation, we get Formula (20). Formula (21) follows from (20) and (18).

Lemma 6 *Under Condition* (A) *and for* $k>0$ *the next limiting equalities hold*

$$\lim_{B\to\infty}B^{-2}S_{[kB]}^{s/B^2}(x)=\frac{1}{s\mathbf{E}\eta}\big(\cosh(k\sqrt{2s}/\sigma)-1\big)=\lim_{B\to\infty}B^{-2}\mathbf{E}S_{\delta+[kB]}^{s/B^2},$$

$$\lim_{B\to\infty}\big[\mathbf{E}Q_{[\delta+kB]}^{s/B^2}-Q_{[kB]}^{s/B^2}\big]=\frac{2\mu\mathbf{E}\varkappa}{\sigma^2}\cosh(k\sqrt{2s}/\sigma), \tag{22}$$

$$\lim_{B\to\infty}B\big[A_0^{[kB]}\big(s/B^2\big)-\mathbf{E}A_0^{[\delta+kB]}\big(s/B^2\big)\big]=\frac{\sqrt{2s}}{(1-\lambda)\sigma}\sinh(k\sqrt{2s}/\sigma),$$

where $S_k^s(x)=\sum_{i=0}^k Q_i^s(x)$, $S_k^s(x)=0$ *for* $k<0$.

The proof of the lemma can be found in the Appendix. Formulae (21), (16) and the latter equalities imply, for $B \to \infty$, $k \in (0,1)$, $r = 1-k$, that

$$\begin{aligned}\mathbf{E}e^{-s\,\underline{\tau}^B_{rB}(x)/B^2} &\to \cosh(k\sqrt{2s}/\sigma) - \frac{2\sinh(k\sqrt{2s}/\sigma)}{\sigma\sqrt{2s}\mathbf{E}\eta}\frac{\sqrt{2s}}{(1-\lambda)\sigma}\\ &\quad\times \sinh(\sqrt{2s}/\sigma)\frac{\sigma^2}{2\mu\varkappa}/\cosh(\sqrt{2s}/\sigma)\\ &= \frac{\cosh(r\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}.\end{aligned}$$

It is obvious that

$$\int_0^\infty e^{-st}\lim_{B\to\infty}\mathbf{P}\big[\underline{\tau}^B_{rB}(x)/B^2 > t\big]\,dt = \frac{1}{s} - \frac{1}{s}\frac{\cosh(r\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}.$$

Inverting the Laplace transforms on both sides of the latter equality, we get the limiting equality of the theorem.

4.2 Increments of the process reflected at its infimum

Let $r \in \mathbb{Z}^+$, and denote by $\underline{D}^0_{-r}(x,t) = D_x(t) - \min\{0, \inf_{[0,t]} D_x(\cdot) + r\} \in [-r,\infty)$, the process reflected from the lower boundary $-r$. The reflections are generated by its infimum. Introduce

$$\begin{aligned}\underline{P}^s_r(x,z) &= \mathbf{E}z^{\underline{D}^0_{-r}(x,\nu_s)}, \quad |z| \le 1,\\ \underline{p}^s_r(x,u) &= \mathbf{P}\big[\underline{D}^0_{-r}(x,\nu_s) \ge u\big], \quad u \in [-r,\infty),\end{aligned}$$

i.e., the generating function and the distribution function of the increments of the process sampled at the exponential time ν_s. The following statement holds.

Theorem 3 *Let $\{\underline{D}^0_{-r}(x,t)\}_{t\ge 0}$ be the process reflected at the lower boundary. Then*

(i) *The following equalities are valid for $r \in \mathbb{Z}^+$, $x \ge 0$, $u \in [-r,\infty)$,*

$$\begin{aligned}\underline{P}^s_r(x,z) &= (1-z)\left[\mathbb{A}^z_x(s) + z^{-r} f_r(x,s)\frac{1-\lambda}{1-c(s)}\,\frac{1-z}{\lambda - z}\,\mathbb{A}^z_0(s)\right],\\ \underline{p}^s_r(x,u) &= 1 - A^{u-1}_x(s) - \frac{1-\lambda}{1-c(s)}\,f_r(x,s)\big[A^{u+r-1}_0(s) - \mathbf{E}A^{\delta+u+r-1}_0(s)\big],\end{aligned} \tag{23}$$

where $A^u_x(s) = 0$, for $u < 0$.

(ii) *Under Condition (A), for $r > 0$, $u \ge -r$,*

$$\lim_{B\to\infty}\mathbf{P}\big[\underline{D}^0_{[-rB]}\big(x,tB^2\big) \ge [uB]\big] = 1 - \frac{1}{\sigma\sqrt{2\pi t}}\int_{-u}^{u+2r} e^{-v^2/2\sigma^2 t}\,dv. \tag{24}$$

(iii) *If* $\rho=(1-\lambda)\mu\mathbf{E}\eta\mathbf{E}\varkappa<1$, *then the ergodic distribution*

$$\underline{p}_r(u)=\lim_{t\to\infty}\mathbf{P}\big[\underline{D}^0_{-r}(x,t)\ge u\big]$$

exists, *and the limiting equality*

$$\underline{p}_r(u)=1-\frac{1-\rho}{\mathbf{E}\eta}\big[A_0^{u+r-1}-\mathbf{E}A_0^{\delta+u+r-1}\big]$$

holds, *where* $A_x^k=\sum_{i=0}^k\rho_i[1-(1-\lambda)^{-1}Q_{k-i}(x)]$, $\rho_i=\int_0^\infty\mathbf{P}[\pi(t)=i]\,dt$.

Proof In view of the total probability law, Markov property of $\tau_r(x)$ and homogeneity property of the process with respect to the first component, we can write the following equation for the function $\underline{P}^s_r(x,z)$:

$$\begin{aligned}\underline{P}^s_r(x,z)&=E_r^-(x,z,s)+\frac{sf_r(x,s)z^{-r}}{s+\mu}\\&\quad+\frac{\mu f_r(x,s)z^{-r}}{1-\tilde{f}(s+\mu)}\int_0^\infty\sum_{i\in\mathbb{N}}a_i(dl)\underline{P}^s_i(l,z)z^i,\qquad(25)\\a_i(dl)&=e^{-l(s+\mu)}\big[1-F(l)\big]\mathbf{P}[\varkappa=i]\,dl,\end{aligned}$$

where the generating function $E_r^-(x,z,s)=\mathbf{E}[z^{D_x(\nu_s)};D_x^-(\nu_s)\ge -r]$, $|z|\le 1$, is determined by (14).

This equation reflects the following fact. The increments of the process $\{\underline{D}^0_{-r}(x,t)\}$ can take place either on the sample paths which do not hit the lower boundary $-r$, (the first term on the right-hand side), or on the sample paths which hit the lower boundary $-r$ and stay there (the second term), or finally, on the sample paths which hit the boundary $-r$ and then are reflected from the boundary (the third term). Denote

$$X(s,z)=\frac{\mu}{1-\tilde{f}(s+\mu)}\int_0^\infty\sum_{i\in\mathbb{N}}a_i(dl)\underline{P}^s_i(l,z)z^i.$$

Setting $x=0$ in (25) and performing necessary calculations, we get

$$X(s,z)=\frac{1-\lambda}{1-c(s)}\bigg[\mu\int_0^\infty\sum_{r\in\mathbb{N}}a_r(dx)E_r^-(x,z,s)z^r+s\frac{1-\tilde{f}(s+\mu)}{s+\mu}\bigg]-\frac{s}{s+\mu}.$$

Inserting this expression for the function $X(s,z)$ into (25) yields

$$\begin{aligned}\underline{P}^s_r(x,z)&=E_r^-(x,z,s)\\&\quad+f_r(x,s)z^{-r}\frac{1-\lambda}{1-c(s)}\bigg[\mu\int_0^\infty\sum_{r\in\mathbb{N}}a_r(dx)E_r^-(x,z,s)z^r\\&\quad+s\frac{1-\tilde{f}(s+\mu)}{s+\mu}\bigg].\end{aligned}$$

Inserting Expression (14) for the function $E_r^-(x,z,s)$ into the latter equality and performing necessary calculations, we find that

$$\underline{P}_r^s(x,z) = (1-z)\left[\mathbb{A}_x^z(s) + z^{-r} f_r(x,s)\frac{1-\lambda}{1-c(s)}\,\frac{1-z}{\lambda - z}\,\mathbb{A}_0^z(s)\right].$$

It is not difficult to derive the following relation

$$\hat{p}_r^s(x,z) = \sum_{u=-r}^{\infty} z^u \underline{p}_r^s(x,u) = \frac{z^{-r}}{1-z} - \frac{z}{1-z}\underline{P}_r^s(x,z), \quad |z| \le 1.$$

The latter and the previous equality imply that

$$\hat{p}_r^s(x,z) = \frac{z^{-r}}{1-z} - z\mathbb{A}_x^z(s) - z^{-r+1} f_r(x,s)\frac{1-\lambda}{1-c(s)}\,\frac{1-z}{\lambda - z}\mathbb{A}_0^z(s).$$

Comparing the coefficients of z^u, $u \in [-r,\infty)$, we get the second formula of (23). We now verify (24). The first formula of (23) and (8) imply that

$$\lim_{B\to\infty} f_{[rB]}\big(x, s/B^2\big) = e^{-r\sqrt{2s}/\sigma}, \quad r > 0.$$

Denote $\tilde{p}_r^t(x,u,B) = \mathbf{P}[\underline{D}^0_{[-rB]}(x,tB^2) \ge [uB]]$, $r > 0$, $u \ge -r$. It is obvious that

$$\lim_{B\to\infty} \int_0^\infty e^{-st}\,\tilde{p}_k^t(x,u,B)\,dt = \frac{1}{s}\lim_{B\to\infty} \underline{p}_{[rB]}^{s/B^2}\big(x,[uB]\big).$$

Employing (16) and Lemma 6, we obtain

$$\begin{aligned}
\lim_{B\to\infty} &\int_0^\infty e^{-st}\,\tilde{p}_k^t(x,u,B)\,dt \\
&= s^{-1}\mathbf{I}_{\{u>0\}}\big(e^{-u\sqrt{2s}/\sigma}/2 + e^{-(2r+u)\sqrt{2s}/\sigma}/2\big) \\
&\quad + s^{-1}\mathbf{I}_{\{u\in[-r,0]\}}\big(1 - e^{u\sqrt{2s}/\sigma}/2 + e^{-(2r+u)\sqrt{2s}/\sigma}/2\big), \quad u \ge -r.
\end{aligned}$$

In view of the formula $s^{-1}\,e^{-a\sqrt{2s}/\sigma} = 2\int_0^\infty e^{-st}\mathbf{P}[w_t \ge a]\,dt$, we invert the Laplace transforms on the right-hand side of this equality, which yields the limiting equality of the theorem. For $\rho < 1$ the mathematical expectation of $\tau_r(x)$ is finite. It follows from (8) that

$$\mathbf{E}\tau_r(x) = \big[\mathbf{E}\eta_x + r(1-\lambda)\mathbf{E}\eta\big](1-\rho)^{-1} < \infty.$$

Moreover, the process $\underline{D}^0_{-r}(x,t)$ is of regenerative type [12]. The instants of the passages of the lower boundary are the regeneration times. Due to this property (see [12]) there exists an ergodic distribution of the process $p_k(u) = \lim_{t\to\infty}\mathbf{P}[\underline{D}_0^k(x,t) \le u]$. To determine this distribution it suffices to apply the Tauberian theorem to Formula (23). Here $p_k(u) = \lim_{s\to 0}\underline{p}_k^s(x,u)$. □

Let $\{\underline{D}^0_{-r}(x,t)\}_{t\geq 0}$ be the process reflected from the lower boundary $-r$, $r,k\in\mathbb{Z}^+$. Introduce the following random variables

$$\underline{\tau}_{r,k}(x)=\inf\{t:\underline{D}^0_{-r}(x,t)>k\}=\underline{\tau},$$
$$\underline{T}_{r,k}(x)=\underline{D}^0_{-r}(x,\underline{\tau})-k,\qquad \underline{L}_{r,k}(x)=\eta^+_x(\underline{\tau}),$$

i.e., the first exit time from the interval $[-r,k]$ by the process $\underline{D}^0_{-r}(x,t)$, the value of the overshoot through the upper boundary k and the value of the linear component at this instant. Since the process X_t is homogeneous with respect to the first component, then the random variables $\{\underline{\tau}_{r,k}(x),\underline{T}_{r,k}(x),\underline{L}_{r,k}(x)\}$ have the same distributions as $\{\underline{\tau}^B_r(x),\underline{T}^B_r(x),\underline{L}_r(x)\}$, $B=k+r$. And hence their joint distribution is given by (20).

Theorem 4 *Let $\{\underline{D}^0_{-r}(x,t)\}_{t\geq 0}$ be the process reflected from the lower boundary $-r$. Denote by $\underline{p}^s_{r,k}(x,u)=\mathbf{P}[\underline{D}^0_{-r}(x,\nu_s)\leq u;\ \underline{\tau}_{r,k}(x)>\nu_s]$, $u\in[-r,k]$, the distribution of the increments of the process sampled at the exponential time ν_s, $s>0$, given that the event $\{\underline{\tau}_{r,k}(x)>\nu_s\}$ takes place. Then*

(i) *The following equality is valid for $r,k\in\mathbb{Z}^+$, $u\in[-r,k]$,*

$$\underline{p}^s_{r,k}(x,u)=A^u_x(s)-Q^s_k(x)\,\frac{\mathbf{E}A_0^{\delta+u+r}(s)-A_0^{u+r}(s)}{\mathbf{E}Q^s_{B+\delta}-Q^s_B}. \tag{26}$$

(ii) *Under Condition (A), the limiting equality*

$$\lim_{B\to\infty}\mathbf{P}\big[\underline{D}_x^{[kB]}\big(tB^2\big)\leq[uB];\ \underline{\tau}_{[rB],[kB]}(x)>tB^2\big]$$
$$\stackrel{\text{def}}{=}p(t)=\frac{4}{\pi}\sum_{n\in\mathbb{Z}^+}\frac{e^{-\frac{t}{2}(\pi(n+\frac12)\sigma)^2}}{2n+1}$$
$$\times\sin\left((r+u)\left(n+\frac12\right)\pi\right)\cos\left(r\left(n+\frac12\right)\pi\right) \tag{27}$$

holds, where $r\in(0,1)$, $k=1-r$, $u\in[-r,k]$.

Proof Introduce the generating function

$$\underline{P}^s_{r,k}(x,z)=\mathbf{E}\big[z^{\underline{D}^0_{-r}(x,\nu_s)};\ \underline{\tau}_{r,k}(x)>\nu_s\big].$$

According to the total probability law, the homogeneity property of the process X_t with respect to the first component, Markov property of $\underline{\tau}_{r,k}(x)$, and properties of the exponential variable ν_s, we can write the following equation

$$\underline{P}^s_r(x,z)=\underline{P}^s_{r,k}(x,z)+\int_0^\infty\sum_{m\in\mathbb{N}}\underline{v}^r_x(dl,m,s)\underline{P}^s_{m+B}(l,z)z^{m+k},\quad |z|\leq 1,$$

where $\underline{v}_x^r(dl,m,s) = \mathbf{E}[e^{-s\underline{\tau}_r^B(x)}; \underline{L} \in dl, \underline{T} = m]$ is determined by (20), and $\underline{P}_r^s(x,z) = \mathbf{E}[z^{\underline{D}^0_{-r}(x,\nu_s)}]$ is found in Theorem 3 by (23). This equation is written using the path decomposition principle. In other words, the increments of the process $\underline{D}^0_{-r}(x,\nu_s)$ can be realized on one of the following self-excluding events: (1) the sample paths do not cross the upper boundary k, (2) the sample paths do intersect the upper boundary and then the evolution of the process is just a probabilistic replica of the process on $[0,\nu_s]$. Inserting Expression (23) for the function $\underline{P}_r^s(x,z)$ into the latter equation, we find that

$$\frac{1}{1-z}\underline{P}_{r,k}^s(x,z) = \mathbb{A}_x^z(s) - z^k\int_0^\infty \tilde{v}_x^r(dl,z,s)\mathbb{A}_l^z(s) + z^{-r}\mathbb{A}_0^z(s)\frac{1-\lambda}{1-c(s)}\frac{1-z}{\lambda-z} \times \left[f_r(x,s) - \sum_{m\in\mathbb{N}}\int_0^\infty \underline{v}_x^r(dl,m,s)f_{m+B}(l,s)\right], \quad (28)$$

where $\tilde{v}_x^r(dl,z,s) = \mathbf{E}[e^{-s\underline{\tau}_r^B(x)}z^{\underline{T}}; \underline{L} \in dl]$. Performing necessary calculations and taking into account (8) and (20) yields

$$\sum_{m\in\mathbb{N}}\int_0^\infty \underline{v}_x^r(dl,m,s)f_{m+B}(l,s) = f_r(x,s) - \frac{1-c(s)}{1-\lambda}\frac{Q_k^s(x)}{\mathbf{E}Q_{B+\delta}^s - Q_B^s}.$$

In view of this equality and (28), we find that

$$\frac{\underline{P}_{r,k}^s(x,z) - z^{k+1}(1-\underline{v}_x^r(s))}{1-z} = \mathbb{A}_x^z(s) - z^k\int_0^\infty \tilde{v}_x^r(dl,z,s)\mathbb{A}_l^z(s) - \frac{z^{k+1}(1-\underline{v}_x^r(s))}{1-z} + z^{-r}\mathbb{A}_0^z(s)\frac{1-z}{\lambda-z}\frac{Q_k^s(x)}{\mathbf{E}Q_{B+\delta}^s - Q_B^s}, \quad |z|\le 1, \quad (29)$$

where $\tilde{v}_x^r(s) = \mathbf{E}[e^{-s\underline{\tau}_r^B(x)}]$. Then it is easily verified that

$$\frac{1}{1-z}\underline{P}_{r,k}^s(x,z) - \frac{z^{k+1}}{1-z}\big(1-\underline{v}_x^r(s)\big) = \sum_{u=-r}^{k} z^u \underline{p}_{r,k}^s(x,u).$$

Comparing the coefficients of z^u, $u \in [-r,k]$, on both sides of (29), we obtain (26). For $r \in (0,1)$, $k = 1-r$, $u \in [-r,k]$, denote

$$\overline{p}_{r,k}^t(x,u,B) = \mathbf{P}\big[\underline{D}_x^{[kB]}\big(tB^2\big) \le [uB];\ \underline{\tau}_{[rB],[kB]}(x) > tB^2\big].$$

Employing Formula (26) and the limiting equality of Lemma 6, we find that

$$\begin{aligned}\frac{1}{s}\lim_{B\to\infty}\underline{p}^{s/B^2}_{[rB],[kB]}\big(x,[uB]\big) &= \lim_{B\to\infty}\int_0^\infty e^{-st}\overline{p}^t_{r,k}(x,u,B)\,dt\\ &= \frac{1-\cosh(u^+\sqrt{2s}/\sigma)}{s}\\ &\quad+\frac{1}{s}\frac{\sinh(k\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}\sinh\big((u+r)\sqrt{2s}/\sigma\big)\\ &\stackrel{\text{def}}{=} p^*(s),\end{aligned} \tag{30}$$

where $u^+ = \max\{0, u\}$. When $u \in [-r, 0]$, we derive from this formula that

$$p^*(s) = \frac{1}{s}\frac{\sinh(k\sqrt{2s}/\sigma)}{\cosh(\sqrt{2s}/\sigma)}\sinh\big((u+r)\sqrt{2s}/\sigma\big), \quad u \in [-r, 0].$$

It is clear that $s = 0$ is not a singular point (pole or point of branching) of the function $p^*(s)$. On the half-plane $\Re(s) < 0$, this function has simple poles at

$$s_n = -\frac{1}{2}\sigma^2\pi^2\left(n+\frac{1}{2}\right)^2, \quad n \in \mathbb{Z}^+,$$

and it is analytic on the whole plane apart from these points. Hence, for $\alpha > 0$,

$$p(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}p^*(s)\,ds = \sum_{n\in\mathbb{Z}^+}\operatorname{Res}_{s=s_n}p^*(s).$$

Calculating the residues of the function $p^*(s)$ at s_n, we obtain the right-hand side of Formula (27) for $u \in [-r, 0]$. On can see that for $u \in (0, k]$ the first term on the right-hand side of (30) is analytic on the entire plane. Applying the inversion formula, we find that the contour integral of this term is equal to zero. The second term of (30) is the same also for $u \in [-r, 0]$. Thus, Formula (27) holds for $u \in [-r, k]$. □

5 Applications for the $G^\delta|M^\varkappa|1|B$ system

Before considering the queueing system of interest, we stress the following facts.

Remark 1 Let $B \in \mathbb{Z}^+$ be fixed, $k \in [0, B]$, $r = B - k$. Introduce the process

$$\begin{aligned}\overline{D}^{B+1}_{k+1}(x,t) &= -\underline{D}^0_{-r}(x,t) + k + 1 \in [-\infty, B+1],\\ \overline{D}^{B+1}_{k+1}(x,0) &= k+1.\end{aligned} \tag{31}$$

This process is reflected at the upper boundary $B + 1$, due to the infimum of the process $D_x(t)$. Introduce the following random variable:

$$\overline{\tau}_{k+1}(x) = \inf\{t : \overline{D}_{k+1}^{B+1}(x,t) < 1\} = \inf\{t : \underline{D}_{-r}^{0}(x,t) > k\}$$
$$= \inf\{t : \underline{D}_{0}^{r}(x,t) > B\} = \underline{\tau}_r^B(x).$$

This defining chain of stochastic equalities implies that $\overline{\tau}_{k+1}(x)$ and $\underline{\tau}_r^B(x)$ are identically distributed. And hence, by (21),

$$\overline{v}_x^k(s) = \mathbf{E}e^{-s\overline{\tau}_k(x)} = 1 - A_x^{k-1}(s) + Q_{k-1}^s(x)\frac{\mathbf{E}A_0^{\delta+B}(s) - A_0^B(s)}{\mathbf{E}\,Q_{\delta+B}^s - Q_B^s}, \quad k \in [1, B+1].$$

Remark 2 Denote by $\overline{p}_{k+1,x}^s(u) = \mathbf{P}[\overline{D}_{k+1}^{B+1}(x,\nu_s) \geq u; \overline{\tau}_{k+1}(x) > \nu_s]$, $u \in [1, B+1]$, the Laplace transform of the increments of the process $\overline{D}_{k+1}^{B+1}(x,t)$ on the event $\{\overline{\tau}_{k+1}(x) > t\}$. The definition of the process and Remark 1 imply that

$$\mathbf{P}[\overline{D}_{k+1}^{B+1}(x,\nu_s) \geq u;\ \overline{\tau}_{k+1}(x) > \nu_s]$$
$$= \mathbf{P}[\underline{D}_{-r}^{0}(x,\nu_s) \leq k+1-u;\ \underline{\tau}_r^B(x) > \nu_s].$$

It follows from the latter equality and from (26) that for $k, u \in [1, B+1]$

$$\overline{p}_{k,x}^s(u) = A_x^{k-u}(s) - Q_{k-1}^s(x)\frac{\mathbf{E}A_0^{\delta+B+1-u}(s) - A_0^{B+1-u}(s)}{\mathbf{E}Q_{B+\delta}^s - Q_B^s}. \tag{32}$$

We now introduce the process describing the functioning of the queueing system. Let $B \in \mathbb{Z}^+$, $k \in [0, B+1]$, $x \geq 0$. Define the Markov process

$$Y_{k,x}(t) = \{d_{k,x}(t), \eta_x^+(t)\} \in [0, B+1] \times \mathbb{R}_+, \qquad Y_{k,x}(0) = (k,x)$$

by means of the following stochastic recurrences

$$Y_{k,x}(t) = \begin{cases} (\overline{D}_k^{B+1}(x,t), \eta_x^+(t)), & 0 \leq t < \overline{\tau}_k(x), \\ Y_{0,\eta_x^+(\overline{\tau}_k(x))}(t - \overline{\tau}_k(x)), & t \geq \overline{\tau}_k(x), \end{cases} \quad k \in [1, B+1],$$

$$Y_{0,x}(t) = \begin{cases} (0, \eta_x^+(t)), & 0 \leq t < \eta_x, \\ Y_{k,0}(t-\eta_x) \text{ with probability } (1-\lambda)\lambda^{k-1}, & k = [1, B],\ t \geq \eta_x, \\ Y_{B+1,0}(t-\eta_x) \text{ with probability } \lambda^B, & t \geq \eta_x. \end{cases}$$

The process $Y_{k,x}(t)_{\{t\geq 0\}}$ serves as a mathematical model of the functioning of the $\mathrm{G}^\delta|\mathrm{M}^\varkappa|1|\mathrm{B}$ system with ($\delta \sim \mathrm{ge}(\lambda)$). Let us describe how this system works.

(i) Customers arrive into the system in batches according to the renewal process $N_x(t)_{\{t\geq 0\}}$. The number of customers in every batch is a random variable identically distributed as $\delta \sim \mathrm{ge}(\lambda) \in \mathbb{N}$.
(ii) The system has a finite buffer whose size is equal to $B + 1 < \infty$. Suppose that upon arrival of a new customer of size δ, there are $k \in [0, B+1]$ occupied places in the waiting room. Then $\min\{r, \delta\}$ customers join the queue, and loss of size $\max\{0, \delta - r\}$ occurs, where $r = B + 1 - k$ is the size of the empty space in the buffer;

(iii) The duration of service completion is exponentially distributed with parameter $\mu > 0$. Suppose that at time t the service cycle is accomplished. Then the occupied space in the buffer is reduced by $\min\{k, \varkappa\}$, where $k \in [1, B+1]$ is the number of the occupied places in the waiting room at time $t-0$. If at the instant of the service completion $k - \min\{k, \varkappa\} > 0$, then a new service cycle starts. If at the instant of the service completion $k - \min\{k, \varkappa\} = 0$, then the new service cycle starts upon arrival of a new customer (batch of customers).

For all $t \geq 0$, the event $\{Y_{k,x}(t) = (i, y)\}$, $i \in [1, B+1]$, $y \geq 0$, means that at time t there are i customers in the buffer and that time y has elapsed since the last arrival up to time t. We assume that (k, x) is an initial state of the system.

The event $\{Y_{k,x}(t) = (0, y)\}$ means that at time t the buffer is empty and the system is idle, and time y has elapsed since the last customer's arrival (up to time t). Hence, η_y is the duration of the idle period (state $(0, y)$).

Thus, $d_{k,x}(t)$ is the number of customers in the buffer at time t, $\eta_x^+(t)$ is the time elapsed since the last arrival of the batch up to time t. Note that the definition of the process $Y_{k,x}(t)$ (homogeneity of the process X_t with respect to the first component) implies that the linear component $\eta_x^+(t)$ does not depend on k.

In the next part of this paper, we will determine the Laplace transforms of the main performance characteristics of the system. The Laplace transform of the busy period of the system is the subject of the next subsection.

5.1 Busy period of the system

Suppose that the system starts functioning at time $t_0 = 0$ from the state (k, x), where $k \in [1, B+1]$ is the number of customers in the buffer, $x \geq 0$ is the time elapsed since the last arrival up to time $t_0 = 0$. Denote by

$$b_k(x) = \inf\{t : d_{k,x}(t) = 0\}, \qquad \eta(x) = \eta_x^+\big(b_k(x)\big)$$

the instant at which the system becomes empty for the first time and the value of the linear component at time $b_k(x)$, respectively. Hence, the interval $[0, b_k(x)]$ is a busy period of the (k, x)-type.

Corollary 1 *Let $b_k^s(x) = \mathbf{E}[e^{-sb_k(x)}; b_k(x) < \infty]$ be the Laplace transform of the busy period of the (k, x)-type. Then*

(i) *The following equality holds*

$$b_k^s(x) = 1 - A_x^{k-1}(s) + Q_{k-1}^s(x) \frac{\mathbf{E}A_0^{\delta+B}(s) - A_0^B(s)}{\mathbf{E}Q_{\delta+B}^s - Q_B^s}, \quad k \in [1, B+1], \quad (33)$$

the random variable $b_k(x)$ is proper ($\mathbf{P}[b_k(x) < \infty] = 1$), *and it has a finite mathematical expectation given by*

$$\mathbf{E}b_k(x) = A_x^{k-1} - Q_{k-1}(x) \frac{\mathbf{E}A_0^{\delta+B} - A_0^B}{\mathbf{E}Q_{\delta+B} - Q_B} < \infty. \quad (34)$$

(ii) *The Laplace transform* $b_k^s(x,dy) = \mathbf{E}[e^{-sb_k(x)};\ \eta(x) \in dy]$ *of the joint distribution of* $\{b_k(x), \eta(x)\}$ *is such that for* $k \in [1, B+1]$

$$b_k^s(x,dy) = f^{k-1}(x,dy,s) - Q_{k-1}^s(x)\frac{\mathbf{E} f^{\delta+B}(0,dy,s) - f^B(0,dy,s)}{\mathbf{E} Q_{\delta+B}^s - Q_B^s}, \quad (35)$$

where the function $f^k(x,dy,s) = \mathbf{E}[e^{-s\tau^k(x)};\ T^k(x) \in dy, \mathfrak{B}^k(x)]$, $k \in \mathbb{Z}^+$, *is determined by* (26).

Proof Formula (33) follows straightforwardly from (21) and Remark 1. Equalities (20) and (17) imply (35). □

5.2 Time of the first loss of a customer

Suppose that the initial state of the system is (k,x), $k \in [0, B+1]$, $x \ge 0$. Introduce $l_k(x)$ as time of the first loss of a customer (group of customers); $i_{k,x}(t)$ as the number of the lost customers on the time interval $[0,t]$; and $i_{k,x} = i_{k,x}(l_k(x))$ as the number of the lost customers at time $l_k(x)$.

Corollary 2 *Let* $l_k^s(x) = \mathbf{E}[e^{-sl_k(x)}; l_k(x) < \infty]$ *be the Laplace transform of* $l_k(x)$. *Then the following relations hold*

$$\begin{aligned} l_k^s(x) &= \frac{\tilde{f}_x(s) + (1-\tilde{f}(s))S_{k-1}^s(x)}{\tilde{f}(s) + (1-\tilde{f}(s))\mathbf{E}\, S_{\delta+B}^s}, \quad k \in [0, B+1], \\ l_k^s(x,m) &= \mathbf{E}\big[e^{-sl_k(x)}; i_{k,x} = m\big] = l_k^s(x)(1-\lambda)\lambda^{m-1}, \quad m \in \mathbb{N}, \end{aligned} \quad (36)$$

where $S_k^s(x) = \sum_{i=0}^k Q_i^s(x)$, $S_k^s(x) = 0$ *for* $k < 0$. *The random variable* $l_k(x)$ *is proper with a finite mathematical expectation*

$$\mathbf{E} l_k(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta\big[\mathbf{E} S_{\delta+B} - S_{k-1}(x)\big] < \infty,$$

where $S_k(x) = S_k^0(x)$, $\mathbf{E} S_{\delta+B} = \mathbf{E} S_{\delta+B}^0$.

Proof The functions $l_k^s(x)$, $k \in [1, B+1]$, $l_0^s(y)$ obey the following system of equations:

$$\begin{aligned} l_k^s(x) &= V_{B+1-k}(x,s) + \int_0^\infty V^{k-1}(x,dy,s)l_0^s(y), \\ l_0^s(y) &= \tilde{f}_y(s)\lambda^{B+1} + \tilde{f}_y(s)\sum_{k=1}^{B+1}(1-\lambda)\lambda^{k-1}l_k^s(0), \end{aligned} \quad (37)$$

where the functions $V_r(x,s)$, $V^k(x,dy,s)$ are given by (17) and (18). Substituting the expression for the function $l_0^s(y)$ from the second equation into the first, we get

$$l_k^s(x) = V_{B+1-k}(x,s) + \lambda^{B+1}\int_0^\infty V^{k-1}(x,dy,s)\tilde{f}_y(s)$$

$$+\int_0^\infty V^{k-1}(x,dy,s)\tilde{f}_y(s)\sum_{k=1}^{B+1}(1-\lambda)\lambda^{k-1}l_k^s(0).$$

Letting $x=0$ in the latter equation, we find for the function $X(s)=\sum_{k=1}^{B+1}(1-\lambda)\lambda^{k-1}l_k^s(0)$ that

$$X(s)+\lambda^{B+1}=\frac{\lambda^{B+1}+\check{V}_B(\lambda,s)}{1-\int_0^\infty \check{V}^B(\lambda,dy,s)\tilde{f}_y(s)}, \tag{38}$$

where $\check{V}_B(\lambda,s)=(1-\lambda)\sum_{k=1}^{B+1}\lambda^{k-1}V_{B+1-k}(0,s)$,

$$\check{V}^B(\lambda,dy,s)=(1-\lambda)\sum_{k=1}^{B+1}\lambda^{k-1}V^{k-1}(0,dy,s).$$

Employing Formulae (11), (17), (18) and performing necessary calculations, we obtain

$$\lambda^{B+1}+\check{V}_B(\lambda,s)=(1-\lambda)\big(\mathbf{E}Q_{\delta+B}^s\big)^{-1},$$
$$1-\int_0^\infty \check{V}^B(\lambda,dy,s)\tilde{f}_y(s)=(1-\lambda)S_B(\lambda,s)\big(\mathbf{E}Q_{\delta+B}^s\big)^{-1},$$

where $S_B(\lambda,s)=\tilde{f}(s)+(1-\tilde{f}(s))\mathbf{E}S_{\delta+B}^s$. These equalities, Formula (38) and the second equality of (37) imply that

$$l_0^s(y)=\tilde{f}_y(s)S_B(\lambda,s)^{-1}.$$

Inserting the right-hand side of this equality into the second equality of (37), we get

$$l_k^s(x)=\frac{Q_{k-1}^s(x)}{\mathbf{E}Q_{\delta+B}^s}+S_B(\lambda,s)^{-1}\int_0^\infty V^{k-1}(x,dy,s)\tilde{f}_y(s),\quad k\in[1,B+1].$$

In view of (11), (17) and (18), we find that

$$\int_0^\infty V^{k-1}(x,dy,s)\tilde{f}_y(s)=\tilde{f}_x(s)+\big(1-\tilde{f}(s)\big)S_{k-1}^s(x)-\frac{Q_{k-1}^s(x)}{\mathbf{E}Q_{\delta+B}^s}S_B(\lambda,s).$$

The latter and the previous equality imply the first equality of (36). We now verify the second equality. Observe that $i_{k,x}\sim \mathrm{ge}(\lambda)$. This can be formally derived from the first formula of (17) and from the following system of equations

$$l_k^s(x,m)=V_{B+1-k}(x,s)(1-\lambda)\lambda^{m-1}+\int_0^\infty V^{k-1}(x,dy,s)l_0^s(y,m),$$
$$l_0^s(y,m)=\tilde{f}_y(s)(1-\lambda)\lambda^{B+m}+\tilde{f}_y(s)\sum_{k=1}^{B+1}(1-\lambda)\lambda^{k-1}l_k^s(0,m).$$

To solve this system, one can apply a similar reasoning as for the system (37). □

Corollary 3 *Let $\nu_s \sim \exp(s)$ be an independent from the process $Y_{k,x}(t)$ exponential variable with parameter $s > 0$. Denote by $I^s_{k,x}(n) = \mathbf{P}[i_{k,x}(\nu_s) = n]$, $n \in \mathbb{Z}^+$, the distribution of the number of the lost customers on the time interval $[0, \nu_s]$. For all $k \in [1, B+1]$, $x \geq 0$, the following equalities are valid*

$$\begin{aligned} &I^s_{k,x}(0) = 1 - l^s_k(x), \\ &I^s_{k,x}(n) = l^s_k(x)(1-\lambda)\big(1 - l^s_{B+1}(0)\big)\big(\lambda + (1-\lambda)l^s_{B+1}(0)\big)^{n-1}, \quad n \in \mathbb{N}. \end{aligned} \tag{39}$$

Proof Let $\tilde{I}^s_{k,x}(z) = \mathbf{E}[z^{i_{k,x}(\nu_s)}]$, $|z| \leq 1$, be the generating function of the distribution of the number of the lost customers. Then it obeys the equation

$$\tilde{I}^s_{k,x}(z) = 1 - l^s_k(x) + \tilde{L}^s_{k,x}(z)\tilde{I}^s_{B+1,0}(z), \tag{40}$$

where (see (36))

$$\tilde{L}^s_{k,x}(z) = \mathbf{E}\big[e^{-sl_k(x)}z^{i_{k,x}}\big] = l^s_k(x)z\frac{1-\lambda}{1-z\lambda}.$$

Letting $k = B+1$, $x = 0$ in (40), we find that

$$\tilde{I}^s_{B+1,0}(z) = \big(1 - l^s_{B+1}(0)\big)\big(1 - \tilde{L}^s_{B+1,0}(z)\big)^{-1}.$$

Inserting the right-hand side of this equality into (40) implies

$$\tilde{I}^s_{k,x}(z) = 1 - l^s_k(x)\frac{1-z}{1 - z\lambda - z(1-\lambda)l^s_{B+1}(0)}.$$

Comparing the coefficients of z^n, $n \in \mathbb{Z}^+$, we obtain (40) of the corollary. □

5.3 Number of customers in the system

Let $\nu_s \sim \exp(s)$ be an exponential r.v. with parameter $s > 0$. Introduce the transient probabilities of the process $d_{k,x}(t)_{\{t \geq 0\}}$, $k \in [0, B+1]$, $x \geq 0$,

$$q^s_{k,x}(0) = \mathbf{P}\big[d_{k,x}(\nu_s) = 0\big], \qquad q^s_{k,x}(u) = \mathbf{P}\big[d_{k,x}(\nu_s) \geq u\big], \quad u \in [1, B+1].$$

Theorem 5 *The distribution of the number of customers in the system sampled at the exponential time ν_s is such that*

$$\begin{aligned} &q^s_{k,x}(0) = 1 - A^{k-1}_x(s) - \big(A^B_0(s) - \mathbf{E}A^{\delta+B}_0(s)\big)\frac{1-\lambda}{\mathbf{E}Q^s_{\delta+B}}C^s_{k-1}(x), \\ &q^s_{k,x}(u) = A^{k-u}_x(s) + \big(A^{B+1-u}_0(s) - \mathbf{E}A^{\delta+B+1-u}_0(s)\big)\frac{1-\lambda}{\mathbf{E}Q^s_{\delta+B}}C^s_{k-1}(x), \end{aligned} \tag{41}$$

where $A^k_x(s) = S^s_k(x) = 0$ for $k < 0$,

$$C^s_k(x) = \tilde{f}_x(s)\big(1 - \tilde{f}(s)\big)^{-1} + S^s_k(x).$$

Under Condition (A), *the following limiting equality holds*

$$\lim_{B\to\infty} \mathbf{P}\big[d_{[kB],x}\big(tB^2\big) \ge [uB]\big]$$
$$\stackrel{\text{def}}{=} q(t) = 1 - u - \frac{2}{\pi} \sum_{n\in\mathbb{N}} \frac{e^{-\frac{t}{2}(\pi\sigma n)^2}}{n} \sin(k\pi n)\sin(u\pi n), \quad k,u \in (0,1). \quad (42)$$

Corollary 4 *Let* $q_0 = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) = 0]$, $q_u = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) \ge u]$, $u \in [1, B+1]$, *be the stationary distribution of the number of customers. Then*

$$q_0 = 1 - \frac{1-\lambda}{\mathbf{E}\eta}\big(A_0^B - \mathbf{E}A_0^{\delta+B}\big)(\mathbf{E}Q_{\delta+B})^{-1},$$
$$q_u = \frac{1-\lambda}{\mathbf{E}\eta}\big(A_0^{B+1-u} - \mathbf{E}A_0^{\delta+B+1-u}\big)(\mathbf{E}Q_{\delta+B})^{-1},$$

where $\rho_i = \lim_{s\to0} s^{-1}\tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}[\pi(t) = i]\,dt < \infty$, $Q_k = Q_k^0$,

$$A_0^u = \lim_{s\to0} \frac{1}{s} A_0^u(s) = \sum_{i=0}^{u} \rho_i \left[1 - \frac{Q_{u-i}}{1-\lambda}\right].$$

Proof In view of the definition of the process $Y_{k,x}(t)$ and Remark 2, we can write the following equations for the function $q_{k,x}^s(u)$, $q_{0,x}^s(u)$ for $u \in [1, B+1]$

$$\begin{aligned} q_{k,x}^s(u) &= p_{k,x}^s(u) + \int_0^\infty b_k^s(x,dy)q_{0,y}^s(u), \quad k \in [1, B+1], \\ q_{0,y}^s(u) &= \tilde{f}_y(s)\left[\lambda^B q_{B+1,0}^s(u) + (1-\lambda)\sum_{k=1}^{B}\lambda^{k-1}q_{k,0}^s(u)\right], \end{aligned} \quad (43)$$

where the function $b_k^s(x,dy) = \mathbf{E}[e^{-sb_k(x)};\ \eta(x) \in dy]$ is given by (35). Inserting the right-hand side of the second equation into the first, we get

$$q_{k,x}^s(u) = p_{k,x}^s(u) + \int_0^\infty b_k^s(x,dy)\tilde{f}_y(s)q_B^s(\lambda,u),$$

where $q_B^s(\lambda,u) = \lambda^B q_{B+1,0}^s(u) + (1-\lambda)\sum_{k=1}^B \lambda^{k-1}q_{k,0}^s(u)$. After some manipulations, we get

$$q_B^s(\lambda,u) = p_B^s(\lambda,u)\big(1 - \tilde{b}(s,\lambda)\big)^{-1},$$

where $p_B^s(\lambda,u) = \lambda^B p_{B+1,0}^s(u) + (1-\lambda)\sum_{k=1}^B \lambda^{k-1}p_{k,0}^s(u)$,

$$\tilde{b}(s,\lambda) = \lambda^B \int_0^\infty b_{B+1}^s(0,dy)\tilde{f}_y(s) + (1-\lambda)\sum_{k=1}^B \lambda^{k-1}\int_0^\infty b_k^s(0,dy)\tilde{f}_y(s).$$

Employing (11), (32), (35) and performing necessary calculations, we obtain

$$\int_0^\infty b_k^s(x,dy)\tilde{f}_y(s) = \big(1-\tilde{f}(s)\big)\bigg(C_{k-1}^s(x) - \frac{Q_{k-1}^s(x)}{1-\lambda}\frac{\mathbf{E}Q_{\delta+B}^s}{\mathbf{E}Q_{\delta+B}^s - Q_B^s}\bigg),$$

$$1-\tilde{b}(s,\lambda) = \big(1-\tilde{f}(s)\big)\frac{\mathbf{E}Q_{\delta+B}}{\mathbf{E}Q_{\delta+B}^s - Q_B^s},$$

$$p_B^s(\lambda,u) = (1-\lambda)\,\frac{A_0^{B+1-u}(s) - \mathbf{E}A_0^{\delta+B+1-u}(s)}{\mathbf{E}Q_{\delta+B}^s - Q_B^s},$$

$$q_B^s(\lambda,u) = \frac{1-\lambda}{1-\tilde{f}(s)}\big(A_0^{B+1-u}(s) - \mathbf{E}A_0^{\delta+B+1-u}(s)\big)\big(\mathbf{E}Q_{\delta+B}^s\big)^{-1}.$$

In view of these equalities and the equations of the system (43), we derive the second formula of (41). Taking into account the definition of the process $Y_{k,x}(t)$ and Remark 2, we find for the functions $q_{k,x}^s(0)$, $q_{0,x}^s(0)$ that

$$\begin{aligned} q_{k,x}^s(0) &= \int_0^\infty b_k^s(x,dy)q_{0,y}^s(0), \\ q_{0,y}^s(0) &= 1-\tilde{f}_y(s) + \tilde{f}_y(s)\Bigg[\lambda^B q_{B+1,0}^s(0) + (1-\lambda)\sum_{k=1}^B \lambda^{k-1} q_{k,0}^s(0)\Bigg]. \end{aligned} \tag{44}$$

Inserting the right-hand side of the second equation into the first, we get

$$q_{k,x}^s(0) = \int_0^\infty b_k^s(x,dy)\big(1-\tilde{f}_y(s)\big) + \int_0^\infty b_k^s(x,dy)\tilde{f}_y(s)q_B^s(\lambda), \tag{45}$$

where $q_B^s(\lambda) = \lambda^B q_{B+1,0}^s(0) + (1-\lambda)\sum_{k=1}^B \lambda^{k-1} q_{k,0}^s(0)$. After some transformations of the latter equation, we obtain

$$q_B^s(\lambda) = 1 - \big(1-b(s,\lambda)\big)\big(1-\tilde{b}(s,\lambda)\big)^{-1},$$

where

$$\begin{aligned} b(s,\lambda) &= \lambda^B b_{B+1}^s(0) + (1-\lambda)\sum_{k=1}^B \lambda^{k-1} b_k^s(0) \\ &= 1-(1-\lambda)\frac{A_0^B(s) - \mathbf{E}A_0^{B+\delta}(s)}{\mathbf{E}Q_{\delta+B}^s - Q_B^s}. \end{aligned}$$

The latter and the previous equality imply that

$$q_B^s(\lambda) = 1 - \frac{1-\lambda}{1-\tilde{f}(s)}\big(A_0^{B+}(s) - \mathbf{E}A_0^{\delta+B}(s)\big)\big(\mathbf{E}Q_{\delta+B}^s\big)^{-1}.$$

Inserting the right-hand side of this equality into (45) yields the first equality of (41).

For $k, u \in (0,1)$, denote $q_k^t(x,u,B) = \mathbf{P}[d_{[kB],x}(tB^2) \geq [uB]]$. Employing (41), the limiting equality (16), as well as (22), we find that

$$\begin{aligned}\frac{1}{s}\lim_{B\to\infty} q_{[kB],x}^{s/B^2}\big([uB]\big) &= \lim_{B\to\infty}\int_0^\infty e^{-st} q_k^t(x,u,B)\,dt \\ &= \frac{1-\cosh((k-u)^+\sqrt{2s}/\sigma)}{s} \\ &\quad + \frac{1}{s}\frac{\cosh(k\sqrt{2s}/\sigma)}{\sinh(\sqrt{2s}/\sigma)}\sinh\big((1-u)\sqrt{2s}/\sigma\big) \stackrel{\text{def}}{=} q^*(s), \qquad (46)\end{aligned}$$

where $u^+ = \max\{0,u\}$. When $u \in [k,1)$, we derive from this formula that

$$q^*(s) = \frac{1}{s}\frac{\cosh(k\sqrt{2s}/\sigma)}{\sinh(\sqrt{2s}/\sigma)}\sinh\big((1-u)\sqrt{2s}/\sigma\big), \quad u \in [k,1).$$

It is clear that $s_0 = 0$ is a simple pole of the function $q^*(s)$. On the half-plane $\Re(s) < 0$, this function has simple poles at $s_n = -\frac{1}{2}(\sigma\pi n)^2$, $n \in \mathbb{N}$, and it is analytic on the entire plane apart from these points. Hence, for $\alpha > 0$,

$$q(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} q^*(s)\,ds = \sum_{n\in\mathbb{Z}^+} \mathrm{Res}_{s=s_n} q^*(s).$$

Calculating the residues of the function $q^*(s)$ in s_n, we obtain the right-hand side of Formula (42) for $u \in [k,1)$. One can see that for $u \in (0,k]$ the first term on the right-hand side of (46) is analytic on the whole plane. Applying the inversion formula, we find that the contour integral of this term is equal to zero. The second term of (46) is the same also for $u \in [k,1)$. Thus, Formula (42) holds for $u \in (0,1)$.

Observe that $\lim_{s\to 0} A_x^u(s) = \lim_{s\to 0}\mathbf{E}A_0^{\delta+u}(s) = 0$, $\lim_{s\to 0} b_k^s(x) = 1$. It follows from (41) and the properties of Laplace transforms that

$$\begin{aligned}\lim_{s\to 0} q_{k,x}^s(u) &= \lim_{s\to 0} q_{0,x}^s(u) = q_u = \lim_{t\to\infty}\mathbf{P}\big[d_{(\cdot)}(t) \geq u\big], \quad u \in [1, B+1], \\ \lim_{s\to 0} q_{k,x}^s(0) &= \lim_{s\to 0} q_{0,x}^s(0) = q_0 = \lim_{t\to\infty}\mathbf{P}\big[d_{(\cdot)}(t) = 0\big].\end{aligned}$$

Calculating the limits on the right-hand sides of (41) as $s \to 0$ yields the equalities of Corollary 4. □

5.4 Virtual waiting time

Suppose that at time $t_0 = 0$ the system is at the state (k,x), $k \in [0, B+1]$, $x \geq 0$. Denote by $W_{k,x}(t)$ time required to serve the customers present in the system at time t. Formally, this random variable can be determined in the following way. Let $\tilde{\tau}(k) = \inf\{t : \pi(t) \geq k\}$, $k \in \mathbb{Z}^+$. Then

$$W_{k,x}(t) = \tilde{\tau}\big(d_{k,x}(t)\big),$$

$$\mathbf{E}e^{-pW_{k,x}(t)} = \sum_{i=0}^{B+1} \mathbf{P}\big[d_{k,x}(t) = i\big]\mathbf{E}e^{-p\tilde{\tau}(i)}, \quad p > 0.$$

Corollary 5 *Let* $k \in [0, B+1]$, $x \geq 0$. *The following equality holds*

$$\mathbf{P}\big[W_{k,x}(\nu_s) \leq v\big] = 1 - \sum_{i=0}^{B} \mathbf{P}\big[\pi(v) = i\big]q^s_{k,x}(i+1),$$

$$\lim_{t\to\infty} \mathbf{P}\big[W_{k,x}(t) \leq v\big] = 1 - \sum_{i=0}^{B} \mathbf{P}\big[\pi(v) = i\big]q_{i+1},$$

where the distributions $q^s_{k,x}(u)$, q_u, $u \in [0, B+1]$ *are given by* (41) *and* (42).

Proof It is clear that $\mathbf{P}[\tilde{\tau}(k) > t] = \mathbf{P}[\pi(t) < k]$. Hence,

$$\mathbf{E}e^{-p\tilde{\tau}(k)} = 1 - \mathbf{P}\big[\pi(\nu_p) < k\big] = 1 - \sum_{i=0}^{k-1} \tilde{\rho}_i(p),$$

where $\tilde{\rho}_i(p) = p\int_0^\infty e^{-pv}\mathbf{P}[\pi(v) = i]\,dv$. Then

$$\mathbf{E}e^{-pW_{k,x}(\nu_s)} = 1 - \sum_{i=0}^{B+1} \mathbf{P}\big[d_{k,x}(\nu_s) = i\big] \sum_{j=0}^{i-1} \tilde{\rho}_j(p)$$

$$= 1 - \sum_{i=0}^{B} \tilde{\rho}_i(p) q^s_{k,x}(i+1).$$

The right-hand side of this equality implies the formulae of Corollary 5. □

5.5 G|M$^\varkappa$|1|B system

In this subsection, we consider a partial case of the G$^\delta$|M$^\varkappa$|1|B system, namely when $\mathbf{P}[\delta = 1] = 1$. This translates to the queueing system where the customers are served one by one. Technically, one has to set the parameter $\lambda = 0$ for the geometrical distribution: $\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}$, $n \in \mathbb{N}$, $\lambda \in [0, 1)$ of the random variable δ. In other words, it means that the process $\{D_x(t)\}_{t\geq 0}$ has unit negative jumps at the time instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$ and $\delta_{N_x(t)} = N_x(t)$. Then, it follows from (2) that

$$D_x(t) = \pi(t) - N_x(t) \in \mathbb{Z}, \quad t \geq 0. \tag{47}$$

We will call this process a difference of a compound Poisson process and a simple renewal process. Setting the parameter $\lambda = 0$ in the statements of Lemma 2 leads to the following result.

Lemma 7 *For $s > 0$ the equation $z = \tilde{f}(s - k(z))$ has a unique solution $c(s)$ inside the circle $|z| < 1$. This solution is positive, $c(s) \in (0, 1)$. If $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$, $\rho = \mu\mathbf{E}[\varkappa]\mathbf{E}[\eta]$, then for $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (0, 1)$; and for $\rho \le 1$, $\lim_{s\to 0} c(s) = 1$.*

Statements of the lemmas and theorems derived in the previous subsections can be reformulated in a similar way. Letting $\lambda = 0$ in the defining Formula (10) for all $s, x \ge 0$, we get

$$Q_k^s(x) = \frac{1}{2\pi i} \oint_{|z|=\alpha} \frac{1}{z^{k+1}} \frac{\tilde{f}_x(s - k(z))}{\tilde{f}(s - k(z)) - z}\, dz, \quad \alpha \in \big(0, c(s)\big), \tag{48}$$

which is a resolvent sequence of the process $\{D_x(t)\}_{t\ge 0}$ defined by (47). This resolvent sequence was introduced in [11]. Setting $\lambda = 0$ in (11) and (12), we obtain

$$\begin{aligned} f^k(x, dl, m, s) &= e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m\big(d(l - x)\big) \\ &\quad + \Phi_0^s(dl, m)\, Q_k^s(x) - e^{-sl}\big[1 - F(l)\big] \sum_{i=0}^{k} Q_i^s(x)\, p_{k-i}^m(dl), \end{aligned} \tag{49}$$

$$\mathbf{E}e^{-s\tau^k(x)} = 1 - \frac{s}{s - k(c(s))} Q_k^s(x) - A_x^k(s),$$

i.e., the Laplace transforms of the upper one-boundary functionals of the process $\{D_x(t)\}_{t\ge 0}$ in (47), where $\tilde{\rho}_i(s) = s\int_0^\infty e^{-st}\mathbf{P}[\pi(t) = i]\, dt$,

$$\begin{aligned} A_x^k(s) &= \sum_{i=0}^{k} \tilde{\rho}_i(s)\big[1 - Q_{k-i}^s(x)\big], \\ \Phi_0^s(dl, m) &= e^{-sl}\big[1 - F(l)\big] \sum_{k\in\mathbb{Z}^+} c(s)^k p_k^m(dl). \end{aligned}$$

We have introduced the auxiliary functions and the resolvent sequence of the process (47); therefore, we can state the following result.

Corollary 6 *Let $\{D_x(t)\}_{t\ge 0}$ be the difference of the compound Poisson process and the renewal process (see (47)), let $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$, $x \ge 0$, be the resolvent sequence of the process given by (48) with $Q_k^s \stackrel{\text{def}}{=} Q_k^s(0)$. The Laplace transforms $V_r^x(m, s)$, $V_x^k(dl, m, s)$ of the joint distribution of $\{\chi, L, T\}$ satisfy the following equalities for all $x, s \ge 0$, $m \in \mathbb{N}$,*

$$\begin{aligned} V_r(x, i, s) &= \frac{Q_k^s(x)}{Q_{B+1}^s} \delta_{i1}, \\ V^k(x, dl, i, s) &= f^k(x, dl, i, s) - \frac{Q_k^s(x)}{Q_{B+1}^s} f^{B+1}(0, dl, i, s), \end{aligned}$$

where δ_{ij} is the Kronecker symbol and $f^k(x, dl, m, s)$ is given by (49).

For the Laplace transforms of the first exit time χ from the interval by the process $\{D_x(t)\}_{t\ge 0}$, the following formulae hold

$$\mathbf{E}\big[e^{-s\chi};\mathfrak{A}_r\big]=\frac{Q_k^s(x)}{Q_{B+1}^s},$$

$$\mathbf{E}\big[e^{-s\chi};\mathfrak{A}^k\big]=1-A_x^k(s)+\frac{Q_k^s(x)}{Q_{B+1}^s}\big(1-A_0^{B+1}(s)\big).$$

In order to prove the corollary, one has to put $\lambda=0$ in the statements of Theorem 1. We now illustrate how the results obtained in Corollaries 1–5 can be applied for studying the queueing system G|M$^{\varkappa}$|1|B ($\mathbf{P}[\delta=1]=1$) with a finite buffer.

Corollary 7 *Let $b_k^s(x)=\mathbf{E}[e^{-sb_r(x)};b_r(x)<\infty]$ be the Laplace transform of the busy period of (k,x) type of the* G|M$^{\varkappa}$|1|B *system. Then*

(i) *The following equality holds*

$$b_k^s(x)=1-A_x^{k-1}(s)+Q_{k-1}^s(x)\frac{A_0^{B+1}(s)-A_0^B(s)}{Q_{B+1}^s-Q_B^s},\quad k\in[1,B+1],$$

the random variable $b_k(x)$ is proper with a finite mathematical expectation

$$\mathbf{E}b_k(x)=A_x^{k-1}-Q_{k-1}(x)\frac{A_0^{B+1}-A_0^B}{Q_{B+1}-Q_B}<\infty.$$

(ii) *The Laplace transform $b_k^s(x,dy)=\mathbf{E}[e^{-sb_k(x)};\ \eta(x)\in dy]$ of the joint distribution of $\{b_k(x),\eta(x)\}$ is such that for $k\in[1,B+1]$*

$$b_k^s(x,dy)=f^{k-1}(x,dy,s)-Q_{k-1}^s(x)\frac{f^{B+1}(0,dy,s)-f^B(0,dy,s)}{Q_{B+1}^s-Q_B^s}.$$

Corollary 8 *Let $l_k(x)$ be the time of the first loss of a customer in the* G|M$^{\varkappa}$|1|B *system, $i_{k,x}(\nu_s)$ be the number of the lost customers on the time interval $[0,\nu_s]$. Then*

(i) *The Laplace transform $l_k^s(x)=\mathbf{E}[e^{-sl_k(x)};l_k(x)<\infty]$ of $l_k(x)$ is such that*

$$l_k^s(x)=\frac{\tilde f_x(s)+(1-\tilde f(s))S_{k-1}^s(x)}{\tilde f(s)+(1-\tilde f(s))S_{B+1}^s},\quad k\in[0,B+1],$$

where $S_k^s(x)=\sum_{i=0}^k Q_i^s(x)$, $S_k^s(x)=0$ for $k<0$. The random variable $l_k(x)$ is proper with a finite mathematical expectation $\mathbf{E}l_k(x)=\mathbf{E}\eta_x-\mathbf{E}\eta+\mathbf{E}\eta[S_{B+1}-S_{k-1}(x)]<\infty$.

(ii) *The distribution $I_{k,x}^s(n)=\mathbf{P}[i_{k,x}(\nu_s)=n]$, $n\in\mathbb{Z}^+$ of the number of the lost customers on the time interval $[0,\nu_s]$ obeys the equality*

$$I_{k,x}^s(n)=\mathbf{I}_{\{n=0\}}\big(1-l_k^s(x)\big)+\mathbf{I}_{\{n\in\mathbb{N}\}}l_k^s(x)\big(1-l_{B+1}^s(0)\big)\big(l_{B+1}^s(0)\big)^{n-1}.$$

Corollary 9 *The distribution of the number of customers in the system* G|M$^{\varkappa}$|1|B *at time* ν_s *is such that*

$$q_{k,x}^s(0) = 1 - A_x^{k-1}(s) - \big(A_0^B(s) - A_0^{B+1}(s)\big)C_{k-1}^s(x)/Q_{B+1}^s,$$
$$q_{k,x}^s(u) = A_x^{k-u}(s) + \big(A_0^{B+1-u}(s) - A_0^{B+2-u}(s)\big)C_{k-1}^s(x)/Q_{B+1}^s,$$

where $A_x^k(s) = S_k^s(x) = 0$ *for* $k < 0$, $C_k^s(x) = \tilde{f}_x(s)(1 - \tilde{f}(s))^{-1} + S_k^s(x)$. *Let* $q_0 = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) = 0]$, $q_u = \lim_{t\to\infty} \mathbf{P}[d_{(\cdot)}(t) \geq u]$, $u \in [1, B+1]$ *be the stationary distribution of the number of customers in the* G|M$^{\varkappa}$|1|B *system. Then*

$$q_0 = 1 - \frac{1}{\mathbf{E}\eta}\big(A_0^B - A_0^{B+1}\big)/Q_{B+1},$$

$$q_u = \frac{1}{\mathbf{E}\eta}\big(A_0^{B+1-u} - A_0^{B+2-u}\big)/Q_{B+1},$$

$$\lim_{t\to\infty} \mathbf{P}\big[W_{k,x}(t) \leq v\big] = 1 - \frac{1}{\mathbf{E}\eta}\sum_{i=0}^{B} \mathbf{P}\big[\pi(v) = i\big]\big(A_0^{B-i} - A_0^{B+1-i}\big)/Q_{B+1},$$

where $\rho_i = \lim_{s\to 0} s^{-1}\tilde{\rho}_i(s) = \int_0^\infty \mathbf{P}[\pi(t) = i]\,dt < \infty$, $Q_k = Q_k^0$,

$$A_0^u = \lim_{s\to 0}\frac{1}{s}A_0^u(s) = \sum_{i=0}^{u} \rho_i[1 - Q_{u-i}].$$

In order to prove Corollaries 7–9, it suffices to set $\lambda = 0$ in the formulae of Theorem 5 and of Corollaries 1–5. The resolvent coefficients Q_k, $k \in \mathbb{Z}^+$, can be obtained form the following recurrence relation

$$Q_0 = 1, \quad Q_k = \frac{1}{f_0}\left(f_k + Q_{k-1} - \sum_{i=0}^{k-1} Q_i f_{k-i}\right), \quad k \in \mathbb{N},$$

where $f_k = \mathbf{P}[\pi(\eta) = k] = \int_0^\infty \mathbf{P}[\eta \in dt]\mathbf{P}[\pi(t) = k]$. For the system G|M|1|B, $\mathbf{P}[\varkappa = 1] = 1$, so that $\mathbf{P}[\pi(t) = k] = (\mu t)^k e^{-\mu t}/k!$.

Appendix

Let us verify Equalities (22) of Lemma 6. Suppose that Condition (A) is satisfied. Then for $s, p \to 0$ the following expansions hold

$$\tilde{f}_x(s) = 1 - s\mathbf{E}\eta_x + \frac{1}{2}s^2\mathbf{E}\eta_x^2 + o\big(s^2\big), \quad x \geq 0,$$
$$\mathbf{E}e^{-p\varkappa} = 1 - p\mathbf{E}\varkappa + \frac{1}{2}p^2\mathbf{E}\varkappa^2 + o\big(p^2\big). \tag{50}$$

We now derive the asymptotic expansions for the function $S^{s/B^2}_{[kB]}(x)$ as $B \to \infty$. The generating function

$$\begin{aligned}\mathbb{S}^s_\theta(x) &= \sum_{k\in\mathbb{Z}^+} \theta^k \sum_{i=0}^{k} Q^s_i(x) \\ &= \frac{1}{1-\theta}\,\frac{(1-\lambda)\tilde{f}_x(s-k(\theta))}{(1-\lambda)\tilde{f}(s-k(\theta))+\lambda-\theta}, \qquad \theta \in \big(0, c(s)\big),\end{aligned}$$

is such that for $\theta = e^{-p}$, $p > -\ln c(s)$,

$$\begin{aligned}\mathbb{S}^s_\theta(x) &= \sum_{k\in\mathbb{Z}^+} \theta^k S^s_k(x)\big|_{\theta=e^{-p}} = \int_0^\infty e^{-p[k]} S^s_{[k]}(x)\,dk \\ &= \int_0^\infty e^{\{k\}p} e^{-pk} S^s_{[k]}(x)\,dk = \mathbb{S}^s_{e^{-p}}(x), \qquad p > -\ln c(s),\end{aligned}$$

where $\{a\}$ is the fractional part of the number a. By $\mathfrak{S}^s_p(x) = \int_0^\infty e^{-pk} S^s_{[k]}(x)\,dk$, $p > -\ln c(s)$, we denote the Laplace transform of the function $S^s_{[k]}(x)$. It is clear that

$$\mathfrak{S}^s_p(x) \le \mathbb{S}^s_{e^{-p}}(x) \le e^p \mathfrak{S}^s_p(x). \tag{51}$$

Employing the limiting equalities (50) and the definition of the function $\mathbb{S}^s_\theta(x)$, we obtain

$$\begin{aligned}\lim_{B\to\infty} \mathbb{S}^{s/B^2}_{e^{-p/B}}(x) B^{-3} &= \lim_{B\to\infty} \frac{B^{-3}}{1-e^{-p/B}}\,\frac{(1-\lambda)\tilde{f}_x(s/B^2 - k(e^{-p/B}))}{(1-\lambda)\tilde{f}(s/B^2 - k(e^{-p/B}))+\lambda - e^{-p/B}} \\ &= \frac{1}{sp\mathbf{E}\eta}\,\frac{1}{\frac{1}{2}p^2\sigma^2 - s}, \qquad p > \sqrt{2s}/\sigma.\end{aligned}$$

It follows from the chain (51) that

$$\lim_{B\to\infty} \frac{1}{B^3} \mathfrak{S}^{s/B^2}_{p/B}(x) = \lim_{B\to\infty} \frac{1}{B^3} \mathbb{S}^{s/B^2}_{e^{-p/B}}(x) = \frac{1}{sp\,\mathbf{E}\eta}\,\frac{1}{\frac{1}{2}p^2\sigma^2 - s}. \tag{52}$$

Inverting the Laplace transforms (with respect to p) on both sides, we obtain

$$\lim_{B\to\infty} \frac{1}{B^2} S^{s/B^2}_{[kB]}(x) = \frac{1}{s\mathbf{E}\eta}\cosh(k\sqrt{2s}/\sigma - 1).$$

In order to invert the Laplace transforms, we have calculated the residues on the right-hand side of (52) at the simple poles $p = 0, \pm\sqrt{2s}/\sigma$. The second part of the first equality (22) can be verified analogously:

$$\lim_{B\to\infty} B^{-2}\mathbf{E}S^{s/B^2}_{\delta+[kB]} = \frac{1}{s\mathbf{E}\eta}\cosh(k\sqrt{2s}/\sigma - 1).$$

We will now verify the second formula of (22). Denote $q_k^s = \mathbf{E}Q_{\delta+k}^s - Q_k^s$, $k \in \mathbb{Z}^+$. Employing (9) and (10), we determine the generating function of this sequence:

$$\tilde{q}_\theta^s = \sum_{k\in\mathbb{Z}^+} \theta^k q_k^s = \frac{(1-\lambda)(1-\tilde{f}(s-k(\theta)))}{(1-\lambda)\tilde{f}(s-k(\theta))+\lambda-\theta}, \quad \theta \in \bigl(0, c(s)\bigr).$$

This generating function is such that for $\theta = e^{-p}$, $p > -\ln c(s)$,

$$\tilde{q}_{e^{-p}}^s = \int_0^\infty e^{-p[k]} q_{[k]}^s \, dk = \int_0^\infty e^{\{k\}p} e^{-pk} q_{[k]}^s \, dk.$$

By $\mathfrak{q}_p^s = \int_0^\infty e^{-pk} q_{[k]}^s \, dk$, $p > -\ln c(s)$, we denote the Laplace transform of the function $q_{[k]}^s$. It is clear that

$$\mathfrak{q}_p^s \le \tilde{q}_{e^{-p}}^s \le e^p \mathfrak{q}_p^s. \tag{53}$$

In view of the limiting equalities (50) and the definition of the function $\tilde{q}_\theta^s$, we find

$$\begin{aligned}\lim_{B\to\infty} \tilde{q}_{e^{-p/B}}^{s/B^2} B^{-1} &= \lim_{B\to\infty} \frac{(1-\lambda)(1-\tilde{f}(s/B^2-k(e^{-p/B})))B^{-1}}{(1-\lambda)\tilde{f}(s/B^2-k(e^{-p/B}))+\lambda-e^{-p/B}} \\ &= \frac{p\mu\mathbf{E}\varkappa}{\frac{1}{2}p^2\sigma^2-s}.\end{aligned}$$

It follows from the chain (53) that

$$\lim_{B\to\infty} \frac{1}{B} \mathfrak{q}_{p/B}^{s/B^2} = \lim_{B\to\infty} \frac{1}{B} \tilde{q}_{e^{-p/B}}^{s/B^2} = \frac{p\,\mu\mathbf{E}\varkappa}{\frac{1}{2}p^2\sigma^2-s}.$$

Inverting the Laplace transforms (with respect to p) on both sides, we obtain

$$\lim_{B\to\infty} q_{[kB]}^{s/B^2} = \lim_{B\to\infty} \bigl[\mathbf{E}Q_{[\delta+kB]}^{s/B^2} - Q_{[kB]}^{s/B^2}\bigr] = \frac{\mu\mathbf{E}\varkappa}{\sigma^2} \cosh(k\sqrt{2s}/\sigma).$$

The third formula of (22) can be verified analogously.

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