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# Exit problems for the difference of a compound Poisson process and a compound renewal process

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**Abstract** In this paper we solve a two-sided exit problem for a difference of a compound Poisson process and a compound renewal process. More specifically, we determine the Laplace transforms of the joint distribution of the first exit time, the value of the overshoot and the value of a linear component at this time instant. The results obtained are applied to solve the two-sided exit problem for a particular class of stochastic processes, i.e. the difference of the compound Poisson process and the renewal process whose jumps are exponentially distributed. The advantage is that these results are in a closed form, in terms of resolvent sequences of the process.

We determine the Laplace transforms of the busy period of the systems $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$, $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ in case when $\delta \sim \exp(\lambda)$. Additionally, we prove the weak convergence of the two-boundary characteristics of the process to the corresponding functionals of the standard Wiener process.

**Keywords** Difference of compound renewal processes · The first exit time · Value of the overshoot · Resolvent

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## 1 Introduction

In this paper we solve the two-sided exit problem for the difference between a compound Poisson process and a compound renewal process. The method that we use is

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based on employing one-boundary characteristics of the process and it is borrowed from [16]. Our motivation stems from the fact that such processes have proven to be appropriate models in many applied fields of probability theory, such as telecommunication networks, computer networks, and in queueing theory. More specifically, such processes can serve for studying the characteristics of certain queueing systems with limited waiting room like $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$, $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$. Before describing the problem of interest, we give a short overview of the relevant literature.

Distributions of the one-boundary functionals for the difference of renewal processes have been studied by Lindley [20], Prabhu [27] and Cohen [5]. A summary of the known results for the G|G|1 type model can be found in [5]. Distributions of the one-boundary functionals for the difference of compound renewal processes have been considered by Nasirova [22] in terms of the solutions of linear integral equations. Special cases of the difference of renewal processes have been studied by many authors [4, 9–12, 26]. One-boundary functionals for the difference of compound Poisson processes and renewal processes have been considered by Korolyuk and Pirliev [19]. One-boundary functionals for the difference of two compound Poisson processes with drifts with various jump distributions have been studied by Perry et al. [24].

A more complicated problem, i.e. the two-sided exit problem, has been considered for Lévy processes and random walks. For an overview of the existing results on the two-boundary problems we refer to [13]. And here we only mention several authors who contributed a lot in the development of this area. Starting from Kemperman (1963), Takacs (1966), Emery (1973), Gihman, Skorohod (1973), Pecherskii (1974), Korolyuk (1975), Suprun, Shurenkov (1975–1988), Lotov, Khodzhibaev, Orlova (1979–2006), and later Bertoin (1996, 1997), Lambert (2000), Doney (2003), Avram, Kyprianou, Pistorius (2004), Pistorius (2003–2004), Kyprianou, Palmowski (2005), and Kadankov, Kadankova (2003–2008) studied one- or two-boundary characteristics employing various methods. The Laplace transforms of the joint distribution of the first exit time and the value of the overshoot at this time instant for general Lévy processes and random walks have been determined in [16]. The Laplace transforms of this joint distribution were found in terms of the Laplace transforms of the one-boundary characteristics of the process. This method for Lévy processes and random walks [16] can be applied for other classes of stochastic processes, such as the difference of compound renewal processes [15, 18] and semi-Markov random walks with linear drift [17].

The rest of the article is structured as follows. First, we will provide necessary definitions and notations (Sect. 2). The two-sided exit problem is solved for the difference of the compound Poisson process and a compound renewal process in Sect. 3. In Sect. 4 we will derive similar results for the difference of the compound Poisson process and a renewal process whose jumps are exponentially distributed. Section 5 deals with the reflected process.

In Sect. 6 in case when $\rho = 1$, $\mathbf{E}\varkappa^2, \mathbf{E}\eta^2 < \infty$, we prove the weak convergence of the two-boundary characteristics of the process to the corresponding distributions of the standard Wiener process. We also show how the results obtained can be applied for the queueing systems $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$, $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ with limited waiting room.

## 2 Preliminaries

Let $\varkappa$, $\delta$, $\eta \in (0,\infty)$ be independent positive random variables. Denote by $F(x) = \mathbf{P}[\eta \le x]$, $x \in \mathbb{R}_+ = [0,\infty)$, the distribution function of $\eta$. Introduce the following sequences: $\{\eta, \eta'_n\}$, $\{\varkappa, \varkappa'_n\}$, $\{\delta, \delta'_n\}$, $n \in \mathbb{N} = \{1,2,\ldots\}$, of independent and identically distributed (inside each sequence) random variables. Define monotone sequences by means of the relations

$$\begin{gathered}\eta_0(x) = 0, \qquad \eta_1(x) = \eta_x, \qquad \eta_{n+1}(x) = \eta_x + \eta'_1 + \cdots + \eta'_n, \quad n \in \mathbb{N}, \\ \varkappa_0 = 0, \qquad \varkappa_n = \varkappa'_1 + \cdots + \varkappa'_n; \quad \delta_0 = 0, \quad \delta_n = \delta'_1 + \cdots + \delta'_n; \quad n \in \mathbb{N},\end{gathered} \tag{1}$$

where $\eta_x \in (0,\infty)$ is a random variable with the distribution function given by

$$F_x(u) = \mathbf{P}[\eta_x \le u] = \bigl[F(x+u) - F(x)\bigr]\bigl(1 - F(x)\bigr)^{-1}, \quad u \ge 0.$$

Denote by $\{\pi_t\}_{t\ge 0} \in \mathbb{R}_+$ the compound Poisson process with the Laplace transform

$$\mathbf{E}e^{-p\pi_t} = e^{tk(p)}, \quad k(p) = \mu\bigl(\mathbf{E}e^{-p\varkappa} - 1\bigr), \quad \Re(p) \ge 0,$$

where $\mu > 0$ is the jump intensity and $\varkappa$ is the jump size. For all $t \ge 0$ define a renewal process induced by the sequence $\{\eta_n(x)\}_{n\in\mathbb{N}\cup 0}$ :

$$N_x(t) = \max\bigl\{n \in \mathbb{N} \cup \{0\} : \eta_n(x) \le t\bigr\}, \quad x \ge 0.$$

For all $x \ge 0$ introduce a right-continuous step process

$$D_x(t) = \pi_t - \delta_{N_x(t)} \in \mathbb{R}, \quad t \ge 0,\ D_x(0) = 0. \tag{2}$$

Note that the inter-arrival times of positive jumps are exponentially distributed with parameter $\mu$, the positive jumps themselves are of a random size $\varkappa$, and that there occur negative jumps of size $\delta'_n$ at time instants $\eta_n(x)$, $n \in \mathbb{N}$. We will call the process $\{D_x(t)\}_{t\ge 0}$ a difference of the compound Poisson process and the compound renewal process. Observe that this process is not a Markov process in general. Therefore, we introduce a right-continuous linear component for all $t \ge 0$

$$\eta_x^+(t) = \begin{cases} t + x, & 0 \le t < \eta_x, \\ t - \eta_{N_x(t)}(x), & t \ge \eta_x \end{cases} \in \mathbb{R}_+, \quad x \ge 0. \tag{3}$$

The process $\{\eta_x^+(t)\}_{t\ge 0}$ increases linearly on the time intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{N} \cup \{0\}$, it is killed to zero at the instants $\eta_n(x)$, $n \in \mathbb{N}$, and the value of the process at the instant $t_0 \ge \eta_x$ is equal to the time elapsed from the last negative jump of the process (2) till $t_0$. Note that this linear component is sometimes referred to as the age process (age since the last renewal). By adding this linear component to the process $\{D_x(t)\}_{t\ge 0}$, we obtain a right-continuous Markov process

$$\{X_t\}_{t\ge 0} = \bigl\{D_x(t), \eta_x^+(t)\bigr\}_{t\ge 0} \in \mathbb{R} \times \mathbb{R}_+, \quad X_0 = \{0, x\},\ x \ge 0, \tag{4}$$

which governs the process $\{D_x(t)\}_{t\geq 0}$. The process (4) is homogeneous with respect to the first component [8]. In other words if $X_{t_0}=\{k,u\}$, $k\in\mathbb{Z}$, $u\geq 0$, then the evolution of the process $\{X_t\}_{t\geq t_0}$ in the sequel does not depend on the value $k$ of the first component, and the first positive jump of the process $\{D_x(t)\}_{t\geq t_0}$ (which is distributed as $\varkappa$) will occur after an exponential time with parameter $\mu$. The first negative jump of the process $\{D_x(t)\}_{t\geq t_0}$ (which is distributed as $\delta$) will take place after elapsing of time $\eta_u$. We now define the one-boundary characteristics of the process. Let $X_0=\{0,x\}$, $x,k\geq 0$, and denote by

$$\tau^k(x)=\inf\bigl\{t: D_x(t)>k\bigr\},\qquad T^k(x)=D_x\bigl(\tau^k(x)\bigr)-k,\qquad \eta^k(x)=\eta_x^+\bigl(\tau^k(x)\bigr)$$

the first crossing time of the upper boundary $k$ by the process $\{D_x(t)\}_{t\geq 0}$, the value of the overshoot and the value of the linear component $\eta_x^+(\cdot)$ at this instant. For $x,k\geq 0$ we define the lower one-boundary functionals in a similar way:

$$\tau_k(x)=\inf\bigl\{t:\ D_x(t)<-k\bigr\},\qquad T_k(x)=-D_x\bigl(\tau_k(x)\bigr)-k,$$
$$\eta_k(x)=\eta_x^+\bigl(\tau_k(x)\bigr),$$

i.e., the first crossing time of the lower boundary $-k$ by the process $\{D_x(t)\}_{t\geq 0}$, the value of the overshoot, and the value of the linear component at the instant of the first crossing. Observe that the crossing of the lower boundary occurs at the time instants $\eta_n(x)$, $n\in\mathbb{N}$, i.e. the times of the negative jumps of the process $\{D_x(t)\}_{t\geq 0}$ and, therefore, in view of (3), $\mathbf{P}[\eta_k(x)=0]=1$. The random variables $\tau^k(x)$, $\tau_k(x)$ are the Markov times of the process $\{X_t\}_{t\geq 0}$ and they take values from the countable sets $\{\xi_n,n\in\mathbb{N}\}$, $\{\eta_n(x),n\in\mathbb{N}\}$ ($\xi_n$ is an instant of the $n$th jump of the process $\{\pi_t\}_{t\geq 0}$). As mentioned above, one-boundary characteristics of the process were considered by many authors. We, thus, assume that the integral transforms of the distributions of the upper and lower one-boundary functionals of the process $\{X_t\}_{t\geq 0}$ are known. We will solve the two-sided exit problem for the process $\{D_x(t)\}_{t\geq 0}$ in terms of the Laplace transforms of these distributions.

## 3 Two-sided exit problem for the process $\{D_x(t)\}_{t\geq 0}$

Let $B\in\mathbb{R}_+$ be fixed, $k\in[0,B]$, $r=B-k$, $X_0=\{0,x\}$, and denote by

$$\chi_x^B(r)=\inf\bigl\{t: D_x(t)\notin[-r,k]\bigr\}\overset{\text{def}}{=}\chi$$

the first exit time from the interval $[-r,k]$ by the process $\{D_x(t)\}_{t\geq 0}$. This random variable takes values from a countable set $\{\xi_n,n\in\mathbb{N}\}\cup\{\eta_n(x),n\in\mathbb{N}\}$, and it is a Markov time of the process $\{X_t\}_{t\geq 0}$. It is clear that the exit from the interval can occur either through the upper boundary $k$, or through the lower boundary $-r$. Define the events:

$\mathfrak{A}^k=\{D_x(\chi)>k\}$, i.e. the process $\{D_x(t)\}_{t\geq 0}$ exits the interval $[-r,k]$ through the upper boundary $k$.

$\mathfrak{A}_r = \{D_x(\chi) < -r\}$, i.e. the exit from the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\geq 0}$ occurs through the lower boundary $-r$, $P[\mathfrak{A}^k + \mathfrak{A}_r] = 1$.

Introduce

$$T = \big(D_x(\chi) - k\big)\mathbf{I}_{\mathfrak{A}^k} + \big(-D_x(\chi) - r\big)\mathbf{I}_{\mathfrak{A}_r}, \qquad L = \eta_x^+(\chi)\mathbf{I}_{\mathfrak{A}^k} + 0 \cdot \mathbf{I}_{\mathfrak{A}_r},$$

the value of the overshoot across the boundaries of the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\geq 0}$ and the value of the linear component at this instant, where $\mathbf{I}_A = \mathbf{I}_A(\omega)$ is the indicator of a set $A$. For all $x, k \geq 0$ denote

$$\begin{aligned}
f_x^k(dl, du, s) &= \mathbf{E}\big[e^{-s\tau^k(x)}; \eta^k(x) \in dl, T^k(x) \in du, \tau^k(x) < \infty\big],\\
f_k^x(du, s) &= \mathbf{E}\big[e^{-s\tau_k(x)}; T_k(x) \in du, \tau_k(x) < \infty\big],\\
F_x^k(dl, du, s) &= f_x^k(dl, du, s) - \int_0^\infty f_r^x(dv, s) f_0^{v+B}(dl, du, s),\\
F_r^x(du, s) &= f_r^x(du, s) - \iint_{\mathbb{R}_+^2} f_x^k(dl, dv, s) f_{v+B}^l(du, s).
\end{aligned}$$

**Theorem 1** *Let $\{D_x(t)\}_{t\geq 0}$ be the difference of the compound Poisson process and the compound renewal process* (2), *$B \in \mathbb{R}_+$, $k \in [0, B]$, $r = B - k$, $X_0 = \{0, x\}$, $x \geq 0$. Then the Laplace transforms*

$$\begin{aligned}
V^k(x, dl, du, s) &= \mathbf{E}\big[e^{-s\chi}; L \in dl, T \in du, \mathfrak{A}^k\big],\\
V_r(x, du, s) &= \mathbf{E}\big[e^{-s\chi}; T \in du, \mathfrak{A}_r\big]
\end{aligned}$$

*of the joint distribution of $\{\chi, L, T\}$ (the first exit time from the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\geq 0}$, the value of the linear component and the value of the overshoot through the boundary at the instant of the first exit) satisfy the following formulae for $s > 0$:*

$$\begin{aligned}
V^k(x, dl, du, s) &= F_x^k(dl, du, s) + \iint_{\mathbb{R}_+^2} F_x^k(dl_1, du_1, s)\mathfrak{K}_{l_1u_1}^+(dl, du, s),\\
V_r(x, du, s) &= F_r^x(du, s) + \int_0^\infty F_r^x(dv, s)\mathfrak{K}_v^-(du, s),
\end{aligned} \tag{5}$$

*where*

$$\mathfrak{K}_{l_1u_1}^+(dl, du, s) = \sum_{n\in\mathbb{N}} K_{l_1u_1}^+(dl, du, s)^{*(n)}, \qquad \mathfrak{K}_v^-(du, s) = \sum_{n\in\mathbb{N}} K_v^-(du, s)^{*(n)} \tag{6}$$

*are the uniformly convergent series of the successive iterations, and*

$$\begin{aligned}
&K_{l_1u_1}^+(dl, du, s)^{*(1)} \overset{\text{def}}{=} K_{l_1u_1}^+(dl, du, s), \qquad K_v^-(du, s)^{*(1)} \overset{\text{def}}{=} K_v^-(du, s),\\
&K_{l_1u_1}^+(dl, du, s)^{*(n+1)} = \iint_{\mathbb{R}_+^2} K_{l_1u_1}^+(dl_2, du_2, s) K_{l_2u_2}^+(dl, du, s)^{*(n)}, \quad n \in \mathbb{N},\\
&K_v^-(du, s)^{*(n+1)} = \int_0^\infty K_v^-(du_1, s) K_{u_1}^-(du, s)^{*(n)}, \quad n \in \mathbb{N}
\end{aligned} \tag{7}$$

*are the successive iterations of the kernels* $K^{+}_{l_1u_1}(dl,du,s)$, $K^{-}_{v}(du,s)$ *which are given by the following formulae*:

$$\begin{aligned} K^{+}_{l_1u_1}(dl,du,s) &= \int_0^\infty f^{l_1}_{u_1+B}(dv,s) f_0^{v+B}(dl,du,s), \\ K^{-}_{v}(du,s) &= \iint_{\mathbb{R}^2_+} f_0^{v+B}(dl,du_1,s) f^{l}_{u_1+B}(du,s). \end{aligned} \tag{8}$$

*Proof* The joint distribution of the first exit time and the value of the overshoot through the boundaries was determined in [16] for Lévy processes and random walks. This distribution was obtained in terms of the joint distributions of the one-boundary functionals. We will employ this idea (use of one-boundary functionals) and the method (setting up a system of integral equations) to solve the two-sided exit problem for the difference of the compound Poisson process and the compound renewal process. Following [16], we derive a system of equations with respect to the functions $V^k(x,dl,du,s)$, $V_r(x,du,s)$, $s>0$:

$$\begin{aligned} f^{k}_{x}(dl,du,s) &= V^k(x,dl,du,s) + \int_0^\infty V_r(x,dv,s) f_0^{v+B}(dl,du,s), \\ f^{x}_{r}(du,s) &= V_r(x,du,s) + \iint_{\mathbb{R}^2_+} V^k(x,dl,dv,s) f^{l}_{v+B}(du,s). \end{aligned} \tag{9}$$

In order to write this system, we have used the total probability law, homogeneity of the process $\{X_t\}_{t\ge 0}$ with respect to the first component, the Markov property of the random variables $\tau^k_x$, $\tau^x_k$, $\chi$, and the following probabilistic reasoning. The first equation of the system (9) stresses the fact that the first overshoot of the upper boundary $k$ by the process $\{D_x(t)\}_{t\ge 0}$ (expression on the left-hand side of the equation) can be realized either on sample paths which do not intersect the lower boundary $-r$ (the first term on the right-hand side of the equation) or on the sample paths which do intersect the boundary $-r$ and then later on intersect the boundary $k$ (the second term on the right-hand side of the equation). The second equation of the system is written analogously. This system is similar to a system of linear equations with two unknown variables. Substituting the expression for the function $V^k(x,dl,du,s)$ from the first equation of system (9) into the second one, we get

$$\begin{aligned} V_r(x,du,s) = f^{x}_{r}(du,s) &- \iint_{\mathbb{R}^2_+} f^{k}_{x}(dl,dv,s) f^{l}_{v+B}(du,s) \\ &+ \iint_{\mathbb{R}^2_+} \int_{v=0}^{\infty} V_r(x,dv,s) f_0^{v+B}(dl,du_1,s) f^{l}_{u_1+B}(du,s). \end{aligned}$$

Changing the order of integration in the third term of the right-hand side of the latter equation, we obtain a linear integral equation for the function $V_r(x,du,s)$:

$$V_r(x,du,s) = F^{x}_{r}(du,s) + \int_0^\infty V_r(x,dv,s) K^{-}_{v}(du,s), \tag{10}$$

where

$$K_v^-(du,s)=\iint_{\mathbb{R}_+^2} f_0^{v+B}(dl,du_1,s)f_{u_1+B}^{l}(du,s)$$

is the kernel of this equation. As mentioned above, $\tau^k(x)\in\{\xi_n,n\in\mathbb{N}\}$, $\tau_k(x)\in\{\eta_n(x),n\in\mathbb{N}\}$, where $\xi_n$ is the instant of the $n$th jump of the Poisson process. It is clear that, for all $x,k\in\mathbb{R}_+$, the following inequalities hold:

$$\mathbf{E}e^{-s\tau^k(x)}\le\frac{\mu}{s+\mu},\qquad \mathbf{E}e^{-s\tau_k(x)}\le\mathbf{E}e^{-s\eta_x}.$$

Therefore, the kernel of equation (10) enjoys the following property for $u,v\in\mathbb{R}_+$, $s>0$:

$$\begin{aligned}K_v^-(du,s)&\le\iint_{\mathbb{R}_+^2} f_0^{v+B}(dl,du_1,s)\mathbf{E}\big[e^{-s\eta_l}\big]\\&\le\iint_{\mathbb{R}_+^2} f_0^{v+B}(dl,du_1,s)\le\frac{\mu}{s+\mu}<1.\end{aligned}$$

Employing the method of mathematical induction and the latter bound for the kernel, one can show that for all $v,u\in\mathbb{R}_+$, $K_v^-(du,s)^{*(n)}\le\mu^n(s+\mu)^{-n}$, $n\in\mathbb{N}$. Hence, the series

$$\mathfrak{K}_v^-(du,s)=\sum_{n\in\mathbb{N}}K_v^-(du,s)^{*(n)}\le\frac{\mu}{s}<\infty,\quad s>0,$$

converges uniformly for all $v,u\in\mathbb{R}_+$. Utilizing the method of successive iterations [25] to solve (10), we get the second equality of Theorem 1. Substituting the expression for the function $V_r(x,du,s)$ from the second equation into the first one, we find

$$\begin{aligned}V^k(x,dl,du,s)&=f_x^k(dl,du,s)-\int_0^\infty f_r^x(dv,s)f_0^{v+B}(dl,du,s)\\&\quad+\int_{v=0}^\infty\iint_{\mathbb{R}_+^2} V^k(x,dl_1,du_1,s)f_{u_1+B}^{l_1}(dv,s)f_0^{v+B}(dl,du,s).\end{aligned}$$

Changing the order of integration in the third term on the right-hand side of this equation, we get for the function $V^k(x,dl,du,s)$:

$$V^k(x,dl,du,s)=F_x^k(dl,du,s)+\iint_{\mathbb{R}_+^2} V^k(x,dl_1,du_1,s)K_{l_1u_1}^+(dl,du,s),\tag{11}$$

i.e. a linear integral equation. The kernel of this equation

$$K_{l_1u_1}^+(dl,du,s)=\int_0^\infty f_{u_1+B}^{l_1}(dv,s)f_0^{v+B}(dl,du,s)$$

can be estimated as follows (for all $u,u_1,l,l_1\in\mathbb{R}_+$, $s>0$):

$$K_{l_1u_1}^+(dl,du,s)\le\frac{\mu}{s+\mu}\int_0^\infty f_{u_1+B}^{l_1}(dv,s)\le\frac{\mu}{s+\mu}\mathbf{E}\big[e^{-s\tau_{u_1+B}(l_1)}\big]\le\frac{\mu}{s+\mu}<1.$$

Therefore, the series of the successive iterations

$$\mathfrak{K}^{+}_{l_1 u_1}(dl, du, s) = \sum_{n\in\mathbb{N}} K^{+}_{l_1 u_1}(dl, du, s)^{*(n)} \le \frac{\mu}{s} < \infty$$

converges uniformly for all $u, u_1, l, l_1 \in \mathbb{R}_+$. Employing the method of successive iterations [25] to solve (11), we derive the first equation of the theorem, which completes the proof. □

It is worth noticing that the results of Theorem 1 can be naturally generalized for the matrix case for Lévy processes defined on the finite Markov chain. The main contribution of Theorem 1 is that the joint distribution of $\{\chi, L, T\}$ is derived in terms of more simple joint distributions of the one-boundary functionals of the process. From practical point of view, however, it is not obvious how to derive the numerical solutions from Theorem 1 in case when these random variables have arbitrary distributions. In case when some of the variables $\varkappa$, $\delta$, and $\eta$ are exponentially distributed, the Laplace transform of the first exit time $\chi$ is given in terms of the roots of a cubic equation.

## 4 The difference of the compound Poisson process and the renewal process with exponential jumps

In this section we will suppose that the random variable $\delta$ is exponentially distributed with parameter $\lambda > 0 : \delta \sim \exp(\lambda)$. This assumption means that the process $\{D_x(t)\}_{t\ge 0}$ has exponentially distributed negative jumps at time instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$. In this case, it is possible to obtain closed-form solutions for the two-sided exit problem introduced in the previous section. These solutions will be given in terms of certain functions which are given below. Our task now is to determine the Laplace transforms of the joint distributions of the upper and lower one-boundary functionals of the process $\{X_t\}_{t\ge 0}$. In the sequel we will use the following result.

**Lemma 1** *Let* $\tilde{f}(s) = \mathbf{E}[e^{-s\eta}]$. *Then*

(i) *For* $s > 0$ *the equation*

$$\lambda \tilde{f}\big(s - k(p)\big) + p - \lambda = 0 \tag{12}$$

*has a unique solution in the semi-plane* $\Re(p) > 0$. *This solution* $p = c(s)$ *is positive and* $c(s) \in (0, \lambda)$.

(ii) *If* $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$, $\rho = \lambda\mu\mathbf{E}\varkappa\mathbf{E}\eta$, *then for* $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (0, \lambda)$; *and for* $\rho \le 1$, $\lim_{s\to 0} c(s) = 0$.

*Proof* Let $s > 0$, $p \ge 0$. Examining the graphs of the real-valued functions $y_1(p) = \lambda - p$, $y_2(s, p) = \lambda \tilde{f}(s - k(p))$ on the interval $[0, \lambda]$ it is easy to establish the existence of a solution $c(s) \in (0, \lambda)$ of (12). Exploiting the behavior of the derivative $\frac{d}{dp} y_2(0, p)|_{p\downarrow 0} = -\rho$, one can assure that the statements of the second part of the lemma hold.

In order to prove the uniqueness of the solution of (12) in the semi-plane $\Re(p) > 0$, we employ Rouché's theorem. Let $\varepsilon$, $R > 0$ be such that for a fixed $s_0 > 0$ the following inequalities hold:

$$\varepsilon < \lambda\bigl(1 - \tilde{f}(s_0)\bigr), \qquad R > \lambda\bigl(1 + \tilde{f}(s_0)\bigr). \tag{13}$$

Observe that these inequalities are valid for all $s > s_0$. Define a closed contour $\Gamma_\varepsilon(R) = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{p : \Re(p) = \varepsilon, |p| \le R\}$ is a vertical segment of the line $\Re(p) = \varepsilon$ and $\Gamma_2 = \{p : |\arg p| \le \arccos(\varepsilon/R), |p| = R\}$ is a segment of a circle with a center in the origin connecting the ends of the segment $\Gamma_1$. Denote by $G_\varepsilon(R)$ the area enclosed inside the contour $\Gamma_\varepsilon(R)$. It is obvious that $\lambda \in G_\varepsilon(R)$.

Let $p \in \mathbb{C}$ and introduce the complex-valued functions $f(p) = \lambda - p$, $g(p) = \lambda\tilde{f}(s - k(p)) + p - \lambda$. It is clear that $f(p) + g(p) = \lambda\tilde{f}(s - k(p))$. According to the first inequality of (13) the following chain of inequalities holds on the segment $\Gamma_1$ for $s > s_0$:

$$\bigl|f(p) + g(p)\bigr| = \lambda\bigl|\tilde{f}\bigl(s - k(p)\bigr)\bigr| \le \lambda\tilde{f}(s) < \lambda - \varepsilon \le \bigl|f(p)\bigr|, \quad p \in \Gamma_1.$$

The following relations are valid on the segment $\Gamma_2$ for $s > s_0$ (in view of the second inequality of (13)):

$$\bigl|f(p) + g(p)\bigr| \le \lambda\tilde{f}(s) < R - \lambda \le \bigl|f(p)\bigr|, \quad p \in \Gamma_2.$$

Therefore, on the contour $\Gamma_\varepsilon(R)$, for $s > s_0$, we have

$$\bigl|f(p) + g(p)\bigr| < \bigl|f(p)\bigr|, \quad p \in \Gamma_\varepsilon(R).$$

According to Rouché's theorem, the number of zeros of the function $g(p) = \lambda\tilde{f}(s - k(p)) + p - \lambda$ in the domain $G_\varepsilon(R)$ is equal to the number of zeros of the function $f(p) = \lambda - p$ in this domain. Consequently, equation (12) has a unique solution in the domain $G_\varepsilon(R)$. Letting $\varepsilon \to 0$, $R \to \infty$, the domain $G_\varepsilon(R)$ becomes $G = \{p : \Re(p) > 0\}$. Note that the number of zeros of the function $f(p)$ will not change as we pass $\varepsilon \to 0$, $R \to \infty$. Hence, the function $g(p)$ has a unique simple zero $p = c(s)$ in the semi-plane $\Re(p) > 0$ for $s > s_0$; moreover, $c(s) \in (0, \lambda)$. Letting $s_0 \to 0$, we get the statement of the first part of Lemma 1. ☐

In the next lemma we derive the Laplace transforms of the joint distribution of the lower one-boundary functionals of the process.

**Lemma 2** *Let* $\{D_x(t)\}_{t\ge0}$ *be the difference of the compound Poisson process and the renewal process* (2), $\delta \sim \exp(\lambda)$. *Then*

(i) *The Laplace transforms of the joint distribution of* $\{\tau_k(x), T_k(x)\}$, $x, k \ge 0$, *satisfy the following formulae for* $s > 0$:

$$\begin{aligned} f_k^x(du, s) &= \mathbf{E}\bigl[e^{-s\tau_k(x)}; T_k(x) \in du, \tau_k(x) < \infty\bigr] \\ &= \lambda\tilde{f}_x\bigl(s - k\bigl(c(s)\bigr)\bigr)e^{-kc(s) - u\lambda}\,du, \end{aligned} \tag{14}$$

*where* $c(s) \in (0, \lambda)$ *is the unique solution of equation* (12) *in the semi-plane* $\Re(p) > 0$, *and* $\tilde{f}_x(s) = \mathbf{E}[e^{-s\eta_x}]$.

(ii) *If* $\rho > 1$, *then* $\mathbf{P}[\tau_k(x) < \infty] = \tilde{f}_x(-k(c))e^{-kc} < 1$, *and* $\tau_k(x)$ *is a defective random variable for all* $x, k \geq 0$. *If* $\rho \leq 1$, *then* $\mathbf{P}[\tau_k(x) < \infty] = 1$, *and* $\tau_k(x)$ *is a proper variable for all* $x, k \geq 0$.

*Proof* Since the random variables $\delta$ and $\varkappa$ are independent, using the properties of the exponential distribution we find that

$$\begin{gathered} \mathbf{E}\big[e^{-s\eta_x - p(\pi_{\eta_x} - \delta)}\big] = \frac{\lambda}{\lambda - p}\tilde{f}_x\big(s - k(p)\big), \quad \Re(p) = 0,\ x \geq 0, \\ \mathbf{P}[\pi_t - \delta \in du] \overset{\text{def}}{=} g(t,u)du = \lambda\, \mathbf{E}\big[e^{-\lambda(\pi_t - u)}; \pi_t > u\big]du, \quad u \in \mathbb{R}. \end{gathered} \tag{15}$$

The mathematical expectations $f_k^x(du, s),\ x, k \geq 0$ obey the following equation:

$$\begin{aligned} f_k^x(du, s) = {} & \int_0^\infty e^{-st}\mathbf{P}[\eta_x \in dt]g\big(t, -(k+u)\big)\, du \\ & + \int_{-k}^\infty \int_0^\infty e^{-st}\mathbf{P}[\eta_x \in dt]g(t, v) f_{k+v}^0(du, s)\, dv. \end{aligned} \tag{16}$$

This equation is written by conditioning on the first renewal instant $\eta_1(x) = \eta_x$ of the renewal process $\{N_x(t)\}_{t \geq 0}$. We also have employed the total probability law, the fact that $\{X_t\}_{t \geq 0}$ is homogeneous with respect to the first component and the Markov property of the random variable $\eta_1(x)$. Setting $x = 0$ in this equation, we get

$$\begin{aligned} f_k^0(du, s) = {} & \lambda \tilde{f}\big(s - k(\lambda)\big)e^{-\lambda(k+u)}\, du \\ & + \int_0^\infty \int_0^\infty e^{-st}\mathbf{P}[\eta \in dt]g(t, v-k) f_v^0(du, s)\, dv, \end{aligned} \tag{17}$$

i.e. an equation for the mathematical expectations $f_k^0(du, s),\ k \geq 0$. This equation can be solved by factorizations methods. But we prefer a more simple reasoning to solve (17), which is as follows. Equation (17) is a linear integral equation with a kernel dependent on the difference of arguments. This kernel,

$$K_s(k, dv) = K_s(v - k)\, dv = \int_0^\infty e^{-st}\mathbf{P}[\eta \in dt]g(t, v-k)\, dv,$$

enjoys the following property for $s > 0$, all $u, v \in \mathbb{R} : K_s(u, dv) \leq \mathbf{E}[e^{-s\eta}] < 1$. Therefore, (17) has a unique solution which can be found by the method of successive iterations [25]. Substituting the function

$$f_k^0(du, s) = \lambda \tilde{f}\big(s - k\big(c(s)\big)\big)e^{-kc(s) - u\lambda}\, du$$

into (17) and performing some calculations (with use of (15)), we assure that this function satisfies (17). Therefore, we conclude that it is a unique solution of this equation. Substituting the expression for the function $f_k^0(du,s)$ into (16) and performing similar calculations, we derive for all $x,k\geq 0$:

$$f_k^x(du,s)=\lambda\tilde{f}_x\bigl(s-k\bigl(c(s)\bigr)\bigr)e^{-kc(s)-u\lambda}\,du.$$

This is the formula (14) of Lemma 2. The statements of the second part of the lemma follow straightforwardly from Lemma 1 and from (14). ☐

In order to proceed further, we introduce a function which will play a crucial role in the sequel. The idea of defining and employing such a function for semi-continuous random walks and spectrally one-sided Lévy processes was due to Takács [28]. It is clear that for all $s,x\geq 0$ the function

$$\mathbb{Q}_p^s(x)=\frac{\lambda\tilde{f}_x(s-k(p))}{\lambda\tilde{f}(s-k(p))+p-\lambda},\quad p\neq c(s),\ \Re(p)>0 \tag{18}$$

is analytical in semi-plane $\Re(p)>c(s)$, and $\lim_{p\to\infty}\mathbb{Q}_p^s(x)=0$. Therefore, it allows the representation in the form of an absolutely convergent Laplace integral [6]:

$$\mathbb{Q}_p^s(x)=\int_0^\infty e^{-pu}\,Q_u^s(x)\,du,\quad \Re(p)>c(s),\ s,x\geq 0. \tag{19}$$

We will call the function $\{Q_u^s(x),u\geq 0\}$, $s,x\geq 0$, defined by its Laplace transform $\mathbb{Q}_p^s(x)$, the resolvent of the process $\{D_x(t)\}_{t\geq 0}$ (2) in case when $\delta\sim\exp(\lambda)$. We also assume that for all $s,x\geq 0$, $Q_u^s(x)=0$, for $u<0$. Observe that $Q_0^s(x)=\lim_{p\to\infty}p\mathbb{Q}_p^s(x)=\lambda\tilde{f}_x(s+\mu)$.

Let us define the upper one-boundary functionals of the process $\{X_t\}_{t\geq 0}$. Let $k\in\mathbb{R}_+$ and $\tilde{\tau}^k=\inf\{t:\pi_t>k\}$, $\tilde{T}^k=\pi_{\tilde{\tau}^k}-k$, be the first crossing time of the upper boundary $k$ by the process $\{\pi_t\}_{t\geq 0}$ and the value of the overshoot at this instant. Denote

$$d\rho_t(u)=d_u\mathbf{P}[\pi_t\leq u],\qquad \int_0^\infty e^{-pu}\,d\rho_t(u)=\mathbf{E}\bigl[e^{-p\pi_t}\bigr]=e^{tk(p)},\quad \Re(p)\geq 0,$$

$$p_k(dt,du)=\mathbf{P}\bigl[\tilde{\tau}^k\in dt,\tilde{T}^k\in du\bigr]=dt\mu\int_{0-}^k d\rho_t(v)\mathbf{P}[\varkappa-k+v\in du].$$

We now derive the joint distributions of the one-boundary characteristics of the process in terms of the resolvent.

**Lemma 3** *Let* $\{D_x(t)\}_{t\geq 0}$ *be the difference of the compound Poisson process and the renewal process* (2), $\delta\sim\exp(\lambda)$, $\{Q_u^s(x)\}_{u\in\mathbb{R}_+}$, $s,x\geq 0$, *be the resolvent of the process* $\{D_x(t)\}_{t\geq 0}$, *given in* (19),

$$\tau^k(x)=\inf\bigl\{t:D_x(t)>k\bigr\},\qquad T^k(x)=D_x\bigl(\tau^k(x)\bigr)-k,\qquad \eta^k(x)=\eta_x^+\bigl(\tau^k(x)\bigr)$$

*be the first crossing time of the upper boundary* $k\in\mathbb{R}_+$ *by the process* $\{D_x(t)\}_{t\geq 0}$, *the value of the overshoot of the upper boundary and the value of the linear component* $\eta_x^+(\cdot)$ *at this instant. Then*

(i) *The Laplace transforms of the joint distribution of* $\{\tau^k(x), \eta^k(x), T^k(x)\}$,

$$f_x^k(dl, du, s) = \mathbf{E}\big[e^{-s\tau^k(x)}; \eta^k(x) \in dl, T^k(x) \in du, \tau^k(x) < \infty\big], \quad s \geq 0,$$

*are such that*

$$\begin{aligned} f_x^k(dl, du, s) = {} & e^{-s(l-x)} \frac{1-F(l)}{1-F(x)} \mathbf{I}\{l > x\} p_k\big(d(l-x), du\big) \\ & + \Phi_\lambda^s(dl, du) Q_k^s(x) - e^{-sl}\big[1 - F(l)\big] \\ & \times \int_0^k Q_v^s(x) p_{k-v}(dl, du)\, dv, \end{aligned} \tag{20}$$

*where* $\Phi_\lambda^s(dl, du) = e^{-sl}[1 - F(l)] \int_0^\infty e^{-kc(s)} p_k(dl, du)\, dk$.

(ii) *For the Laplace transforms* $f_x^k(s) = \mathbf{E}[e^{-s\tau^k(x)}; \tau^k(x) < \infty]$ *of the first crossing time of the upper boundary k by the process* $\{D_x(t)\}_{t\geq 0}$ *the following equalities hold for all* $s, x, k \geq 0$:

$$f_x^k(s) = 1 - \frac{s\lambda^{-1}}{s - k(c(s))} Q_k^s(x) + \int_{0-}^k d\tilde{\rho}_s(v)\big[\lambda^{-1} Q_{k-v}^s(x) - 1\big], \tag{21}$$

*where* $d\tilde{\rho}_s(v) = s \int_0^\infty e^{-st} d\rho_t(dv)\, dt$.

(iii) *If* $\rho < 1$, *then* $\tau^k(x)$ *is a defective random variable and*

$$\mathbf{P}\big[\tau^k(x) < \infty\big] = 1 - (1-\rho)\lambda^{-1} Q_k(x) < 1, \quad x, k \geq 0,$$

*where* $Q_k(x) \overset{\text{def}}{=} Q_k^0(x)$, $x, k \geq 0$; *if* $\rho \geq 1$, *then for* $x, k \geq 0$ $\tau^k(x)$ *is a proper variable.*

*Proof* The mathematical expectations $f_x^k(dl, du, s)$, $x, k \geq 0$, obey the following equation:

$$\begin{aligned} f_x^k(dl, du, s) = {} & e^{-s(l-x)} \frac{1-F(l)}{1-F(x)} \mathbf{I}\{l > x\} p_k\big(d(l-x), du\big) \\ & + \int_0^k \mathbf{E}\big[e^{-s\eta_x}; \pi_{\eta_x} \in dv\big] \int_0^\infty \lambda e^{-\lambda v_1} f_0^{k+v_1-v}(dl, du, s)\, dv_1. \end{aligned} \tag{22}$$

It holds due to the total probability law, the Markov property of the random variables $\{\eta_n(x), n \in \mathbb{N}\}$ and the homogeneity of the process $\{X_t\}_{t\geq 0}$ with respect to the first component. Introduce the integral transform for $\Re(p) > 0$:

$$\Phi_p^s(x, dl, du) = \int_0^\infty e^{-pk} f_x^k(dl, du, s)\, dk, \qquad \Phi_p^s(dl, du) \overset{\text{def}}{=} \Phi_p^s(0, dl, du).$$

Multiplying (22) by $e^{-pk}$ and integrating it with respect to $k \in \mathbb{R}_+$, we derive an equation for the generating function

$$\begin{aligned}\Phi_p^s(x, dl, du) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)}\, I\{l > x\} \Pi_p\big(d(l - x), du\big)\\ + \frac{\lambda}{\lambda - p} \tilde{f}_x\big(s - k(p)\big)\big[\Phi_p^s(dl, du) - \Phi_\lambda^s(dl, du)\big],\end{aligned} \qquad (23)$$

where $\Pi_p(dl, du) = \int_0^\infty e^{-pk} p_k(dl, du)\, dk$, $\Re(p) > 0$. Letting $x = 0$ in this equation, we obtain for $\Re(p) > 0$:

$$\begin{aligned}\Phi_p^s(dl, du) = \big\{\lambda \tilde{f}\big(s - k(p)\big) + p - \lambda\big\}^{-1}\\ \times \big\{\lambda \tilde{f}\big(s - k(p)\big)\Phi_\lambda^s(dl, du) - (\lambda - p)e^{-sl}\big[1 - F(l)\big]\Pi_p(dl, du)\big\}.\end{aligned} \qquad (24)$$

In order to determine the function $\Phi_\lambda^s(dl, du)$, we proceed as follows. According to Lemma 1, the first term on the right-hand side of (24) has a simple pole in $p = c(s) \in (0, \lambda)$. The function which enters the left-hand side of this equality is analytical for $\Re(p) > 0$. Hence, the second term on the right-hand side must have a simple zero in $p = c(s)$. This implies that $\Phi_\lambda^s(dl, du) = e^{-sl}[1 - F(l)]\Pi_{c(s)}(dl, du)$. Substituting the expression (24) for the Laplace transform $\Phi_p^s(dl, du)$ into (23), we find

$$\begin{aligned}\Phi_p^s(x, dl, du) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} I\{l > x\} \Pi_p\big(d(l - x), du\big)\\ + \Phi_\lambda^s(dl, du)\mathbb{Q}_p^s(x) - e^{-sl}\big[1 - F(l)\big]\\ \times \Pi_p(dl, du)\mathbb{Q}_p^s(x), \quad \Re(p) > 0.\end{aligned} \qquad (25)$$

Employing the definition of the resolvent $\{Q_u^s(x)\}_{u \in \mathbb{R}_+}$, $s, x \geq 0$ (19) and inverting the Laplace transforms in both sides of (25), we obtain the formulae (20) of Lemma 3. Integrating (20) with respect to $l, u \in \mathbb{R}_+$, we get (21). Statements of the third part of the lemma follow from Lemma 1 and (21). □

Thus, the Laplace transforms of the one-boundary functionals of the process $D_x(t)$ are determined by means of the formulae (14), (20). The following statement will be used in the sequel. Denote, for $x, k \geq 0$,

$$i^k(x) = \inf\big\{t > \tau^k(x) : D_x(t) < k - B\big\}, \qquad I^k(x) = k - B - D_x\big(i^k(x)\big) \in (0, \infty)$$

the first downward crossing time of the interval $[k - B, k]$ by the process $\{D_x(t)\}_{t \geq 0}$ and the value of the overshoot through the lower boundary $k - B$ at the instant of the first crossing. We will use the convention that $\inf \emptyset = \infty$, and $I^k(x) = \infty$ on the event $\{i^k(x) = \infty\}$. It is clear that $\eta_x^+(i^k(x)) = 0$.

**Lemma 4** *Let* $\{D_x(t)\}_{t\ge 0}$ *be the difference of the compound Poisson process and the renewal process* (2), $\delta \sim \exp(\lambda)$, $\{Q_u^s(x)\}_{u\in\mathbb{R}_+}$, $s, x \ge 0$, *be the resolvent of the process* $\{D_x(t)\}_{t\ge 0}$, *given by* (19), $Q_u(x) \overset{\text{def}}{=} Q_u^0(x)$. *Then*

(i) *The Laplace transforms of the joint distribution of* $\{i^k(x), I^k(x)\}$,

$$\varphi_x^k(du, s) = \mathbf{E}\big[\exp\big\{-si^k(x)\big\}; I^k(x) \in du, i^k(x) < \infty\big],$$

*satisfy the following equality for all* $x, k \ge 0$:

$$\varphi_x^k(du, s) = \big[c_x(s)e^{-c(s)(B-k)} - e^{-c(s)B}r(s)\lambda^{-1}Q_k^s(x)\big]\lambda e^{-\lambda u}\, du, \tag{26}$$

*where* $c_x(s) = \tilde{f}_x(s - k(c(s)))$, $r(s) = 1 - \lambda k'(c(s))\tilde{f}'(s - k(c(s)))$.

(ii) *If* $\rho < 1$, *then* $i^k(x)$ *is a defective random variable and*

$$\mathbf{P}\big[i^k(x) < \infty\big] = 1 - (1-\rho)\lambda^{-1}Q_k(x);$$

*if* $\rho > 1$, *then* $i^k(x)$ *is a defective random variable and*

$$\mathbf{P}\big[i^k(x) < \infty\big] = \tilde{f}_x\big(-k(c)\big)e^{-c(B-k)} - e^{-cB}\big(\lambda^{-1} - k'(c)\tilde{f}'\big(-k(c)\big)\big)Q_k(x);$$

*if* $\rho = 1$, *then* $i^k(x)$ *is a proper random variable.*

*Proof* It is not difficult to show that

$$\varphi_x^k(du, s) = \iint_{\mathbb{R}_+^2} f_x^k(dl, dv, s)\mathbf{E}\big[e^{-s\tau_{v+B}(l)}; T_{v+B}(l) \in du, \tau_{v+B}(l) < \infty\big].$$

In order to derive this equation, we again use the total probability law, space homogeneity of the process $\{D_x(t)\}_{t\ge 0}$, and the Markov property of the random variable $\tau^k(x)$. In view of (14), it follows from the latter equality that for $\Re(p) > 0$,

$$\tilde{\varphi}_x^p(du, s) = \int_0^\infty e^{-pk}\varphi_x^k(du, s)\, dk = \int_0^\infty \tilde{\Phi}_p^s\big(x, dl, c(s)\big)c_l(s)e^{-c(s)B}\lambda e^{-\lambda u}\, du, \tag{27}$$

where

$$\tilde{\Phi}_p^s(x, dl, z) = \int_0^\infty e^{-zu}\Phi_p^s(x, dl, du), \quad \Re(p) > 0,\ \Re(z) \ge 0$$

is the double integral transform $f_x^k(dl, du, s)$ of the joint distribution of $\{\tau^k(x), \eta^k(x), T^k(x)\}$. In view of (25) we derive the following expression for this function:

$$\begin{aligned}\tilde{\Phi}_p^s(x, dl, z) &= e^{-s(l-x)}\frac{1-F(l)}{1-F(x)}I\{l > x\}\tilde{\Pi}_p\big(d(l-x), z\big)\\ &\quad + e^{-sl}\big[1-F(l)\big]\mathbb{Q}_p^s(x)\big(\tilde{\Pi}_{c(s)}(dl, z) - \tilde{\Pi}_p(dl, z)\big), \quad \Re(p) > 0,\end{aligned} \tag{28}$$

where

$$\tilde{\Pi}_p(dl, z) = \int_0^\infty e^{-zu} \Pi_p(dl, du) = e^{lk(p)} \frac{k(z) - k(p)}{p - z} \, dl, \quad \Re(z) \geq 0.$$

We now substitute the expression (28) for the function $\tilde{\Phi}_p^s(x, dl, z)$ for $z = c(s)$ into (27). Integrating the equality obtained with respect to $l \in \mathbb{R}_+$, we find

$$\tilde{\varphi}_x^p(du, s) = e^{-c(s)B} \left[ \frac{c_x(s)}{p - c(s)} - r(s)\lambda^{-1} \mathbb{Q}_p^s(x) \right] \lambda e^{-\lambda u} \, du, \quad \Re(p) > 0,$$

where $r(s) = 1 - \lambda k'(c(s)) \tilde{f}'(s - k(c(s)))$. Utilizing the definition of the resolvent (19) and inverting the Laplace transforms on both sides of this equality, implies

$$\varphi_x^k(du, s) = \left[ c_x(s) e^{-c(s)(B-k)} - e^{-c(s)B} r(s) \lambda^{-1} Q_k^s(x) \right] \lambda e^{-\lambda u} \, du, \quad s, x \geq 0,$$

which is the formula (26). Letting $s \to 0$ in (26) and taking into account statements of Lemma 1, we derive the second part of the lemma. □

We now consider the two-sided exit problem for the process studied. The following result is true.

**Theorem 2** *Let $\{D_x(t)\}_{t \geq 0}$ be the difference of the compound Poisson process and the renewal process* (2), $\delta \sim \exp(\lambda)$ *and* $\{Q_u^s(x)\}_{u \in \mathbb{R}_+}$, $s, x \geq 0$, *be the resolvent of the process* $\{D_x(t)\}_{t \geq 0}$, *given by* (19), $Q_u^s \overset{\text{def}}{=} Q_u^s(0)$. *Then*

(i) *The Laplace transforms of the joint distribution of* $\{\chi, L, T\}$ *satisfy the following equalities for* $x, s \geq 0$:

$$\begin{aligned} V_r(x, du, s) &= \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} \lambda e^{-\lambda u} \, du, \\ V^k(x, dl, du, s) &= f_x^k(dl, du, s) - \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} \mathbf{E} f_0^{\delta+B}(dl, du, s), \end{aligned} \tag{29}$$

*where the function* $f_x^k(dl, du, s)$ *is given by* (20),

$$\begin{aligned} \mathbf{E} Q_{\delta+B}^s = \int_0^\infty \lambda e^{-\lambda v} Q_{v+B}^s \, dv, \mathbf{E} f_0^{\delta+B}(dl, du, s) \\ = \int_0^\infty \lambda e^{-\lambda v} f_0^{v+B}(dl, du, s) \, dv. \end{aligned}$$

(ii) *The Laplace transforms of the first exit time* $\chi$ *by the process* $\{D_x(t)\}_{t \geq 0}$ *are such that*

$$\begin{aligned} \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_r\right] &= \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s}, \\ \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^k\right] &= 1 - A_x^k(s) - \frac{Q_k^s(x)}{\mathbf{E} Q_{\delta+B}^s} \left(1 - \mathbf{E} A_0^{\delta+B}(s)\right), \end{aligned} \tag{30}$$

*where* $\mathbf{E}A_0^{\delta+B}(s) = \int_0^\infty \lambda e^{-\lambda v} A_0^{v+B}(s)\,dv$,

$$A_x^k(s) = \int_{0-}^k d\tilde{\rho}_s(v)\big[1 - \lambda^{-1} Q_{k-v}^s(x)\big], \quad d\tilde{\rho}_s(v) = s\int_0^\infty e^{-st} d\rho_t(v)\,dt.$$

(iii) *The probabilities of the first exit by the process* $\{D_x(t)\}_{t\ge 0}$ *are given by*

$$\mathbf{P}[\mathfrak{A}_r] = \frac{Q_k(x)}{\mathbf{E}Q_{\delta+B}}, \qquad P[\mathfrak{A}^k] = 1 - \frac{\mathbf{Q}_k(x)}{\mathbf{E}Q_{\delta+B}},$$

*where* $Q_k(x) \overset{\text{def}}{=} Q_k^0(x)$, $x, k \ge 0$; $Q_k \overset{\text{def}}{=} Q_k(0)$.

*Proof* Theorem 2 follows straightforwardly from Theorem 1. We employ formulae (5) of Theorem 1, which take a simplified form for the case when $\delta \sim \exp(\lambda)$. First of all, we must calculate the kernels (8), the iterations of the kernels (7), and the series (6) for the process $\{D_x(t)\}_{t\ge 0}$ (2) in case when $\delta \sim \exp(\lambda)$.

Let us verify (29). Employing the statements of the lemmas and the defining formulae (6)–(8), we find for $v, u > 0$ $n \in \mathbb{N}$, that

$$\begin{aligned} K_v^-(du, s) &= \varphi_0^{v+B}(du, s), \\ K_v^-(du, s)^{*(n)} &= \varphi_0^{v+B}(du, s)\big(\mathbf{E}\varphi_0^{\delta+B}(s)\big)^{n-1}, \\ \mathfrak{K}_v^-(du, s) &= \varphi_0^{v+B}(du, s)\big(1 - \mathbf{E}\varphi_0^{\delta+B}(s)\big)^{-1}, \\ F_r^x(du, s) &= c_x(s) e^{-c(s)r} \lambda e^{-\lambda u}\,du - \varphi_x^k(du, s), \end{aligned} \tag{31}$$

where $c_x(s) = \tilde{f}_x(s - k(c(s)))$, and the mathematical expectations $\varphi_x^k(du, s)$, $x, k \ge 0$, are given by (26), $\varphi_x^k(s) = \int_0^\infty \varphi_x^k(du, s)$. Substituting the expressions from (31) into the second formula of (5) we obtain the first equality of (29) of Theorem 2.

Let us clarify the second formula of (29). Employing the statements of the lemmas and the defining formulae (6)–(8), we find for $s, x, \nu, v \in \mathbb{R}_+$, $n \in \mathbb{N}$, that

$$\begin{aligned} K_{\nu v}^+(dl, du, s) &= c_\nu(s) e^{-c(s)(v+B)} \mathbf{E} f_0^{\delta+B}(dl, du, s), \\ K_{\nu v}^+(dl, du, s)^{*(n)} &= c_\nu(s) e^{-c(s)(v+B)} \big(\mathbf{E}\varphi_0^{\delta+B}(s)\big)^{n-1} \mathbf{E} f_0^{\delta+B}(dl, du, s), \\ \mathfrak{K}_{\nu v}^+(dl, du, s) &= c_\nu(s) e^{-c(s)(v+B)} \big(1 - \mathbf{E}\varphi_0^{\delta+B}(s)\big)^{-1} \mathbf{E} f_0^{\delta+B}(dl, du, s), \\ F_x^k(dl, du, s) &= f_x^k(dl, du, s) - c_x(s) e^{-c(s)r} \mathbf{E} f_0^{\delta+B}(dl, du, s), \end{aligned} \tag{32}$$

where the mathematical expectations $f_x^k(dl, du, s)$, $x, k \ge 0$, are given by (20) of Lemma 3. Substituting (32) into the first equality of (5) of Theorem 1, we obtain the second formula of (29) of Theorem 2.

Integrating the first equality of (29) with respect to $u \in \mathbb{R}_+$ and integrating the second equality of (29) with respect to $l, u \in \mathbb{R}_+$ we derive formulae (30). Letting $s \to 0$ in these formulae, we find the probabilities of the first exit through the boundaries by the process $\{D_x(t)\}_{t\ge 0}$.

We now state another proof of Theorem 2. The system of equations (9) will play a key role in this proof. Let us rewrite this system for the difference of the compound Poisson process and the renewal process for the case when $\delta \sim \exp(\lambda)$. In view of (14), we have

$$\begin{aligned} f_x^k(dl, du, s) &= V^k(x, dl, du, s) + \int_0^\infty V_r(x, dv, s) f_0^{v+B}(dl, du, s), \\ c_x(s)e^{-c(s)r}\lambda e^{-\lambda u} du &= V_r(x, du, s) \\ &\quad + \iint_{\mathbb{R}_+^2} V^k(x, dl, dv, s) c_l(s) e^{-c(s)(v+B)} \lambda e^{-\lambda u}\, du. \end{aligned} \tag{33}$$

Note that in order to derive the functions $V_r(x, du, s)$, $V^k(x, dl, du, s)$, it is sufficient to determine the function

$$\tilde{V}_x^k(s) = \iint_{\mathbb{R}_+^2} V^k(x, dl, dv, s) c_l(s) e^{-c(s)(v+B)} = \mathbf{E}\big[e^{-si^k(x)}; i^k(x) < \infty, \mathfrak{A}^k\big].$$

Substituting the expression for the function $V_r(x, du, s)$ from the second equation (33) into the first one, we find

$$\begin{aligned} f_x^k(dl, du, s) = V^k(x, dl, du, s) &+ c_x(s)e^{-c(s)r}\mathbf{E} f_0^{\delta+B}(dl, du, s) \\ &- \tilde{V}_x^k(s)\mathbf{E} f_0^{\delta+B}(dl, du, s). \end{aligned}$$

Multiplying the latter equality by $c_l(s)e^{-c(s)(u+B)}$ and integrating it over $l, u \in \mathbb{R}_+$ yields

$$\varphi_x^k(s) = \tilde{V}_x^k(s) + c_x(s)e^{-c(s)r}\mathbf{E}\varphi_0^{\delta+B}(s) - \tilde{V}_x^k(s)\mathbf{E}\varphi_0^{\delta+B}(s),$$

where $\varphi_x^k(s) = \int_0^\infty \varphi_x^k(du, s)$. This is a linear equation with respect to the function $\tilde{V}_x^k(s)$. Taking into account (26), we find that

$$\tilde{V}_x^k(s) = c_x(s)e^{-c(s)r} - \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s}.$$

Substituting the right-hand side of this equality into the second equation of (33) we get

$$V_r(x, du, s) = \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s}\lambda e^{-\lambda u}\, du,$$

i.e. the first formula of (29). Substituting the right-hand side of this equality into the first equation of (33), we obtain

$$V^k(x, dl, du, s) = f_x^k(dl, du, s) - \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s}\mathbf{E} f_0^{\delta+B}(dl, du, s),$$

the second formula of (29). □

## 5 Reflected process

Denote by $D_x^r(t) = r + D_x(t)$, $t \geq 0$, the process starting from $r \in \mathbb{R}$ when $\eta_x^+(0) = x \geq 0$. Let $B \in \mathbb{R}_+$ and for all $t \geq 0$ let define right-continuous processes reflected at the boundaries as follows:

$$\begin{aligned} &\underline{D}_x^r(t) = D_x^r(t) - \min\Big\{0, \inf_{[0,t]} D_x^r(\cdot)\Big\} \in \mathbb{R}_+, \quad r \in \mathbb{R}_+, \\ &\overline{D}_x^r(t) = D_x^r(t) - \max\Big\{0, \sup_{[0,t]} D_x^r(\cdot) - B\Big\} \in (-\infty, B], \quad r \in (-\infty, B]. \end{aligned} \tag{34}$$

The first reflection from the lower boundary 0 of the process $\underline{D}_x^r(t)$ occurs at time $\tau_r(x)$. Subsequent reflections occur at times which constitute a renewal process. The inter-arrival times of the switches are identically distributed as $\tau_0(0)$. Sample paths of the process $\underline{D}_x^r(t)$ on the time interval $[0, \tau_r(x))$ coincide with those of the process $D_x^r(t)$, and on the rest of the intervals between the renewal points the paths coincide with those of the process $D_0^0(t)$.

The first reflection from the upper boundary $B$ of the process $\overline{D}_x^r(t)$ takes place at $\tau^{B-r}(x)$. Then the process stays at the boundary for some random time $\eta_l$, where $l = \eta_x^+(\tau^{B-r}(x))$. At the instant $t = \tau^{B-r}(x) + \eta_l$ the process is reflected to a random state $B - \delta$. In the sequel the evolution of the process $\overline{D}_x^r(t)$ is a probabilistic copy of its evolution on $[0, \tau^{B-r}(x) + \eta_l)$.

Reflections from the boundaries were introduced by Lévy for a standard Wiener process. Applying the symmetry principle and mirror reflection principle, Lévy determined the distributions of the boundary functionals of the reflected standard Wiener process. It appears that these distributions are the limit distributions for our reflected process after an appropriate scaling of time and space.

We now define the boundary functionals for the reflected difference of the compound Poisson process and the renewal process (34). Denote by

$$\begin{aligned} &\overline{\tau}_x^B(r) = \inf\big\{t : \underline{D}_x^r(t) > B\big\} \stackrel{\text{def}}{=} \overline{\tau}, \qquad \overline{L}_x^B(r) = \eta_x^+(\overline{\tau}) \stackrel{\text{def}}{=} \overline{L}, \\ &\overline{T}_x^B(r) = \underline{D}_x^r(t) - B \end{aligned}$$

the first crossing time of the upper level $B$ by the process $\underline{D}_x^r(t)$, the value of the linear component, and the value of the overshoot at this instant. For the process $\overline{D}_x^r(t)$, denote

$$\underline{\tau}_x^B(r) = \inf\big\{t : \overline{D}_x^r(t) < 0\big\} \stackrel{\text{def}}{=} \underline{\tau}, \qquad \underline{T}_x^B(r) = -\overline{D}_x^r(\underline{\tau}) \stackrel{\text{def}}{=} \underline{T}$$

the first crossing time of the lower level 0 by the process $\underline{D}_x^r(t)$ and the value of the overshoot at this instant.

**Theorem 3** *Let* $\underline{D}_x^r(t)$, $\overline{D}_x^r(t)$, $t \geq 0$, *be reflected processes defined by* (34), $x$, $B \in \mathbb{R}_+$, $r \in [0, B]$, $k = B - r$, *and denote by*

$$\begin{aligned} V^k(x, dl, du, s) &= \mathbf{E}\big[e^{-s\chi}; L \in dl, T \in du, \mathfrak{A}^k\big], \\ V_r(x, du, s) &= \mathbf{E}\big[e^{-s\chi}; T \in du, \mathfrak{A}_r\big] \end{aligned}$$

*the Laplace transforms of the joint distribution of* $\{\chi_x^B(r), L, T\}$ *of the process* $\{D_x^r(t)\}_{t\geq 0}$ (5), *and by*

$$\overline{v}_x^r(dl, du, s) = \mathbf{E}\bigl[e^{-s\overline{\tau}_x^B(r)}; \overline{L}\in dl, \overline{T}\in du\bigr], \qquad \underline{v}_x^r(du, s) = \mathbf{E}\bigl[e^{-s\underline{\tau}_x^B(r)}; \underline{T}\in du\bigr]$$

*the Laplace transforms of the joint distributions of* $\{\overline{\tau}_x^B(r), \overline{L}, \overline{T}\}$, $\{\underline{\tau}_x^B(r), \underline{T}\}$. *Then for* $s>0$ *the following equalities hold*:

$$\overline{v}_x^r(dl, du, s) = V^k(x, dl, du, s) + \frac{V_r(x,s)}{1-V_0(0,s)} V^B(0, dl, du, s), \tag{35}$$

*where* $V_r(x,s) = \int_0^\infty V_r(x, du, s)$;

$$\begin{aligned}\underline{v}_x^r(du, s) &= V_r(x, du, s)\\ &\quad + \frac{a^k(x)}{1-A(0)}\biggl[\mathbf{P}[\delta - B\in du] + \int_0^B \mathbf{P}[\delta\in dv] V_{B-v}(0, du, s)\biggr],\end{aligned} \tag{36}$$

*where* $V^k(x, dl, s) = \int_0^\infty V^k(x, dl, du, s)$,

$$a^k(x) = \int_0^\infty V^k(x, dl, s, B)\tilde{f}_l(s), \qquad A(x) = \int_0^B \mathbf{P}[\delta\in dk] a^k(x).$$

*Proof* Let us verify the formula (35). It follows from the definition of the process $\underline{D}_x^r(t)$ (34), the total probability law, and the Markov property of $\chi$ that the following equation is valid:

$$\overline{v}_x^r(dl, du, s) = V^k(x, dl, du, s) + V_r(x,s)\overline{v}_0^0(dl, du, s).$$

This equation is well known for the reflected Lévy processes, generated by the infimum (supremum), see [21]. It is worth noticing that in this article the asymptotic expansions for the distributions of the characteristics of the process are determined for the reflected Lévy processes obeying the two-boundary Cramer's conditions. The reflected spectrally one-sided Lévy processes generated by the infimum (supremum) of the process were considered in [1, 23]. An interesting application in queueing theory for the spectrally one-sided Lévy process reflected by its infimum was given in [3].

Letting $x=r=0$ in this equation, we find that

$$\overline{v}_0^0(dl, du, s) = V^B(0, dl, du, s)\bigl(1-V_0(0,s)\bigr)^{-1}.$$

Substituting the expression for the function $\overline{v}_0^0(dl, du, s)$ into the previous equation, we get formula (35). Let us prove (36). It follows from the definition of the process $\overline{D}_x^r(t)$ (34), the total probability law and the Markov property of $\chi$, $\eta_n(x)$, that the following system of linear integral equations holds:

$$\underline{v}_x^r(du,s) = V_r(x,du,s) + \int_0^\infty V^k(x,dl,s)\underline{v}_l^B(du,s),$$

$$\underline{v}_x^B(du,s) = \tilde{f}_x(s)\mathbf{P}[\delta - B \in du] + \tilde{f}_x(s)\int_0^B \mathbf{P}[\delta \in dv]v_0^{B-v}(du,s).$$

This system is similar to a system of linear equations with two unknowns and can be solved analogously. Substituting the expression for $\underline{v}_x^B(du,s)$ from the second equation into the first one, we find that

$$\underline{v}_x^r(du,s) = V_r(x,du,s) + a^k(x)\mathbf{P}[\delta - B \in du] + a^k(x)\int_0^B \mathbf{P}[\delta \in dv]\underline{v}_0^{B-v}(du,s).$$

Letting $x = 0$ in the latter equation yields

$$\begin{aligned}&\int_0^B \mathbf{P}[\delta \in dv]\underline{v}_0^{B-v}(du,s)\\ &\quad = -\mathbf{P}[\delta - B \in du]\\ &\qquad + \left[\mathbf{P}[\delta - B \in du] + \int_0^B \mathbf{P}[\delta \in dv]V_{B-v}(0,du,s)\right]\big(1 - A(0)\big)^{-1}.\end{aligned}$$

Inserting the right-hand side of this equality in the previous one, we get (36). $\square$

**Theorem 4** *Let* $\underline{D}_x^r(t)$, $\overline{D}_x^r(t)$, $t \geq 0$, *be the reflected processes* (34), $\delta \sim \exp(\lambda)$, $x, B \in \mathbb{R}_+$, $r \in [0,B]$, $k = B - r$. *Then*

(i) *The Laplace transform* $\overline{v}_x^r(dl,du,s)$ *of the joint distribution of* $\{\overline{\tau}_x^B(r), \overline{L}, \overline{T}\}$ *satisfies the following equality for* $s > 0$:

$$\overline{v}_x^r(dl,du,s) = V^k(x,dl,du,s) + \frac{V_r(x,s)}{1 - V_0(0,s)}V^B(0,dl,du,s),$$

*where* $V_r(x,s) = \int_0^\infty V_r(x,du,s)$, *and the functions* $V^k(x,dl,du,s)$, $V_r(x,du,s)$ *are determined by* (29). *In particular,*

$$\overline{v}_x^r(s) = \mathbf{E}e^{-s\overline{\tau}_x^B(r)} = 1 - A_x^k(s) + Q_k^s(x)\frac{\mathbf{E}A_0^{\delta+B}(s) - A_0^B(s)}{\mathbf{E}Q_{\delta+B}^s - Q_B^s(0)}, \tag{37}$$

*where* $d\tilde{\rho}_s(dv) = s\int_0^\infty e^{-st}\,d\rho_t(v)\,dt$,

$$A_x^k(s) = \int_{0-}^k d\tilde{\rho}_s(v)\big[1 - \lambda^{-1}Q_{k-v}^s(x)\big],$$

$$\mathbf{E}A_0^{\delta+B}(s) = \int_0^\infty \lambda e^{-\lambda v}A_0^{v+B}(s)\,dv.$$

(ii) *The Laplace transforms* $\underline{v}_x^r(du, s)$ *of the joint distribution of* $\{\underline{\tau}_x^B(r), \underline{T}\}$ *are such that for* $s > 0$

$$\underline{v}_x^r(du, s) = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_k^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E}S_{\delta+B}^s} \lambda e^{-\lambda u}\, du, \tag{38}$$

*where* $S_k^s(x) = \int_0^k Q_u^s(x)\, du$, $\mathbf{E}S_{\delta+B}^s = \lambda \int_0^\infty e^{-\lambda v} S_{v+B}^s(0)\, dv$.

*Proof* The formula (37) follows from (35) of Theorem 3 after substituting the expressions for the functions $V_r(x, s)$, $V^k(x, dl, du, s)$ (29) into this equality. Let us verify (39). To do this, we have to calculate $a^k(x)$, $A(0)$ in case when $\delta \sim \exp(\lambda)$. Employing the formulae (20), (25), (29) and performing the necessary calculations, we find that

$$a^k(x) = \tilde{f}_x(s) + \bigl(1 - \tilde{f}(s)\bigr)S_k^s(x) - \frac{Q_k^s(x)}{\mathbf{E}Q_{\delta+B}^s}\bigl[\tilde{f}(s) + \bigl(1 - \tilde{f}(s)\bigr)\mathbf{E}S_{\delta+B}^s\bigr],$$

$$1 - A(0) = \bigl[\tilde{f}(s) + \bigl(1 - \tilde{f}(s)\bigr)\mathbf{E}S_{\delta+B}^s\bigr]\lambda\bigl(\mathbf{E}Q_{\delta+B}^s\bigr)^{-1},$$

$$\mathbf{P}[\delta - B \in du] + \int_0^B \mathbf{P}[\delta \in dv] V_{B-v}(0, du, s) = \lambda\bigl(\mathbf{E}Q_{\delta+B}^s\bigr)^{-1} \lambda e^{-\lambda u}\, du.$$

Substituting the right-hand sides of these equalities into (36) yields (38). □

The formulae for the Laplace transforms of the two-boundary functionals of the process given in Theorems 2, 4 have a simple form. This fact allows us to determine the limiting behavior of these characteristics of the process, which is the subject of the next section.

## 6 Asymptotic results

Assume that the following condition holds:

$$\text{(A)} \quad \rho = \lambda\mu\mathbf{E}\eta\mathbf{E}\varkappa = 1, \qquad \sigma^2 = \mu\bigl[\mathbf{E}\varkappa^2 + \mathbf{E}\varkappa\mathbf{E}\eta^2/\lambda(\mathbf{E}\eta)^2\bigr] < \infty.$$

In this case, the time and space scaling are well known [2]. We first establish the asymptotic properties of the functions that enter the formulae of Theorems 2, 4. The following statement is true.

**Lemma 5** *Assume that the conditions* $(A)$ *are satisfied. Then the following equalities hold for all* $k > 0$:

$$
\begin{aligned}
\lim_{B\to\infty} B^{-1} Q_{kB}^{s/B^2}(x) &= \frac{2sh(k\sqrt{2s}/\sigma)}{\sigma\sqrt{2s}\mathbf{E}\eta} \\
&= \lim_{B\to\infty} B^{-1}\mathbf{E}Q_{\delta+kB}^{s/B^2}, \\
\lim_{B\to\infty}\bigl[\mathbf{E}Q_{\delta+kB}^{s/B^2} - Q_{kB}^{s/B^2}(0)\bigr] &= \frac{2\mu\mathbf{E}\varkappa}{\sigma^2}\operatorname{ch}(k\sqrt{2s}/\sigma), \\
\lim_{B\to\infty} A_x^{kB}\bigl(s/B^2\bigr) &= 1-\operatorname{ch}(k\sqrt{2s}/\sigma) \\
&= \lim_{B\to\infty} A_0^{\delta+kB}\bigl(s/B^2\bigr), \\
\lim_{B\to\infty} B\bigl[A_0^{kB}\bigl(s/B^2\bigr) - \mathbf{E}A_0^{\delta+kB}\bigl(s/B^2\bigr)\bigr] &= \frac{2s\operatorname{sh}(k\sqrt{2s}/\sigma)}{\lambda\sigma\sqrt{2s}}, \\
\lim_{B\to\infty} B^{-2} S_{kB}^{s/B^2}(x) &= \frac{1}{s\mathbf{E}\eta}\bigl(\operatorname{ch}(k\sqrt{2s}/\sigma)-1\bigr) \\
&= \lim_{B\to\infty} B^{-2}\mathbf{E}S_{kB-\delta}^{s/B^2}.
\end{aligned}
\tag{39}
$$

*Proof* Let $\zeta$ be a positive random variable with a finite second moment. Then its Laplace transform obeys the following expansion for small enough values of $s>0$:

$$
\mathbf{E}e^{-s\zeta} = 1 - s\mathbf{E}\zeta + \frac{1}{2}s^2\mathbf{E}\zeta^2 + o\bigl(s^2\bigr). \tag{40}
$$

In view of the latter equality, we derive the following relations for large enough values of $B$:

$$
\begin{aligned}
s/B^2 - k(p/B) &= \mu\mathbf{E}\varkappa p B^{-1} + B^{-2}\bigl(s - p^2\mu\mathbf{E}\varkappa^2/2\bigr) + o\bigl(B^{-2}\bigr), \\
\lambda\tilde{f}\bigl(s/B^2 - k(p/B)\bigr) + p/B - \lambda &= \lambda\mathbf{E}\eta\bigl(\sigma^2 p^2/2 - s\bigr)B^{-2} + o\bigl(B^{-2}\bigr).
\end{aligned}
$$

We now verify the first formula of (39). It follows from the definition of resolvent function (18) and from the second equality that

$$
\lim_{B\to\infty} B^{-2}\mathbb{Q}_{p/B}^{s/B^2}(x) = \frac{1}{\mathbf{E}\eta}\bigl(\sigma^2p^2/2 - s\bigr)^{-1}. \tag{41}
$$

The right-hand side of the latter equality is a Laplace transform for $p>\sqrt{2s}/\sigma$. Thus, the inversion formula is valid:

$$
\int_{\alpha-i\infty}^{\alpha+i\infty} e^{kp}\frac{dp}{\frac{1}{2}\sigma^2p^2 - s} = \frac{2}{\sigma\sqrt{2s}}\operatorname{sh}(k\sqrt{2s}/\sigma), \quad \alpha > \sqrt{2s}/\sigma.
$$

The linearity property of the Laplace transform and (41) imply that

$$
\lim_{B\to\infty} B^{-1}Q_{kB}^{s/B^2}(x) = \frac{1}{\mathbf{E}\eta}\frac{2}{\sigma\sqrt{2s}}\operatorname{sh}(k\sqrt{2s}/\sigma).
$$

Other equalities of (39) can be verified analogously. □

**Corollary 1** *Let* $\delta \sim \exp(\lambda)$, $r \in (0,1)$, $k = 1 - r$ *and assume that* $(A)$ *is satisfied. Then*

(i) *The Laplace transforms of* $\chi_x^B(rB)$, $\overline{\tau}_x{}^B(rB)$, $\underline{\tau}_x^B(rB)$ *are such that for* $B \to \infty$

$$\begin{aligned}
\mathbf{E}\big[e^{-\frac{s}{B^2}\chi_x^B(rB)}; \mathfrak{A}^{kB}\big] &\to \frac{\operatorname{sh}(r\sqrt{2s}/\sigma)}{\operatorname{sh}(\sqrt{2s}/\sigma}, \\
\mathbf{E}\big[e^{-\frac{s}{B^2}\chi_x^B(rB)}; \mathfrak{A}_{rB}\big] &\to \frac{\operatorname{sh}(k\sqrt{2s}/\sigma)}{\operatorname{sh}(\sqrt{2s}/\sigma)}, \\
\mathbf{E}e^{-\frac{s}{B^2}\overline{\tau}_x^B(rB)} &\to \frac{\operatorname{ch}(r\sqrt{2s}/\sigma)}{\operatorname{ch}(\sqrt{2s}/\sigma)}, \\
\mathbf{E}e^{-\frac{s}{B^2}\underline{\tau}_x^B(rB)} &\to \frac{\operatorname{ch}(k\sqrt{2s}/\sigma)}{\operatorname{ch}(\sqrt{2s}/\sigma)}.
\end{aligned} \tag{42}$$

(ii) *The distributions of* $\chi_x^B(rB)$, $\overline{\tau}_x{}^B(rB)$, $\underline{\tau}_x{}^B(rB)$ *obey the following expansions for* $B \to \infty$:

$$\begin{aligned}
\mathbf{P}\left[\frac{\chi_x^B(rB)}{B^2} \in dt; \mathfrak{A}^{kB}\right] &\to \pi\sigma^2 \sum_{n\in\mathbb{N}} n e^{-\frac{t}{2}(\sigma\pi n)^2} \sin(k\pi n)\, dt, \\
\mathbf{P}\left[\frac{\overline{\tau}_x^B(rB)}{B^2} > t\right] &\to \frac{4}{\pi} \sum_{n\in\mathbb{Z}^+} \frac{e^{-\frac{t}{2}(\pi(n+\frac{1}{2})\sigma)^2}}{2n+1} \sin\big(k(2n+1)\pi/2\big), \\
\mathbf{P}\left[\frac{\underline{\tau}_x^B(rB)}{B^2} > t\right] &\to \frac{4}{\pi} \sum_{n\in\mathbb{Z}^+} \frac{e^{-\frac{t}{2}(\pi(n+\frac{1}{2})\sigma)^2}}{2n+1} \sin\big(r(2n+1)\pi/2\big).
\end{aligned} \tag{43}$$

*Proof* The second formula of (30) and the equality (39) imply the following chain of the equalities for $B \to \infty$:

$$\begin{aligned}
\mathbf{E}\big[e^{-\frac{s}{B^2}\chi_x^B(rB)}; \mathfrak{A}^{kB}\big] &= 1 - A_x^{kB}\big(s/B^2\big) - \frac{Q_{kB}^{s/B^2}(x)}{\mathbf{Q}_{\delta+B}^{s/B^2}}\big(1 - \mathbf{E}A_0^{\delta+B}\big(s/B^B\big)\big) \\
&= \operatorname{ch}(k\sqrt{2s}/\sigma) - \frac{\operatorname{sh}(k\sqrt{2s}/\sigma)}{\operatorname{sh}(\sqrt{2s}/\sigma)}\operatorname{ch}(\sqrt{2s}/\sigma) = \frac{\operatorname{sh}(r\sqrt{2s}/\sigma)}{\operatorname{sh}(\sqrt{2s}/\sigma)}.
\end{aligned}$$

It follows from (37) and (39) that for $B \to \infty$:

$$\begin{aligned}
\mathbf{E}e^{-\frac{s}{B^2}\overline{\tau}_x^B(rB)} &\to \operatorname{ch}(k\sqrt{2s}/\sigma) \\
&\quad - \frac{2\operatorname{sh}(k\sqrt{2s}/\sigma)}{\sigma\sqrt{2s}\mathbf{E}\eta}\frac{2s\operatorname{sh}(\sqrt{2s}/\sigma)}{\lambda\sigma\sqrt{2s}}\left(\frac{2\mu\mathbf{E}\varkappa}{\sigma^2}\operatorname{ch}(\sqrt{2s}/\sigma)\right)^{-1} \\
&= \frac{\operatorname{ch}(r\sqrt{2s}/\sigma)}{\operatorname{ch}(\sqrt{2s}/\sigma)}.
\end{aligned}$$

The rest of the equalities (42) can be verified analogously. Inverting the Laplace transforms in (42) [6] we obtain (43). □

Consider the following one-server queueing system $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$.

(i) Customers of size $\varkappa$ arrive to the system according to a Poisson process with intensity $\mu > 0$.

(ii) The system has a finite waiting room (buffer) whose size equals $B < \infty$. Suppose that upon the arrival of a new customer of size $\varkappa$ it finds $r \in [0, B]$ occupied space in the waiting room. Then $\min\{k, \varkappa\}$ joins the queue, and loss of size $\max\{0, \varkappa - k\}$ occurs, where $k = B - r$ is the size of empty space in the waiting room.

(iii) The duration of service completion is arbitrarily distributed as $\eta > 0$. Suppose that at a time $t$ the service cycle is accomplished. Then the occupied space in the buffer is reduced by $\min\{r, \delta\}$, where $r \in (0, B]$ is the value of occupied space in the waiting room at a time $t - 0$. If at the instant of the service completion $r - \min\{r, \delta\} > 0$, then a new service cycle starts. If at the instant of the service completion $r - \min\{r, \delta\} = 0$, then the new service cycle starts upon arrival of a new customer (after exponential time with parameter $\mu > 0$).

Suppose that the queueing system $\mathrm{M}^{\varkappa}|\mathrm{G}^{\delta}|1|\mathrm{B}$ with buffer of size $B$ starts performing from the state $(r, x)$, where $r \in [0, B]$ is the initial volume of the system, and $x \ge 0$ is time elapsed since the last service completion. Denote by $b_r(x)$ the first time at which the waiting room becomes empty (the interval $[0, b_r(x)]$ is the busy period of type $(r, x)$). It follows from the definition of the process $\overline{D}_x^{\,r}(t)$ (34) and the queueing system, that the process $\overline{D}_x^{\,r}(t)$, $t \in [0, \underline{\tau}_x^{\,B}(r)]$, describes the functioning of the system on the busy period $[0, b_r(x)]$, and the random variable $b_r(x)$ is identically distributed as $\underline{\tau}_x^{\,B}(r)$. Hence

$$\mathbf{E}\big[e^{-sb_r(x)}; b_r(x) < \infty\big] = \frac{\tilde{f}_x(s) + (1 - \tilde{f}(s))S_k^s(x)}{\tilde{f}(s) + (1 - \tilde{f}(s))\mathbf{E}S_{\delta+B}^s}, \quad k = B - r.$$

In case when the state space is discrete (i.e. $B, \varkappa \in \mathbb{N}$, $\delta = 1$), this formula was derived in [14] for the system $\mathrm{M}^{\varkappa}|\mathrm{G}|\mathrm{B}|1$. The random variable $b_r(x)$ is proper, $\mathbf{P}[b_r(x) < \infty] = 1$, and

$$\mathbf{E}b_r(x) = \mathbf{E}\eta_x - \mathbf{E}\eta + \mathbf{E}\eta\big[\mathbf{E}S_{\delta+B} - S_k(x)\big] < \infty,$$

where $S_k(x) = S_k^0(x)$, $\mathbf{E}S_{\delta+B} = \mathbf{E}S_{\delta+B}^0$.

Let $r \in [0, B]$, $k = B - r$ and for all $t \ge 0$ introduce the process reflected from the upper boundary of the interval as follows:

$$\tilde{D}_x^r(t) = -\underline{D}_x^{\,k}(t) + B \in (-\infty, B], \quad \tilde{D}_x^r(0) = r. \tag{44}$$

Observe that the reflections are generated by the running infimum of the process $D_x(t)$. One can verify validity of the following chain of stochastic equalities:

$$\begin{aligned}\tilde{\tau}_x^B(r) &= \inf\big\{t : \tilde{D}_x^r(t) < 0\big\} = \inf\big\{t : -\underline{D}_x^{\,k}(t) + B < 0\big\} \\ &= \inf\big\{t : \underline{D}_x^{\,k}(t) > B\big\} = \overline{\tau}_x^{\,B}(k).\end{aligned}$$

Hence, the random variables $\tilde{\tau}_x^B(r)$, $\overline{\tau}_x^{\,B}(k)$ are identically distributed. Consider the one-server queueing system $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ with the following properties.

- (i) Customers of size $\delta$ arrive to the system at the renewal times of the process $N_x(t)$.
- (ii) The system has a finite waiting room (buffer) whose size is equal to $B < \infty$. If upon arrival of the customer (of size $\delta$) it meets $r \in [0, B]$ occupied space in the waiting room, then $\min\{k, \delta\}$ joins the waiting room, and there occurs loss of size $\max\{0, \delta - k\}$, where $k = B - r$ is the value of empty place in the waiting room at the instant of the customer's arrival.
- (iii) Duration of the service completion is an exponential variable with parameter $\mu > 0$. Suppose that at a time $t$ there is $r \in (0, B]$ occupied space, and let $x \geq 0$ denote the time elapsed from the last arrival of the customer. If at $t + 0$ the service is accomplished, then the buffer size is reduced by $\min\{r, \varkappa\}$. If at the instant of the service completion $r - \min\{r, \varkappa\} > 0$, then a new service cycle starts. If at this instant $r - \min\{r, \varkappa\} = 0$, then a new service cycle starts only upon arriving of the next customer (after time $\eta_x$).

Suppose the system $\mathrm{G}^{\delta}|\mathrm{M}^{\varkappa}|1|\mathrm{B}$ with the buffer size $B$ starts from the state $(r, x)$, where $r \in [0, B]$ is an initial workload of the system, and $x \geq 0$ is the time elapsed since the last arrival of the customer. Denote by $\tilde{b}_r(x)$ the first time at which the waiting room becomes empty (then the interval $[0, \tilde{b}_r(x)]$ is a busy period of type $(r, x)$). It follows from the definition of the process $\tilde{D}_x^r(t)$ (44) and the queueing system, that the process $\tilde{D}_x^r(t)$, $t \in [0, \tilde{\tau}_x^B(r)]$ describes the functioning of the system on the busy interval $[0, \tilde{b}_r(x)]$, and the random variable $\tilde{b}_r(x)$ is identically distributed as $\overline{\tau}_x^B(k)$. Hence

$$\mathbf{E}\big[e^{-s\tilde{b}_r(x)}; \tilde{b}_r(x) < \infty\big] = 1 - A_x^r(s) + Q_r^s(x)\frac{\mathbf{E}A_0^{\delta+B}(s) - A_0^B(s)}{\mathbf{E}Q_{\delta+B}^s - Q_B^s(0)}.$$

The random variable $\tilde{b}_r(x)$ is proper, $\mathbf{P}[\tilde{b}_r(x) < \infty] = 1$, and

$$\mathbf{E}\overline{\tau}_r^B(x) = A_x^r + Q_r(x)\frac{\mathbf{E}A_0^{\delta+B} - A_0^B}{\mathbf{E}Q_{\delta+B} - Q_B(0)} < \infty,$$

where $Q_k(x) = Q_k^0(x)$, $\mathbf{E}Q_{\delta+B} = \mathbf{E}Q_{\delta+B}^0$,

$$A_x^k = \int_{0-}^{k} d\rho^*(v)\big[1 - \lambda^{-1}Q_{k-v}(x)\big], \quad d\rho^*(v) = \int_0^{\infty} d_v\mathbf{P}[\pi_t \leq v]\,dt.$$

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