A two-sided exit problem for a difference of a compound Poisson process and a compound renewal process with a discrete phase space

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Abstract

A two-sided exit problem is solved for a difference of a compound Poisson process and a compound renewal process. The Laplace transforms of the joint distribution of the first exit time, the value of the overshoot and the value of a linear component at this instant are determined. The results obtained are applied to solve the two-sided exit problem for a particular case of this process, namely, the difference of the compound Poisson process and the renewal process whose jumps are geometrically distributed. The advantage is that these results are in a closed form, in terms of resolvent sequences of the process.

Introduction

In this paper we solve the two-sided exit problem for the difference of a compound Poisson process and a compound renewal process. Such processes have proven to be appropriate models in many applied fields of the probability theory, in particular in queueing theory. First we give a short overview of the existing literature related to our problem. The Laplace transform of the joint distribution of the first exit time and the value of the overshoot at this instant for Lévy processes and random walks has been determined in [8]. This joint distribution plays a key role for solving another two-boundary problems for Lévy processes and random walks. To solve this problem the authors used a probabilistic approach based on employing one-boundary functionals of the process. Following the approach from [8], Kadankov [9] solved the two-sided exit problem for another class of random processes, namely for a difference of renewal processes.

Distributions of the one-boundary functionals for the difference of renewal processes have been studied by Lindley [13], Prabhu [19] and Cohen [2]. The

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two-sided exit problem for such processes is closely related to the G|G|1 type queueing models. A summary of known results for the G|G|1 type model can be found in the book [2]. The joint distributions of the one-boundary functionals for the difference of compound renewal processes have been considered by Nasirova [14] in terms of the solutions of linear integral equations. Special cases of the difference of renewal processes have been studied by many authors. Ezhov and Kadankov [3]—[6], for instance, have employed probability-factorization methods, while Pirdzhanov and Bratiychuk [18], [1] have used factorization methods. One-boundary functionals for the difference of two compound Poisson processes with drifts with various jump distributions have been studied by Perry, Stadje, Zacks et. al [15], [16]. Two-boundary problems for the difference of two Poisson processes with exponential negative jumps were solved in [10].

The rest of the article is structured as follows. First we will provide necessary definitions and notations. The two-sided exit problem will be solved for the difference of the compound Poisson process and a compound renewal process in Section 2. In Section 3 we will derive similar results for the difference of the compound Poisson process and a renewal process whose jumps are geometrically distributed. Finally, we will obtain the exact formulae for the probabilities of the first exit from the interval for the difference of the compound Poisson process and the renewal process.

1 Main definitions

Let $\varkappa,\, \delta \in \mathbb{N} = \{1, 2, \dots\}$ be positive independent integer random variables, and $\eta \in (0, \infty)$ be a positive random variable independent of $\varkappa, \delta$ with the distribution function $F(x) = \mathbf{P}[\eta \le x]$, $x \ge 0$. We will assume that $\mathbf{E}\varkappa$, $\mathbf{E}\delta$, $\mathbf{E}\eta < \infty$. Introduce the sequences $\{\eta, \eta_n'\}$, $\{\varkappa, \varkappa_n'\}$, $\{\delta, \delta_n'\}$, $n \in \mathbb{N}$ of independent identically distributed (inside of each sequence) variables and define the monotone sequences

$$\begin{aligned} &\eta_0(x) = 0, \quad \eta_1(x) = \eta_x, \quad \eta_{n+1}(x) = \eta_x + \eta_1' + \dots + \eta_n', \qquad n \in \mathbb{N}, \\ &\varkappa_0 = 0, \quad \varkappa_n = \varkappa_1' + \dots + \varkappa_n'; \qquad \delta_0 = 0, \quad \delta_n = \delta_1' + \dots + \delta_n'; \qquad n \in \mathbb{N}, \end{aligned} \tag{1}$$

where $\eta_x \in (0, \infty)$ is a random variable with the following distribution function

$$F_x(u) = \mathbf{P}[\eta_x \le u] = [F(x+u) - F(x)](1 - F(x))^{-1} \qquad u \ge 0.$$

Denote by $\{\pi(t)\}_{t\ge 0} \in \mathbb{Z}^+ = \{0, 1, \dots\}$ a compound Poisson process with the generating function of the form

$$\mathbf{E}\,\theta^{\pi(t)} = e^{tk(\theta)}, \qquad k(\theta) = \mu\,(\mathbf{E}\,\theta^{\varkappa} - 1)\,, \quad |\theta| \le 1,$$

where $\mu > 0$ is the intensity of the jumps and $\varkappa$ is a jump size. For all $t \ge 0$ define a renewal process generated by the random sequence $\{\eta_n(x)\}_{n\in\mathbb{Z}^+}$:

$$N_x(t) = \max\{n \in \mathbb{Z}^+ : \eta_n(x) \le t\} \in \mathbb{Z}^+, \qquad x \ge 0.$$

Introduce a right-continuous step process for all $x \geq 0$

$$D_x(t) = \pi(t) - \delta_{N_x(t)} \in \mathbb{Z} = \{0, \pm 1, \dots\}, \qquad t \geq 0; \quad D_x(0) = 0. \tag{2}$$

Note, that inter-arrival times of the positive jumps are exponentially distributed with parameter μ, the positive jumps themselves are of a random size $\varkappa$, and there occur negative jumps of size δ'_n at time instants $\eta_n(x)$, $n \in \mathbb{N}$. We will call the process $\{D_x(t)\}_{t \geq 0}$ a difference of the compound Poisson process and a compound renewal process. Observe, that this process is not a Markov process in general. For all $t \geq 0$ introduce a right-continuous linear component

$$\eta_x^+(t) = \begin{cases} t + x, & 0 \leq t < \eta_x, \\ t - \eta_{N_x(t)}(x), & t \geq \eta_x \end{cases} \in \mathbb{R}_+ = [0, \infty), \qquad x \geq 0. \tag{3}$$

The process $\{\eta_x^+(t)\}_{t \geq 0}$ increases linearly on the intervals $[\eta_n(x), \eta_{n+1}(x))$, $n \in \mathbb{Z}^+$, it is killed to zero at the points $\eta_n(x)$, $n \in \mathbb{N}$, and the value of the process at the instant $t_0 \geq \eta_x$ is equal to the time elapsed from the moment of the last negative jump of the process (2) till t_0. We will call the process (3) a linear component. Note that this linear component is sometimes referred to as the age process (age since the last renewal).

By adding this linear component to the process $\{D_x(t)\}_{t \geq 0}$ we obtain a right-continuous Markov process

$$\{X_t\}_{t \geq 0} = \left\{D_x(t), \eta_x^+(t)\right\}_{t \geq 0} \in \mathbb{Z} \times \mathbb{R}_+, \qquad X_0 = \{0, x\}, \quad x \geq 0, \tag{4}$$

which governs the process $\{D_x(t)\}_{t \geq 0}$. The process defined in (4) is a Markov process. Note, that it is homogeneous with respect to the first component [7]. This fact will be used constantly when setting up the equations. If $X_{t_0} = \{k, u\}$, $k \in \mathbb{Z}$, $u \geq 0$, then the evolution of the process $\{X_t\}_{t \geq t_0}$ in the sequel does not depend on the value k of the first component, and the first positive jump of the process $\{D_x(t)\}_{t \geq t_0}$ (which is distributed as $\varkappa$) will occur after an exponential period of time with parameter μ. The first negative jump of the process $\{D_x(t)\}_{t \geq t_0}$ (which is distributed as δ) will take place after elapsing of time η_u.

Let $X_0 = \{0, x\}$, $x \geq 0$, $k \in \mathbb{Z}^+$ and introduce upper one-boundary functionals of the process $\{X_t\}_{t \geq 0}$:

$$\tau^k(x) = \inf\{t : D_x(t) > k\}, \quad T^k(x) = D_x(\tau^k(x)) - k, \quad \eta^k(x) = \eta_x^+(\tau^k(x))$$

i.e. the instant of the first crossing of the level k by the process $\{D_x(t)\}_{t \geq 0}$, the value of the overshoot across the upper level and the value of the linear component $\eta_x^+(\cdot)$ at the instant of the first crossing (the time since the last renewal). We define similarly for $k \in \mathbb{Z}^+$, $x \geq 0$ lower one-boundary functionals

$$\tau_k(x) = \inf\{t : D_x(t) < -k\}, \quad T_k(x) = -D_x(\tau_k(x)) - k, \quad \eta_k^x = \eta_x^+(\tau_k(x))$$

i.e the instant of the first crossing of the level $-k$ by the process $\{D_x(t)\}_{t \geq 0}$, the value of the overshoot across the lower level and the value of the linear component at that instant. Observe, that the overshoot of the lower level can only occur at arrival times $\eta_n(x)$, $n \in \mathbb{N}$, of the negative jumps of the process

$\{D_x(t)\}_{t\geq 0}$ and, therefore, (3), $\mathbf{P}[\eta_k^x = 0] = 1$. We will not use this random variable in the sequel. The random variables $\tau^k(x)$, $\tau_k(x)$ take values from countable sets $\{\xi_n,\, n \in \mathbb{N}\}$, $\{\eta_n(x),\, n \in \mathbb{N}\}$ (here ξ_n is an instant of the n-th jump of the process $\{\pi(t)\}_{t\geq 0}$) and they are Markov times of the process $\{X_t\}_{t\geq 0}$. Per definition we set $\inf\{\emptyset\} = \infty$. On the event $\{\tau^k(x) = \infty\}$ we assume that $T^k(x) = \eta^k(x) = \infty$. Similarly, on the event $\{\tau_k(x) = \infty\}$ we set $T_k(x) = \infty$.

We will assume that the integral transforms of the distributions of the upper and lower one-boundary functionals of the process $\{X_t\}_{t\geq 0}$ are known, and we will solve the two-sided exit problem for the process $\{D_x(t)\}_{t\geq 0}$ in terms of the Laplace transforms of these distributions.

2 Two-sided exit problem for the process $\{D_x(t)\}_{t\geq 0}$

Let $B \in \mathbb{Z}^+$ be fixed, $k \in \{0, \dots, B\}$, $r = B - k$, $X_0 = \{0, x\}$, $x \geq 0$, and introduce the random variable

$$\chi = \inf\{t :\, D_x(t) \notin [-r, k]\}$$

the first exit time from the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\geq 0}$. This random variable takes values from a countable set $\{\xi_n,\, n \in \mathbb{N}\} \cup \{\eta_n(x),\, n \in \mathbb{N}\}$ and it is a Markov time of the process $\{X_t\}_{t\geq 0}$. Exit from the interval can occur either through the upper boundary k, or through the lower boundary $-r$. Introduce the events
$\mathfrak{A}^k = \{D_x(\chi) > k\}$ i.e. the process $\{D_x(t)\}_{t\geq 0}$ exits the interval $[-r, k]$ through the upper boundary k;
$\mathfrak{A}_r = \{D_x(\chi) < -r\}$ i.e. the process $\{D_x(t)\}_{t\geq 0}$ exits the interval $[-r, k]$ through the lower boundary $-r$. Denote by

$$T = (D_x(\chi) - k)\mathbf{I}_{\mathfrak{A}^k} + (-D_x(\chi) - r)\mathbf{I}_{\mathfrak{A}_r}, \quad L = \eta_x^+(\chi)\mathbf{I}_{\mathfrak{A}^k} + 0\cdot\mathbf{I}_{\mathfrak{A}_r}, \quad \mathbf{P}[\mathfrak{A}^k + \mathfrak{A}_r] = 1$$

the value of the overshoot through the boundaries of the interval $[-r, k]$ by the process $\{D_x(t)\}_{t\geq 0}$ and the value of a linear component at the instant of the first exit (the time since the last renewal), where $\mathbf{I}_{\mathfrak{A}} = \mathbf{I}_{\mathfrak{A}}(\omega)$ is the indicator function of the event $\mathfrak{A}$. For all $k \in \mathbb{Z}^+$, $m \in \mathbb{N}$, $x \geq 0$ denote

$$\begin{aligned}
f_x^k(dl, m, s) &= \mathbf{E}\left[e^{-s\tau^k(x)}; \eta^k(x) \in dl, T^k(x) = m, \tau^k(x) < \infty\right], \\
f_k^x(m, s) &= \mathbf{E}\left[e^{-s\tau_k(x)}; T_k(x) = m, \tau_k(x) < \infty\right], \\
F_x^k(dl, m, s) &= f_x^k(dl, m, s) - \sum_{i\in\mathbb{N}} f_r^x(i, s) f_0^{i+B}(dl, m, s), \\
F_r^x(m, s) &= f_r^x(m, s) - \sum_{i\in\mathbb{N}} \int_0^\infty f_x^k(dl, i, s) f_{i+B}^l(m, s).
\end{aligned}$$

Theorem 1. *Let $\{D_x(t)\}_{t\geq 0}$ be a difference of the compound Poisson process and the compound renewal process (2) and $B \in \mathbb{Z}^+$, $k \in \{0, \dots, B\}$, $r = B - k$,*

$X_0=\{0,x\}$, $x\geq 0$. Then the Laplace transforms

$$V_x^k(dl,m,s)=\mathbf{E}\left[e^{-s\chi};L\in dl,T=m,\mathfrak{A}^k\right],\quad V_r^x(m,s)=\mathbf{E}\left[e^{-s\chi};T=m,\mathfrak{A}_r\right]$$

of the joint distribution of $\{\chi,L,T\}$, i.e. of the first exit time from the interval $[-r,k]$ by the process $\{D_x(t)\}_{t\geq 0}$, the value of the linear component and the value of the overshoot through the boundary at the instant of the first exit satisfy the following formulae for $s>0$, $m\in\mathbb{N}$

$$\begin{aligned} V_x^k(dl,m,s)&=F_x^k(dl,m,s)+\sum_{i\in\mathbb{N}}\int_0^\infty F_x^k(d\nu,i,s)\mathfrak{K}^+_{\nu,i}(dl,m,s),\\ V_r^x(m,s)&=F_r^x(m,s)+\sum_{i\in\mathbb{N}}F_r^x(i,s)\,\mathfrak{K}^-_i(m,s), \end{aligned}\tag{5}$$

where

$$\mathfrak{K}^+_{\nu,i}(dl,m,s)=\sum_{n\in\mathbb{N}}K^+_{\nu,i}(dl,m,s)^{*(n)},\quad \mathfrak{K}^-_i(m,s)=\sum_{n\in\mathbb{N}}K^-_i(m,s)^{*(n)}\tag{6}$$

are uniformly convergent series of the iterations, and

$$\begin{aligned} &K^+_{\nu,i}(dl,m,s)^{*(1)}\overset{\text{def}}{=}K^+_{\nu,i}(dl,m,s),\qquad K^-_i(m,s)^{*(1)}\overset{\text{def}}{=}K^-_i(m,s),\\ &K^+_{\nu,i}(dl,m,s)^{*(n+1)}=\sum_{j\in\mathbb{N}}\int_0^\infty K^+_{\nu,i}(du,j,s)K^+_{u,j}(dl,m,s)^{*(n)},\quad n\in\mathbb{N}\\ &K^-_i(m,s)^{*(n+1)}=\sum_{j\in\mathbb{N}}K^-_i(j,s)\,K^-_j(m,s)^{*(n)},\quad n\in\mathbb{N} \end{aligned}\tag{7}$$

are the successive iterations of the kernels $K^+_{\nu,i}(dl,m,s)$, $K^-_i(m,s)$, which are given by the following defining formulae

$$\begin{aligned} K^+_{\nu,i}(dl,m,s)&=\sum_{j\in\mathbb{N}}f^\nu_{i+B}(j,s)\,f_0^{j+B}(dl,m,s),\\ K^-_i(m,s)&=\sum_{j\in\mathbb{N}}\int_0^\infty f_0^{i+B}(du,j,s)\,f^u_{j+B}(m,s). \end{aligned}\tag{8}$$

Proof. The joint distribution of the first exit time and the value of the overshoot through the boundaries at that instant for processes with independent increments and random walks has been determined in [8]. This distribution was obtained in terms of the joint distributions of one-boundary functionals. We will employ this idea (use of the one-boundary functionals) and the method (setting up a system of integral equations) to solve the two-sided exit problem for the difference of a compound Poisson process and a compound renewal process (see also [9]). It is worth noting that the two-sided exit problem for the semi-Markov random walk with a drift was solved by means of these methods as well (see [11]. Following [8], we derive a system of equations with respect to the functions $V_x^k(dl,m,s)$,

$V_r^x(m,s), \ \ s>0, \ \ m \in \mathbb{N}$,

$$f_x^k(dl,m,s) = V_x^k(dl,m,s) + \sum_{i \in \mathbb{N}} V_r^x(i,s)\, f_0^{i+B}(dl,m,s),$$

$$f_r^x(m,s) = V_r^x(m,s) + \sum_{i \in \mathbb{N}} \int_0^\infty V_x^k(d\nu,i,s)\, f_{i+B}^{\nu}(m,s). \tag{9}$$

In order to write this system, we have used the law of total probability, homogeneity of the process $\{X_t\}_{t \geq 0}$ with respect to the first component, Markov property of the random variables $\tau^k(x), \ \tau_k(x), \ \chi$ and the following probabilistic reasoning. The first equation of the system (9) stresses the fact that the first overshoot of the upper level k by the process $\{D_x(t)\}_{t \geq 0}$ (expression in the left-hand side of the equation) can be realized either on sample paths which do not intersect the lower level $-r$ (the first term of the right-hand side of the equation) or on the sample paths which do intersect the level $-r$ and then later on intersect the level k (the second term of the right-hand side of the equation). The second equation of the system is written analogously. This system is similar to a system of linear equations with two unknown variables. Substituting the expression for the function $V_x^k(dl,m,s)$ from the first equation of system (9) into the second one, we get

$$V_r^x(m,s) = f_r^x(m,s) - \sum_{i \in \mathbb{N}} \int_0^\infty f_x^k(d\nu,i,s)\, f_{i+B}^{\nu}(m,s) +$$
$$+ \sum_{j \in \mathbb{N}} \int_0^\infty \sum_{i \in \mathbb{N}} V_r^x(i,s)\, f_0^{i+B}(du,j,s)\, f_{j+B}^{u}(m,s).$$

Changing the order of summation in the third term of the right-hand of the latter equation we obtain a discrete analog of a linear integral equation for the function $V_r^x(m,s)$

$$V_r^x(m,s) = F_r^x(m,s) + \sum_{i \in \mathbb{N}} V_r^x(i,s)\, K_i^-(m,s), \tag{10}$$

where the function

$$K_i^-(m,s) = \sum_{j \in \mathbb{N}} \int_0^\infty f_0^{i+B}(du,j,s)\, f_{j+B}^{u}(m,s)$$

is a kernel of this equation. As mentioned above $\tau^k(x) \in \{\xi_n, \, n \in \mathbb{N}\}, \ \tau_k(x) \in \{\eta_n(x), \, n \in \mathbb{N}\}$, where ξ_n is the instant of the n-th jump of the compound Poisson process. It follows from the properties of the compound Poisson process that for all $k \in \mathbb{Z}+, \ \ x,s \geq 0$ the following inequalities hold:

$$\mathbf{E} e^{-s\tau^k(x)} \leq \frac{\mu}{s+\mu}, \qquad \mathbf{E} e^{-s\tau_k(x)} \leq \mathbf{E} e^{-s\eta_x}.$$

Therefore, the kernel of the equation (10) enjoys the following property for all $i,m \in \mathbb{N}, \ \ s>0$

$$K_i^-(m,s) \leq \sum_{j \in \mathbb{N}} \int_0^\infty f_0^{i+B}(du,j,s)\, \mathbf{E} e^{-s\eta_u} \leq \sum_{j \in \mathbb{N}} \int_0^\infty f_0^{i+B}(du,j,s) \leq \frac{\mu}{s+\mu} < 1.$$

Using the method of mathematical induction and the latter bound for the kernel, one can show that for all $i, m \in \mathbb{N}$, $K_i^-(m,s)^{*(n)} \le \mu^n (s+\mu)^{-n}$, $n \in \mathbb{N}$. Hence, the series

$$\mathfrak{K}_i^-(m,s) = \sum_{n\in\mathbb{N}} K_i^-(m,s)^{*(n)} \le \frac{\mu}{s} < \infty, \qquad s>0$$

converges uniformly for all $i, m \in \mathbb{N}$. Utilizing the method of successive iterations [17] to solve (10), we get the second equality of Theorem 1. Substituting the expression for the function $V_r^x(m,s)$ from the second equation into the first one, we find

$$V_x^k(dl,m,s) = f_x^k(dl,m,s) - \sum_{i\in\mathbb{N}} f_r^x(i,s)\, f_0^{i+B}(dl,m,s) + \\ + \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \int_0^\infty V_x^k(d\nu,i,s)\, f_{i+B}^{\nu}(j,s)\, f_0^{j+B}(dl,m,s).$$

Changing the order of summation in the third term in the right-hand side of this equation, we obtain for the function $V_x^k(dl,m,s)$

$$V_x^k(dl,m,s) = F_x^k(dl,m,s) + \sum_{i\in\mathbb{N}} \int_0^\infty V_x^k(d\nu,i,s)\, K_{\nu,i}^+(dl,m,s), \qquad (11)$$

i.e. a discrete-continuous analog of a linear integral equation. The kernel of this equation is given by

$$K_{\nu,i}^+(dl,m,s) = \sum_{j\in\mathbb{N}} f_{i+B}^{\nu}(j,s)\, f_0^{j+B}(dl,m,s)$$

and for all $\nu, l > 0$ $i, m \in \mathbb{N}$, $s > 0$ it satisfies the following inequality

$$K_{\nu,i}^+(dl,m,s) \le \frac{\mu}{s+\mu} \sum_{j\in\mathbb{N}} f_{i+B}^{\nu}(j,s) \le \frac{\mu}{s+\mu}\, \mathbf{E} e^{-s\tau_{i+B}(\nu)} \le \frac{\mu}{s+\mu} < 1.$$

Hence, the series of the successive iterations

$$\mathfrak{K}_{\nu,i}^+(dl,m,s) = \sum_{n\in\mathbb{N}} K_{\nu,i}^+(dl,m,s)^{*(n)} \le \frac{\mu}{s} < \infty$$

converges uniformly for all $\nu, l > 0$ $i, m \in \mathbb{N}$. Employing the method of successive iterations [17] for the equation (11) we derive the first equality of Theorem 1. □

3 Exit problem for the process $\{D_x(t)\}_{t\ge 0}$ with geometrically distributed negative jumps

In this section we assume that the random variable $\delta \in \mathbb{N}$ is geometrically distributed with parameter $\lambda \in [0,1)$: $\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}$, $n \in \mathbb{N}$. This assumption means that the process $\{D_x(t)\}_{t\ge 0}$ has geometrically distributed negative jumps at time instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$. In this case it is possible to obtain closed form solutions for the two-sided exit problem introduced in the previous section. Our task now is to determine the Laplace transforms of the joint distributions of the upper and lower one-boundary functionals of the process $\{X_t\}_{t\ge 0}$. In the sequel we will use the following result.

Lemma 1. *Let* $\tilde{f}(s) = \mathbf{E}e^{-s\eta}$*. Then for* $s > 0$ *the equation*

$$\theta - \lambda = (1-\lambda)\tilde{f}(s - k(\theta)), \qquad |\theta| < 1 \tag{12}$$

has a unique solution $c(s)$ *inside the circle* $|\theta| < 1$*. This solution is positive and* $c(s) \in (\lambda, 1)$*. If* $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$*,* $\rho = \mu(1-\lambda)\mathbf{E}[\varkappa]\,\mathbf{E}[\eta]$*, then for* $\rho > 1$*,* $\lim\limits_{s\to 0} c(s) = c \in (\lambda, 1)$*; and for* $\rho \le 1$*,* $\lim\limits_{s\to 0} c(s) = 1$*.*

A detailed proof of an analogous proposition for semi-continuous random walks can be found in the monograph of Spitzer [20]. The reasoning in that proof can be applied to the equation (12) as well. The Laplace transforms of the joint distribution of the lower one-boundary functionals are determined by means of the following lemma.

Lemma 2. *Let* $\{D_x(t)\}_{t\ge 0}$ *be a difference of the compound Poisson process and the compound renewal process with geometrically distributed jumps. Then*

(i) *the Laplace transform of the joint distribution of* $\{\tau_k(x), T_k(x)\}$*,* $k \in \mathbb{Z}^+$*,* $x \ge 0$ *satisfies the following equality for* $s > 0$*,* $m \in \mathbb{N}$

$$f_k^x(m,s) = \tilde{f}_x(s - k(c(s)))\, c(s)^k\, (1-\lambda)\lambda^{m-1}, \tag{13}$$

where $c(s) \in (\lambda, 1)$ *is the unique solution of the equation (12) inside the circle* $|\theta| < 1$*,* $\tilde{f}_x(s) = \mathbf{E}\, e^{-s\eta_x}$*,* $\tilde{f}(s) = \mathbf{E}\, e^{-s\eta} = \tilde{f}_0(s)$*;*

(ii) *if* $\rho > 1$*, then* $\mathbf{P}[\tau_k(x) < \infty] = \tilde{f}_x(-k(c))\, c^k < 1$*, and* $\tau_k(x)$ *for all* $k \in \mathbb{Z}^+$*,* $x \ge 0$ *is a defective random variable; if* $\rho \le 1$*, then* $\mathbf{P}[\tau_k(x) < \infty] = 1$*, and* $\tau_k(x)$ *is a proper variable for all* $k \in \mathbb{Z}^+$*,* $x \ge 0$*.*

Proof. Taking into account that δ and $\varkappa$ are independent and using the properties of the geometrical distribution it is not difficult to establish that

$$\begin{aligned}
&\mathbf{E}\left[e^{-s\eta_x}\theta^{\pi(\eta_x)-\delta}\right] = \frac{1-\lambda}{\theta-\lambda}\,\tilde{f}_x(s-k(\theta)), \qquad |\theta| = 1, \quad x \ge 0,\\
&\mathbf{P}[\pi(t) - \delta = i] = (1-\lambda)\mathbf{E}\left[\lambda^{\pi(t)-(i+1)}; \pi(t) > i\right], \qquad i \in \mathbb{Z}.
\end{aligned} \tag{14}$$

The mathematical expectations $f_k^x(m,s)$, $k \in \mathbb{Z}^+$, $x \ge 0$ satisfy the following equation

$$\begin{aligned}
f_k^x(m,s) = {}&\mathbf{E}\left[e^{-s\eta_x}; \pi(\eta_x) - \delta = -(k+m)\right]\\
&+ \sum_{i\in\mathbb{Z}^+} \mathbf{E}\left[e^{-s\eta_x}; \pi(\eta_x) - \delta = i - k\right] f_i^0(m,s).
\end{aligned} \tag{15}$$

This equation is written by conditioning on the first regeneration point $\eta_1(x) = \eta_x$ of the renewal process $\{N_x(t)\}_{t\ge 0}$. We have used the total probability formula, homogeneity of the process $\{X_t\}_{t\ge 0}$ with respect to the first component and the Markov property of the random time $\eta_1(x)$. Setting $x = 0$ in this equation we obtain

$$f_k^0(m,s) = (1-\lambda)\tilde{f}(s - k(\lambda))\lambda^{k+m-1} + \sum_{i\in\mathbb{Z}^+} K_s(k,i)\, f_i^0(m,s), \tag{16}$$

where

$$K_s(k,i) = K_s(i-k) = \mathbf{E}\left[e^{-s\eta}; \pi(\eta) - \delta = i-k\right], \qquad i,k \in \mathbb{Z}.$$

In general this equation can be solved by factorization methods. In this case this method seems to be cumbersome, therefore we will use a more straightforward reasoning to solve (16). The equation (16) is a discrete analog of a linear integral equation with a kernel $K_s(i-k)$, depending on the difference of the arguments. This kernel enjoys the following property for $s > 0$, $i,k \in \mathbb{Z}$: $K_s(i-k) \leq \mathbf{E}[e^{-s\eta}] < 1$. Therefore, (16) has a unique solution which can be found by means of the method of successive iterations [17]. It is easy to check (after some simplifications and use of (14)) that the function

$$f_k^0(m,s) = \tilde{f}(s - k(c(s)))\, c(s)^k\, (1-\lambda)\lambda^{m-1}$$

satisfies the equation (16). Therefore, it is a unique solution of (16). Substituting the expression for the function $f_k^0(m,s)$ into the equation (15) and performing analogous calculations for all $k, m-1 \in \mathbb{Z}^+$, $x \geq 0$, we find

$$f_k^x(m,s) = \tilde{f}_x(s - k(c(s)))\, c(s)^k\, (1-\lambda)\lambda^{m-1}, \tag{17}$$

which is the formula (13) of Lemma 2. Validity of the second statement follows from Lemma 1 and (13). □

We now introduce a sequence which will be used to obtain the results in the sequel. The idea to employ this sequence for semi-continuous random walks and semi-continuous Lévy processes is due to Takács [21]. Since the function

$$\tilde{f}_x(s - k(\theta)) = \mathbf{E}\left[e^{-s\eta_x}\theta^{\pi(\eta_x)}\right] = \sum_{i \in \mathbb{Z}^+} \theta^i \int_0^\infty e^{-st}\mathbf{P}[\eta_x \in dt,\, \pi(t) = i], \quad |\theta| \leq 1,$$

is analytical inside the unit circle for all $s, x \geq 0$, then the function

$$\mathbb{Q}_\theta^s(x) = (1-\lambda)\tilde{f}_x(s - k(\theta))\left((1-\lambda)\tilde{f}(s - k(\theta)) + \lambda - \theta\right)^{-1}, \quad s, x \geq 0$$

is analytical on the open set $|\theta| < c(s)$. In this region it can be represented as a powers series

$$\mathbb{Q}_\theta^s(x) = \sum_{k \in \mathbb{Z}^+} \theta^k Q_k^s(x), \quad s, x \geq 0.$$

The coefficients of this expansion can be calculated by means of the inversion formula

$$Q_k^s(x) = \frac{1}{2\pi i}\oint_{|\theta|=\alpha} \frac{1}{\theta^{k+1}}\, \frac{(1-\lambda)\tilde{f}_x(s - k(\theta))}{(1-\lambda)\tilde{f}(s - k(\theta)) + \lambda - \theta}\, d\theta, \quad \alpha \in (0, c(s)). \tag{18}$$

We will call the sequence $\{Q_k^s(x)\}_{k \in \mathbb{Z}^+}$, $x \geq 0$, defined by the formula (18) the resolvent sequence of the process $\{D_x(t)\}_{t \geq 0}$.

We now define upper one-boundary functionals of the process $\{D_x(t)\}_{t\geq 0}$. Let $k \in \mathbb{Z}^+$ and

$$\tilde{\tau}^k = \inf\{t : \pi(t) > k\}, \qquad \tilde{T}^k = \pi(\tilde{\tau}^k) - k$$

be the first crossing time through the upper level k by the compound Poisson process $\{\pi(t)\}_{t\geq 0}$ and the value of the overshoot at this instant. Denote by

$$\rho_k(t) = \mathbf{P}[\pi(t) = k], \quad \sum_{k=0}^{\infty} \theta^k \rho_k(t) = \mathbf{E}\,\theta^{\pi(t)} = e^{tk(\theta)}, \quad |\theta| \leq 1,$$

$$p_k^m(dt) = \mathbf{P}[\tilde{\tau}^k \in dt,\, \tilde{T}^k = m] = \mu \sum_{i=0}^{k} \rho_i(t) \mathbf{P}[\varkappa = k - i + m]\, dt, \quad m \in \mathbb{N}.$$

Lemma 3. *Let $\{D_x(t)\}_{t\geq 0}$ be a difference of the compound Poisson process and the compound renewal process whose jumps are geometrically distributed, $x \geq 0$, $k \in \mathbb{Z}^+$ and $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$ be a resolvent sequence of the process $\{D_x(t)\}_{t\geq 0}$, given by (18). Then*

(i) *the Laplace transforms of the joint distribution of $\{\tau^k(x), \eta^k(x), T^k(x)\}$ satisfy the following equality*

$$f_x^k(dl, m, s) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m(d(l - x))$$
$$+ \Phi_\lambda^s(0, dl, m)\, Q_k^s(x) - e^{-sl}\, [1 - F(l)] \sum_{i=0}^{k} Q_i^s(x)\, p_{k-i}^m(dl), \tag{19}$$

where $\Phi_\lambda^s(0, dl, m) = e^{-sl}[1 - F(l)] \sum_{k=0}^{\infty} c(s)^k p_k^m(dl)$;

(ii) *for the Laplace transform of the first crossing time through the upper level k by the process $\{D_x(t)\}_{t\geq 0}$ for all $k \in \mathbb{Z}^+$, $s, x \geq 0$ the following formula holds*

$$\mathbf{E} e^{-s\tau^k(x)} = 1 - \frac{s}{s - k(c(s))} \frac{Q_k^s(x)}{1 - \lambda} + \sum_{i=0}^{k} \tilde{\rho}_i(s) \left[\frac{Q_{k-i}^s(x)}{1 - \lambda} - 1 \right], \tag{20}$$

where $\tilde{\rho}_k(s) = s \int_0^\infty e^{-st} \rho_k(t)\, dt$;

(iii) *for $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$ and $\rho < 1$, $\tau^k(x)$ is a defective random variable and*

$$\mathbf{P}[\tau^k(x) < \infty] = 1 - (1 - \rho)(1 - \lambda)^{-1} Q_k(x) < 1, \qquad k \in \mathbb{Z}^+ \quad x \geq 0,$$

where $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$ is the resolvent sequence of the process $\{D_x(t)\}_{t\geq 0}$, given by (18) and for $s = 0$:

$$Q_k(x) = \frac{1}{2\pi i} \oint_{|\theta|=\alpha} \frac{d\theta}{\theta^{k+1}} \frac{(1 - \lambda)\tilde{f}_x(-k(\theta))}{(1 - \lambda)\tilde{f}(-k(\theta)) + \lambda - \theta}, \quad \alpha \in (0, c(0)); \tag{21}$$

if $\rho \geq 1$, then for all $k \in \mathbb{Z}^+$, $x \geq 0$ $\tau^k(x)$ is a proper random variable.

Proof. Due to the total probability formula, the Markov property of the random times $\{\eta_n(x),\, n \in \mathbb{N}\}$ and the space homogeneity of the process $\{D_x(t)\}_{t\geq 0}$ the mathematical expectations $f_x^k(dl, m, s),\ \ k \in \mathbb{Z}^+,\ \ x \geq 0$ satisfy the following equation

$$f_x^k(dl, m, s) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} p_k^m(d(l-x))$$

$$+ \sum_{i=0}^{k} \mathbf{E}[e^{-s\eta_x}; \pi(\eta_x) = i] \sum_{r=1}^{\infty} (1-\lambda)\lambda^r f_0^{k+r-i}(dl, m, s), \qquad m \in \mathbb{N}. \tag{22}$$

Introduce the generating functions

$$\Phi_\theta^s(x, dl, m) = \sum_{k\in\mathbb{Z}^+} \theta^k f_x^k(dl, m, s), \qquad |\theta| < 1.$$

Multiplying the equation (22) by θ^k and summing with respect to $k \in \mathbb{Z}^+$, we derive the equation for the introduced functions

$$\Phi_\theta^s(x, dl, m) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} \Pi_\theta^m(d(l-x))$$

$$+ \frac{1-\lambda}{\theta - \lambda} \tilde{f}_x(s - k(\theta)) \left[\Phi_\theta^s(0, dl, m) - \Phi_\lambda^s(0, dl, m)\right], \tag{23}$$

where $\Pi_\theta^m(dl) = \sum\limits_{k\in\mathbb{Z}^+} \theta^k p_k^m(dl),\ \ |\theta| < 1$. Setting $x = 0$ in this equation, we get

$$\Phi_\theta^s(0, dl, m) = \Big((1-\lambda)\tilde{f}(s - k(\theta)) + \lambda - \theta\Big)^{-1} \tag{24}$$

$$\times \Big((1-\lambda)\tilde{f}(s - k(\theta))\Phi_\lambda^s(0, dl, m) - (\theta - \lambda)e^{-sl}[1 - F(l)]\Pi_\theta^m(dl)\Big), \quad |\theta| < 1.$$

In order to determine the function $\Phi_\lambda^s(0, dl, m)$ we will use the following reasoning. According to Lemma 1 the first term in the right-hand side of the equality has a simple pole in $\theta = c(s) \in (\lambda, 1)$. The function which enters the left-hand side of the equality is analytical inside the circle $|\theta| < 1$. Therefore, the second term in the right-hand side of the equality must have a simple zero in $\theta = c(s)$. The latter statement implies that $\Phi_\lambda^s(0, dl, m) = e^{-sl}[1 - F(l)]\Pi_{c(s)}^m(dl)$. Substituting the expression (24) for the function $\Phi_\theta^s(0, dl, m)$ into (23), we get

$$\Phi_\theta^s(x, dl, m) = e^{-s(l-x)} \frac{1 - F(l)}{1 - F(x)} \mathbf{I}\{l > x\} \Pi_\theta^m(d(l-x))$$

$$+ \Phi_\lambda^s(0, dl, m)\mathbb{Q}_\theta^s(x) - e^{-sl}[1 - F(l)]\Pi_\theta^m(dl)\mathbb{Q}_\theta^s(x), \qquad |\theta| < 1. \tag{25}$$

Employing the definition of the resolvent sequence $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+},\ \ x \geq 0$ (18), and comparing the coefficients of $\theta^k,\ \ k \in \mathbb{Z}^+$ in both sides of the equation (25) we get the formula (19) of the lemma. Integrating (19) with respect to $l \geq 0$ and summing over $m \in \mathbb{N}$, we derive (20). The third statement of Lemma 3 follows from Lemma 1 and (20). $\square$

Equalities (13), (19) determine the Laplace transforms of the one-boundary functionals of the process $\{D_x(t)\}_{t\geq 0}$. We state another lemma which is essential in our derivations. For all $x \geq 0$, $k \in \mathbb{Z}^+$ denote by

$$i_x^k = \inf\{t > \tau^k(x) : D_x(t) < k - B\}, \qquad I_x^k = k - B - D_x(i_x^k) \in \mathbb{N}$$

the first downward crossing time of the interval $[k-B, k]$ by the process $\{D_x(t)\}_{t\geq 0}$ after the first upward crossing time and the value of the overshoot through the lower level $k - B$ at this instant. As mentioned before, we use the convention that $\inf\{\emptyset\} \stackrel{\text{def}}{=} \infty$ and $I_x^k = \infty$ on the event $\{i_x^k = \infty\}$. It is obvious that $\eta_x^+(i_x^k) = 0$.

Lemma 4. *Let $\{D_x(t)\}_{t\geq 0}$ be a difference of the compound Poisson process and the renewal process whose jumps are geometrically distributed, $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$, $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$ be the resolvent sequences of the process given by (18), (21). Then*

(i) *the Laplace transforms of the joint distribution of $\{i_x^k, I_x^k\}$*

$$\varphi_x^k(m, s) = \mathbf{E}\left[e^{-si_x^k}; I_x^k = m, i_x^k < \infty\right]$$

satisfy the following equality for all $k \in \mathbb{Z}^+$, $x \geq 0$

$$\varphi_x^k(m, s) = \left[c_x(s)c(s)^{B-k} - c(s)^{B+1}r(s)\frac{Q_k^s(x)}{1-\lambda}\right](1-\lambda)\lambda^{m-1}, \qquad (26)$$

where $c_x(s) = \tilde{f}_x(s - k(c(s)))$, $r(s) = 1 + (1-\lambda)k'(c(s))\tilde{f}'(s - k(c(s)))$;

(ii) *if $\rho < 1$, then i_x^k is a defective random variable and*

$$\mathbf{P}\left[i_x^k < \infty\right] = 1 - (1-\rho)(1-\lambda)^{-1}Q_k(x);$$

if $\rho > 1$, then i_x^k is a defective random variable and

$$\mathbf{P}\left[i_x^k < \infty\right] = \tilde{f}_x(-k(c))c^{B-k} - c^{B+1}\left[(1-\lambda)^{-1} + k'(c)\tilde{f}'(-k(c)\right]Q_k(x);$$

if $\rho = 1$, then i_x^k a proper random variable.

Proof. Observe that the following equality holds

$$\varphi_x^k(m, s) = \sum_{i\in\mathbb{N}} \int_0^\infty f_x^k(dl, i, s) f_{i+B}^l(m, s).$$

To obtain this formula, it is necessary to use the total probability law, space homogeneity of the process a $\{D_x(t)\}_{t\geq 0}$ and the Markov property of the random time $\tau^k(x)$. Employing the formula (13), it follows from the latter equality that

$$\begin{aligned}\tilde{\varphi}_x^\theta(m, s) &= \sum_{k\in\mathbb{Z}^+} \theta^k \varphi_x^k(m, s)\\ &= \int_0^\infty \tilde{\Phi}_\theta^s(x, dl, c(s))\tilde{f}_l(s - k(c(s)))c(s)^B(1-\lambda)\lambda^{m-1}, \qquad (27)\end{aligned}$$

where $|\theta| < 1$,

$$\tilde{\Phi}^s_\theta(x, dl, z) = \sum_{m\in\mathbb{N}} z^m \Phi^s_\theta(x, dl, m), \qquad |z| \le 1,$$

is a generating function of the Laplace transform $f^k_x(dl, m, s)$ of the joint distribution of $\{\tau^k(x), \eta^k(x), T^k(x)\}$. From (25) we have

$$\begin{aligned}\tilde{\Phi}^s_\theta(x, dl, z) = {} & e^{-s(l-x)}\,\frac{1-F(l)}{1-F(x)}\,\mathbf{I}\{l > x\}\tilde{\Pi}^z_\theta(d(l-x)) \\ & + e^{-sl}[1-F(l)]\mathbb{Q}^s_\theta(x)\left(\tilde{\Pi}^z_{c(s)}(dl) - \tilde{\Pi}^z_\theta(dl)\right), \quad |\theta| < 1, \end{aligned} \tag{28}$$

where

$$\tilde{\Pi}^z_\theta(dl) = \sum_{m\in\mathbb{N}} z^m \Pi^m_\theta(dl) = e^{lk(\theta)}\,\frac{k(z) - k(\theta)}{1-\theta/z}\,dl, \qquad |z| \le 1.$$

Substituting the expression for the function $\tilde{\Phi}^s_\theta(x, dl, z)$ for $z = c(s)$ into (27), and integrating it with respect to $l \ge 0$, we find

$$\tilde{\varphi}^\theta_x(m, s) = \left[\frac{\tilde{f}_x(s - k(c(s)))}{1-\theta/c(s)}\,c(s)^B - c(s)^{B+1}\,r(s)(1-\lambda)^{-1}\mathbb{Q}^s_\theta(x)\right](1-\lambda)\lambda^{m-1},$$

where $|\theta| < 1$, $r(s) = 1 + (1-\lambda)k'(c(s))\tilde{f}'(s - k(c(s)))$. Comparing the coefficients of θ^k, $k \in \mathbb{Z}^+$ in both sides of this equality, we get for all $s, x \ge 0$

$$\varphi^k_x(m, s) = [c_x(s)c(s)^{B-k} - c(s)^{B+1}r(s)(1-\lambda)^{-1}Q^s_k(x)](1-\lambda)\lambda^{m-1},$$

which is (26). Letting $s \to 0$ and using Lemma 1 we obtain the statements of the second part of Lemma 4. □

Theorem 2. *Let* $\{D_x(t)\}_{t\ge0}$ *be a difference of the compound Poisson process and the renewal process (2),* $\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}$, $\lambda \in [0, 1]$, $n \in \mathbb{N}$; $\{Q^s_k(x)\}_{k\in\mathbb{Z}^+}$, $x \ge 0$ *be a resolvent sequence of the process given by (18),* $Q^s_k \overset{\text{def}}{=} Q^s_k(0)$. *Then*

(i) *the Laplace transforms of the joint distribution of* $\{\chi, L, T\}$ *satisfy the following equalities for all* $x, s \ge 0$, $m \in \mathbb{N}$

$$\begin{aligned} V^x_r(m, s) &= \frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}}(1-\lambda)\lambda^{m-1}, \\ V^k_x(dl, m, s) &= f^k_x(dl, m, s) - \frac{Q^s_k(x)}{\mathbf{E}\,Q^s_{\delta+B}}\,\mathbf{E}f^{\delta+B}_0(dl, m, s), \end{aligned} \tag{29}$$

where the function $f^k_x(dl, m, s)$ *is given by (19),*

$$\begin{aligned} \mathbf{E}\,Q^s_{\delta+B} &= \sum_{k\in\mathbb{N}}(1-\lambda)\lambda^{k-1}Q^s_{k+B}, \\ \mathbf{E}f^{\delta+B}_0(dl, m, s) &= \sum_{k\in\mathbb{N}}(1-\lambda)\lambda^{k-1}f^{k+B}_0(dl, m, s); \end{aligned}$$

(ii) *for the Laplace transforms of the first exit time* χ *from the interval by the process* $\{D_x(t)\}_{t\geq 0}$ *the following formulae hold*

$$\begin{aligned} \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}_r\right] &= \frac{Q_k^s(x)}{\mathbf{E}\, Q^s_{\delta+B}}, \\ \mathbf{E}\left[e^{-s\chi}; \mathfrak{A}^k\right] &= 1 - \frac{Q_k^s(x)}{\mathbf{E}\, Q^s_{\delta+B}} + A_x^k(s) - \frac{Q_k^s(x)}{\mathbf{E}\, Q^s_{\delta+B}} \mathbf{E}\, A_0^{\delta+B}(s), \end{aligned} \tag{30}$$

where

$$A_x^k(s) = \sum_{i=0}^{k} \tilde{\rho}_i(s) \left[\frac{Q^s_{k-i}(x)}{1-\lambda} - 1\right], \quad \tilde{\rho}_i(s) = s\int_0^\infty e^{-st}\mathbf{P}[\pi(t) = i]\, dt;$$

(iii) *the probabilities of the exit from the interval through the upper and the lower boundary by the process* $\{D_x(t)\}_{t\geq 0}$ *are given by*

$$\mathbf{P}[\mathfrak{A}_r] = \frac{Q_k(x)}{\mathbf{E}\, Q_{\delta+B}}, \qquad \mathbf{P}[\mathfrak{A}^k] = 1 - \frac{Q_k(x)}{\mathbf{E}\, Q_{\delta+B}},$$

where the resolvent sequence of the process $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $x \geq 0$, $Q_k \overset{\text{def}}{=} Q_k(0)$ *is defined by (21).*

Proof. Proof of Theorem 2 is straightforward from Theorem 1. We clarify the formulae of Theorem 2 by employing equalities (5) of Theorem 1 which take a simple form for the difference of the compound Poisson process and the renewal process whose jumps are geometrically distributed. To use (5), we have to determine the kernels (8), successive iterations (7) and the series of the successive iterations (6) for the process $\{D_x(t)\}_{t\geq 0}$ (2) in case when $\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}$, $n \in \mathbb{N}$. Let us verify the first formula of (29). Employing Lemma's and (6)-(8), for all $i, m, n \in \mathbb{N}$ one can derive

$$\begin{aligned} &K_i^-(m,s) = \varphi_0^{i+B}(m,s), \\ &K_i^-(m,s)^{*(n)} = \varphi_0^{i+B}(m,s)\left(\mathbf{E}\varphi_0^{\delta+B}(s)\right)^{n-1}, \\ &\mathfrak{K}_i^-(m,s) = \varphi_0^{i+B}(m,s)\left(1 - \mathbf{E}\varphi_0^{\delta+B}(s)\right)^{-1}, \\ &F_r^x(m,s) = c_x(s)c(s)^r(1-\lambda)\lambda^{m-1} - \varphi_x^k(m,s), \end{aligned} \tag{31}$$

where $c_x(s) = \tilde{f}_x(s - k(c(s)))$, and the mathematical expectations $\{\varphi_x^k(m,s),\ k \in \mathbb{Z}^+\}$, $x \geq 0$ are given by (26),

$$\varphi_x^k(s) = \mathbf{E}\left[e^{-si_x^k}; i_x^k < \infty\right] = \sum_{m\in\mathbb{N}} \varphi_x^k(m,s).$$

Substituting the expressions (31) into the second equality of (5), we obtain the first formula (29) of Theorem 2.

Let us verify the second formula of (29). Employing the statements of lemma's and (6)-(8) we can calculate for all $s, \nu, x \geq 0,\ i, m, n \in \mathbb{N}$ that

$$\begin{aligned}
&K^{+}_{\nu,i}(dl,m,s) = c_\nu(s)c(s)^{i+B}\mathbf{E}\, f_0^{\delta+B}(dl,m,s),\\
&K^{+}_{\nu,i}(dl,m,s)^{*(n)} = c_\nu(s)c(s)^{i+B}\left(\mathbf{E}\,\varphi_0^{\delta+B}(s)\right)^{n-1}\mathbf{E}\, f_0^{\delta+B}(dl,m,s),\\
&\mathfrak{K}^{+}_{\nu,i}(dl,m,s) = c_\nu(s)c(s)^{i+B}\left(1-\mathbf{E}\,\varphi_0^{\delta+B}(s)\right)^{-1}\mathbf{E}\, f_0^{\delta+B}(dl,m,s),\\
&F^k_x(dl,m,s) = f^k_x(dl,m,s) - c_x(s)c(s)^r\mathbf{E}\, f_0^{\delta+B}(dl,m,s),
\end{aligned} \tag{32}$$

where the mathematical expectations $f^k_x(dl,m,s),\ k \in \mathbb{Z}^+,\ x \geq 0$ are given by (19) of Lemma 3. Substituting the expressions (32) into the first equality of (5) we obtain the second formula of (29) of Theorem 2.

Summing both sides of the first equality of (29) with respect to $m \in \mathbb{N}$, and summing the second equality of (29) over $m \in \mathbb{N}$, and then integrating it with respect to $l \geq 0$, we get the formulae (30). Letting $s \to 0$, in both sides of these equalities, we derive the probabilities of the exit from the interval by the process $\{D_x(t)\}_{t\geq 0}$.

We now state another approach to prove formulae of Theorem 2. The key issue of this proof is the system (9). For the difference of the compound Poisson process and the renewal process in case $\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1}$, this system takes the following form (after taking into account (13))

$$\begin{aligned}
&f^k_x(dl,m,s) = V^k_x(dl,m,s) + \sum_{i\in\mathbb{N}} V^x_r(i,s)\, f_0^{i+B}(dl,m,s), \qquad (33)\\
&c_x(s)c(s)^r(1-\lambda)\lambda^{m-1} = V^x_r(m,s) + \sum_{i\in\mathbb{N}}\int_0^\infty V^k_x(d\nu,i,s)c_\nu(s)c(s)^{i+B}(1-\lambda)\lambda^{m-1}.
\end{aligned}$$

Observe, that it is sufficient to determine the function

$$\tilde{V}^k_x(s) = \sum_{i\in\mathbb{N}}\int_0^\infty V^k_x(d\nu,i,s)\, c_\nu(s)c(s)^{i+B} = \mathbf{E}\left[e^{-si^k_x}; i^k_x < \infty, \mathfrak{A}^k\right],$$

in order to find the functions $V^x_r(m,s),\ V^k_x(dl,m,s)$. Substituting the expressions for $V^x_r(m,s)$ from the second equation of the system (33) into the first one, we find

$$f^k_x(dl,m,s) = V^k_x(dl,m,s) + c_x(s)c(s)^r\mathbf{E}\, f_0^{\delta+B}(dl,m,s) - \tilde{V}^k_x(s)\mathbf{E}\, f_0^{\delta+B}(dl,m,s).$$

After multiplying the first this equation by $c_l(s)c(s)^{m+B}$, summing the both sides of the equation with respect to $m \in \mathbb{N}$, and integrating over $l \geq 0$ we find

$$\varphi^k_x(s) = \tilde{V}^k_x(s) + c_x(s)c(s)^r\mathbf{E}\,\varphi_0^{\delta+B}(s) - \tilde{V}^k_x(s)\mathbf{E}\,\varphi_0^{\delta+B}(s), \quad \varphi^k_x(s) = \sum_{m\in\mathbb{N}}\varphi^k_x(m,s)$$

i.e. a linear equation with respect to $\tilde{V}^k_x(s)$. Taking into account (26), we find

$$\tilde{V}^k_x(s) = c_x(s)c(s)^r - \frac{Q^s_k(x)}{\mathbf{E}\, Q^s_{\delta+B}}.$$

Substituting the right-hand side of this equality into the second equation of (33), we obtain

$$V_r^x(m,s) = \frac{Q_k^s(x)}{\mathbf{E}\,Q_{\delta+B}^s}\,(1-\lambda)\lambda^{m-1}$$

the first formula of (29). Substituting the right-hand side of this equation into the first equation of (33) yields the second formula of (29)

$$V_x^k(dl,m,s) = f_x^k(dl,m,s) - \frac{Q_k^s(x)}{\mathbf{E}\,Q_{\delta+B}^s}\,\mathbf{E}f_0^{\delta+B}(dl,m,s).$$

☐

4 Two-sided exit problem for the difference of the compound Poisson process and the renewal process

Let us stress the following fact. If we set parameter $\lambda = 0$ in the geometrical distribution $\mathbf{P}[\delta = n] = (1-\lambda)\lambda^{n-1},\ n \in \mathbb{N},\ \lambda \in [0,1)$ of the random variable δ, then $\mathbf{P}[\delta = 1] = 1$. With other words it means that the process $\{D_x(t)\}_{t\geq 0}$ has unit negative jumps at the times instants $\{\eta_n(x)\}_{n\in\mathbb{N}}$ and $\delta_{N_x(t)} = N_x(t)$. Then, it follows from (2) that

$$D_x(t) = \pi(t) - N_x(t) \in \mathbb{Z}, \qquad t \geq 0. \tag{34}$$

We will call this process a difference of the compound Poisson process and a simple renewal process. Setting the parameter $\lambda = 0$ in the statements of Lemma 1 leads to the following result.

Lemma 5. *For $s > 0$ the equation $\theta = \tilde{f}(s - k(\theta))$ has a unique solution $c(s)$ inside the circle $|\theta| < 1$. This solution is positive, $c(s) \in (0,1)$. If $\mathbf{E}[\varkappa], \mathbf{E}[\eta] < \infty$, $\rho = \mu\mathbf{E}[\varkappa]\,\mathbf{E}[\eta]$, then for $\rho > 1$, $\lim_{s\to 0} c(s) = c \in (0,1)$; and for $\rho \leq 1$, $\lim_{s\to 0} c(s) = 1$.*

Statements of Lemma's 2-4 can be reformulated in a similar way. Letting $\lambda = 0$ in the defining formula (18) for all $s, x \geq 0$ we get

$$Q_k^s(x) = \frac{1}{2\pi i}\oint_{|\theta|=\alpha} \frac{1}{\theta^{k+1}}\,\frac{\tilde{f}_x(s-k(\theta))}{\tilde{f}(s-k(\theta)) - \theta}\,d\theta, \quad \alpha \in (0, c(s)) \tag{35}$$

a resolvent sequence of the process $\{D_x(t)\}_{t\geq 0}$, which is given by (34). This resolvent sequence has been introduced in [12]. Setting $\lambda = 0$ in (19), (20), we obtain

$$f_x^k(dl,m,s) = e^{-s(l-x)}\,\frac{1-F(l)}{1-F(x)}\mathbf{I}\{l > x\}p_k^m(d(l-x))$$

$$+\,\Phi_0^s(dl,m)\,Q_k^s(x) - e^{-sl}\,[1-F(l)]\sum_{i=0}^{k} Q_i^s(x)\,p_{k-i}^m(dl),$$

$$\mathbf{E}e^{-s\tau^k(x)} = 1 - \frac{s}{s-k(c(s))}\,Q_k^s(x) + \sum_{i=0}^{k}\tilde{\rho}_i(s)\,\left[Q_{k-i}^s(x) - 1\right] \tag{36}$$

the Laplace transforms of the upper one-boundary functionals of the process $\{D_x(t)\}_{t\geq 0}$ (34), where

$$\Phi_0^s(dl,m) = e^{-sl}[1-F(l)]\sum_{k\in\mathbb{Z}^+} c(s)^k p_k^m(dl), \quad \tilde{\rho}_i(s) = s\int_0^\infty e^{-st}\mathbf{P}[\pi(t)=i]\,dt.$$

We have introduced the functions and the resolvent sequence of the process (34), therefore we can state the following result.

Corollary 1. *Let* $\{D_x(t)\}_{t\geq 0}$ *be a difference of the compound Poisson process and a renewal process (34),* $\{Q_k^s(x)\}_{k\in\mathbb{Z}^+}$, $x\geq 0$ *be the resolvent sequence of the process given by (35),* $Q_k^s \overset{\text{def}}{=} Q_k^s(0)$. *Then*

(i) *the Laplace transforms* $V_r^x(m,s)$, $V_x^k(dl,m,s)$ *of the joint distribution of* $\{\chi, L, T\}$ *satisfy the following equalities for all* $x,s\geq 0$, $m\in\mathbb{N}$

$$V_r^x(m,s) = \frac{Q_k^s(x)}{Q_{B+1}^s}\,\delta_{m1}, \quad V_x^k(dl,m,s) = f_x^k(dl,m,s) - \frac{Q_k^s(x)}{Q_{B+1}^s} f_0^{B+1}(dl,m,s),$$

where δ_{ij} *is the Kronecker symbol and the function* $f_x^k(dl,m,s)$ *is given by (36);*

(ii) *for the Laplace transforms of the first exit time* χ *from the interval by the process* $\{D_x(t)\}_{t\geq 0}$ *the formulae hold*

$$\mathbf{E}\left[e^{-s\chi};\mathfrak{A}_r\right] = \frac{Q_k^s(x)}{Q_{B+1}^s}, \qquad \mathbf{E}\left[e^{-s\chi};\mathfrak{A}^k\right] = 1-$$

$$-\frac{Q_k^s(x)}{Q_{B+1}^s} + \sum_{i=0}^{k}\tilde{\rho}_i(s)[Q_{k-i}^s(x)-1] - \frac{Q_k^s(x)}{Q_{B+1}^s}\sum_{i=0}^{B}\tilde{\rho}_i(s)[Q_{B+1-i}^s - 1];$$

(iii) *the exit probabilities from the interval by the process* $\{D_x(t)\}_{t\geq 0}$ *satisfy the equalities*

$$\mathbf{P}[\mathfrak{A}_r] = \frac{Q_k(x)}{Q_{B+1}}, \qquad \mathbf{P}[\mathfrak{A}^k] = 1 - \frac{Q_k(x)}{Q_{B+1}},$$

where the resolvent sequence of the process $\{Q_k(x)\}_{k\in\mathbb{Z}^+}$, $x\geq 0$, $Q_k \overset{\text{def}}{=} Q_k(0)$ *is given by (35) for* $s=0$.

To prove the corollary one has to put $\lambda = 0$ in statements of Theorem 2. The results obtained in Corollary 1 can be applied for studying the queueing systems M|G|1|N with a limited number of waiting places.

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