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On the distribution of duration of stay in an interval of the semi-continuous process with independent increments

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Abstract—For a semicontinuous homogeneous process $\xi(t)$ with independent increments the distribution of the its total duration of stay in an interval is obtained. In the case $\mathbf{E}\,\xi(1) = 0, \quad \mathbf{E}\,\xi(1)^2 < \infty$, the limit theorem on a weak convergence of the time of duration of stay in an interval of the process to distribution of the time of duration of stay of Wiener process in the interval$(0, 1)$ is obtained. For Wiener process the distribution of the total duration of stay in an interval is found.

1. THE JOINT DISTRIBUTION OF THE MOMENT OF ACHIEVEMENT OF LOW BORDER AND TIME OF DURATION OF STAY OF PROCESS IN AN INTERVAL

Let $\xi(t) \in R, \quad t \geq 0$ be homogeneous process with independent increments [1], and let it be semicontinues from below with the cumulant

$$k(p) = \frac{1}{2}\,p^2\sigma^2 - \alpha p + \int_0^\infty \left(e^{-px} - 1 + \frac{px}{1+x^2} \right) \Pi(dx).$$

We shall assume that sample trajectories of process $\xi(t), \quad t \geq 0$ are continuous from the right.

Let us fix a real number $B > 0$ and introduce functions

$$g(x) = \frac{1}{2}\,\{\,\mathrm{sgn}\; x + \mathrm{sgn}\;(B - x)\},$$
$$g^*(x) = 1 - g(x), \qquad x \in R$$

and introduce random variables

$$\sigma_y(t) = \int_0^t g(y + \xi(u))\;du,$$
$$\sigma_y^*(t) = \int_0^t g^*(y + \xi(u))\;du, \qquad y \in R.$$

[1]*Translated by A. I. Vladimirova*

Therefore, $\sigma_y(t)$ is the total time of duration of stay of process $y+\xi(\cdot)$ in the interval $(0,B)$ till the moment t, $\sigma_y^*(t)$ is the total time of duration of stay of process $y+\xi(\cdot)$ till the moment t outside of the interval $(0,B)$.

Let $y>0$ and

$$\tau_{-y} = \inf\{t>0:\ y+\xi(t)=0\}, \qquad \sigma_y = \sigma_y(\tau_{-y}), \qquad \sigma_y^* = \sigma_y^*(\tau_{-y}).$$

We denote in this section

$$\mathbf{E}\exp\{-s\tau_{-y} - a\sigma_y - b\sigma_y^*\} \stackrel{\text{def}}{=} D_y^s(a,b), \qquad a,b,s\geq 0$$

is the integral transformation of joint distribution of random variables $\{\tau_{-y},\ \sigma_y,\ \sigma_y^*\}$, where σ_y is the response time of process $y+\xi(\cdot)$ in the interval $(0,B)$ until the first passage time τ_{-y} of the lower border, σ_y^* is the response time of process $y+\xi(\cdot)$ over the interval $(0,B)$ until the first passage time of the lower border.

According [3],[4] we introduce the resolvent of semicontinues process with independent increments

$$R_x(s) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{xp}\,\frac{1}{k(p)-s}\,dp, \qquad \gamma > c(s), \tag{1}$$

where $c(s)>0$, $s>0$ is a unique root [1] in the half-plane Re $p>0$ of the equation $k(p)-s=0$.

We formulate the following theorem and corollary.

Theorem 1.1. *Let $\xi(t)\in R$, $t\geq 0$ be a semecontinues from below homogeneous process with independent increments.*

Then for the integral transformation of joint distribution of random variables

$$\{\tau_{-y},\ \sigma_y,\ \sigma_y^*\}$$

the following equalities are valid

$$D_y^s(a,b) = \frac{1+(a-b)\int\limits_0^{B-y} e^{-uc(s+b)}R_u(s+a)\,du}{1+(a-b)\int\limits_0^{B} e^{-uc(s+b)}R_u(s+a)\,du}\,\exp\{-y\,c(s+b)\}, \qquad y\in(0,B), \tag{2}$$

$$D_y^s(a,b) = \frac{1}{1+(a-b)\int\limits_0^{B} e^{-uc(s+b)}R_u(s+a)\,du}\,\exp\{-y\,c(s+b)\}, \qquad y\geq B. \tag{3}$$

Corollary 1.2. *Let $\xi(t)$, $t\geq 0$ be symmetric Wiener process with the cumulant $k(p)=\frac{1}{2}\,p^2$ and $y\in(0,B)$, $x=B-y$.*

Then

1) the following equality is valid for the integral transformation of joint distribution of random variables $\{\tau_{-y},\ \sigma_y,\ \sigma_y^*\}$

$$D_y^s(a,b) = \frac{\sqrt{s+a}\,\mathrm{ch}\,(x\sqrt{2(s+a)}) + \sqrt{s+b}\,\mathrm{sh}\,(x\sqrt{2(s+a)})}{\sqrt{s+a}\,\mathrm{ch}\,(B\sqrt{2(s+a)}) + \sqrt{s+b}\,\mathrm{sh}\,(B\sqrt{2(s+a)})}, \tag{4}$$

where $\mathrm{sh}\,x,\ \mathrm{ch}\,x$ *are hyperbolic sine and cosine X.*

2) The Laplace transformation of random variables $\sigma_y,\ \sigma_y^*$ *has the following form*

$$\mathbf{E}\,e^{-a\sigma_y} = \frac{\mathrm{ch}\,x\sqrt{2a}}{\mathrm{ch}\,B\sqrt{2a}},$$

$$\mathbf{E}\,e^{-b\sigma_y^+} = \frac{1+x\sqrt{2b}}{1+B\sqrt{2b}}.$$

3) The following equalities are valid for random variables $\sigma_y,\ \sigma_y^*$

$$\mathbf{P}[\sigma_y > t] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\ \exp\left(-t\frac{(2n+1)^2\pi^2}{8B^2}\right)\ \cos\frac{x\pi}{2B}(2n+1), \tag{5}$$

$$\mathbf{P}[\sigma_y^* = 0] = \frac{x}{B}, \qquad \mathbf{P}[\sigma_y^* > t] = \frac{y}{B}\ e^{t/2B^2}\ \sqrt{\frac{2}{\pi}}\int_{\sqrt{t}/B}^{\infty} e^{-u^2/2}\,du, \tag{6}$$

4)The random variable σ_y^* *has not moments and for the moments of random variable* σ_y *the following formula is valid*

$$\mathbf{E}\,\sigma_y^n = \frac{B^{2n}}{(2n-1)!!}\sum_{k=0}^{n}(-1)^k\ \left(\frac{x}{B}\right)^{2k}\ \binom{2n}{2k}\ \mathrm{E_{n-k}}, \qquad n\in N=\{1,2,...\} \tag{7}$$

which in particular equals

$$\mathbf{E}\,\sigma_B^n = \frac{B^{2n}}{(2n-1)!!}\ \mathrm{E_n}, \qquad n\in N,$$

where $\mathrm{E_n},\quad n\in N$ *are Euler numbers,,* $\quad \mathrm{E_1}=1,\ \mathrm{E_2}=5,\ \dots\ .$

The proof of Theorem 1.1 and Corollary 1.1. Let $\xi(0)=0,\quad x>0$ and

$$\tau_x = \inf\{t>0:\ \xi(t)>x\},\quad T_x = \xi(\tau_x)-x,\qquad \xi^{\pm}(t) = \pm\sup_{u\le t}\ \pm\xi(u).$$

The distribution of these and others one-boundary functionals for the homogeneous process with independent increments are obtained in the sixtieth in the paper of B. A. Rogozin, A. A. Borovkov, E. A. Pechersky, E. C. Statland and others. In particular, the following equalities are valid for the lower semicontinuous process

$$\mathbf{E}\,e^{-s\tau_{-y}} = e^{-yc(s)},$$

$$\mathbf{P}[\xi^-(\nu_s) > -y] = 1-e^{-yc(s)},$$

$$\mathbf{E}\,[e^{-p\xi^+(\nu_s)}] = \frac{p-c(s)}{c(s)}\ \frac{s}{k(p)-s};$$

$$\int_0^\infty e^{-px}\mathbf{E}\left[\exp\{-s\tau_x-\lambda\xi(\tau_x)\}\right]dx=\frac{1}{p}\left(1-\frac{p+\lambda-c(s)}{k(p+\lambda)-s}\;\frac{k(\lambda)-s}{\lambda-c(s)}\right),\qquad \operatorname{Re}p,\lambda\ge 0, \tag{8}$$

where ν_s is the exponentially distributed random varible with parameter $s>0$, independent of process $\xi(t),\ t\ge 0$.

Let $\xi(0)=0,\quad x,y>0\quad B=x+y$, and

$$\chi_x=\inf\{t>0:\ \xi(t)\notin(-y,x)\}$$

are the moment of the first exit of semicontinuous process from the interval $(-y,x)$. The following equalities were obtained in the paper [6] on the base of resolvent methods [5]

$$\begin{aligned}\varphi_{-y}(s)&=\mathbf{E}\left[e^{-s\chi_x};\ \xi(\chi_x)=-y\right]=\frac{R_x(s)}{R_B(s)},\\ \varphi_x(s)&=\mathbf{E}\left[e^{-s\chi_x};\ \xi(\chi_x)\ge x\right]\\ &=1-\frac{R_x(s)}{R_B(s)}-s\frac{R_x(s)}{R_B(s)}\int_0^B R_u(s)\,du+s\int_0^x R_u(s)\,du.\end{aligned}\tag{9}$$

Let us define the random variable $X_x=\xi(\chi_x)-x$ for the random event $\{\xi(\chi_x)\ge x\}$. Along with equalities (9) we need the function

$$\varphi_x(s,\lambda)=\mathbf{E}\,\exp\{-s\chi_x-\lambda X_x\}$$

and the function $\varphi_{-y}(s),\ \ \varphi_x(s)$ in the "factorizing" form. Therefore, we consider the other method of definition of $\chi_x,\ \ \{\chi_x,X_x\}$ which is based on the direct probability calculations.

Lemma 1.3. *Let* $x,y>0,\quad x+y=B$.

Then 1) The following equalities are valid for the integral transformations of random variables $\chi_x,\ \ \{\chi_x,X_x\}$

$$\begin{aligned}\varphi_{-y}(s)&=\mathbf{E}\left[e^{-s\chi_x};\ \xi(\chi_x)=-y\right]\\ &=\frac{1-G_x^s(c(s))}{1-G_B^s(c(s))}\,e^{-yc(s)},\\ \varphi_x(s,\lambda)&=\mathbf{E}\,\exp\{-s\chi_x-\lambda X_x\}\\ &=G_x^s(\lambda)\,e^{\lambda x}-\varphi_{-y}(s)\,G_B^s(\lambda)\,e^{\lambda B},\end{aligned}\tag{10}$$

where

$$G_x^s(\lambda)=\mathbf{E}\,\exp\{-s\tau_x-\lambda\xi(\tau_x)\},\qquad x>0$$

and the integral transformation of this function is defined by equality (8).
2) The following resolvent representation is valid for the function $G_x^{s+a}(c(s+b))$

$$\begin{aligned}G_x^{s+a}(c(s+b))&=1-\frac{a-b}{c(s+a)-c(s+b)}\,R_x(s+a)\,e^{-xc(s+b)}\\ &\quad+(a-b)\int_0^x e^{-uc(s+b)}R_u(s+a)\,du.\end{aligned}\tag{11}$$

According the formula of the total probability we consider the following equalities for the justification of equalities (10)

$$\begin{aligned} &\mathbf{E}\, e^{-s\tau_{-y}} = \varphi_{-y}(s) + \int_0^\infty \mathbf{E}\,[e^{-s\chi_x};\ X_x \in du]\, \mathbf{E}\, e^{-s\tau_{-(u+B)}}, \\ &\mathbf{E}\,[e^{-s\tau_x};\ T_x \in du] = \mathbf{E}\,[e^{-s\chi_x};\ X_x \in du] + \varphi_{-y}(s)\, \mathbf{E}\,[e^{-s\tau_B};\ T_B \in du]. \end{aligned} \tag{12}$$

The first equality means that starting from zero process $\xi(t),\quad t \geq 0$, can visit the lower bound $-y$ either without intersection of upper bound, nor with an intersection of upper bound with the next excitement of lower bound. Similarly the second equation is derived.

Multiplying the second equality from (12) by $e^{-uc(s)}$ and integrating along $u > 0$ we obtain from (12) the system of equations with respect to functions $\varphi_{-y}(s),\ \ \varphi_x(s, c(s))$,

$$\begin{aligned} &e^{-yc(s)} = \varphi_{-y}(s) + \varphi_x(s, c(s))\; e^{-Bc(s)} \\ &G_x^s(c(s))\; e^{xc(s)} = \varphi_x(s, c(s)) + \varphi_{-y}(s)\; G_B^s(c(s))\; e^{Bc(s)} \end{aligned}$$

Solving it we find

$$\varphi_{-y}(s) = \frac{1 - G_x^s(c(s))}{1 - G_B^s(c(s))}\; e^{-yc(s)},$$

$$\varphi_x(s, c(s)) = \frac{G_x^s(c(s)) - G_B^s(c(s))}{1 - G_B^s(c(s))}\; e^{xc(s)}.$$

Multiplying the second equality from (12) by $e^{-\lambda u}$ and integrating it along $u > 0$ we obtain the second equality (10) of the lemma.

The equality (11) follows from the definition of resolvent and the formula (8).

Now we begin to prove the Theorem 1.

We remind that interval $(0, B)$ is fixed and

$$D_y^s(a, b) = \mathbf{E}\,[e^{-s\tau_{-y}} \exp\{-a\sigma_y - b\sigma_y^*\}], \qquad y > 0.$$

According to the formula of the total probability the following system of equations is valid for the functions $D_y^s(a, b)$

$$\begin{aligned} D_y^s(a, b) &= \varphi_{-y}(s + a) \\ &\quad + \int_0^\infty \mathbf{E}\,[e^{-(s+a)\chi_x};\ X_x \in du]\; e^{-uc(s+b)}\; D_B^s(a, b), \qquad y \in (0, B), \\ D_y^s(a, b) &= \exp\{-(y - B)c(s + b)\}\; D_B^s(a, b), \qquad y \geq B. \end{aligned} \tag{13}$$

The first equation means that the total time of the stay of process in the interval $(0, B)$ (over interval $(0, B)$) till the moment of achievement of the lower bound can be either on the trajectories of process which do not intersect the upper bound, or on the trajectories which exit from the interval $(0, B)$ through upper bound. The second equality is evident.

Using the second equality from (10) as $s \to s + a$, and $\quad \lambda = c(s + b)$, the resolvent representations (9) and (11) of functions $\varphi_{-y}(s + a)$ and $G_x^{s+a}(c(s + b))$, we obtain the system of equations (13)

$$\begin{aligned} D_y^s(a,b) &= \frac{R_x(s+a)}{R_B(s+a)} \\ &\quad - D_B^s(a,b)\,\frac{R_x(s+a)}{R_B(s+a)}\,\left(1+(a-b)\int_0^B e^{-uc(s+b)}R_u(s+a)\,du\right)\,e^{Bc(s+b)}, \\ &\quad + D_B^s(a,b)\,\left(1+(a-b)\int_0^x e^{-uc(s+b)}R_u(s+a)\,du\right)\,e^{xc(s+b)},\ y\in(0,B), \\ D_y^s(a,b) &= \exp\{-(y-B)c(s+b)\}\,D_B^s(a,b),\ y\geq B, \qquad x=B-y. \end{aligned} \tag{14}$$

¿From the results of the manuscript [1, \$19, \$20] it follows that function $D_y^s(a,b),\quad y>0$ and its first derivative are continues functions in y. Differentiating the equalities (14) and letting $y=B$, we obtain the following system of equations

$$\begin{aligned} -\frac{d}{dy}D_y^s(a,b)\Big|_{y=B} &= \frac{R_0'(s+a)}{R_B(s+a)} + c(s+b)D_B^s(a,b) \\ &\quad - D_B^s(a,b)\,\frac{R_0'(s+a)}{R_B(s+a)} \\ &\quad \times\left(1+(a-b)\int_0^B e^{-uc(s+b)}R_u(s+a)\,du\right)\,e^{Bc(s+b)}, \\ \frac{d}{dy}D_y^s(a,b)\Big|_{y=B} &= -c(s+b)D_B^s(a,b), \end{aligned}$$

where

$$R_0'(s+a) = \frac{d}{dx}R_x(s+a)\Big|_{x=0}.$$

Solving this system we find

$$D_B^s(a,b) = \frac{1}{1+(a-b)\int_0^B e^{-uc(s+b)}R_u(s+a)\,du}\ \exp\{-Bc(s+b)\}.$$

Substituting this expression into equalities (14) we obtain (2), (3). Thus, Theorem 1.1 is proved.

Now we obtain the formulas given in Corollary 1.1. Using the main equality (1), let us calculate the resolvent of Wiener process. By $\gamma > c(s) = \sqrt{2s}$ we have

$$R_x(s) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{xp}\ \frac{1}{\frac{1}{2}\,p^2-s}\ dp = \sqrt{\frac{2}{s}}\ \mathrm{sh}\ \left(x\sqrt{2s}\right). \tag{15}$$

Substituting this expression for the resolvent into equality (2) and making a necessary calculations we obtain formula (4) of Corollary 1.1.

Further, letting step by step in (4) $s=0,\quad b=0$ and $s=0,\quad a=0$ we obtain the

Laplace transformations of random variables σ_y, σ_y^*

$$\mathbf{E}\, e^{-a\sigma_y} = \frac{\operatorname{ch} x\sqrt{2a}}{\operatorname{ch} B\sqrt{2a}},$$
$$\mathbf{E}\, e^{-b\sigma_y^*} = \frac{1 + x\sqrt{2b}}{1 + B\sqrt{2b}}, \quad x = B - y.$$

Calculating the boundary integrals [7]

$$\frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{at}\, \frac{1}{a} \left(1 - \frac{\operatorname{ch} x\sqrt{2a}}{\operatorname{ch} B\sqrt{2a}}\right)\, da,$$

$$\frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{bt}\, \frac{1}{b} \left(1 - \frac{1 + x\sqrt{2b}}{1 + B\sqrt{2b}}\right)\, db, \ (\gamma > 0)$$

we obtain the distributions of random variables σ_y, σ_y^*

$$\mathbf{P}[\,\sigma_y > t\,] = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, \exp\left(-t\frac{(2n+1)^2\pi^2}{8B^2}\right)\, \cos\frac{x\pi}{2B}(2n+1), \tag{16}$$

$$\mathbf{P}[\,\sigma_y^* = 0\,] = \frac{x}{B}, \quad \mathbf{P}[\,\sigma_y^* > t\,] = \frac{y}{B}\, e^{t/2B^2}\, \sqrt{\frac{2}{\pi}} \int_{\sqrt{t}/B}^{\infty} e^{-u^2/2}\, du. \tag{17}$$

The formula (16) is also the asymptotic expansion for the probability from the left part. For instance, choosing the first summand in the right part of (16), we write under $t \to \infty$

$$\mathbf{P}[\,\sigma_B > t\,] = \frac{4}{\pi}\, \exp\left(-t\frac{\pi^2}{8B^2}\right) + o\left(e^{-t\frac{\pi^2}{8B^2}}\right).$$

Using asymptotic expansion of the probability integral [8], we obtain from formula (17) under $t \to \infty$

$$\mathbf{P}[\,\sigma_y^* > t\,] = \frac{y}{\sqrt{2\pi t}} \left(1 + \sum_{k=1}^{n} (-1)^k\, \frac{B^{2k}}{t^k}\, (2k-1)!!\right) + o\left(\frac{1}{t^{n+\frac{1}{2}}}\right), \quad y \in (0, B).$$

To obtain formula (7) for the calculation of the moments of random variable σ_y with Laplace transformation

$$\mathbf{E}\, e^{-a\sigma_y} = \frac{\operatorname{ch} x\sqrt{2a}}{\operatorname{ch} B\sqrt{2a}}, \quad x = B - y$$

it is necessary to use the expansion in the row [8] of the functions $\operatorname{ch}(\cdot)$ $\operatorname{sech}(\cdot) = (\operatorname{ch}(\cdot))^{-1}$.

The theorem and the corollary are proved. □

2. ON THE JOINT DISTRIBUTION OF THE TIME OF STAY OF PROCESS IN AN INTERVAL AND OVER AN INTERVAL TILL FIRST PASSAGE TIME OF A LOWER BOUND

In this section we defined

$$Q_y^s(a,b) = \mathbf{E}\,[\exp\{-a\sigma_y(\nu_s) - b\sigma_y^*(\nu_s)\};\ \tau_{-y} > \nu_s], \qquad y > 0$$

is the integral transformation of joint distribution of random variables $\sigma_y(t),\ \ \sigma_y^*(t)$ till first passage time $\tau_{-y} = \inf\{t > 0:\ \xi(t) = 0\}$ of a lower bound.

The following theorem is valid

Theorem 2.1. *Let $\xi(t) \in R,\quad t \geq 0$ be the lower semicontinues homogeneous process with independent increments and*

$$\sigma_y(t) = \int_0^t g(y + \xi(u))\,du, \quad \sigma_y^*(t) = \int_0^t g^*(y + \xi(u))\,du, \quad y \in R.$$

Then

$$\begin{aligned} Q_y^s(a,b) = {} & \frac{s}{s+b}\left(1 + (a-b)\int_0^{B-y} R_u(s+a)\,du\right) \\ & - \frac{s}{s+b}\left(1 + (a-b)\int_0^{B} R_u(s+a)\,du\right)\, D_y^s(a,b), \quad y \in (0,B), \end{aligned} \tag{18}$$

$$Q_y^s(a,b) = \frac{s}{s+b} - \frac{s}{s+b}\left(1 + (a-b)\int_0^{B} R_u(s+a)\,du\right)\, D_y^s(a,b), \quad y \geq B, \tag{19}$$

where the functions $D_y^s(a,b),\quad y > 0$ are defined by equalities (2), (3) *of Theorem 1.1.*

Proof. According to the formula of the total probability the following system of equations for the functions $Q_y^s(a,b),\quad y > 0$ is valid

$$\begin{aligned} Q_y^s(a,b) = {} & \frac{s}{s+a}\,(1 - \mathbf{E}\,e^{-(s+a)\chi_x}) \\ & + \frac{s}{s+b}\int_0^\infty \mathbf{E}\,[e^{-(s+a)\chi_x};\ X_x \in du]\,(1 - e^{-uc(s+b)}) \\ & + \int_0^\infty \mathbf{E}\,[e^{-(s+a)\chi_x};\ X_x \in du]\,e^{-uc(s+b)}\,Q_B^s(a,b), \quad y \in (0,B),\ x = B - y, \\ Q_y^s(a,b) = {} & \frac{s}{s+b}\left(1 - e^{-(y-B)c(s+b)}\right) + Q_B^s(a,b)\,e^{-(y-B)c(s+b)},\ y \geq B. \end{aligned} \tag{20}$$

The three summands staying in the right part of the first equation mean that the total time of staying the process in the interval $(0,B)$ (over the interval $(0,B)$) without intersection of the lower bound can be realized on the trajectories of process which either do not leave the interval $(0,B)$, or leave the interval through upper bound and then return. The second equation is evident. To avoid complication presentation we introduce notations

$$u_x = 1 + (a-b)\int_0^x R_u(s+a)du,$$
$$U_x = \left(1 + (a-b)\int_0^x e^{-uc(s+b)}\ R_u(s+a)du\right) e^{xc(s+b)}.$$

Using the second equation (10) for the function $\mathbf{E}\ \exp\{-s\chi_x - \lambda X_x\}$ as $s \to s+a$, and $\lambda = c(s+b)$, resolvent representations (9) and (11) of functions $\varphi_{-y}(s)$ and $G_x^{s+a}(c(s+b))$, we obtain the following system from the equations (18)

$$\begin{aligned} Q_y^s(a,b) &= \frac{s}{s+b}\left(u_x - U_x - (u_B - U_B)\frac{R_x(s+a)}{R_B(s+a)}\right) \\ &\quad + Q_B^s(a,b)\left(U_x - \frac{R_x(s+a)}{R_B(s+a)}\ U_B\right), \quad y \in (0,B), \\ Q_y^s(a,b) &= \frac{s}{s+b}\left(1 - e^{-(y-B)c(s+b)}\right) + Q_B^s(a,b)\ e^{-(y-B)c(s+b)}, \quad y \geq B. \end{aligned} \tag{21}$$

It is follows from the results of the manuscript [1, \$19, \$28] that functions $Q_y^s(a,b)$, $y > 0$ are continuous in y together with its first derivative. Differentiating equalities (21) and then letting $y = B$, we obtain the following system

$$\begin{aligned} \frac{d}{dy}Q_y^s(a,b)\Big|_{y=B} &= \frac{s}{s+b}\left(c(s+b) + (u_B - U_B)\frac{R_0'(s+a)}{R_B(s+a)}\right) \\ &\quad - Q_B^s(a,b)\left(c(s+b) - U_B\ \frac{R_0'(s+a)}{R_B(s+a)}\right), \\ \frac{d}{dy}Q_y^s(a,b)\Big|_{y=B} &= \frac{s}{s+b}\ c(s+b) - Q_B^s(a,b)\ c(s+b). \end{aligned}$$

Solving this system of equation we find

$$\begin{aligned} Q_B^s(a,b) &= \frac{s}{s+b}\ \left(1 - \frac{u_B}{U_B}\right) \\ &= \frac{s}{s+b}\left(1 - \frac{1 + (a-b)\int_0^B R_u(s+a)du}{1 + (a-b)\int_0^B e^{-uc(s+b)}\ R_u(s+a)du}\ e^{-Bc(s+b)}\right). \end{aligned}$$

Substituting this expression for function $Q_B^s(a,b)$ in the equality (21) we obtain

$$\begin{aligned} Q_y^s(a,b) &= -\frac{s}{s+b}\ \left(1 + (a-b)\int_0^B R_u(s+a)\ du\right) \\ &\quad \times \frac{1 + (a-b)\int_0^x e^{-uc(s+b)}\ R_u(s+a)\ du}{1 + (a-b)\int_0^B e^{-uc(s+b)}\ R_u(s+a)\ du}\ e^{-yc(s+b)} \\ &\quad + \frac{s}{s+b}\ \left(1 + (a-b)\int_0^x R_u(s+a)\ du\right), \quad y \in (0,B), \\ Q_y^s(a,b) &= \frac{s}{s+b}\left(1 - \frac{1 + (a-b)\int_0^B R_u(s+a)\ du}{1 + (a-b)\int_0^B e^{-uc(s+b)}\ R_u(s+a)\ du}\ e^{-yc(s+b)}\right), \quad y \geq B. \end{aligned}$$

Using the expressions for functions $D_y^s(a,b)$ from theorem 1.1, we obtain formulas (18), (19) from these equations. Thus, Theorem 2.1 is proved. □

3. ON THE OCCUPATION TIME OF SEMICONTINUES PROCESS WITH INDEPENDENT INCREMENTS IN AN INTERVAL

Now we can defined the integral transformation of the joint distribution of random variables $\{\sigma_y(t),\ \sigma_y^*(t)\},\quad y\in R$.

Denote

$$\begin{aligned} v(y) &= 1+(a-b)\int_0^y R_u(s+a)\,du,\\ V(y) &= 1+(a-b)\int_0^y e^{-uc(s+b)}\,R_u(s+a)\,du,\quad y>0. \end{aligned} \tag{22}$$

$$C_y^s(a,b)=s\int_0^\infty e^{-st}\,\mathbf{E}\exp\{-a\sigma_y(t)-b\sigma_y^*(t)\}\,dt,\qquad y\in R$$

and formulate the following theorem.

Theorem 3.1. *Let $\xi(t),\quad t\ge 0$ be the lower semicontinues process with independent increments and $\sigma_y(t),\ \sigma_y^*(t)$ are the total times of the stay o process $y+\xi(\cdot),\quad y\in R$ in the interval and outside the interval $(0,B)$ till moment of the time t.*

Then the following equalities are valid for the integral transformation of the joint distribution of random variables $\{\sigma_y(t),\ \sigma_y^*(t)\}$

$$C_y^s(a,b)=\frac{s}{s+b}\left(1-C(B)\,e^{-yc(s+b)}\right),\qquad y>B,$$

$$C_y^s(a,b)=\frac{s}{s+b}\left(v(B-y)-C(B)\,V(B-y)\,e^{-yc(s+b)}\right),\qquad y\in(0,B),$$

$$\begin{aligned} C_{-y}^s(a,b) &= \frac{s}{s+b}\left(1+(a-b)\int_0^B \mathbf{E}\,[e^{-(s+b)\tau_y};\ T_y\in du]\int_0^{B-u} R_v(s+a)\,dv\right)\\ &-\frac{s}{s+b}\,C(B)\,\mathbf{E}\exp\{-(s+b)\tau_y-c(s+b)T_y\}\\ &-\frac{s(a-b)}{s+b}\,C(B)\int_0^B e^{-yc(s+b)}\,\mathbf{E}\,[e^{-(s+b)\tau_y};\ T_y\in du]\\ &\times\int_0^{B-u} e^{-vc(s+b)}\,R_v(s+a)\,dv,\quad y>0, \end{aligned}$$

where

$$C(B)=\frac{a-b}{c(s+b)}\,\frac{V(B)\,e^{Bc(s+b)}-v(B)}{k'(c(s+b))+(a-b)\int_0^B V(x)\,dx},$$

$$\tau_y=\inf\{t>0:\ \xi(t)>y\},$$

$$T_y=\xi(\tau_y)-y,\quad y>0.$$

Proof. According to the formula of the total probability the following equations are valid for the functions $C_y^s(a,b),\quad y\in R$,

$$
\begin{aligned}
C_y^s(a,b) &= \frac{s}{s+b}\left(1-e^{-(y-B)c(s+b)}\right) + C_B^s(a,b)\, e^{-(y-B)c(s+b)}, \quad y>B,\\
C_y^s(a,b) &= Q_y^s(a,b) + D_y^s(a,b)\, C_0^s(a,b), \quad y\in(0,B),\\
C_{-y}^s(a,b) &= \frac{s}{s+b}\left(1-\mathbf{E}\, e^{-(s+b)\tau_y}\right)\\
&\quad + \int_0^\infty \mathbf{E}\,[e^{-(s+b)\tau_y};\ T_y\in du]\, C_u^s(a,b), \quad y>0,
\end{aligned}
\tag{23}
$$

where the functions $D_y^s(a,b),\ \ Q_y^s(a,b)$ are defined in the previous sections by equalities (2),(16).

Therefore, since the function $C_0^s(a,b)$ is defined by equality

$$
C_B^s(a,b) = Q_B^s(a,b) + D_B^s(a,b)\, C_0^s(a,b).
$$

we need to defined the function $C_B^s(a,b)$ for the solutions of these equations.

Denote

$$
C(B) = \left(\frac{s+b}{s}\, C_B^s(a,b) - 1\right) e^{Bc(s+b)}.
$$

Using equalities (2),(18) and functions $v(x),\quad V(x),\quad x>0$ (20), we find from the first two equations (23)

$$
\begin{aligned}
C_y^s(a,b) &= \frac{s}{s+b}\left(1 - C(B)\, e^{-yc(s+b)}\right), \qquad y>B,\\
C_y^s(a,b) &= \frac{s}{s+b}\left(v(B-y) - C(B)\, V(B-y)\, e^{-yc(s+b)}\right), \qquad y\in(0,B).
\end{aligned}
\tag{24}
$$

Substituting these expressions for function $C_y^s(a,b)$ in the last equation (21), we obtain

$$
\begin{aligned}
C_{-y}^s(a,b) &= \frac{s}{s+b}\left(1 + (a-b)\int_0^B M_e^{s+b}(du)\int_0^{B-u} R_v(s+a)\, dv\right)\\
&\quad - \frac{s}{s+b}\, C(B)(a-b)\int_0^B e^{-uc(s+b)} M_y^{s+b}(du)\int_0^{B-u} e^{-vc(s+b)} R_v(s+a)\, dv\\
&\quad - \frac{s}{s+b}\, C(B)\mathbf{E}\,[e^{-(s+b)\tau_y - c(s+b)T_y}], \quad y>0,
\end{aligned}
\tag{25}
$$

where

$$
M_y^{s+b}(du) = \mathbf{E}\,[\exp\{-(s+b)\tau_y\};\ T_y\in du].
$$

It is follows from the results of the manuscript [1, §19,§28] that function $C_y^s(a,b),\quad y\in R$ is continuous in y together with its first derivative and therefore, the following equality is valid

$$
\left.\frac{d}{dy} C_y^s(a,b)\right|_{y=0} = -\left.\frac{d}{dy} C_{-y}^s(a,b)\right|_{y=0}.
$$

Differentiating of equalities (23),(24), and calculating the derivatives in the point $y = 0$ we obtain the equality with respect to the function $C(B)$

$$
\begin{aligned}
&-(a-b)R_B(s+a) + \left(c(s+b)V(B) + (a-b)R_B(s+a)e^{-Bc(s+b)}\right)C(B) \\
&= -(a-b)R_B(s+a) + \frac{a-b}{c(s+b)}\, R_0'(s+a)\left(V(B)\, e^{Bc(s+b)} - v(B)\right) \\
&+ C(B)\left(c(s+b)V(B) + (a-b)R_B(s+a)e^{-Bc(s+b)}\right) \\
&- C(B)\left(k'(c(s+b)) + (a-b)\int_0^B V(x)\, dx\right) R_0'(s+a)
\end{aligned}
\tag{26}
$$

where

$$
R_0'(s+a) = \frac{d}{dx}\, R_x(s+a)\bigg|_{x=0},
$$
$$
k\prime(c(s+b)) = \frac{d}{dp}\, k(p)\bigg|_{p=c(s+b)}.
$$

We use equalities (1),(8),(11) for the calculations of derivatives in the right part of (24).

Solving equation (26), we find the constant $C(B)$:

$$
C(B) = \frac{a-b}{c(s+b)}\, \frac{V(B)\, e^{Bc(s+b)} - v(B)}{k'(c(s+b)) + (a-b)\int_0^B V(x)\, dx}. \tag{27}
$$

Therefore, the function $C_y^s(a,b), \quad y \in R$ is defined by equalities (23),(24) and theorem 3.1 is proved. □

Remark. If $\xi(t), \quad t \geq 0$ is the Poisson process with the positive jumps and negative flow with the cumulant

$$
k(p) = \alpha p + \lambda\left(\mathbf{E}\, e^{-p\eta} - 1\right), \qquad \alpha,\ \lambda > 0, \quad \eta \in (0,\infty),
$$

then for the solution of the system of equations (21), (23), (24), (22) we do not need to differentiate the corresponding equations. For the definition of $D_B^s(a,b),\ Q_B^s(a,b),\ C(B)$, it is sufficient to use the continuity of the functions $D_y^s(a,b),\ Q_y^s(a,b),\ C_y^s(a,b)$ and passage to the limit by $y \to B$ in the equations (14), (21), and to the limit $y \to 0$ in the equation (24).

Naturally, that such obtained solutions will be coincide with analytic expressions for functions $D_y^s(a,b),\ Q_y^s(a,b),\ C_y^s(a,b)$ in the theorems 1-3 for the lower semicontinuous homogeneous process with independent increasers under stipulation that functions $R_x(s),\ c(s)$ correspond to the Poisson process with the negative flow.

4. ON THE DISTRIBUTION OF THE TIME OF STAY OF WIENER PROCESS IN AN INTERVAL

Here we apply the results obtained in the previous sections for the finding of distribution of the total time of stay of symmetric Wiener process $w(t), \quad t \geq 0$ in the fixed interval.

Corollary 4.1. *Let $w(t)$, $t \geq 0$ be symmetric Wiener process with the cumulant $k(p) = \frac{1}{2}p^2$.*
Then
1) The following equality is valid for the integral transformation

$$C^s_y(a,b) = s\int_0^\infty e^{-st}\mathbf{E}\,\exp\{-a\sigma_y(t) - b\sigma^*_y(t)\}\,dt, \qquad y \in R$$

of joint distribution of random variables
$\{\sigma_y(t),\ \sigma^*_y(t)\}$

$$C^s_y(a,b) = \frac{s}{s+b}\left(1 - \frac{a-b}{\sqrt{s+a}}\,\frac{\operatorname{sh}(\frac{B}{\sqrt{2}}\sqrt{s+a})\,\exp\{-(y-B)\sqrt{2(s+b)}\}}{\sqrt{s+a}\,\operatorname{sh}(\frac{B}{\sqrt{2}}\sqrt{s+a}) + \sqrt{s+b}\,\operatorname{ch}(\frac{B}{\sqrt{2}}\sqrt{s+a})}\right), \quad y > B, \tag{28}$$

$$C^s_y(a,b) = \frac{s}{s+a}\left(1 + \frac{a-b}{\sqrt{s+b}}\,\frac{\operatorname{ch}(\frac{B-2y}{\sqrt{2}}\sqrt{s+a})}{\sqrt{s+a}\,\operatorname{sh}(\frac{B}{\sqrt{2}}\sqrt{s+a}) + \sqrt{s+b}\,\operatorname{ch}(\frac{B}{\sqrt{2}}\sqrt{s+a})}\right), \quad y \in (0,B), \tag{29}$$

$$C^s_y(a,b) = \frac{s}{s+b}\left(1 - \frac{a-b}{\sqrt{s+a}}\,\frac{\operatorname{sh}(\frac{B}{\sqrt{2}}\sqrt{s+a})\,\exp\{y)\sqrt{2(s+b)}\}}{\sqrt{s+a}\,\operatorname{sh}(\frac{B}{\sqrt{2}}\sqrt{s+a}) + \sqrt{s+b}\,\operatorname{ch}(\frac{B}{\sqrt{2}}\sqrt{s+a})}\right), \quad y < 0. \tag{30}$$

2) The following formulas are valid for the mean values of random variables $\sigma_y(t)$, $\sigma^*_y(t)$

$$\begin{aligned}
\mathbf{E}\,\sigma^*_y(t) &= \int_0^t \mathbf{P}[w(u) > y]\,du + \int_0^t \mathbf{P}[w(u) > B-y]\,du,\\
\mathbf{E}\,\sigma_y(t) &= t - \mathbf{E}\,\sigma^*_y(t), \quad y \in (0,B),\\
\mathbf{E}\,\sigma_y(t) &= \int_0^t \mathbf{P}[w(u) > y-B]\,du - \int_0^t \mathbf{P}[w(u) > y]\,du,\\
\mathbf{E}\,\sigma^*_y(t) &= t - \mathbf{E}\,\sigma_y(t), \quad y \geq B,
\end{aligned} \tag{31}$$

where

$$\int_0^t \mathbf{P}[w(u) > x]\,du = \frac{t+x^2}{\sqrt{2\pi t}}\int_x^\infty e^{-\frac{u^2}{2t}}\,du - \frac{x\sqrt{t}}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2t}}.$$

3) The asymptotic formulas are valid for the mean values of random variables $\sigma_y(t)$, $\sigma^*_y(t)$ as $t \to \infty$ and ($y \in (0,B)$, $x = B - y$)

$$\begin{aligned}
\mathbf{E}\,\sigma_y(t) &= \sqrt{\frac{2}{\pi}}\,B\sqrt{t} - \frac{1}{2}(x^2+y^2) + \frac{1}{3}(x^3+y^3)\frac{1}{\sqrt{2\pi t}} + o\left(\frac{1}{\sqrt{t}}\right),\\
\mathbf{E}\,\sigma^*_y(t) &= t - \sqrt{\frac{2}{\pi}}\,B\sqrt{t} + \frac{1}{2}(x^2+y^2) - \frac{1}{3}(x^3+y^3)\frac{1}{\sqrt{2\pi t}} + o\left(\frac{1}{\sqrt{t}}\right).
\end{aligned} \tag{32}$$

Converting the Laplace transform in the equalities (28) – (30) it can be fined the distribution of random variables $\sigma_y(t)$, $\sigma_y^*(t)$. In the next theorem we give these distributions in the case when $y \in (0, B)$.

Theorem 4.2. *Let* $w(t)$, $t \geq 0$ *be the symmetric Wiener process,*

$$g(x) = \frac{1}{2}\{\,\mathrm{sgn}\; x + \mathrm{sgn}\;(B - x)\}, \qquad g^*(x) = 1 - g(x), \quad x \in R$$

are indicator functions of the interval $(0, B)$,

$$\sigma_y(t) = \int_0^t g(\,y + w(u))\,du, \qquad \sigma_y^*(t) = \int_0^t g^*(\,y + w(u))\,du, \quad y \in (0, B)$$

the total times of stay of process $y + w(\cdot)$ *in the interval and outside of the interval* $(0, B)$ *till the moment of time* t.

Then the following equalities are valid for the distribution of random variables $\sigma_y(t)$, $\sigma_y^*(t)$

$$\begin{aligned}\mathbf{P}[\,\sigma_y(t) < u\,] = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^u \exp\left(-\frac{(y+nB)^2}{2v}\right)\,\mathbf{K}_n\left(\frac{t-u}{t-v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}} \\ + \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^u \exp\left(-\frac{(x+nB)^2}{2v}\right)\,\mathbf{K}_n\left(\frac{t-u}{t-v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}}, \\ u \in (0, t),\end{aligned}$$

$$\begin{aligned}\mathbf{P}[\,\sigma_y^*(t) < u\,] = 1 - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^{t-u} \exp\left(-\frac{(y+nB)^2}{2v}\right)\,\mathbf{K}_n\left(\frac{u}{t-v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}} \\ - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^{t-u} \exp\left(-\frac{(x+nB)^2}{2v}\right)\,\mathbf{K}_n\left(\frac{u}{t-v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}}, \\ u \in (0, t),\end{aligned}$$

where $x = B - y$,

$$\mathbf{K}_n(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}^2 x^k\,(1-x)^{n-k}, \qquad x \in [0, 1].$$

In particular,

$$\begin{aligned}\mathbf{P}[\,\sigma_y^*(t) = 0\,] = \mathbf{P}[\,\sigma_y(t) = t\,] \\ = 1 - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^t \exp\left(-\frac{(y+nB)^2}{2v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}} \\ - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^t \exp\left(-\frac{(x+nB)^2}{2v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}}.\end{aligned}$$

The proof of Theorem 4.1 and Corollary 4.1. Let us obtain the equalities of corollary. Using the resolvent (15) for the symmetric Wiener process we find from equality (27)

$$C(B) = \frac{a-b}{\sqrt{s+a}}\;\frac{\operatorname{sh}\left(\frac{B}{\sqrt{2}}\sqrt{s+a}\right)\,\exp\{B\sqrt{2(s+b)}\}}{\sqrt{s+a}\,\operatorname{sh}\left(\frac{B}{\sqrt{2}}\sqrt{s+a}\right)+\sqrt{s+b}\,\operatorname{ch}\left(\frac{B}{\sqrt{2}}\sqrt{s+a}\right)}.$$

Substituting this expression into equalities (24), we find the formulas (28), (29) of the corollary 4.1. The formula (30) follows from (26) due to the symmetry of Wiener process.

After calculation of the derivatives

$$-\frac{d}{da}\,C_y^s(a,0)\Big|_{a=0}, \qquad -\frac{d}{db}\,C_y^s(0,b)\Big|_{b=0}, \qquad y\in(0,B)$$

we obtain from formula (29), where ($y\in(0,B)$, $x=B-y$)

$$\int_0^\infty e^{-st}\mathbf{E}\,\sigma_y(t)\,dt = \frac{1}{s^2}-\frac{1}{2s^2}e^{-y\sqrt{2s}}-\frac{1}{2s^2}e^{-x\sqrt{2s}},$$
$$\int_0^\infty e^{-st}\mathbf{E}\,\sigma_y^*(t)\,dt = \frac{1}{2s^2}\left(e^{-y\sqrt{2s}}+e^{-x\sqrt{2s}}\right).$$

Converting these Laplace transformations we obtain the first equality (31). Similarly we establish the validity of the second row of equations from (31).

Using the row expansion of the probability integral [8]

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du = \frac{2x}{\sqrt{\pi}}\left(1-\frac{x^2}{1!\,3}+\frac{x^4}{2!\,5}-\frac{x^6}{3!\,7}+\dots\right), \qquad x^2<\infty,$$

we obtain the asymptotic formulas (32) from (31). Corollary 4.1 is proved.

Let us obtain the equalities of Theorem 4.1. Letting in (29) $a=0$, we find

$$\int_0^\infty e^{-st}\mathbf{E}\,\exp\{-b\sigma_y^*(t)\}\,dt = \frac{1}{s}\left(1-\frac{b}{\sqrt{s+b}}\;\frac{\operatorname{ch}\left(\frac{B-2y}{\sqrt{2}}\sqrt{s}\right)}{\sqrt{s}\,\operatorname{sh}\left(\frac{B}{\sqrt{2}}\sqrt{s}\right)+\sqrt{s+b}\,\operatorname{ch}\left(\frac{B}{\sqrt{2}}\sqrt{s}\right)}\right). \tag{33}$$

Since

$$\sqrt{s}\,\operatorname{sh}\left(\frac{B}{\sqrt{2}}\sqrt{s}\right)+\sqrt{s+b}\,\operatorname{ch}\left(\frac{B}{\sqrt{2}}\sqrt{s}\right) = \sqrt{b}\,\operatorname{ch}\left(\frac{B}{\sqrt{2}}\sqrt{s}+\ln\left(\sqrt{1+\frac{s}{b}}+\sqrt{\frac{s}{b}}\right)\right)$$

and

$$(\operatorname{ch}x)^{-1} = 2\sum_{n=0}^\infty(-1)^n\exp\{-(2n+1)x\},$$

then we obtain from (33)

$$
\begin{aligned}
\int_0^\infty e^{-st}\mathbf{E}\exp\{-b\sigma_y^*(t)\}\,dt = {} & \frac{1}{s} - \frac{1}{s}\,\frac{b}{\sqrt{s+b}} \\
& \times \sum_{n=0}^{\infty}(-1)^n \frac{b^n}{(\sqrt{s+b}+\sqrt{s})^{2n+1}}\ \exp(-(y+nB)\sqrt{2s}) \\
& - \frac{1}{s}\,\frac{b}{\sqrt{s+b}} \\
& \times \sum_{n=0}^{\infty}(-1)^n \frac{b^n}{(\sqrt{s+b}+\sqrt{s})^{2n+1}}\ \exp(-(x+nB)\sqrt{2s}).
\end{aligned}
\tag{34}
$$

According [7] the following relations are valid between original functions and its Laplace transformation

$$
\begin{aligned}
\frac{1}{\sqrt{\pi t}}\ \exp\left(-\frac{\lambda^2}{4t}\right) &\stackrel{\mathbf{L}}{\longleftrightarrow} \frac{1}{\sqrt{s}}\ \exp(-\lambda\sqrt{s}), \\
\exp\left(-\frac{bt}{2}\right)\ J_{n+\frac{1}{2}}\left(\frac{bt}{2}\right) &\stackrel{\mathbf{L}}{\longleftrightarrow} \frac{1}{\sqrt{s(s+b)}}\ \frac{b^{n+\frac{1}{2}}}{(\sqrt{s+b}+\sqrt{s})^{2n+1}},
\end{aligned}
$$

where $J_{n+\frac{1}{2}}(x)$ is the Bessel function with the semiinteger index.

Using these relations we obtain from (34)

$$
\begin{aligned}
\frac{1}{b}\,\mathbf{E}\exp\{-b\sigma_y^*(t)\} = {} & \frac{1}{b} \\
& - \frac{1}{\sqrt{b}}\sum_{n=0}^{\infty}(-1)^n\left(\exp\left(-\frac{bt}{2}\right)\ J_{n+\frac{1}{2}}\left(\frac{bt}{2}\right)\right) \\
& \quad * \left(\frac{1}{\sqrt{\pi t}}\ \exp\left(-\frac{(y+nB)^2}{2t}\right)\right) \\
& - \frac{1}{\sqrt{b}}\sum_{n=0}^{\infty}(-1)^n\left(\exp\left(-\frac{bt}{2}\right)\ J_{n+\frac{1}{2}}\left(\frac{bt}{2}\right)\right) \\
& \quad * \left(\frac{1}{\sqrt{\pi t}}\ \exp\left(-\frac{(x+nB)^2}{2t}\right)\right),
\end{aligned}
\tag{35}
$$

where the symbol $*$ denotes the operation of convolution of corresponding functions.

According [7] the following relation is valid

$$
\frac{1}{b^{n+\frac{1}{2}}}\ \exp\left(-\frac{bt}{2}\right)\ J_{n+\frac{1}{2}}\left(\frac{bt}{2}\right) \stackrel{\mathbf{L}}{\longleftrightarrow} \begin{cases} 0, & u > t, \\ \dfrac{1}{\sqrt{\pi t}}\,\dfrac{(tu-u^2)^n}{n!\,t^n}, & 0 < u < t. \end{cases}
$$

It is easy to establish for functions

$$
k_n(u,t) = \frac{(tu-u^2)^n}{n!\,t^n}, \qquad u \in (0,t)
$$

the following properties

$$\frac{d^m}{du^m}\, k_n(u,t)\Big|_{u=0} = 0, \quad m = 0, 1, ..., n-1;$$
$$\frac{d^n}{du^n}\, k_n(u,t) = K_n\left(\frac{u}{t}\right), \quad u \in [0,t],$$

where

$$K_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^k\,(1-x)^{n-k},$$
$$x \in [0,1], \quad K_n(0) = 1, \quad K_n(1) = (-1)^n.$$

Using the theorem of the differentiation of the original function and properties of function $k_n(u,t)$ we find new relation

$$\frac{1}{\sqrt{b}}\, \exp\left(-\frac{bt}{2}\right)\, J_{n+\frac{1}{2}}\left(\frac{bt}{2}\right) \overset{\mathbf{L}}{\longleftrightarrow} \begin{cases} 0, & u > t, \\ \frac{1}{\sqrt{\pi t}}\, K_n\left(\frac{u}{t}\right), & u \in [0,t]. \end{cases}$$

Passing in equality (35) to the original functions and accomplishing the operation of the convolution we obtain the distribution of random variable $\sigma_y^*(t)$

$$\begin{aligned} \mathbf{P}[\sigma_y^*(t) < u] = 1 & \\ & - \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^{t-u} \exp\left(-\frac{(y+nB)^2}{2v}\right) K_n\left(\frac{u}{t-v}\right) \frac{dv}{\sqrt{v(t-v)}} \\ & - \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^{t-u} \exp\left(-\frac{(x+nB)^2}{2v}\right) K_n\left(\frac{u}{t-v}\right) \frac{dv}{\sqrt{v(t-v)}}, \\ & u \in (0,t). \end{aligned} \tag{36}$$

Since

$$\mathbf{P}[\,\sigma_y(t) < u\,] = \mathbf{P}[\,\sigma_y^*(t) > t-u\,] = 1 - \mathbf{P}[\,\sigma_y^*(t) < t-u\,]$$

we obtain from (36)

$$\begin{aligned} \mathbf{P}[\,\sigma_y(t) < u\,] = & \\ & = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^{u} \exp\left(-\frac{(y+nB)^2}{2v}\right) K_n\left(\frac{t-u}{t-v}\right) \frac{dv}{\sqrt{v(t-v)}} \\ & + \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_0^{u} \exp\left(-\frac{(x+nB)^2}{2v}\right) K_n\left(\frac{t-u}{t-v}\right) \frac{dv}{\sqrt{v(t-v)}}, \\ & u \in (0,t). \end{aligned}$$

In particular, if random process $w(t), \quad t \geq 0$ begins evolution from the middle of interval $(0,B)$, then

$$\mathbf{P}[\,\sigma^*_{B/2}(t)<u\,]=1-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{t-u}\exp\left(-\frac{B^2(2n+1)^2}{8v}\right)K_n\left(\frac{u}{t-v}\right)\frac{dv}{\sqrt{v(t-v)}},\quad u\in(0,t),$$

$$\mathbf{P}[\,\sigma_{B/2}(t)<u\,]=\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{u}\exp\left(-\frac{B^2(2n+1)^2}{8v}\right)K_n\left(\frac{t-u}{t-v}\right)\frac{dv}{\sqrt{v(t-v)}},\quad u\in(0,t).$$

Calculating the limits in the both parts of (36) as $u\to 0$ and keeping in mind that $K_n(0)=1$, we find

$$\mathbf{P}[\,\sigma^*_y(t)=0\,]=1-\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{t}\exp\left(-\frac{(y+nB)^2}{2v}\right)\frac{dv}{\sqrt{v(t-v)}}-\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{t}\exp\left(-\frac{(x+nB)^2}{2v}\right)\frac{dv}{\sqrt{v(t-v)}}.$$

Theorem 4.1 is proved. □

Remark. It is valid also equality for the probability $\mathbf{P}[\sigma^*_y(t)=0]$ [9]

$$\mathbf{P}[\,\sigma^*_y(t)=0\,]=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\exp\left(-\frac{1}{2}t\left(\pi(2n+1)/B\right)^2\right)\sin\left(\frac{x}{B}(2n+1)\pi\right),\quad x=B-y$$

from which in particular it is followed that under $t\to\infty$

$$\mathbf{P}[\,\sigma^*_y(t)=0\,]=\frac{4}{\pi}\exp\left(-t\,\frac{\pi^2}{2B^2}\right)\sin\left(\frac{x}{B}\pi\right)+o\left(e^{-t\,\pi^2/2B^2}\right).$$

Formulate the limit theorem in the case when

$$\mathbf{E}\,\xi(1)=0,\quad \mathbf{E}\,\xi(1)^2=\sigma^2<\infty.$$

Without loss of generalization we assume that $\sigma=1$.

Theorem 4.3. *Let $\xi(t),\quad t\ge 0$ be the lower semicontinues homogenous process with independent increasers such that* $\mathbf{E}\,\xi(1)=0$, $\mathbf{E}\,\xi(1)^2=1$ *and*

$$\sigma_y(t,B)=\sigma_{yB}(tB^2)=\int_0^{tB^2}g(\,yB+\xi(u))\,du,\quad y\in[0,1]$$

is the total time of the stay of process $yB+\xi(\cdot)$ in the interval $(0,B)$ till the moment tB^2.

Then

$$\lim_{B\to\infty} \mathbf{P}[\,\sigma_y(t,B) < uB^2\,] = \mathbf{P}[\,\hat\sigma_y(t) < u\,]$$
$$= \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^u \exp\left(-\frac{(y+n)^2}{2v}\right)\mathbf{K}_n\left(\frac{t-u}{t-v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}}$$
$$+\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^u \exp\left(-\frac{(x+n)^2}{2v}\right)\mathbf{K}_n\left(\frac{t-u}{t-v}\right)\,d_v \arcsin\sqrt{\frac{v}{t}},$$
$$u\in(0,t),$$

where $x = 1-y$ *and*

$$\hat\sigma_y(t) = \int_0^t \hat g(\,y + w(u))\,du, \quad \left(\hat g(x) = \frac{1}{2}\{\,\mathrm{sgn}\,x + \mathrm{sgn}\,(1-x)\}\right)$$

is the total time of stay of process $y + w(\cdot)$ *in the interval* $(0,1)$ *till time moment* t.

Proof. It is followed formula From the second equality of Theorem 3 under $b = 0$ we have

$$C^s_{yB}(a) = v_{xB}(s,a) - C_B(s,a)V_{xB}(s,a)\,e^{-yBc(s)}, \quad x = 1-y, \tag{37}$$

where

$$v_{xB}(s,a) = 1 + a\int_0^{xB} R_u(s+a)\,du,$$
$$V_{xB}(s,a) = 1 + a\int_0^{xB} e^{-uc(s)}R_u(s+a)\,du,$$
$$C_B(s,a) = \frac{a}{c(s)}\,\frac{V_B(s,a)\,e^{Bc(s)} - v_B(s,a)}{k'(c(s)) + a\int_0^B V_u(s,a)\,du}.$$

Noting that

$$C^{s/B^2}_{yB}(a/B^2) = \frac{s}{B^2}\int_0^\infty e^{-su/B^2}\mathbf{E}\,\exp\left(-a\,\frac{\sigma_{yB}(u)}{B^2}\right)du$$
$$= s\int_0^\infty e^{-st}\mathbf{E}\,\exp\left(-a\,\frac{\sigma_{yB}(tB^2)}{B^2}\right)dt$$

and let us calculate the limit $\lim_{B\to\infty} C^{s/B^2}_{yB}(a/B^2)$. To do this we need the following equalities [4]

$$\lim_{B\to\infty} Bc\left(s/B^2\right) = \sqrt{2s},$$
$$\lim_{B\to\infty}\frac{1}{B}R_{xB}(s/B^2) = \sqrt{\frac{2}{s}}\,\mathrm{sh}\,x\sqrt{2s}.$$

Using these formulas we calculate

$$\begin{aligned}
\lim_{B\to\infty} V_{xB}\left(\frac{s}{B^2},\frac{a}{B^2}\right) &= 1 + a\lim_{B\to\infty}\frac{1}{B^2}\int_0^{xB} e^{-uc(s)}R_u\left(\frac{s+a}{B^2}\right)du\\
&= 1 + a\lim_{B\to\infty}\frac{1}{B}\int_0^{x} e^{-uBc(s)}R_{uB}\left(\frac{s+a}{B^2}\right)du\\
&= 1 + a\sqrt{\frac{2}{s+a}}\int_0^{x} e^{-u\sqrt{2s}}\left(\operatorname{sh} u\sqrt{2(s+a)}\right)du\\
&= \left(\operatorname{ch}\left(x\sqrt{2(s+a)}\right) + \sqrt{\frac{s+b}{s+a}}\operatorname{sh}\left(x\sqrt{2(s+a)}\right)\right)e^{-x\sqrt{2s}},\\
\lim_{B\to\infty} v_{xB}\left(\frac{s}{B^2},\frac{a}{B^2}\right) &= 1 + a\lim_{B\to\infty}\frac{1}{B^2}\int_0^{xB} R_u\left(\frac{s+a}{B^2}\right)du\\
&= 1 + a\lim_{B\to\infty}\frac{1}{B}\int_0^{x} R_{uB}\left(\frac{s+a}{B^2}\right)du\\
&= 1 + a\sqrt{\frac{2}{s+a}}\int_0^{x}\operatorname{sh}\left(u\sqrt{2(s+a)}\right)du\\
&= \frac{s}{s+a} + \frac{a}{s+a}\operatorname{ch}\left(x\sqrt{2(s+a)}\right).
\end{aligned}$$

Using these limits we find

$$\begin{aligned}
&a\lim_{B\to\infty}\frac{1}{B}\int_0^{B} V_u\left(\frac{s}{B^2},\frac{a}{B^2}\right)du = a\lim_{B\to\infty}\int_0^{1} V_{uB}\left(\frac{s}{B^2},\frac{a}{B^2}\right)du\\
&= -\sqrt{2s} + \left(\left(\frac{s}{\sqrt{2(s+a)}} + \frac{1}{2}\sqrt{2(s+a)}\right)\operatorname{sh}\sqrt{2(s+a)} + \sqrt{2s}\operatorname{ch}\sqrt{2(s+a)}\right)e^{-\sqrt{2s}},
\end{aligned}$$

$$\begin{aligned}
&a\lim_{B\to\infty}\left(V_B\left(\frac{s}{B^2},\frac{a}{B^2}\right)e^{Bc(s/B^2)} - v_B\left(\frac{s}{B^2},\frac{a}{B^2}\right)\right)\\
&\quad= \frac{s}{s+a}\left(\operatorname{ch}\sqrt{2(s+a)} - 1\right) + \sqrt{\frac{s}{s+a}}\operatorname{sh}\sqrt{2(s+a)},
\end{aligned}$$

$$\lim_{B\to\infty} C_B\left(\frac{s}{B^2},\frac{a}{B^2}\right) = \frac{a}{\sqrt{s+a}}\,\frac{\operatorname{sh}\sqrt{\frac{s+a}{2}}}{\sqrt{s+a}\operatorname{sh}\sqrt{\frac{s+a}{2}} + \sqrt{s}\operatorname{ch}\sqrt{\frac{s+a}{2}}}e^{\sqrt{2s}}$$

Due to these limits and (37) we obtain

$$\begin{aligned}
\lim_{B\to\infty} C_{yB}^{s/B^2}\left(\frac{a}{B^2}\right) &= \lim_{B\to\infty} s\int_0^{\infty} e^{-st}\mathbf{E}\exp\left(-a\,\frac{\sigma_y(t,B)}{B^2}\right)dt\\
&= \frac{s}{s+a}\left(1 + \frac{a}{\sqrt{s}}\,\frac{\operatorname{ch}\left((1-2y)\sqrt{\frac{s+a}{2}}\right)}{\sqrt{s+a}\operatorname{sh}\sqrt{\frac{s+a}{2}} + \sqrt{s}\operatorname{ch}\sqrt{\frac{s+a}{2}}}\right)
\end{aligned}$$

The right part of this formula coincides with the right part of equality (29) by $b = 0, \quad B = 1$.

Comparing the left parts of these equalities we find

$$\lim_{B\to\infty} s \int_0^\infty e^{-st}\mathbf{E} \exp\left(-a\,\frac{\sigma_y(t,B)}{B^2}\right) dt = s\int_0^\infty e^{-st}\mathbf{E}\exp\left(-a\,\hat{\sigma}_y(t)\right) dt,$$

where

$$\hat{\sigma}_y(t) = \int_0^t \hat{g}(y + w(u))\, du, \qquad \left(\hat{g}(x) = \frac{1}{2}\left\{\operatorname{sgn} x + \operatorname{sgn}(1-x)\right\}\right)$$

is the total time of stay of the process $y + w(\cdot)$ in the interval $(0,1)$ till time moment t.

Therefore, the random variable $\sigma_y(t,B)/B^2$ under $B\to\infty$ weakly converges to $\hat{\sigma}_y(t)$ and according to the first equality of Theorem 4.1 under $B = 1$

$$\begin{aligned}\lim_{B\to\infty} \mathbf{P}[\sigma_y(t,B) < uB^2] &= \mathbf{P}[\hat{\sigma}_y(t) < u] \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^u \exp\left(-\frac{(y+n)^2}{2v}\right) \\ &\quad\times \mathbf{K}_n\left(\frac{t-u}{t-v}\right) d_v \arcsin\sqrt{\frac{v}{t}} \\ &+ \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^u \exp\left(-\frac{(x+n)^2}{2v}\right) \\ &\quad\times \mathbf{K}_n\left(\frac{t-u}{t-v}\right) d_v \arcsin\sqrt{\frac{v}{t}}, \quad u\in(0,t),\end{aligned}$$

Theorem 4.2 is proved. □

Remark. We explain other method of obtaining of formulas (28) – (30).

Let

$$g(x) = \frac{1}{2}\left\{\mathbf{sgn}\, x + \mathbf{sgn}\,(B - x)\right\}, \qquad x\in R.$$

According to the theorem of the additive functionals [1, §19] the function

$$u(a,s,x) = \int_0^\infty e^{-st}\mathbf{E}\exp\left\{-a\int_0^t g(x + w(y))\,dy\right\} dt$$

satisfies the system of differential equations

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2}{\partial x^2}\,u(a,s,x) - s\,u(a,s,x) = -1, \qquad && x<0, \quad x>B,\\ &\frac{1}{2}\frac{\partial^2}{\partial x^2}\,u(a,s,x) - (s+a)\,u(a,s,x) = -1, && x\in(0,B)\end{aligned}$$

and the general solution of these equations has a form

$$\begin{aligned}&u(a,s,x) = \frac{1}{s} + C_1\,e^{x\sqrt{2s}} + C_2\,e^{-x\sqrt{2s}}, && x<0,\quad x>B,\\ &u(a,s,x) = \frac{1}{s+a} + C_3\,e^{x\sqrt{2(s+a)}} + C_4\,e^{-x\sqrt{2(s+a)}}, && x\in(0,B).\end{aligned} \tag{38}$$

We consider the solutions of the system (36) for $x > 0$, since due to the symmetry of Wiener process the function $u(a, s, x)$ for $x < 0$ can be easily reconstruct on the base of obtained solution of the system (36) for $x > 0$.

In this case we obtain the following system from (36)

$$\begin{aligned} u(a,s,x) &= \frac{1}{s} + C_2\, e^{-x\sqrt{2s}}, && x > B,\\ u(a,s,x) &= \frac{1}{s+a} + C_3\, e^{x\sqrt{2(s+a)}} + C_4\, e^{-x\sqrt{2(s+a)}}, && x \in (0,B). \end{aligned} \tag{39}$$

since in virtue of the boundedness of function $u(a, s, x) < \frac{1}{s}$, $\quad C_1 = 0$.

Using the fact, that according to theorem from [1] the function $u(a, s, x)$ is continuous in x jointly with its first derivative, and that Wiener process is symmetric we obtain the system of equations for the definition of the constants $C_2,\ C_3,\ C_4$

$$\begin{aligned} &\frac{1}{s} + C_2\, e^{-B\sqrt{2s}} = \frac{1}{s+a} + C_3\, e^{B\sqrt{2(s+a)}} + C_4\, e^{-B\sqrt{2(s+a)}},\\ &-C_2\,\sqrt{2s}\, e^{-B\sqrt{2s}} = C_3\,\sqrt{2(s+a)}\, e^{B\sqrt{2(s+a)}} - C_4\,\sqrt{2(s+a)}\, e^{-B\sqrt{2(s+a)}},\\ &\frac{1}{s+a} + C_3 + C_4 = \frac{1}{s} + C_2\, e^{-B\sqrt{2s}}. \end{aligned}$$

Solving this system of equations we find

$$\begin{aligned} C_2 &= -\frac{a}{s\sqrt{s+a}}\ \frac{\operatorname{sh}(\frac{B}{\sqrt2}\sqrt{s+a})}{\sqrt{s+a}\operatorname{sh}(\frac{B}{\sqrt2}\sqrt{s+a}) + \sqrt{s}\operatorname{ch}(\frac{B}{\sqrt2}\sqrt{s+a})}\ \exp\{B\sqrt{2s}\},\\ C_3 &= \frac{a}{2(s+a)\sqrt{s}}\ \frac{1}{\sqrt{s+a}\operatorname{sh}(\frac{B}{\sqrt2}\sqrt{s+a}) + \sqrt{s}\operatorname{ch}(\frac{B}{\sqrt2}\sqrt{s+a})}\ \exp\{-\frac{B}{\sqrt2}\sqrt{s+a}\},\\ C_4 &= \frac{a}{2(s+a)\sqrt{s}}\ \frac{1}{\sqrt{s+a}\operatorname{sh}(\frac{B}{\sqrt2}\sqrt{s+a}) + \sqrt{s}\operatorname{ch}(\frac{B}{\sqrt2}\sqrt{s+a})}\ \exp\{\frac{B}{\sqrt2}\sqrt{s+a}\}. \end{aligned}$$

Substituting obtained constants $C_2,\ C_3,\ C_4$ in the system of equations (39), we obtain

$$\begin{aligned} s\,u(a,s,x) &= \frac{s}{s+a}\left(1 + \frac{a}{\sqrt{s}}\ \frac{\operatorname{ch}(\frac{B-2x}{\sqrt2}\sqrt{s+a})}{\sqrt{s+a}\operatorname{sh}(\frac{B}{\sqrt2}\sqrt{s+a}) + \sqrt{s}\operatorname{ch}(\frac{B}{\sqrt2}\sqrt{s+a})}\right),\\ &\quad y \in (0,B),\\ s\,u(a,s,x) &= 1 - \frac{a}{\sqrt{s+a}}\ \frac{\operatorname{sh}(\frac{B}{\sqrt2}\sqrt{s+a})\ \exp\{(B-x)\sqrt{2s}\}}{\sqrt{s+a}\operatorname{sh}(\frac{B}{\sqrt2}\sqrt{s+a}) + \sqrt{s}\operatorname{ch}(\frac{B}{\sqrt2}\sqrt{s+a})},\\ &\quad x > B. \end{aligned}$$

and equalities (28), (29) by $b = 0$.

Remark. Let

$$g_0(x) = \lim_{B\to\infty} g(x) = \frac{1}{2}\{1 + \operatorname{sgn} x\}, \quad g_0^*(x) = \frac{1}{2}\{1 - \operatorname{sgn} x\}, \quad x \in R$$

and

$$\alpha_y(t) = \lim_{B\to\infty} \sigma_y(t) = \int_0^t g_0(y+w(u))\,du,$$
$$\alpha_y^*(t) = \lim_{B\to\infty} \sigma_y^*(t) = \int_0^t g_0^*(y+w(u))\,du$$

is the total time of stay of process $y+w(\cdot)$ in the upper half-plain till the moment t and time of stay of process $y+w(\cdot)$ in the lower half-plain till the moment t accordingly. Then calculating the limits under $B\to\infty$, we obtain from the equalities of Theorem 4.1

$$\begin{aligned}
&\mathbf{P}[\,\alpha_y(t) < u\,] = \frac{2}{\pi}\int_0^u \exp\left(-\frac{y^2}{2v}\right) d_v \arcsin\sqrt{\frac{v}{t}}, && u \in (0,t),\\
&\mathbf{P}[\,\alpha_y^*(t) < u\,] = 1-\frac{2}{\pi}\int_0^{t-u} \exp\left(-\frac{y^2}{2v}\right) d_v \arcsin\sqrt{\frac{v}{t}}, && u \in (0,t).
\end{aligned} \tag{40}$$

In particular, letting in the first formula $y=0$, we obtain

$$\mathbf{P}[\,\alpha(t) < u\,] = \frac{2}{\pi}\arcsin\sqrt{\frac{u}{t}}, \qquad u\in(0,t)$$

arcsine law obtained by P. Levy [2] for the distribution of the time of stay of Wiener process in the upper half-plain.

We can also obtain the equalities (40) using the formula (29). Indeed, letting in (27) $b=0$ and calculating the limits under $B\to\infty$, we find

$$\int_0^\infty e^{-st}\mathbf{E}\exp\left(-a\,\alpha_y(t)\right)dt = \frac{1}{s+a}\left(1-e^{-y\sqrt{2(s+a)}}\right) + \frac{1}{\sqrt{s(s+a)}}\,e^{-y\sqrt{2(s+a)}}$$

Converting the Laplace transform [7], in the right part of this equality we obtain the distribution of the time of stay of process $y+w(t)$, $y\ge 0$, $t\ge 0$ in the upper half-plain.

$$\begin{aligned}
&\mathbf{P}[\,\alpha_y(t) = t\,] = \frac{2}{\sqrt{2\pi t}}\int_0^y \exp\left(-\frac{u^2}{2t}\right)du,\\
&\mathbf{P}[\,\alpha_y(t) < u\,] = \frac{2}{\pi}\int_0^u \exp\left(-\frac{y^2}{2v}\right) d_v \arcsin\sqrt{\frac{v}{t}}, \qquad u\in(0,t)
\end{aligned}$$

Performing similar calculations we also find from formula (29) the distribution of the time of stay of process $y+w(t)$, $y\ge 0$, $t\ge 0$ in the lower half-plain

$$\begin{aligned}
&\mathbf{P}[\,\alpha_y^*(t) = 0\,] = \frac{2}{\sqrt{2\pi t}}\int_0^y \exp\left(-\frac{u^2}{2t}\right)du,\\
&\mathbf{P}[\,\alpha_y^*(t) < u\,] = 1-\frac{2}{\pi}\int_0^{t-u} \exp\left(-\frac{y^2}{2v}\right) d_v \arcsin\sqrt{\frac{v}{t}}, \qquad u\in(0,t)
\end{aligned}$$

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