Unitals with many Baer secants through a fixed point

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The Hermitian curve in $\text{PG}(2, q^2)$

- $\text{PG}(2, q^2)$: Desarguesian projective plane over the finite field $\mathbb{F}_{q^2}$.

- A **Hermitian curve** $\mathcal{H}$ in $\text{PG}(2, q^2)$ is a set of $q^3 + 1$ points projectively equivalent to the set of points whose coordinates $(x_0, x_1, x_2)$ satisfy equation
  \[
x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0.
  \]

Every line of $\text{PG}(2, q^2)$ intersects $\mathcal{H}$ in exactly $1$ point or in $q + 1$ points.

- A Hermitian curve is the classical example of a **unital**.
A Hermitian curve is the classical example of a unital.

- A unital $\mathcal{U}$ of $\text{PG}(2, q^2)$ is a set of $q^3 + 1$ points such that every line of $\text{PG}(2, q^2)$ contains exactly 1 point of $\mathcal{U}$ (tangent line) or $q + 1$ points of $\mathcal{U}$ (secant line).

In $\text{PG}(2, 4)$ all unitals are classical.
The classification of unitals in $\text{PG}(2, q^2)$, $q > 2$, is an open problem.
See e.g. *Unitals in Projective Planes* by G. Ebert and S. Barwick.

All known unitals arise as *ovoidal Buekenhout-Metz unitals*, for short BM-unitals.

- Characterisations of BM-unital?
A **line spread** $S$ in $\text{PG}(3, q)$ is a set of $q^2 + 1$ disjoint lines such that every point of $\text{PG}(3, q)$ is contained in exactly one line of $S$.

A **translation plane** $\mathbb{P}(S)$ of order $q^2$, with points $\mathcal{P}$ and lines $\mathcal{L}$, can be obtained from a line spread $S$ as follows.

- $H_\infty = \text{PG}(3, q)$
- $\mathcal{P}$: $q^2 + 1$ lines of spread $S$,
  - $q^4$ affine points of $\Sigma \setminus H_\infty$
- $\mathcal{L}$: $q^4 + q^2$ planes containing a line of $S$,
  - $H_\infty$
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  - $H_\infty$
If $S$ is a **Desarguesian line spread**, then $\mathbb{P}(S)$ is isomorphic to $\text{PG}(2, q^2)$. 

\[
H_\infty = \text{PG}(3, q) \\
L_\infty = \text{PG}(1, q^2)
\]
An ovoidal Buekenhout-Metz unital in $\text{PG}(2, q^2)$

- An **ovoid** $\mathcal{O}$ in $\text{PG}(3, q)$, $q > 2$, is a set of $q^2 + 1$ points, no three of which are collinear. Every plane intersects $\mathcal{O}$ in 1 or $q + 1$ points.

$H_\infty = \text{PG}(3, q)$

**Construct a BM-unital:**

1. Fix a point $Q$ of a spread line $T$ of $S$.
2. Consider an ovoid $\mathcal{O}$ through $Q$ (in a 3-space intersecting $T$ in $Q$).
3. Take a point $V$ on $T \setminus \{Q\}$ and consider the cone $VO$.

$\text{PG}(4, q)$
An ovoidal Buekenhout-Metz unital in $\text{PG}(2, q^2)$

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$\text{PG}(4, q)$
The ovoidal cone $\mathcal{V}\mathcal{O}$ in $\text{PG}(4, q)$ is a \textbf{BM-unital $\mathcal{U}$} in $\text{PG}(2, q^2)$.

The point $P$ (corresponding to $T$) is called the \textbf{special point} of $\mathcal{U}$. The line $L_\infty$ is a tangent line to $\mathcal{U}$ at $P$. 

$H_\infty = \text{PG}(3, q)$  
$P_L \infty = \text{PG}(1, q^2)$  
$\mathcal{V}\mathcal{O}$  

$\text{PG}(4, q)$  

$\text{PG}(2, q^2)$
A **Baer subline** of $\text{PG}(2, q^2)$ is a set of $q + 1$ points on a line whose homogeneous coordinates are in the subfield $\mathbb{F}_q \leq \mathbb{F}_{q^2}$, with respect to a well-chosen frame of $\text{PG}(2, q^2)$.

A **Baer secant** to a unital $\mathcal{U}$ is a secant intersecting $\mathcal{U}$ in a Baer subline.

Every secant to a classical unital $\mathcal{H}$ is a Baer secant.

Every secant to a BM-unital $\mathcal{U}$, that contains its special point $P$, is a Baer secant.

Characterisations of BM-unital in terms of Baer secants?
Known characterisations

**Theorem (Lefèvre-Percsy, 1982)**

Let $\mathcal{U}$ be a unital in $\text{PG}(2, q^2)$ such that all secants are Baer secants, then $\mathcal{U}$ is classical.

Note that a unital in $\text{PG}(2, q^2)$ has in total $(q^2 - q + 1)q^2$ secants.

**Theorem (Ball, Blokhuis & O’Keefe, 1999)**

Let $\mathcal{U}$ be a unital in $\text{PG}(2, p^2)$, $p$ prime, such that at least $(p^2 - 2)p$ secants are Baer secants, then $\mathcal{U}$ is classical.
Known characterisations

Theorem (Quinn & Casse, 1995; Casse, O’Keefe & Penttila, 1996)

Let $\mathcal{U}$ be a unital in $\operatorname{PG}(2, q^2)$, $q > 2$, such that all secants through a fixed point $P$ are Baer secants, then $\mathcal{U}$ is a BM-unital with special point $P$.

Theorem (Barwick & Quinn, 2001)

Let $\mathcal{U}$ be a BM-unital in $\operatorname{PG}(2, q^2)$ with special point $P$. If $\mathcal{U}$ contains a Baer secant not through $P$, then $\mathcal{U}$ is classical.
Our characterisation

De Clerck and Durante (Chapter in *Current Research Topics in Galois Geometry*, 2014) posed the question:

- What is the minimum required number of secants being Baer secants, to conclude that a unital is a BM-unital?

Main Theorem (S.R. & G. Van de Voorde, 2015)

Consider a unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$ containing a point $P$ such that at least $q^2 - \epsilon$ secants through $P$ are Baer secants.

If $\epsilon \approx 2q$ for $q \geq 128$ even or $\epsilon \approx q^{3/2}/2$ for $q \geq 17$ odd, then $\mathcal{U}$ is an ovoidal Buekenhout-Metz unital with special point $P$. 

Sara Rottey (VUB-UGent)
Rough sketch of proof: Step 1

Given unital $\mathcal{U}$: $q^2 - \epsilon$ Baer secants through $P$.
View representation in $\text{PG}(4, q)$.

$q^2 - \epsilon$ “good” secants
$\epsilon$ “bad” secants

$H_\infty = \text{PG}(3, q)$

$\text{PG}(2, q^2)$

$\text{PG}(4, q)$
Given unital $U$: $q^2 - \epsilon$ Baer secants through $P$.

View representation in $\text{PG}(4, q)$.

$q^2 - \epsilon$ “good” secants

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$H_\infty = \text{PG}(3, q)$

$\text{PG}(4, q)$
Rough sketch of proof: Step 2

Given unital $U$: $q^2 - \epsilon$ Baer secants through $P$.
View representation in $\text{PG}(4, q)$.

$q^2 - \epsilon$ “good” secants
$\epsilon$ “bad” secants

$H_\infty = \text{PG}(3, q)$

$\text{PG}(4, q)$
Rough sketch of proof: Step 3

Given unital $\mathcal{U} : q^2 - \epsilon$ Baer secants through $P$.
View representation in $\text{PG}(4, q)$.

$q^2 - \epsilon$ “good” secants are all lines through point $V$

- Awful counting arguments
- Properties on intersections of unitals with subplanes

Restriction $\epsilon \leq \frac{q^3}{2}/2$

$H_\infty = \text{PG}(3, q)$

$\text{PG}(4, q)$
Given unital $\mathbf{u} : q^2 - \epsilon$ Baer secants through $P$.
View representation in $\text{PG}(4, q)$.

unique ovoidal cone $V\mathcal{O}$ containing these $q^2 - \epsilon$ lines

\[ \epsilon \approx 2q, \: q \geq 128 \text{ even} \]
\[ \epsilon \approx q^{3/2}/2, \: q \geq 17 \text{ odd} \]

$V\mathcal{O}$ corresponds to BM-unital $\text{PG}(4, q)$

$q^2 - \epsilon$ “good” secants are all lines through point $V$

$H_\infty = \text{PG}(3, q)$
Rough sketch of proof: Step 5


Consider an ovoidal Buekenhout-Metz unital \( U' \) of \( \text{PG}(2, q^2) \) with special point \( P \). Consider a unital \( U \) of \( \text{PG}(2, q^2) \) containing \( P \) and having \( q^2 - \epsilon \) secants through \( P \) in common with \( U' \).

If \( \epsilon \leq \frac{(q-1)q}{2} \), then \( U \) and \( U' \) coincide.

Main Theorem (S.R. & G. Van de Voorde, 2015)

Suppose \( \epsilon \approx 2q \) for \( q \geq 128 \) even, \( \epsilon \approx q^{3/2}/2 \) for \( q \geq 17 \) odd.

Let \( U \) be a unital in \( \text{PG}(2, q^2) \) containing a point \( P \) such that at least \( q^2 - \epsilon \) secants through \( P \) are Baer secants, then \( U \) is an ovoidal Buekenhout-Metz unital with special point \( P \).

Consider an ovoidal Buekenhout-Metz unital $U'$ of $\text{PG}(2, q^2)$ with special point $P$. Consider a unital $U$ of $\text{PG}(2, q^2)$ containing $P$ and having $q^2 - \epsilon$ secants through $P$ in common with $U'$. If $\epsilon \leq \frac{(q-1)q}{2}$, then $U$ and $U'$ coincide.

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Thank you for your attention!
<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>Conditions</th>
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</thead>
<tbody>
<tr>
<td>( \epsilon \leq q - 3 )</td>
<td>( q ) even, ( q \geq 16 )</td>
</tr>
<tr>
<td>( \epsilon \leq 2q - 7 )</td>
<td>( q ) even, ( q \geq 128 )</td>
</tr>
<tr>
<td>( \epsilon \leq \frac{\sqrt{q}q}{4} - \frac{39q}{64} - O(\sqrt{q}) + 1 )</td>
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| \( \epsilon \leq \frac{\sqrt{q}q}{2} - 2q \) | \( q \) odd, \( q \geq 17 \), \( q = p^{2e+1} \), \( e \geq 1 \)  \\
| | or \( q \) prime                      |
| \( \epsilon \leq \frac{\sqrt{q}q}{2} - \frac{67q}{16} + \frac{5\sqrt{q}}{4} - \frac{1}{12} \) | \( q \) odd, \( q \geq 17 \), \( q = p^h \), \( p \geq 5 \) |
| \( \epsilon \leq \frac{\sqrt{q}q}{2} - \frac{35q}{16} - O(\sqrt{q}) + 1 \) | \( q \) odd, \( q \geq 23^2 \), \( q = p^h \), \( h \) even for \( p = 3 \), \( q \neq 5^5, 3^6 \) |