Two-sided two-cosided Hopf modules and Doi-Hopf modules for quasi-Hopf algebras*

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Abstract

Let H be a finite dimensional quasi-Hopf algebra over a field k and $\mathfrak{A}$ a right H-comodule algebra. We introduce the category of two-sided Hopf modules, and prove that it is isomorphic to a module category. We also show that two-sided Hopf modules are coalgebra over a certain comonad. We introduce Doi-Hopf modules, and show that they are comodules over a certain coring. If the underlying H-module coalgebra is finite dimensional, then Doi-Hopf modules are modules over a certain smash products. A similar result holds for two-sided two-cosided Hopf modules.

Introduction

Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [17] in connection with the Knizhnik-Zamolodchikov equations [21]. Let k be a field, H an associative algebra and $\Delta:\ H\rightarrow H\otimes H$ and $\varepsilon:\ H\rightarrow k$ two algebra morphisms. Roughly speaking, H is a quasi-bialgebra if the category ${}_H\mathcal{M}$ of left H-modules, equipped with the tensor product of vector spaces endowed with the diagonal H-module structure given via Δ, and with unit object k viewed as a left H-module via ε, is a monoidal category. The comultiplication Δ is not coassociative but only quasi-coassociative, in the sense that it is coassociative up to conjugation by an invertible element $\Phi\in H\otimes H\otimes H$. Moreover, H is a quasi-Hopf algebra if and only if each finite dimensional left H-module has a dual H-module. Note that the definition of a quasi-bialgebra is not self dual.

From an algebraic point of view, quasi-bialgebras and quasi-Hopf algebras appear naturally. They can be obtained by twisting the comultiplication on a bialgebra H by an invertible element $F\in H\otimes H$ satisfying $(\varepsilon\otimes id)(F)=(id\otimes\varepsilon)(F)=1$: a new comultiplication Δ_F making H a quasi-bialgebra is given by $\Delta_F(h)=F\Delta(h)F^{-1}$. Another important example is the Dijkgraaf-Pasquier-Roche quasi-Hopf algebra $D^{\omega}(G)$, where G is a finite group and ω a normalized 3-cocycle. The representations of $D^{\omega}(G)$ are important in physics (see [13]). Altschuler and Coste [3] used them to construct invariants for knots, links and 3-manifolds. In [8], this construction was generalized to finite dimensional cocommutative Hopf algebras, and an even more general construction is the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra, see [23], [18], [19]. Albuquerque and Majid [1] showed recently that the octonions

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are a twisting of the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in the monoidal category of representations of a quasi-Hopf algebra associated to a group 3-cocycle. In particular, they shown that the octonions are quasi-algebras associative up to a 3-cocycle isomorphism. They provide new quasi-associative algebras beyond the octonions and also introduce a suitable quasi-Hopf algebra of "authomorphisms" associated to any quasi-algebra of the type presented above. More examples of quasi-algebras, where the non-associativity constraint is induced by a $\mathbb{Z}_n$-grading and a nontrivial 3-cocycle, were given in [2].
Let H be a bialgebra, A and H-comodule algebra and C an H-module coalgebra. We can consider several types of modules, such as modules, comodules, (relative) Hopf modules, Long dimodules and Yetter-Drinfeld modules. Doi [15] and Koppinen [22] introduced Doi-Hopf modules, and it turned out that they generalize and unify all the types of modules mentioned above. Basically, we obtain the definition of a Doi-Hopf module, by combining the definitions of a relative (A,H)-module and its dual notion, a relative $[H,C]$-module: a (H,A,C)-module is a k-linear space together with an A-action and a C-coaction satisfying an appropriate compatibility relation. We recover the two types of relative Hopf modules taking respectively $C = H$ and $A = H$. At the end of last century, Takeuchi [30] observed that $A \otimes C$ is in a canonical way an A-coring, and that Doi-Hopf modules are nothing else then comodules over the coring $A \otimes C$. This observation was the reason for a revived interest in corings and comodules (see for example [5]); actually corings were considered already by Sweedler in 1965 [28], but then forgotten by Hopf algebra theorists.

The aim of this paper is to introduce the quasi-bialgebraic versions of these categories, including interpretations in terms of monoidal categories, and to give duality Theorems in the finite dimensional case. The conceptual problem that arises comes from the fact that the definition of a quasi-bialgebra H is not self-dual: an immediate consequence is that we cannot consider H-comodules, because a quasi-bialgebra is not coassociative. H-module (co)algebras can be introduced as (co)algebras in the monoidal category of H-modules, but we cannot introduce H-comodule algebras as algebras in the category of comodules. A formal definition of H-comodule algebras was given by Hausser and Nill [18]; we propose the following interpretation: if H is a bialgebra, and $\mathfrak{A}$ is a right H-comodule algebra, then $\mathfrak{A} \otimes H$ is an $\mathfrak{A}$-coring, which means that it is a coalgebra in the category of $\mathfrak{A}$-bimodules. The quasi-bialgebra analog of this property is the following: let H be a quasi-bialgebra, and $\mathfrak{A}$ an algebra. Then the category of $(\mathfrak{A} \otimes H, \mathfrak{A})$-bimodules is monoidal. If $\mathfrak{A}$ is a right H-comodule algebra in the sense of [18], then $\mathfrak{A} \otimes H$ is a coalgebra in the category ${}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$. This coalgebra induces a comonad, and the two-sided Hopf modules that are introduced in Section 3.1 are precisely the coalgebras over this comonad. This will be discussed in detail in Section 3.3.
Given a finite dimensional quasi-bialgebra H and a right H-comodule algebra $\mathfrak{A}$, we can introduce the quasi-smash product $\mathfrak{A} \,\overline{\#}\, H^*$, which reduces to the usual smash product in the situation where H is a bialgebra. $\mathfrak{A} \,\overline{\#}\, H^*$ is then a left H-module algebra, and we can consider the category $\mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*}$ of relative Hopf modules (see Section 2). In Section 3, we introduce the category ${}_H\mathcal{M}^H_{\mathfrak{A}}$ of two-sided $(H,\mathfrak{A})$-Hopf modules; the main result of Section 3 is Theorem 3.5, stating that these two categories are isomorphic if H is a quasi-Hopf algebra. This generalizes [12, Proposition 2.3]. Appying results from [6], we find that the category $\mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*}$ is isomorphic to the category of right modules over the smash product algebra (in the sense of [9]) of $\mathfrak{A} \,\overline{\#}\, H^*$ and H. In the case where $\mathfrak{A} = H$, we recover a result of Nill announced in [20] stating that ${}_H\mathcal{M}^H_H$ is isomorphic to the category of right modules over the two-sided crossed product $H \rtimes\!\!\!\lhd H^* \ltimes\!\!\!\rhd H$. In Section 4, we will prove that the two-sided crossed product constructed in [18] is in fact a generalized smash product. As a consequence, $(H \,\overline{\#}\, H^*)\# H$ is just the two-sided crossed product $H \rtimes\!\!\!\lhd H^* \ltimes\!\!\!\rhd H$ (as an algebra).

The second part of this paper is devoted to the study of the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$. Here C is a coalgebra in the monoidal category of (H,H)-bimodules ${}_H\mathcal{M}_H$ (i.e. an H-bimodule coalgebra), and $\mathbb{A}$ is an H-bicomodule algebra in the sense of [18]. Roughly speaking, an object in ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is a two-sided $(H,\mathbb{A})$-Hopf module which is also an "almost" left C-comodule such that the left C-coaction is compatible with the other structure maps. In Section 5 we will show that if C and H are finite dimensional then ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is isomorphic to a category of right modules. To this end we will describe first ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ as a category of Doi-Hopf modules. If $\mathfrak{B}$ is a left H-comodule algebra and C is a right H-module coalgebra then the category of right-left $(H,\mathfrak{B},C)$-Doi-Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a straightforward generalization of the category of relative Hopf modules ${}^C\mathcal{M}_H$. When C is finite dimensional, ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category of right modules over the generalized smash product $C^*\blacktriangleright\!\!<\mathfrak{B}$. We also have an interpretation in terms of monoidal categories: $\mathfrak{B}\otimes C$ is a coring, and the Doi-Hopf modules are comodules over this coring. Now, returning to the category ${}^C_H\mathcal{M}^H_{\mathbb{A}}$, if H is finite dimensional then we will show that $(\mathbb{A}\,\overline{\#}\,H^*)\#H$ is a left $H\otimes H^{\rm op}$-comodule algebra (here "op" means the opposite multiplication on H) so, it makes sense to consider the category of Doi-Hopf modules ${}^C\mathcal{M}(H\otimes H^{\rm op})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}$. The main result states that ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is isomorphic to ${}^C\mathcal{M}(H\otimes H^{\rm op})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}$, generalizing [4, Proposition 2.3]. In particular, if C is finite dimensional, then ${}^C_H\mathcal{M}^H_{\mathbb{A}}$ is isomorphic to the category of right modules over the generalized smash product $\mathcal{A}=C^*\blacktriangleright\!\!<((\mathbb{A}\,\overline{\#}\,H^*)\#H)$. In the Hopf case, the left-handed version of this result was first obtained by Cibils and Rosso [11]. More precisely, they define an algebra X having the property that the category ${}^{H^*}_{H^*}\mathcal{M}^{H^*}_{H^*}$ is isomorphic to the category of left X-modules. Recently, Panaite [25] introduced two other algebras Y and Z with the same property as X. Y is the two-sided crossed product $H^*\#(H\otimes H^{\rm op})\#H^{*\rm op}$ and Z is the diagonal crossed product (in the sense of [18]) $(H^*\otimes H^{*\rm op})\bowtie(H\otimes H^{\rm op})$.

1 Preliminary results

Quasi-Hopf algebras

We work over a field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [17], a quasi-bialgebra is a fourtuple $(H,\Delta,\varepsilon,\Phi)$ where H is an associative algebra with unit, Φ is an invertible element in $H\otimes H\otimes H$, and $\Delta:\ H\to H\otimes H$ and $\varepsilon:\ H\to k$ are algebra homomorphisms satisfying the identities

$$(id\otimes\Delta)(\Delta(h))=\Phi(\Delta\otimes id)(\Delta(h))\Phi^{-1}, \tag{1.1}$$

$$(id\otimes\varepsilon)(\Delta(h))=h,\quad (\varepsilon\otimes id)(\Delta(h))=h, \tag{1.2}$$

for all $h\in H$, and Φ has to be a normalized 3-cocycle, in the sense that

$$(1\otimes\Phi)(id\otimes\Delta\otimes id)(\Phi)(\Phi\otimes 1)=(id\otimes id\otimes\Delta)(\Phi)(\Delta\otimes id\otimes id)(\Phi), \tag{1.3}$$

$$(id\otimes\varepsilon\otimes id)(\Phi)=1\otimes 1. \tag{1.4}$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. We use the Sweedler-Heyneman notation $\Delta(h)=\sum h_1\otimes h_2$. Since Δ is only quasi-coassociative we will write

$$(\Delta\otimes id)(\Delta(h))=\sum h_{(1,1)}\otimes h_{(1,2)}\otimes h_2,\quad (id\otimes\Delta)(\Delta(h))=\sum h_1\otimes h_{(2,1)}\otimes h_{(2,2)},$$

for all $h\in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\Phi=\sum X^1\otimes X^2\otimes X^3=\sum T^1\otimes T^2\otimes T^3=\sum V^1\otimes V^2\otimes V^3=\cdots$$
$$\Phi^{-1}=\sum x^1\otimes x^2\otimes x^3=\sum t^1\otimes t^2\otimes t^3=\sum v^1\otimes v^2\otimes v^3=\cdots$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-automorphism S of the algebra H and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{1.5}$$

$$\sum X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad \sum S(x^1)\alpha x^2\beta S(x^3) = 1. \tag{1.6}$$

For a quasi-Hopf algebra the antipode is determined uniquely up to a transformation $\alpha \mapsto U\alpha$, $\beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, where $U \in H$ is invertible. The axioms for a quasi-Hopf algebra imply that $\varepsilon \circ S = \varepsilon$ and $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β, we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$. The identities (1.2), (1.3) and (1.4) also imply that

$$(\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1. \tag{1.7}$$

Recall that the definition of a quasi-Hopf algebra is “twist coinvariant” in the following sense. An invertible element $F \in H \otimes H$ is called a *gauge transformation* or *twist* if $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$. If H is a quasi-Hopf algebra and $F = \sum F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = \sum G^1 \otimes G^2$, then we can define a new quasi-Hopf algebra H_F by keeping the multiplication, unit, counit and antipode of H and replacing the comultiplication, reassociator and the elements α and β by

$$\Delta_F(h) = F\Delta(h)F^{-1}, \tag{1.8}$$

$$\Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1), \tag{1.9}$$

$$\alpha_F = \sum S(G^1)\alpha G^2, \quad \beta_F = \sum F^1\beta S(F^2). \tag{1.10}$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f \in H \otimes H$ such that

$$f\Delta(S(h))f^{-1} = \sum (S \otimes S)(\Delta^{\mathrm{op}}(h)), \quad \text{for all } h \in H, \tag{1.11}$$

where $\Delta^{\mathrm{op}}(h) = \sum h_2 \otimes h_1$. f can be computed explicitly. First set

$$\sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (1 \otimes \Phi^{-1})(id \otimes id \otimes \Delta)(\Phi), \tag{1.12}$$

$$\sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1) \tag{1.13}$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = \sum S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = \sum B^1\beta S(B^4) \otimes B^2\beta S(B^3). \tag{1.14}$$

f and f^{-1} are then given by the formulas

$$f = \sum (S \otimes S)(\Delta^{\mathrm{op}}(x^1))\gamma\Delta(x^2\beta S(x^3)), \tag{1.15}$$

$$f^{-1} = \sum \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\mathrm{op}}(x^3)). \tag{1.16}$$

f satisfies the following relations:

$$f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta. \tag{1.17}$$

Furthermore the corresponding twisted reassociator (see (1.9)) is given by

$$\Phi_f = \sum (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1). \tag{1.18}$$

In a Hopf algebra H, we obviously have the identity

$$\sum h_1 \otimes h_2 S(h_3) = h \otimes 1, \text{ for all } h \in H.$$

We will need the generalization of this formula to the quasi-Hopf algebra setting. Following [18, 19], we define

$$p_R = \sum p_R^1 \otimes p_R^2 = \sum x^1 \otimes x^2 \beta S(x^3), \quad q_R = \sum q_R^1 \otimes q_R^2 = \sum X^1 \otimes S^{-1}(\alpha X^3) X^2, \tag{1.19}$$

$$p_L = \sum p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}(X^1 \beta) \otimes X^3, \quad q_L = \sum q_L^1 \otimes q_L^2 = \sum S(x^1) \alpha x^2 \otimes x^3. \tag{1.20}$$

For all $h \in H$, we then have

$$\sum \Delta(h_1) p_R [1 \otimes S(h_2)] = p_R [h \otimes 1], \quad \sum [1 \otimes S^{-1}(h_2)] q_R \Delta(h_1) = (h \otimes 1) q_R, \tag{1.21}$$

$$\sum \Delta(h_2) p_L [S^{-1}(h_1) \otimes 1] = p_L (1 \otimes h), \quad \sum [S(h_1) \otimes 1] q_L \Delta(h_2) = (1 \otimes h) q_L, \tag{1.22}$$

and

$$\sum \Delta(q_R^1) p_R [1 \otimes S(q_R^2)] = 1 \otimes 1, \quad \sum [1 \otimes S^{-1}(p_R^2)] q_R \Delta(p_R^1) = 1 \otimes 1, \tag{1.23}$$

$$\sum [S(p_L^1) \otimes 1] q_L \Delta(p_L^2) = 1 \otimes 1, \quad \sum \Delta(q_L^2) p_L [S^{-1}(q_L^1) \otimes 1] = 1 \otimes 1, \tag{1.24}$$

$$\begin{aligned} &(q_R \otimes 1)(\Delta \otimes id)(q_R)\Phi^{-1} \\ &\quad = \sum [1 \otimes S^{-1}(X^3) \otimes S^{-1}(X^2)][1 \otimes S^{-1}(f^2) \otimes S^{-1}(f^1)](id \otimes \Delta)(q_R \Delta(X^1)), \end{aligned} \tag{1.25}$$

$$\begin{aligned} &\Phi(\Delta \otimes id)(p_R)(p_R \otimes id) \\ &\quad = \sum (id \otimes \Delta)(\Delta(x^1) p_R)(1 \otimes f^{-1})(1 \otimes S(x^3) \otimes S(x^2)), \end{aligned} \tag{1.26}$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (1.15).

The smash product

Suppose that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra. If U, V, W are left (right) H-modules, define $a_{U,V,W}, \mathbf{a}_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)),$$

$$\mathbf{a}_{U,V,W}((u \otimes v) \otimes w) = (u \otimes (v \otimes w)) \cdot \Phi^{-1}.$$

Then the category ${}_H\mathcal{M}$ ($\mathcal{M}_H$) of left (right) H-modules becomes a monoidal category (see [21, 24] for the terminology) with tensor product $\otimes$ given via Δ, associativity constraints $a_{U,V,W}$ ($\mathbf{a}_{U,V,W}$), unit k as a trivial H-module and the usual left and right unit constraints.

Now, let H be a quasi-bialgebra. We say that a k-vector space A is a left H-module algebra if it is an algebra in the monoidal category ${}_H\mathcal{M}$, that is, A has a multiplication and a usual unit 1_A satisfying the following conditions:

$$(aa')a'' = \sum (X^1 \cdot a)[(X^2 \cdot a')(X^3 \cdot a'')], \tag{1.27}$$

$$h \cdot (aa') = \sum (h_1 \cdot a)(h_2 \cdot a'), \tag{1.28}$$

$$h \cdot 1_A = \varepsilon(h) 1_A, \tag{1.29}$$

for all $a, a', a'' \in A$ and $h \in H$, where $h \otimes a \mapsto h \cdot a$ is the H-module structure of A. Following [9], we define the smash product $A\#H$ as follows: as a vector space $A\#H$ is $A \otimes H$ ($a \otimes h$ viewed as an element of $A\#H$ will be written $a\#h$) with multiplication given by

$$(a\#h)(a'\#h') = \sum (x^1 \cdot a)(x^2 h_1 \cdot a')\# x^3 h_2 h', \tag{1.30}$$

for all $a, a' \in A$, $h, h' \in H$. $A\#H$ is an associative algebra and it is defined by a universal property (as Heyneman and Sweedler did for Hopf algebras, see [9]). It is easy to see that H is a subalgebra of $A\#H$ via $h \mapsto 1\#h$, A is a k-subspace of $A\#H$ via $a \mapsto a\#1$ and the following relations hold:

$$(a\#h)(1\#h') = a\#hh', \quad (1\#h)(a\#h') = \sum h_1 \cdot a\#h_2 h', \tag{1.31}$$

for all $a \in A$, $h, h' \in H$.
We will also need the notion right H-module coalgebra. This is a coalgebra C in the monoidal category of right modules over a quasi-bialgebra H. This means that C is a right H-module together with a comultiplication $\underline{\Delta}: C \to C \otimes C$ and a counit $\underline{\varepsilon}: C \to k$, satisfying the following relations

$$(\underline{\Delta} \otimes id_C)(\underline{\Delta}(c))\Phi^{-1} = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)) \;\; \forall\, c \in C, \tag{1.32}$$
$$\underline{\Delta}(c \cdot h) = \sum c_{\underline{1}} \cdot h_1 \otimes c_{\underline{2}} \cdot h_2 \;\; \forall\, c \in C, h \in H, \tag{1.33}$$
$$\underline{\varepsilon}(c \cdot h) = \underline{\varepsilon}(c)\varepsilon(h) \;\; \forall\, c \in C, h \in H, \tag{1.34}$$

where we used the Sweedler-type notation

$$\underline{\Delta}(c) = c_{\underline{1}} \otimes c_{\underline{2}}, \;\; (\underline{\Delta} \otimes id_C)(\underline{\Delta}(c)) = \sum c_{(\underline{1},\underline{1})} \otimes c_{(\underline{1},\underline{2})} \otimes c_{\underline{2}} \quad \text{etc.}$$

2 The quasi-smash product

The category of H-modules is monoidal, and an H-module (co)algebra is a (co)algebra in this category. This categorical definition cannot be used to introduce H-comodule algebras, since we do not have H-comodules. Hausser and Nill [18] gave a purely algebraic definition of an H-comodule algebra. We will show in Section 3.3 how their definition can be justified from a categorical point of view.

Definition 2.1 *[18] Let H be a quasi-bialgebra. A unital associative algebra $\mathfrak{A}$ is called a right H-comodule algebra if there exists an algebra morphism $\rho : \mathfrak{A} \to \mathfrak{A} \otimes H$ and an invertible element $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ such that*

$$\Phi_\rho(\rho \otimes id)(\rho(\mathfrak{a})) = (id \otimes \Delta)(\rho(\mathfrak{a}))\Phi_\rho, \;\; \text{for all } \mathfrak{a} \in \mathfrak{A}, \tag{2.1}$$
$$(1_{\mathfrak{A}} \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (id \otimes id \otimes \Delta)(\Phi_\rho)(\rho \otimes id \otimes id)(\Phi_\rho), \tag{2.2}$$
$$(id \otimes \varepsilon) \circ \rho = id, \tag{2.3}$$
$$(id \otimes \varepsilon \otimes id)(\Phi_\rho) = 1_{\mathfrak{A}} \otimes 1_H. \tag{2.4}$$

Similarly, a unital associative algebra $\mathfrak{B}$ is called a left H-comodule algebra if there exists an algebra morphism $\lambda : \mathfrak{B} \to H \otimes \mathfrak{B}$ and an invertible element $\Phi_\lambda \in H \otimes H \otimes \mathfrak{B}$ such that the following relations hold

$$(id \otimes \lambda)(\lambda(\mathfrak{b}))\Phi_\lambda = \Phi_\lambda(\Delta \otimes id)(\lambda(\mathfrak{b})), \;\; \text{for all } \mathfrak{b} \in \mathfrak{B}, \tag{2.5}$$
$$(1_H \otimes \Phi_\lambda)(id \otimes \Delta \otimes id)(\Phi_\lambda)(\Phi \otimes 1_{\mathfrak{B}}) = (id \otimes id \otimes \lambda)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_\lambda), \tag{2.6}$$
$$(\varepsilon \otimes id) \circ \lambda = id, \tag{2.7}$$
$$(id \otimes \varepsilon \otimes id)(\Phi_\lambda) = 1_H \otimes 1_{\mathfrak{B}}. \tag{2.8}$$

We notice that, when $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra we also have

$$(id \otimes id \otimes \varepsilon)(\Phi_\rho) = 1_{\mathfrak{A}} \otimes 1_H.$$

Similarly, if $(\mathfrak{B},\lambda,\Phi_\lambda)$ is a left H-comodule algebra then

$$(\varepsilon\otimes id\otimes id)(\Phi_\lambda)=1_H\otimes 1_{\mathfrak{B}}.$$

When H is a quasi-bialgebra, particular examples of left and right H-comodule algebras are given by $\mathfrak{A}=\mathfrak{B}=H$ and $\rho=\lambda=\Delta$, $\Phi_\rho=\Phi_\lambda=\Phi$.
For a right H-comodule algebra $(\mathfrak{A},\rho,\Phi_\rho)$ we will denote

$$\rho(\mathfrak{a})=\sum\mathfrak{a}_{<0>}\otimes\mathfrak{a}_{<1>},\ \ (\rho\otimes id)(\rho(\mathfrak{a}))=\sum\mathfrak{a}_{<0,0>}\otimes\mathfrak{a}_{<0,1>}\otimes\mathfrak{a}_{<1>}\ \text{ etc.}$$

for any $\mathfrak{a}\in\mathfrak{A}$. Similarly, for a left H-comodule algebra $(\mathfrak{B},\lambda,\Phi_\lambda)$, if $\mathfrak{b}\in\mathfrak{B}$ then we will denote

$$\lambda(\mathfrak{b})=\sum\mathfrak{b}_{[-1]}\otimes\mathfrak{b}_{[0]},\ \ (id\otimes\lambda)(\lambda(\mathfrak{b}))=\sum\mathfrak{b}_{[-1]}\otimes\mathfrak{b}_{[0,-1]}\otimes\mathfrak{b}_{[0,0]}\ \text{ etc.}$$

In analogy with the notation for the reassociator Φ of H, we will write

$$\Phi_\rho=\sum\tilde{X}^1_\rho\otimes\tilde{X}^2_\rho\otimes\tilde{X}^3_\rho=\sum\tilde{Y}^1_\rho\otimes\tilde{Y}^2_\rho\otimes\tilde{Y}^3_\rho=\text{ etc.}$$

and

$$\Phi_\rho^{-1}=\sum\tilde{x}^1_\rho\otimes\tilde{x}^2_\rho\otimes\tilde{x}^3_\rho=\sum\tilde{y}^1_\rho\otimes\tilde{y}^2_\rho\otimes\tilde{y}^3_\rho=\text{ etc.}$$

A similar notation is used for the element Φ_λ of a left H-comodule algebra $\mathfrak{B}$. If no confusion is possible, we will omit the subscripts ρ or λ in the tensor components of the $\Phi_\rho,\Phi_\lambda,\Phi_\rho^{-1}$ and Φ_λ^{-1}.
Recall that, if H is an algebra, then H^* is an (H,H)-bimodule, with left and right action given by $\langle h\rightharpoonup\varphi\leftharpoonup h',h''\rangle=\langle\varphi,h'h''h\rangle$, for all $h,h',h''\in H$ and $\varphi\in H^*$. If H is finite dimensional, then H^* is a coalgebra.
Now let H be a bialgebra and $\mathfrak{A}$ be a right H-comodule algebra. Then we can consider the smash product $\mathfrak{A}\#H^*$, with multiplication

$$(a\#\varphi)(a'\#\psi)=\sum aa'_{<0>}\#(\varphi\leftharpoonup a'_{<1>})\psi.$$

We will now generalize this construction to quasi-bialgebras. In this situation, the convolution product on H^* is not associative, but only quasi-associative, namely

$$[\varphi\psi]\xi=\sum(X^1\rightharpoonup\varphi\leftharpoonup x^1)[(X^2\rightharpoonup\psi\leftharpoonup x^2)(X^3\rightharpoonup\xi\leftharpoonup x^3)],\ \text{ for all }\varphi,\psi,\xi\in H^*.\tag{2.9}$$

In addition, for all $h\in H$ and $\varphi,\psi\in H^*$ we have that

$$h\rightharpoonup(\varphi\psi)=\sum(h_1\rightharpoonup\varphi)(h_2\rightharpoonup\psi)\ \text{ and }\ (\varphi\psi)\leftharpoonup h=\sum(\varphi\leftharpoonup h_1)(\psi\leftharpoonup h_2).\tag{2.10}$$

In other words, H^* is an algebra in the monoidal category of (H,H)-bimodules ${}_H\mathcal{M}_H$. Let $(\mathfrak{A},\rho,\Phi_\rho)$ be a right H-comodule algebra. We define a multiplication on $\mathfrak{A}\otimes H^*$ by

$$(\mathfrak{a}\,\overline{\#}\,\varphi)(\mathfrak{a}'\,\overline{\#}\,\psi)=\sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\,\overline{\#}\,(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(\psi\leftharpoonup\tilde{x}^3)\tag{2.11}$$

for all $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$ and $\varphi,\psi\in H^*$, where we write $\mathfrak{a}\,\overline{\#}\,\varphi$ for $\mathfrak{a}\otimes\varphi$, $\rho(\mathfrak{a})=\sum\mathfrak{a}_{<0>}\otimes\mathfrak{a}_{<1>}$, and $\Phi_\rho^{-1}=\sum\tilde{x}^1\otimes\tilde{x}^2\otimes\tilde{x}^3$. We denote this structure on $\mathfrak{A}\otimes H^*$ by $\mathfrak{A}\,\overline{\#}\,H^*$. In the next Proposition, we prove that $\mathfrak{A}\,\overline{\#}\,H^*$ is an algebra in the category of left H-modules, and this is why we call $\mathfrak{A}\,\overline{\#}\,H^*$ the quasi-smash product.

Proposition 2.2 *Let H be a quasi-bialgebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. Then $\mathfrak{A}\,\overline{\#}\,H^*$ is an H-module algebra with unit $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ and with left H-action given by*

$$h\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)=\mathfrak{a}\,\overline{\#}\,h\rightharpoonup\varphi\tag{2.12}$$

for all $h\in H,\ \mathfrak{a}\in\mathfrak{A}$ and $\varphi\in H^*$.

Proof. Since H^* is a left H-module via the action $\rightharpoonup$, it is easy to see that $\mathfrak{A}\,\overline{\#}\,H^*$ is a left H-module via the action (2.12). Now, we will prove that $\mathfrak{A}\,\overline{\#}\,H^*$ is an algebra in ${}_H\mathcal{M}$ with unit $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$. Indeed, for all $\mathfrak{a},\mathfrak{a}',\mathfrak{a}''\in\mathfrak{A}$ and $\varphi,\psi,\chi\in H^*$

$$
\begin{aligned}
&[X^1\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)]\{[X^2\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)][X^3\cdot(\mathfrak{a}''\,\overline{\#}\,\chi)]\}\\
&\quad= \sum(\mathfrak{a}\,\overline{\#}\,X^1\rightharpoonup\varphi)[(\mathfrak{a}'\,\overline{\#}\,X^2\rightharpoonup\psi)(\mathfrak{a}''\,\overline{\#}\,X^3\rightharpoonup\chi)]\\
&\quad= \sum(\mathfrak{a}\,\overline{\#}\,X^1\rightharpoonup\varphi)[\mathfrak{a}'\mathfrak{a}''_{<0>}\tilde{x}^1\,\overline{\#}\,(X^2\rightharpoonup\psi\leftharpoonup\mathfrak{a}''_{<1>}\tilde{x}^2)(X^3\rightharpoonup\chi\leftharpoonup\tilde{x}^3)]\\
(2.10)&\quad= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\mathfrak{a}''_{<0,0>}\tilde{x}^1_{<0>}\tilde{y}^1\,\overline{\#}\,(X^1\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{<1>}\mathfrak{a}''_{<0,1>}\tilde{x}^1_{<1>}\tilde{y}^2)\\
&\qquad[(X^2\rightharpoonup\psi\leftharpoonup\mathfrak{a}''_{<1>}\tilde{x}^2\tilde{y}^3_1)(X^3\rightharpoonup\chi\leftharpoonup\tilde{x}^3\tilde{y}^3_2)]\\
(2.9,2.2)&\quad= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\mathfrak{a}''_{<0,0>}\tilde{x}^1\tilde{y}^1\,\overline{\#}\,[(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\mathfrak{a}''_{<0,1>}\tilde{x}^2\tilde{y}^2_1)(\psi\leftharpoonup\mathfrak{a}''_{<1>}\tilde{x}^3\tilde{y}^2_2)]\\
&\qquad(\chi\leftharpoonup\tilde{y}^3)\\
(2.1,2.10)&\quad= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\mathfrak{a}''_{<0>}\tilde{y}^1\,\overline{\#}\,\{[(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(\psi\leftharpoonup\tilde{x}^3)]\leftharpoonup\mathfrak{a}''_{<1>}\tilde{y}^2\}(\chi\leftharpoonup\tilde{y}^3)\\
&\quad= \sum[\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\,\overline{\#}\,(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(\psi\leftharpoonup\tilde{x}^3)](\mathfrak{a}''\,\overline{\#}\,\chi)\\
&\quad= [(\mathfrak{a}\,\overline{\#}\,\varphi)(\mathfrak{a}'\,\overline{\#}\,\psi)](\mathfrak{a}''\,\overline{\#}\,\chi).
\end{aligned}
$$

It is not hard to see that $1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ is the unit of $\mathfrak{A}\,\overline{\#}\,H^*$ and that $h\cdot(1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon)=\varepsilon(h)1_{\mathfrak{A}}\,\overline{\#}\,\varepsilon$ for all $h\in H$. Finally, for all $h\in H$, $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$ and $\varphi,\psi\in H^*$, we calculate:

$$
\begin{aligned}
\sum[h_1\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)][h_2\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)] &= \sum(\mathfrak{a}\,\overline{\#}\,h_1\rightharpoonup\varphi)(\mathfrak{a}'\,\overline{\#}\,h_2\rightharpoonup\psi)\\
&= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\,\overline{\#}\,(h_1\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(h_2\rightharpoonup\psi\leftharpoonup\tilde{x}^3)\\
(2.10)\quad &= \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1\,\overline{\#}\,h\rightharpoonup[(\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2)(\psi\leftharpoonup\tilde{x}^3)]\\
(2.12)\quad &= h\cdot[(\mathfrak{a}\,\overline{\#}\,\varphi)(\mathfrak{a}'\,\overline{\#}\,\psi)].
\end{aligned}
$$

□

(H,Δ,Φ) is a right H-comodule algebra, so it makes sense to consider the quasi-smash product $H\,\overline{\#}\,H^*$. In this case where H is a Hopf algebra, $H\#H^*$ is called the Heisenberg double of H, and we will keep the same terminology for quasi-Hopf algebras. $\mathcal{H}(H)=H\,\overline{\#}\,H^*$ is not an associative algebra but it is an algebra in the monoidal category ${}_H\mathcal{M}$. If H is a finite dimensional Hopf algebra then $\mathcal{H}(H)$ is isomorphic to the algebra $\mathrm{End}_k(H)$. In order to prove a similar result for a finite dimensional quasi-Hopf algebra, we first have to deform the algebra structure of $\mathrm{End}_k(H)$.

Proposition 2.3 *Let H be a finite dimensional quasi-Hopf algebra. Define*

$$\mu:\ H\,\overline{\#}\,H^*\to\mathrm{End}_k(H),\quad \mu(h\,\overline{\#}\,\varphi)(h')=\sum\varphi(h'_2p^2_L)hh'_1p^1_L$$

for all $h,h'\in H$ and $\varphi\in H^*$, where $p_L=\sum p^1_L\otimes p^2_L$ is the element defined by (1.20). Then μ is a bijection, and therefore there exists a unique H-module algebra structure on $\mathrm{End}_k(H)$ such that μ becomes an H-module algebra isomorphism. The multiplication, the unit and the H-module structure of $\mathrm{End}_k(H)$ are given by

$$(u\overline{\circ}v)(h)=\sum u(v(hx^3X^3_2)S^{-1}(S(x^1X^2)\alpha x^2X^3_1))S^{-1}(X^1) \tag{2.13}$$

$$\mathbf{1}_{\mathrm{End}_k(H)}(h)=hS^{-1}(\beta)\quad;\quad(h\cdot u)(h')=\sum u(h'h_2)S^{-1}(h_1) \tag{2.14}$$

for all $u,v\in\mathrm{End}_k(H)$ and $h,h'\in H$.

Proof. Let $\{e_i\}_{i=\overline{1,n}}$ be a basis of H and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^*. We claim that the inverse of μ is $\mu^{-1}:\ \mathrm{End}_k(H)\to H\,\overline{\#}\,H^*$ given by

$$\mu^{-1}(u)=\sum u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i$$

for all $u\in \mathrm{End}_k(H)$, where $q_L=\sum q_L^1\otimes q_L^2$ is the element defined by (1.20). Indeed, for any $h\in H$ and $\varphi\in H^*$ we have:

$$\begin{aligned}
(\mu^{-1}\circ\mu)(h\,\overline{\#}\,\varphi) &= \sum_{i=1}^n \mu(h\,\overline{\#}\,\varphi)(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i\\
&= \sum_{i=1}^n \varphi((q_L^2)_2(e_i)_{(2,2)}p_L^2)h(q_L^2)_1(e_i)_{(2,1)}p_L^1S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i\\
\overset{(1.22)}{=}& \sum_{i=1}^n \varphi((q_L^2)_2p_L^2e_i)h(q_L^2)_1p_L^1S^{-1}(q_L^1)\,\overline{\#}\,e^i\\
\overset{(1.24)}{=}& \sum_{i=1}^n \varphi(e_i)h\,\overline{\#}\,e^i=h\,\overline{\#}\,\varphi
\end{aligned}$$

and, in a similar way, for $u\in \mathrm{End}_k(H)$ and $h\in H$ we have that $(\mu\circ\mu^{-1})(u)(h)=u(h)$. Using the bijection μ, we transport the H-module algebra structure from $H\,\overline{\#}\,H^*$ to $\mathrm{End}_k(H)$. First we compute the transported multiplication $\overline{\circ}$: for all $u,v\in \mathrm{End}_k(H)$, we find

$$\begin{aligned}
u\overline{\circ}v &= \sum \mu(\mu^{-1}(u)\mu^{-1}(v))\\
&= \sum_{i,j=1}^n \mu((u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)\,\overline{\#}\,e^i)(v(Q_L^2(e_j)_2)S^{-1}(Q_L^1(e_j)_1)\,\overline{\#}\,e^j))\\
\overset{(2.11)}{=}& \sum_{i,j=1}^n \mu(u(q_L^2(e_i)_2)S^{-1}(q_L^1(e_i)_1)[v(Q_L^2(e_j)_2)S^{-1}(Q_L^1(e_j)_1)]_1x^1\\
&\qquad \overline{\#}\,(e^i\leftharpoonup[v(Q_L^2(e_j)_2)S^{-1}(Q_L^1(e_j)_1)]_2x^2)(e^j\leftharpoonup x^3))
\end{aligned}$$

where $\sum Q_L^1\otimes Q_L^2$ is another copy of q_L. Note that (1.3) and (1.20) imply

$$\sum S(x^1)q_L^1x_1^2\otimes q_L^2x_2^2\otimes x^3=\sum q_L^1X^1\otimes(q_L^2)_1X^2\otimes(q_L^2)_2X^3. \tag{2.15}$$

Using the above arguments, a long but straightforward computation shows that

$$(u\overline{\circ}v)(h)=\sum u(v(hx^3X_2^3)S^{-1}(S(x^1X^2)\alpha x^2X_1^3))S^{-1}(X^1),$$

for all $h\in H$. Thus, we have obtained (2.13). Similar computations show that the tranported unit and the H-action on $\mathrm{End}_k(H)$ are given by (2.14). □

Remarks 2.4 Let H be a finite dimensional quasi-Hopf algebra, $\{e_i\}_{i=\overline{1,n}}$ a basis of H and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^*.

1) The bijection μ defined in Proposition 2.3 induces an associative algebra structure on the k-vector space $H\otimes H^*$: it suffices to transport the composition on $\mathrm{End}_k(H)$ to $H\otimes H^*$.

2) Let $(\mathfrak{A},\rho,\Phi_\rho)$ be a right H-comodule algebra. As in the Hopf case, it is possible to associate different (quasi)smash products to $\mathfrak{A}$. Observe first that the map $\nu:\ \mathfrak{A}\,\overline{\#}\,H^*\to \mathrm{Hom}_k(H,\mathfrak{A})$ given by $\nu(\mathfrak{a}\,\overline{\#}\,\varphi)(h)=\varphi(h)\mathfrak{a}$, for all $\mathfrak{a}\in\mathfrak{A}$, $\varphi\in H^*$ and $h\in H$, is a k-linear isomorphism. The inverse of ν is given by the formula

$$\nu^{-1}(w)=\sum_{i=1}^n w(e_i)\,\overline{\#}\,e^i$$

for $w \in \mathrm{Hom}_k(H,\mathfrak{A})$. Secondly, by transporting the quasi-smash algebra structure from $\mathfrak{A}\,\overline{\#}\,H^*$ to $\mathrm{Hom}_k(H,\mathfrak{A})$ via the isomorphism ν, we obtain that $\mathrm{Hom}_k(H,\mathfrak{A})$ is an H-module algebra. So, if H is an arbitrary quasi-Hopf algebra and $(\mathfrak{A},\rho,\Phi_\rho)$ is a right H-comodule algebra, then we can define the quasi-smash product $\overline{\#}\,(H,\mathfrak{A})$ as follows: $\overline{\#}\,(H,\mathfrak{A})$ is the k-vector space $\mathrm{Hom}_k(H,\mathfrak{A})$ with multiplication given by

$$(v*w)(h)=\sum v(w(\tilde{x}^3h_2)_{<1>}\tilde{x}^2h_1)w(\tilde{x}^3h_2)_{<0>}\tilde{x}^1 \tag{2.16}$$

for $v,w\in\overline{\#}\,(H,\mathfrak{A})$ and $h\in H$. The unit is $1_{\overline{\#}\,(H,\mathfrak{A})}(h)=\varepsilon(h)1_{\mathfrak{A}}$ and the H-module structure is given by $(h\cdot v)(h')=v(h'h)$, $h,h'\in H$, $v\in \mathrm{Hom}_k(H,\mathfrak{A})$. Of course, if H is finite dimensional then $\mathfrak{A}\,\overline{\#}\,H^*\simeq\overline{\#}\,(H,\mathfrak{A})$ as H-module algebras.

3 Two-sided Hopf modules and relative Hopf modules

3.1 Two-sided Hopf modules

The fact that a quasi-bialgbra is not coassociative entails that it makes no sense to consider comodules over quasi-bialgebras. Nevertheless, we can associate monoidal categories to quasi-bialgebras, in which we can consider coalgebras, and comodules over these coalgebras. This point of view has been used in [6], [20] and [26] in order to define relative Hopf modules, quasi-Hopf bimodules and two-sided two-cosided Hopf modules. In the sequel, we will study all these categories in a more general context. The categorical background will be presented in Section 3.3.

Definition 3.1 *Let H be a quasi-bialgebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. A two-sided $(H,\mathfrak{A})$-Hopf module is an $(H,\mathfrak{A})$-bimodule M together with a k-linear map*

$$\rho_M:\ M\to M\otimes H,\ \ \rho_M(m)=\sum m_{(0)}\otimes m_{(1)}$$

satisfying the following relations, for all $m\in M$, $h\in H$ and $\mathfrak{a}\in\mathfrak{A}$. The actions of $h\in H$ and $\mathfrak{a}\in\mathfrak{A}$ on $m\in M$ are denoted by $h\succ m$ and $m\prec\mathfrak{a}$.

$$(id_M\otimes\varepsilon)\circ\rho_M=id_M, \tag{3.1}$$
$$\Phi\cdot(\rho_M\otimes id_H)(\rho_M(m))=(id_M\otimes\Delta)(\rho_M(m))\cdot\Phi_\rho, \tag{3.2}$$
$$\rho_M(h\succ m)=\sum h_1\succ m_{(0)}\otimes h_2m_{(1)}, \tag{3.3}$$
$$\rho_M(m\prec\mathfrak{a})=\sum m_{(0)}\prec\mathfrak{a}_{<0>}\otimes m_{(1)}\mathfrak{a}_{<1>}. \tag{3.4}$$

The category of two-sided $(H,\mathfrak{A})$-Hopf modules and left H-linear, right $\mathfrak{A}$-linear and right H-colinear maps is denoted by ${}_H\mathcal{M}_{\mathfrak{A}}^H$.
Observe that the category of two-sided (H,H)-Hopf bimodules is nothing else then the category of right quasi-Hopf H-bimodules introduced in [20].
We will use the following notation, similar to the notation for the comultiplication on a quasi-bialgebra:

$$\begin{aligned}(\rho_M\otimes id_H)(\rho_M(m)) &= \sum m_{(0,0)}\otimes m_{(0,1)}\otimes m_{(1)},\\ (id_M\otimes\Delta_H)(\rho_M(m)) &= \sum m_{(0)}\otimes m_{(1)_1}\otimes m_{(1)_2}.\end{aligned}$$

Examples 3.2 Let H be a quasi-Hopf algebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra.
1) $\mathcal{V}=\mathfrak{A}\otimes H\in{}_H\mathcal{M}_{\mathfrak{A}}^H$. The structure maps are

$$h\succ(\mathfrak{a}\otimes h')=\mathfrak{a}\otimes hh'\quad;\quad(\mathfrak{a}\otimes h)\prec\mathfrak{a}'=\sum\mathfrak{a}\mathfrak{a}'_{<0>}\otimes h\mathfrak{a}'_{<1>}$$

and

$$\rho_{\mathcal{V}}(\mathfrak{a}\otimes h)=\sum \mathfrak{a}\tilde{X}^1\otimes h_1\tilde{X}^2\otimes h_2\tilde{X}^3$$

for all $h,h'\in H$ and $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$. Verification of the details is left to the reader.
2) $\mathcal{U}=H\otimes\mathfrak{A}\in{}_H\mathcal{M}_{\mathfrak{A}}^H$. Now the structure maps are given by the following formulas, for all $h,h'\in H$ and $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$:

$$h\succ(h'\otimes\mathfrak{a})=hh'\otimes\mathfrak{a}\quad;\quad(h\otimes\mathfrak{a})\prec\mathfrak{a}'=h\otimes\mathfrak{a}\mathfrak{a}'$$

and

$$\rho_{\mathcal{U}}(h\otimes\mathfrak{a})=\sum h_1S^{-1}(q_L^2\tilde{X}_2^3g^2)\otimes\tilde{X}^1\mathfrak{a}_{<0>}\otimes h_2S^{-1}(q_L^1\tilde{X}_1^3g^1)\tilde{X}^2\mathfrak{a}_{<1>}. \tag{3.5}$$

Here $q_L=\sum q_L^1\otimes q_L^2$ and $f^{-1}=\sum g^1\otimes g^2$ are the elements defined by the formulas (1.20) and (1.16).
To this end, consider $\theta:\ \mathcal{V}\to\mathcal{U}$ given by

$$\theta(\mathfrak{a}\otimes h)=\sum hS^{-1}(\mathfrak{a}_{<1>}\tilde{p}_\rho^2)\otimes\mathfrak{a}_{<0>}\tilde{p}_\rho^1$$

for all $h\in H$ and $\mathfrak{a}\in\mathfrak{A}$, where we use the notation

$$\tilde{p}_\rho=\sum\tilde{p}_\rho^1\otimes\tilde{p}_\rho^2=\sum\tilde{x}^1\otimes\tilde{x}^2\beta S(\tilde{x}^3)\in\mathfrak{A}\otimes H. \tag{3.6}$$

We claim that θ is bijective; its inverse $\theta^{-1}:\ \mathcal{U}\to\mathcal{V}$ is defined as follows

$$\theta^{-1}(h\otimes\mathfrak{a})=\sum\tilde{q}_\rho^1\mathfrak{a}_{<0>}\otimes h\tilde{q}_\rho^2\mathfrak{a}_{<1>}$$

with the notation

$$\tilde{q}_\rho=\sum\tilde{q}_\rho^1\otimes\tilde{q}_\rho^2=\sum\tilde{X}^1\otimes S^{-1}(\alpha\tilde{X}^3)\tilde{X}^2\in\mathfrak{A}\otimes H. \tag{3.7}$$

Furthermore, θ is a morphism of two-sided $(H,\mathfrak{A})$-Hopf bimodules, and we conclude that $\mathcal{U}=H\otimes\mathfrak{A}$ and $\mathfrak{A}\otimes H=\mathcal{V}$ are isomorphic in ${}_H\mathcal{M}_{\mathfrak{A}}^H$.
To prove this, we proceed as follows. First, by [18], we have the following relations, for all $\mathfrak{a}\in\mathfrak{A}$:

$$\sum\rho(\mathfrak{a}_{<0>})\tilde{p}_\rho[1_{\mathfrak{A}}\otimes S(\mathfrak{a}_{<1>})]=\tilde{p}_\rho[\mathfrak{a}\otimes 1_H], \tag{3.8}$$
$$\sum[1_{\mathfrak{A}}\otimes S^{-1}(\mathfrak{a}_{<1>})]\tilde{q}_\rho\rho(\mathfrak{a}_{<0>})=[\mathfrak{a}\otimes 1_H]\tilde{q}_\rho, \tag{3.9}$$
$$\sum\rho(\tilde{q}_\rho^1)\tilde{p}_\rho[1_{\mathfrak{A}}\otimes S(\tilde{q}_\rho^2)]=1_{\mathfrak{A}}\otimes 1_H, \tag{3.10}$$
$$\sum[1_{\mathfrak{A}}\otimes S^{-1}(\tilde{p}_\rho^2)]\tilde{q}_\rho\rho(\tilde{p}_\rho^1)=1_{\mathfrak{A}}\otimes 1_H, \tag{3.11}$$
$$\Phi_\rho(\rho\otimes id_H)(\tilde{p}_\rho)\tilde{p}_\rho=\sum(id\otimes\Delta)(\rho(\tilde{x}^1)\tilde{p}_\rho)(1_{\mathfrak{A}}\otimes g^1S(\tilde{x}^3)\otimes g^2S(\tilde{x}^2)), \tag{3.12}$$
$$(\tilde{q}_\rho\otimes 1_H)(\rho\otimes id_H)(\tilde{q}_\rho)\Phi_\rho^{-1}=\sum[1_{\mathfrak{A}}\otimes S^{-1}(f^2\tilde{X}^3)\otimes S^{-1}(f^1\tilde{X}^2)](id_{\mathfrak{A}}\otimes\Delta)(\tilde{q}_\rho\rho(\tilde{X}^1)). \tag{3.13}$$

Here $f=\sum f^1\otimes f^2$ is the element defined in (1.15) and $f^{-1}=\sum g^1\otimes g^2$. Using (3.8-3.11) we can show easily that θ and θ^{-1} are inverses, and that $\mathcal{U}$ is an $(H,\mathfrak{A})$-bimodule via the actions $\succ$ and $\prec$. One can finally compute the right H-coaction on $\mathcal{U}$ transported from the coaction on $\mathcal{V}$ using θ, and then see that it coincides with (3.5). For, observe that (3.6,2.2) and (2.4) imply

$$\sum\tilde{X}_{<1>}^1\tilde{p}_\rho^2S(\tilde{X}^2)\otimes\tilde{X}_{<0>}^1\tilde{p}_\rho^1\otimes\tilde{X}^3=\sum\tilde{x}^2S(\tilde{x}_1^3p_L^1)\otimes\tilde{x}^1\otimes\tilde{x}_2^3p_L^2, \tag{3.14}$$

where $p_L=\sum p_L^1\otimes p_L^2$ is the element defined in (1.20). We also mention that the computation uses the formula (3.13); the details are left to the reader.

3.2 Two-sided Hopf modules and relative Hopf modules

Our aim is to prove a duality Theorem for two-sided Hopf modules: if H is a finite dimensional quasi-Hopf algebra, then the category ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to a category of relative Hopf modules as introduced in [6]. Recall that a right (H^*, A)-Hopf module M is a k-vector space M which is also a right H^*-comodule and a right A-module in the monoidal category of right H^*-comodules $\mathcal{M}^{H^*}$. In terms of H this means:

- M is a left H-module; denote the action of $h \in H$ on $m \in M$ by $h \bullet m$;
- A acts on M from the right; denote the action of $a \in A$ on $m \in M$ by $m \bullet a$;
- for all $m \in M$, $h \in H$ and $a, a' \in A$, we have

$$m \bullet 1_A = m,$$
$$(m \bullet a) \bullet a' = \sum (X^1 \bullet m) \bullet [(X^2 \cdot a)(X^3 \cdot a')], \tag{3.15}$$
$$h \bullet (m \bullet a) = \sum (h_1 \bullet m) \bullet (h_2 \cdot a). \tag{3.16}$$

$\mathcal{M}_A^{H^*}$ will be the category of right (H^*, A)-Hopf modules and A-linear H^*-colinear maps. Before we can establish the claimed isomorphism of categories, we need some Lemmas.

Lemma 3.3 *Let H be a finite dimensional quasi-Hopf algebra and $(\mathfrak{A}, \rho, \Phi_\rho)$ a right H-comodule algebra. We have a functor*

$$F: {}_H\mathcal{M}_{\mathfrak{A}}^H \to \mathcal{M}^{H^*}_{\mathfrak{A} \overline{\#} H^*}.$$

For $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$, $F(M) = M$, with structure maps

- *M is a left H-module via $h \bullet m = S^2(h) \succ m$, $m \in M$, $h \in H$;*
- *$\mathfrak{A} \overline{\#} H^*$ acts on M from the right by*

$$m \bullet (\mathfrak{a} \overline{\#} \varphi) = \sum \langle \varphi, S^{-1}(S(U^1) f^2 m_{(1)} \mathfrak{a}_{<1>} \tilde{p}_\rho^2) \rangle S(U^2) f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>} \tilde{p}_\rho^1 \tag{3.17}$$

where we denote

$$U = \sum U^1 \otimes U^2 = \sum g^1 S(q_R^2) \otimes g^2 S(q_R^1). \tag{3.18}$$

Proof. The most difficult part of the proof is to show that $F(M)$ satisfies the relations (3.15) and (3.16). It is then straightforward to show that a map in ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is also a map in $\mathcal{M}^{H^*}_{\mathfrak{A} \overline{\#} H^*}$, and that F is a functor. By [20, Lemma 3.13] we have, for all $h \in H$:

$$U[1 \otimes S(h)] = \sum \Delta(S(h_1)) U (h_2 \otimes 1), \tag{3.19}$$
$$\Phi^{-1}(id \otimes \Delta)(U)(1 \otimes U) = \sum (\Delta \otimes id)(\Delta(S(X^1))U)(X^2 \otimes X^3 \otimes 1). \tag{3.20}$$

Write $f = \sum f^1 \otimes f^2 = \sum F^1 \otimes F^2$, $f^{-1} = \sum g^1 \otimes g^2$, $\tilde{p}_\rho = \sum \tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2 = \sum \tilde{P}_\rho^1 \otimes \tilde{P}_\rho^1$, and $U = \sum U^1 \otimes U^2 = \sum \mathbf{U}^1 \otimes \mathbf{U}^2$. For all $m \in M$, $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, and $\varphi, \psi \in H^*$, we compute that

$$\begin{aligned}
&(X^1 \bullet m) \bullet \{[X^2 \cdot (\mathfrak{a} \overline{\#} \varphi)][X^3 \cdot (\mathfrak{a}' \overline{\#} \psi)]\} \\
&\quad = \sum \langle (X^2 \rightharpoonup \varphi \leftharpoonup \mathfrak{a}'_{<1>} \tilde{x}^2)(X^3 \rightharpoonup \psi \leftharpoonup \tilde{x}^3), S^{-1}(S(U^1) f^2 S^2(X^1)_2 m_{(1)} \\
&\qquad (\mathfrak{a}\mathfrak{a}'_{<0>} \tilde{x}^1)_{<1>} \tilde{p}_\rho^2) \rangle S(U^2) f^1 S^2(X^1)_1 \succ m_{(0)} \prec (\mathfrak{a}\mathfrak{a}'_{<0>} \tilde{x}^1)_{<0>} \tilde{p}_\rho^1 \\
(1.11) &\quad = \sum \langle \varphi, S^{-1}(F^2 S(U^1)_2 S(S(X^1)_1)_2 f_2^2 m_{(1)_2} \mathfrak{a}_{<1>_2} \mathfrak{a}'_{<0,1>_2} \tilde{x}^1_{<1>_2} (\tilde{p}_\rho^2)_2 \\
&\qquad g^2 S(\mathfrak{a}'_{<1>} \tilde{x}^2)) X^2 \rangle \langle \psi, S^{-1}(F^1 S(U^1)_1 S(S(X^1)_1)_1 f_1^2 m_{(1)_1} \mathfrak{a}_{<1>_1}
\end{aligned}$$

$$
\begin{aligned}
& \mathfrak{a}'_{<0,1>_1}\tilde{x}^1_{<1>_1}(\tilde{p}^2_\rho)_1 g^1 S(\tilde{x}^3))X^3\rangle \\
& S(S(X^1)_2U^2)f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\mathfrak{a}'_{<0,0>}\tilde{x}^1_{<0>}\tilde{p}^1_\rho \\
(1.11, 3.13, 2.1) \quad = \quad & \sum\langle \varphi, S^{-1}(S(S(X^1)_{(1,1)}U^1_1X^2)F^2f^2_2 m_{(1)_2}\mathfrak{a}_{<1>_2}\tilde{X}^3\mathfrak{a}'_{<0,1>}\tilde{p}^2_\rho S(\mathfrak{a}'_{<1>})))\rangle \\
& \langle \psi, S^{-1}(S(S(X^1)_{(1,2)}U^1_2X^3)F^1f^2_1 m_{(1)_1}\mathfrak{a}_{<1>_1}\tilde{X}^2(\mathfrak{a}'_{<0,0>}\tilde{p}^1_\rho)_{<1>}\tilde{P}^2_\rho)\rangle \\
& S(S(X^1)_2U^2)f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{X}^1(\mathfrak{a}'_{<0,0>}\tilde{p}^1_\rho)_{<0>}\tilde{P}^1_\rho \\
(3.20, 3.8) \quad = \quad & \sum\langle \varphi, S^{-1}(S(x^1U^1)F^2f^2_2 m_{(1)_2}\mathfrak{a}_{<1>_2}\tilde{X}^3\tilde{p}^2_\rho)\rangle \\
& \langle \psi, S^{-1}(S(x^2U^2_1\mathbf{U}^1)F^1f^2_1 m_{(1)_1}\mathfrak{a}_{<1>_1}\tilde{X}^2(\tilde{p}^1_\rho\mathfrak{a}')_{<1>}\tilde{P}^2_\rho)\rangle \\
& S(x^3U^2_2\mathbf{U}^2)f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{X}^1(\tilde{p}^1_\rho\mathfrak{a}')_{<0>}\tilde{P}^2_\rho \\
(1.9, 1.18, 2.1) \quad = \quad & \sum\langle \varphi, S^{-1}(S(U^1)F^2 m_{(1)}\mathfrak{a}_{<1>}\tilde{p}^2_\rho)\rangle\langle \psi, S^{-1}(S(U^2_1\mathbf{U}^1)f^2F^1_2 m_{(0,1)}\mathfrak{a}_{<0,1>} \\
& (\tilde{p}^1_\rho\mathfrak{a}')_{<1>}\tilde{P}^2_\rho)\rangle S(U^2_2\mathbf{U}^2)f^1F^1_1 \succ m_{(0,0)} \prec \mathfrak{a}_{<0,0>}(\tilde{p}^1_\rho\mathfrak{a}')_{<0>}\tilde{P}^1_\rho \\
(1.11, 3.17) \quad = \quad & \sum\langle \varphi, S^{-1}(S(U^1)F^2 m_{(1)}\mathfrak{a}_{<1>}\tilde{p}^2_\rho)\rangle(S(U^2)F^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>}\tilde{p}^1_\rho)\bullet(\mathfrak{a}'\,\overline{\#}\,\psi) \\
(3.17) \quad = \quad & [m\bullet(\mathfrak{a}\,\overline{\#}\,\varphi)]\bullet(\mathfrak{a}'\,\overline{\#}\,\psi).
\end{aligned}
$$

Similar computations show that

$$\sum(h_1\bullet m)\bullet(h_2\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)) = h\bullet[m\bullet(\mathfrak{a}\,\overline{\#}\,\varphi)],$$

for all $h\in H$, $\mathfrak{a}\in\mathfrak{A}$ and $\varphi\in H^*$, so the proof is complete. □

Let us next discuss the construction in the converse direction.

Lemma 3.4 *Let H be a finite dimensional quasi-Hopf algebra, $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra and M a right $(H^*,\mathfrak{A}\,\overline{\#}\,H^*)$-Hopf module. Then we have a functor*

$$G:\ \mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*} \to {}_H\mathcal{M}^H_{\mathfrak{A}}.$$

For $M\in\mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*}$, $G(M)=M$, with structure maps ($h\in H$, $m\in M$, $\mathfrak{a}\in\mathfrak{A}$):

- *$h{\succ}m = S^{-2}(h)\bullet m$;*
- *$m{\prec}\mathfrak{a} = m\bullet(\mathfrak{a}\,\overline{\#}\,\varepsilon)$;*
- *$\rho_M:\ M\to M\otimes H$ is given by*

$$\rho_M(m)=\sum m_{\{0\}}\otimes m_{\{1\}}=\sum_{i=1}^{n}[S^{-1}(V^2g^2)\bullet m]\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\otimes e_i \tag{3.21}$$

where $\{e_i\}_{i=\overline{1,n}}$ and $\{e^i\}_{i=\overline{1,n}}$ are dual bases and

$$V=\sum V^1\otimes V^2=\sum S^{-1}(f^2p^2_R)\otimes S^{-1}(f^1p^1_R). \tag{3.22}$$

Proof. As in the previous part, the main thing to show is that $G(M)$ is an object of ${}_H\mathcal{M}^H_{\mathfrak{A}}$. It is then straightforward to show that G behaves well on the level of the morphisms (G is the identity on the morphisms).
From the fact that S^{-2} is an algebra map, it follows that M is a left H-module via the action $h\succ m = S^{-2}(h)\bullet m$. Take the map

$$i:\ \mathfrak{A}\to\mathfrak{A}\,\overline{\#}\,H^*,\quad i(\mathfrak{a})=\mathfrak{a}\,\overline{\#}\,\varepsilon$$

for all $\mathfrak{a}\in\mathfrak{A}$. Then i is injective map, $i(1_{\mathfrak{A}})=1_{\mathfrak{A}\,\overline{\#}\,H^*}$, and $i(\mathfrak{a}\mathfrak{a}')=i(\mathfrak{a})i(\mathfrak{a}')$, for all $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$. Therefore, M becomes a right $\mathfrak{A}$-module by setting $m\prec\mathfrak{a}=m\bullet i(\mathfrak{a})=m\bullet(\mathfrak{a}\,\overline{\#}\,\varepsilon)$, $m\in M$, $\mathfrak{a}\in\mathfrak{A}$. Moreover, it is not hard to see that, with this structure, M is an $(H,\mathfrak{A})$-bimodule. In order to check the relations (3.1-3.3) we need some formulas due to Hausser and Nill [18, Lemma 3.13], namely

$$[1\otimes S^{-1}(h)]V=\sum(h_2\otimes 1)V\Delta(S^{-1}(h_1)), \tag{3.23}$$

$$(\Delta\otimes id)(V)\Phi^{-1}=\sum(X^2\otimes X^3\otimes 1)(1\otimes V)(id\otimes\Delta)(V\Delta(S^{-1}(X^1))). \tag{3.24}$$

Also, it is clear that

$$(\varphi\leftharpoonup h)S=S^{-1}(h)\rightharpoonup\varphi S\quad;\quad(h\rightharpoonup\varphi)S=\varphi S\leftharpoonup S^{-1}(h) \tag{3.25}$$

for all $h\in H$ and $\varphi\in H^*$. Using (1.11), it follows that

$$(\varphi S)(\psi S)=\sum[(g^1\rightharpoonup\psi\leftharpoonup f^1)(g^2\rightharpoonup\varphi\leftharpoonup f^2)]S \tag{3.26}$$

for all $\varphi,\psi\in H^*$. Now, for any $h\in H$ and $m\in M$ we compute that

$$\begin{aligned}
&\sum h_1\succ m_{\{0\}}\otimes h_2m_{\{1\}}\\
&=\sum_{i=1}^{n}S^{-2}(h_1)\bullet[(S^{-1}(V^2g^2)\bullet m)\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)]\otimes h_2e_i\\
\stackrel{(3.16)}{=}&\sum_{i=1}^{n}[S^{-2}(h_1)_1S^{-1}(V^2g^2)\bullet m]\\
&\quad\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-2}(h_1)_2S^{-1}(V^1g^1)\rightharpoonup(e^i\leftharpoonup h_2)S\leftharpoonup\tilde{q}^2_\rho)\otimes e_i\\
\stackrel{(1.11,3.25)}{=}&\sum_{i=1}^{n}[S^{-1}(V^2S^{-1}(h_1)_2g^2)\bullet m]\\
&\quad\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(h_2V^1S^{-1}(h_1)_1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\otimes e_i\\
\stackrel{(3.23)}{=}&\sum_{i=1}^{n}[S^{-1}(V^2g^2)S^{-2}(h)\bullet m]\bullet(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\otimes e_i\\
&=\rho_M(S^{-2}(h)\bullet m)=\rho_M(h\succ m),
\end{aligned}$$

and similarly, for any $m\in M$ and $\mathfrak{a}\in\mathfrak{A}$ one can show that

$$\sum m_{\{0\}}\prec\mathfrak{a}_{<0>}\otimes m_{\{1\}}\mathfrak{a}_{<1>}=\rho_M(m\prec\mathfrak{a}),$$

so the relations (3.3) hold. (3.1) is obviously satisfied, thus remain to check (3.2) for our structures. This fact is left to the reader since it is a similar computation as above. □

We are now able to prove the main result of this Section, generalizing [12, Proposition 2.3].

Theorem 3.5 *Let H be a finite dimensional quasi-Hopf algebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. Then the category of two-sided $(H,\mathfrak{A})$-Hopf modules ${}_H\mathcal{M}^H_{\mathfrak{A}}$ is isomorphic to the category of right $(H^*,\mathfrak{A}\,\overline{\#}\,H^*)$-Hopf modules $\mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*}$.*

Proof. It suffices to show that the functors F and G from Lemmas 3.3 and 3.4 are inverses.
First, let $M\in{}_H\mathcal{M}^H_{\mathfrak{A}}$. The structures on $G(F(M))$ (using first Lemma 3.3 and then Lemma 3.4) are denoted by $\succ'$, $\prec'$ and ρ'_M. For any $m\in M$, $h\in H$ and $\mathfrak{a}\in\mathfrak{A}$ we have that

$$\begin{aligned}
h\succ' m&=S^{-2}(h)\bullet m=S^2(S^{-2}(h))\succ m=h\succ m\\
m\prec'\mathfrak{a}&=m\bullet(\mathfrak{a}\,\overline{\#}\,\varepsilon)=m\prec\mathfrak{a}
\end{aligned}$$

because $\sum\varepsilon(U^1)U^2=\sum\varepsilon(f^2)f^1=1$ and $\sum\varepsilon(m_{(1)})m_{(0)}=m$, $\sum\varepsilon(\mathfrak{a}_{<1>})\mathfrak{a}_{<0>}=\mathfrak{a}$. In order to prove that $\rho'_M=\rho_M$, observe first that

$$\sum g^1S(g^2\alpha)=\beta, \tag{3.27}$$

where we write $f^{-1}=\sum g^1\otimes g^2$. The proof of (3.27) can be found in [6, Lemma 2.6 (i)] (in the equivalent form $\sum g^2\alpha S^{-1}(g^1)=S^{-1}(\beta)$). (3.27) together with (3.18,1.9) and (1.18) implies

$$\sum g_2^2U^2\otimes g^1S(g_1^2U^1)=\sum p_L^2\otimes S(p_L^1) \tag{3.28}$$

where $p_L=\sum p_L^1\otimes p_L^2$ is the element defined by (1.20). Secondly, by $\sum S^{-1}(f^2)\beta f^1=S^{-1}(\alpha)$, (1.9) and (1.18) we have that

$$\sum S(p_L^2)f^1F_1^1\otimes S^{-1}(F^2)S(p_L^1)f^2F_2^1=q_R \tag{3.29}$$

were $\sum F^1\otimes F^2$ is another copy of f, and q_R is the element defined by (1.19). Finally, from (3.28,3.29) and (1.23), it follows that

$$\sum S(g_2^2U^2)f^1F_1^1(p_R^1)_1\otimes S^{-1}(F^2p_R^2)g^1S(g_1^2U^1)f^2F_2^1(p_R^1)_2=1\otimes 1. \tag{3.30}$$

We now compute for $m\in M$ that

$$\begin{aligned}
\rho'_M(m) &= \sum_{i=1}^n[S^{-1}(V^2g^2)\bullet m]\bullet(\tilde{q}_\rho^1\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}_\rho^2)\otimes e_i\\
&= \sum_{i=1}^n[S(V^2g^2)\succ m]\bullet(\tilde{q}_\rho^1\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}_\rho^2)\otimes e_i\\
(3.17)\quad &= \sum_{i=1}^n<S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}_\rho^2,S^{-1}(S(U^1)f^2S(V^2g^2)_2m_{(1)}(\tilde{q}_\rho^1)_{<1>}\tilde{p}_\rho^2)>\\
&\quad S(U^2)f^1S(V^2g^2)_1\succ m_{(0)}\prec(\tilde{q}_\rho^1)_{<0>}\tilde{p}_\rho^1\otimes e_i\\
(1.11)\quad &= \sum S(V_2^2g_2^2U^2)f^1\succ m_{(0)}\prec(\tilde{q}_\rho^1)_{<0>}\tilde{p}_\rho^1\otimes V^1g^1S(V_1^2g_1^2U^1)f^2\\
&\quad m_{(1)}(\tilde{q}_\rho^1)_{<1>}\tilde{p}_\rho^2S(\tilde{q}_\rho^2)\\
(3.10)\quad &= \sum S(V_2^2g_2^2U^2)f^1\succ m_{(0)}\otimes V^1g^1S(V_1^2g_1^2U^1)f^2m_{(1)}\\
(3.22,1.11)\quad &= \sum S(g_2^2U^2)f^1F_1^1(p_R^1)_1\succ m_{(0)}\otimes S^{-1}(F^2p_R^2)g^1S(g_1^2U^1)f^2F_2^1(p_R^1)_2m_{(1)}\\
(3.30)\quad &= \sum m_{(0)}\otimes m_{(1)}=\rho_M(m)
\end{aligned}$$

and this finishes the proof of the fact that $G(F(M))=M$.

Conversely, take $M\in\mathcal{M}^{H^*}_{\mathfrak{A}\,\overline{\#}\,H^*}$. We want to show that $F(G(M))=M$. Denote the left H-action and the right $\mathfrak{A}\,\overline{\#}\,H^*$-action on $F(G(M))$ by $\bullet'$. Using Lemmas 3.3 and 3.4, we find, for all $h\in H$ and $m\in M$:

$$h\bullet' m=S^2(h)\succ m=S^{-2}(S^2(h))\bullet m=h\bullet m.$$

The proof of the fact that the right $\mathfrak{A}\,\overline{\#}\,H^*$-actions $\bullet$ and $\bullet'$ on M coincide is somewhat more complicated. Since $\sum f^2S^{-1}(f^1\beta)=\alpha$, (1.9) and (1.18) imply

$$\sum F^1f_1^1p_R^1\otimes f^2S^{-1}(F^2f_2^1p_R^2)=\sum S(q_L^2)\otimes q_L^1 \tag{3.31}$$

where $q_L=\sum q_L^1\otimes q_L^2$ is the element defined by (1.20). Also, by (1.9), (1.18) and using $\sum S(g^1)\alpha g^2=S(\beta)$ we can prove the following relation

$$\sum S(G^1)q_L^1G_1^2g^1\otimes q_L^2G_2^2g^2=\sum S(p_R^2)\otimes S(p_R^1) \tag{3.32}$$

where $\sum G^1\otimes G^2$ is another copy of f^{-1}. Now, from (3.18,1.11,3.31,3.32) and (1.23) it follows that

$$\sum S^{-1}(F^1 f_1^1 p_R^1)U_2^2 g^2\otimes S(U^1)f^2 S^{-1}(F^2 f_2^1 p_R^2)U_1^2 g^1 = 1\otimes 1. \tag{3.33}$$

Therefore, for all $m\in M$, $\mathfrak{a}\in\mathfrak{A}$ and $\varphi\in H^*$ we have that

$$\begin{aligned}
& m\bullet'(\mathfrak{a}\,\overline{\#}\,\varphi)\\
\overset{(3.17)}{=}\ & \sum\langle\varphi, S^{-1}(S(U^1)f^2 m_{\{1\}}\mathfrak{a}_{<1>}\tilde{p}_\rho^2)\rangle S(U^2)f^1\succ m_{\{0\}}\prec\mathfrak{a}_{<0>}\tilde{p}_\rho^1\\
\overset{(3.21,3.15,2.11)}{=}\ & \sum_{i=1}^n\langle\varphi, S^{-1}(S(U^1)f^2 e_i\mathfrak{a}_{<1>}\tilde{p}_\rho^2)\rangle S^{-2}(S(U^2)f^1)\bullet\{[S^{-1}(V^2g^2)\bullet m]\\
& \bullet[\tilde{q}_\rho^1\mathfrak{a}_{<0,0>}(\tilde{p}_\rho^1)_{<0>}\,\overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup e^iS\leftharpoonup\tilde{q}_\rho^2\mathfrak{a}_{<0,1>}(\tilde{p}_\rho^1)_{<1>}]\}\\
=\ & \sum_{i=1}^n\varphi(e_i)S^{-2}(S(U^2)f^1)\bullet\{[S^{-1}(V^2g^2)\bullet m]\bullet[\tilde{q}_\rho^1\mathfrak{a}_{<0,0>}(\tilde{p}_\rho^1)_{<0>}\\
& \overline{\#}\,S^{-1}(V^1g^1)\rightharpoonup(\mathfrak{a}_{<1>}\tilde{p}_\rho^2\rightharpoonup e^iS^{-1}\leftharpoonup S(U^1)f^2)S\\
& \leftharpoonup\tilde{q}_\rho^2\mathfrak{a}_{<0,1>}(\tilde{p}_\rho^1)_{<1>}]\}\\
\overset{(3.25,3.8,3.10)}{=}\ & \sum S^{-2}(S(U^2)f^1)\bullet\{[S^{-1}(V^2g^2)\bullet m]\\
& \bullet[\mathfrak{a}\,\overline{\#}\,S^{-1}(S(U^1)f^2V^1g^1)\rightharpoonup\varphi]\}\\
\overset{(3.16,1.11)}{=}\ & \sum[S^{-1}(V^2S^{-1}(S(U^2)f^1)_2g^2)\bullet m]\\
& \bullet[\mathfrak{a}\,\overline{\#}\,S^{-1}(S(U^1)f^2V^1S^{-1}(S(U^2)f^1)_1g^1))\rightharpoonup\varphi]\\
\overset{(3.22,1.11)}{=}\ & \sum[S^{-1}(S^{-1}(F^1f_1^1p_R^1)U_2^2g^2)\bullet m]\\
& \bullet[\mathfrak{a}\,\overline{\#}\,S^{-1}(S(U^1)f^2S^{-1}(F^2f_2^1p_R^2)U_1^2g^1)\rightharpoonup\varphi]\\
\overset{(3.33)}{=}\ & m\bullet(\mathfrak{a}\,\overline{\#}\,\varphi)
\end{aligned}$$

and this finishes our proof. □

If H is a finite dimensional quasi-Hopf algebra and A is a left H-module algebra then the category $\mathcal{M}_A^{H^*}$ is isomorphic to the category of right modules over the smash product $A\#H$ ([6, Proposition 2.7]). Let M be a right $A\#H$-module, and denote the right action of $a\#h\in A\#H$ on $m\in M$ by $m\leftarrow(a\#h)$. Following [6], M is a right (H^*,A)-Hopf module, with structure maps

$$h\bullet m = m\leftarrow(1\#S(h)),\ \ m\bullet a=\sum m\leftarrow[g^1S(q_R^2)\cdot a\#g^2S(q_R^1)] \tag{3.34}$$

for all $m\in M$, $a\in A$ and $h\in H$. Conversely, if M is a right (H^*,A)-Hopf module then M is a right $A\#H$-module, with $A\#H$-action

$$m\leftarrow(a\#h)=\sum S^{-1}(h)\bullet[(S^{-1}(q_L^2g^2)\bullet m)\bullet(S^{-1}(q_L^1g^1)\cdot a)]. \tag{3.35}$$

Here $q_R=\sum q_R^1\otimes q_R^2$, $q_L=\sum q_L^1\otimes q_L^2$ and $f^{-1}=\sum g^1\otimes g^2$ are the elements defined by (1.19), (1.20) and (1.16). Combining this with Theorem 3.5, we obtain the following result.

Corollary 3.6 *Let H be a finite dimensional quasi-Hopf algebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. Then the category ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to the category of right $(\mathfrak{A}\,\overline{\#}\,H^*)\#H$-modules, $\mathcal{M}_{(\mathfrak{A}\,\overline{\#}\,H^*)\#H}$.*

For later use, we describe the isomorphism of Corollary 3.6 explicitely, leaving verification of the details to the reader.

First take $M \in \mathcal{M}_{(\mathfrak{A} \overline{\#} H^*)\#H}$. The following structure maps make $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$:

$$h \succ m = m \leftharpoonup ((1_{\mathfrak{A}} \overline{\#} \varepsilon)\#S^{-1}(h)) \tag{3.36}$$
$$m \prec \mathfrak{a} = m \leftharpoonup ((\mathfrak{a} \overline{\#} \varepsilon)\#1) \tag{3.37}$$
$$\rho_M(m) = \sum_{i=1}^{n} m \leftharpoonup [(\tilde{q}^1_\rho \overline{\#} S^{-1}(g^2) \rightharpoonup e^i S \leftharpoonup \tilde{q}^2_\rho)\#S^{-1}(g^1)] \otimes e_i \tag{3.38}$$

for all $m \in M$, $h \in H$ and $\mathfrak{a} \in \mathfrak{A}$. $\tilde{q}_\rho = \sum \tilde{q}^1_\rho \otimes \tilde{q}^2_\rho$ is the element defined in (3.7), $\{e_i\}$ is a basis of H and $\{e^i\}$ is the corresponding dual basis of H^*.
Now take $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$. Then M is a right $(\mathfrak{A} \overline{\#} H^*)\#H$-module via the action

$$m \leftharpoonup [(\mathfrak{a} \overline{\#} \varphi)\#h] = \sum < \varphi, S^{-1}(f^2 m_{(1)} \mathfrak{a}_{<1>} \tilde{p}^2_\rho) > S(h) f^1 \succ m_{(0)} \prec \mathfrak{a}_{<0>} \tilde{p}^1_\rho. \tag{3.39}$$

In [20], it is announced that, for a finite dimensional quasi-Hopf algebra H, the category of right quasi-Hopf H-bimodules ${}_H\mathcal{M}_H^H$ naturally coincides with the category of representations of the two-sided crossed product $H \rtimes H^* \ltimes H$ constructed in [18]. We will show in Section 4 that the algebras $H \rtimes H^* \ltimes H$ and $(H \overline{\#} H^*)\#H$ are equal.

3.3 Two-sided Hopf modules and coalgebras over comonads

Now, let H be a quasi-bialgebra and $\mathfrak{A}$ a right H-comodule algebra. We will show that the category ${}_H\mathcal{M}_{\mathfrak{A}}^H$ is isomorphic to the category of $\mathbb{U}$-coalgebras, where $\mathbb{U}$ is a suitable comonad. Recall that if $\mathcal{D}$ is a category then a comonad on $\mathcal{D}$ is a threetuple $\mathbb{U} = (U, \Delta, \varepsilon)$, where $U : \mathcal{D} \to \mathcal{D}$ is a functor, and $\Delta : U \to U \circ U$ and $\varepsilon : U \to 1_{\mathcal{D}}$ are natural transformations, such that

$$U(\Delta_M) \circ \Delta_M = \Delta_{U(M)} \circ \Delta_M, \tag{3.40}$$
$$U(\varepsilon_M) \circ \Delta_M = \varepsilon_{U(M)} \circ \Delta_M = id_{U(M)} \tag{3.41}$$

for all $M \in \mathcal{D}$. A morphism between two $\mathcal{D}$-comonads $\mathbb{U} = (U, \Delta, \varepsilon)$ and $\mathbb{U}' = (U', \Delta', \varepsilon')$ is a natural transformation $\vartheta : U \to U'$ such that

$$\varepsilon' \circ \vartheta = \varepsilon \text{ and } (\vartheta * \vartheta) \circ \Delta = \Delta' \circ \vartheta \tag{3.42}$$

for all $M \in \mathcal{D}$, where $*$ is the Godement product

$$(\vartheta * \vartheta)_M = \vartheta_{U'(M)} \circ U(\vartheta_M).$$

We denote by $\underline{\underline{\text{Comonad}}}(\mathcal{D})$ the category of comonads on $\mathcal{D}$.
For $\mathbb{U}$ a comonad on $\mathcal{D}$, a $\mathbb{U}$-coalgebra is a pair (M, ξ), with $M \in \mathcal{D}$ and $\xi : M \to U(M)$ a morphism in $\mathcal{D}$ such that

$$\varepsilon_M \circ \xi = id_M \text{ and } \Delta_M \circ \xi = U(\xi) \circ \xi. \tag{3.43}$$

A morhism between two $\mathbb{U}$-coalgebras (M, ξ) and (M', ξ') consists of a morphism $\upsilon : M \to M'$ in $\mathcal{D}$ such that

$$U(\upsilon) \circ \xi = \xi' \circ \upsilon. \tag{3.44}$$

The category of $\mathbb{U}$-coalgebras is denoted by $\mathcal{D}^{\mathbb{U}}$.
If H is a quasi-bialgebra and $\mathfrak{A}$ an algebra then we define $\mathcal{C} := {}_{\mathfrak{A} \otimes H}\mathcal{M}_{\mathfrak{A}}$. Thus, an object of $\mathcal{C}$ is an $\mathfrak{A}$-bimodule and an $(H, \mathfrak{A})$-bimodule such that $h(\mathfrak{a}m) = \mathfrak{a}(hm)$, for all $\mathfrak{a} \in \mathfrak{A}$, $h \in H$ and $m \in M$. Morphisms

are left H-linear maps which are also $\mathfrak{A}$-bimodule maps. We claim that $\mathcal{C}$ is a monoidal category. Indeed, it is not hard to see that $\mathcal{C}$ becomes a monoidal category with tensor product $\otimes_{\mathfrak{A}}$ given via Δ, in the sense that

$$(\mathfrak{a}\otimes h)(m\otimes_{\mathfrak{A}} n)\mathfrak{a}^{'} := \sum \mathfrak{a}h_1 m\otimes_{\mathfrak{A}} h_2 n\mathfrak{a}^{'}$$

for all $M,N\in\mathcal{C}$, $m\in M$, $n\in N$, $\mathfrak{a},\mathfrak{a}^{'}\in\mathfrak{A}$ and $h\in H$, associativity constraints

$$\underline{a}_{M,N,P} : (M\otimes_{\mathfrak{A}} N)\otimes_{\mathfrak{A}} P \to M\otimes_{\mathfrak{A}} (N\otimes_{\mathfrak{A}} P),$$
$$\underline{a}_{M,N,P}((m\otimes_{\mathfrak{A}} n)\otimes_{\mathfrak{A}} p) = \sum X^1 m\otimes_{\mathfrak{A}} (X^2 n\otimes_{\mathfrak{A}} X^3 p),$$

unit $\mathfrak{A}$ as a trivial left H-module, and the usual left and right unit constraints. We denote by $\mathcal{C}$-$\underline{\underline{\text{Coalgebra}}}$ the category of coalgebras in $\mathcal{C}$. We are able now to prove the claimed isomorphism.

Theorem 3.7 *Let H be a quasi-bialgebra, $\mathfrak{A}$ an algebra, $\mathcal{C} = {}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$ and $\mathcal{D} := {}_H\mathcal{M}_{\mathfrak{A}}$. Then there exists a functor*

$$F : \mathcal{C}\text{-}\underline{\underline{\text{Coalgebra}}} \to \underline{\underline{\text{Comonad}}}(\mathcal{D}).$$

In addition, if $\mathfrak{A}$ is a right H-comodule algebra then $\mathfrak{C} := \mathfrak{A}\otimes H$ is a coalgebra in $\mathcal{C}$ and, in this particular case, we have an isomorphism of categories

$$\mathcal{D}^{F(\mathfrak{C})} \cong {}_H\mathcal{M}_{\mathfrak{A}}^H.$$

Proof. If $\mathfrak{C}$ is a coalgebra in $\mathcal{C}$ then it is an $(H,\mathfrak{A})$-bimodule and an $\mathfrak{A}$-bimodule so, we have a functor $U = (-)\otimes_{\mathfrak{A}}\mathfrak{C} : \mathcal{D}\to\mathcal{D}$ (for any $M\in\mathcal{D}$, the left H-module structure of $U(M)$ is given via Δ and the right $\mathfrak{A}$-action on $U(M)$ is induced by the one on $\mathfrak{C}$). For all $M\in\mathcal{D}$ we define

$$\Delta_M : M\otimes_{\mathfrak{A}}\mathfrak{C} = U(M) \to U(U(M)) = (M\otimes_{\mathfrak{A}}\mathfrak{C})\otimes_{\mathfrak{A}}\mathfrak{C},$$
$$\Delta_M(m\otimes_{\mathfrak{A}} c) = \sum (x^1 m\otimes_{\mathfrak{A}} x^2 c_{\underline{1}})\otimes_{\mathfrak{A}} x^3 c_{\underline{2}},$$
$$\varepsilon_M := id_M\otimes_{\mathfrak{A}}\underline{\varepsilon}_{\mathfrak{C}} : M\otimes_{\mathfrak{A}}\mathfrak{C} = U(M)\to M\cong M\otimes_{\mathfrak{A}} A$$

for all $m\in M$ and $c\in\mathfrak{C}$, where $\underline{\Delta}_{\mathfrak{C}}(c) := \sum c_{\underline{1}}\otimes c_{\underline{2}}$ is the comultiplication of $\mathfrak{C}$, and $\underline{\varepsilon}_{\mathfrak{C}}$ is the counit of $\mathfrak{C}$. It is not hard to see that $F(\mathfrak{C}) := (U,\Delta_M,\varepsilon_M)$ is a comonad on $\mathcal{D}$. It is also straightforward to check that a morphism κ in $\mathcal{C}$-$\underline{\underline{\text{Coalgebra}}}$ provides a morphism $U(\kappa)$ in $\underline{\underline{\text{Comonad}}}(\mathcal{D})$, and that F is a functor.
Suppose now that $(\mathfrak{A},\rho,\Phi_\rho)$ is a right H-comodule algebra and let $\mathfrak{C} = \mathfrak{A}\otimes H$. If we define

$$(\mathfrak{a}\otimes h)(\mathfrak{a}^{'}\otimes h^{'})\mathfrak{a}^{''} := \sum \mathfrak{a}\mathfrak{a}^{'}\mathfrak{a}^{''}_{<0>}\otimes hh^{'}\mathfrak{a}^{''}_{<1>} \tag{3.45}$$

for all $\mathfrak{a},\mathfrak{a}^{'},\mathfrak{a}^{''}\in\mathfrak{A}$ and $h,h^{'}\in H$, then one can easily check that with this structure $\mathfrak{C}\in\mathcal{C}$. Moreover, we claim that $\mathfrak{C}$ with the structure given by

$$\underline{\Delta}_{\mathfrak{C}}(\mathfrak{a}\otimes h) := \sum (\mathfrak{a}\tilde{X}^1\otimes h_1\tilde{X}^2)\otimes_{\mathfrak{A}} (1_{\mathfrak{A}}\otimes h_2\tilde{X}^3), \tag{3.46}$$
$$\underline{\varepsilon}_{\mathfrak{C}}(\mathfrak{a}\otimes h) := \varepsilon(h)\mathfrak{a}, \tag{3.47}$$

for all $\mathfrak{a}\in\mathfrak{A}$ and $h\in H$, becomes a coalgebra in $\mathcal{C}$. Indeed, the fact that $\underline{\Delta}_{\mathfrak{C}}$ and $\underline{\varepsilon}_{\mathfrak{C}}$ are morphisms in $\mathcal{C}$ and that $\underline{\varepsilon}_{\mathfrak{C}}$ is the counit for $\underline{\Delta}_{\mathfrak{C}}$ follow from straightforward computations (all these verifications are left to the reader). We only show that the comultiplication $\underline{\Delta}_{\mathfrak{C}}$ is coassociative up to the associativity constraints of $\mathcal{C}$. Indeed, we compute that

$$
\begin{aligned}
&(\underline{\Delta}_{\mathfrak{C}} \otimes_{\mathfrak{A}} id)(\underline{\Delta}_{\mathfrak{C}}(\mathfrak{a}\otimes h)) \\
&\quad = \sum \underline{\Delta}_{\mathfrak{C}}(\mathfrak{a}\tilde{X}^1 \otimes h_1\tilde{X}^2) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes h_2\tilde{X}^3) \\
&\quad = \sum (\mathfrak{a}\tilde{X}^1\tilde{Y}^1 \otimes h_{(1,1)}\tilde{X}^2_1\tilde{Y}^2) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes h_{(1,2)}\tilde{X}^2_2\tilde{Y}^3) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes h_2\tilde{X}^3) \\
(2.2)\ &\quad = \sum (\mathfrak{a}\tilde{X}^1\tilde{Y}^1_{<0>} \otimes h_{(1,1)}x^1\tilde{X}^2\tilde{Y}^1_{<1>}) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes h_{(1,2)}x^2\tilde{X}^3_1\tilde{Y}^2) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes h_2x^3\tilde{X}^3_2\tilde{Y}^3) \\
(1.1)\ &\quad = \sum x^1(\mathfrak{a}\tilde{X}^1 \otimes h_1\tilde{X}^2)\tilde{Y}^1 \otimes_{\mathfrak{A}} x^2(1_{\mathfrak{A}} \otimes h_{(2,1)}\tilde{X}^3_1\tilde{Y}^2) \otimes_{\mathfrak{A}} x^3(1_{\mathfrak{A}} \otimes h_{(2,2)}\tilde{X}^3_2\tilde{Y}^3) \\
&\quad = \Phi^{-1}\sum (\mathfrak{a}\tilde{X}^1 \otimes h_1\tilde{X}^2) \otimes_{\mathfrak{A}} (\tilde{Y}^1 \otimes h_{(2,1)}\tilde{X}^3_1\tilde{Y}^2) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes h_{(2,2)}\tilde{X}^3_2\tilde{Y}^3) \\
&\quad = \Phi^{-1}\sum (\mathfrak{a}\tilde{X}^1 \otimes h_1\tilde{X}^2) \otimes_{\mathfrak{A}} \underline{\Delta}_{\mathfrak{C}}(1_{\mathfrak{A}} \otimes h_2\tilde{X}^3) \\
&\quad = \Phi^{-1}(id \otimes_{\mathfrak{A}} \underline{\Delta}_{\mathfrak{C}})(\underline{\Delta}_{\mathfrak{C}}(\mathfrak{a}\otimes h)),
\end{aligned}
$$

for all $\mathfrak{a} \in \mathfrak{A}$ and $h \in H$, as needed.

Consider now the comonad $F(\mathfrak{C}) = (U, \Delta, \varepsilon)$ and $(M, \xi) \in \mathcal{D}^{F(\mathfrak{C})}$. That means, $M \in \mathcal{D} = {}_H\mathcal{M}_{\mathfrak{A}}$ and $\xi : M \to U(M) = M \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes H)$ is a morphism in $\mathcal{D}$ such that $\Delta_M \circ \xi = U(\xi) \circ \xi$ and $\varepsilon_M \circ \xi = id_M$, for all $M \in \mathcal{D}$. In other words, if we write

$$\xi(m) = \sum m_{(0)} \otimes_{\mathfrak{A}} (m_{(1)^{\mathfrak{A}}} \otimes m_{(1)^H}),\ \ \forall\, m \in M$$

then $(M, \xi) \in \mathcal{D}^{F(\mathfrak{C})}$ if and only if the following relations hold:

$$\xi(hm) = \sum h_1 m_{(0)} \otimes_{\mathfrak{A}} (m_{(1)^{\mathfrak{A}}} \otimes h_2 m_{(1)^H}), \tag{3.48}$$

$$\xi(m\mathfrak{a}) = \sum m_{(0)} \otimes_{\mathfrak{A}} (m_{(1)^{\mathfrak{A}}}\mathfrak{a}_{<0>} \otimes m_{(1)^H}\mathfrak{a}_{<1>}), \tag{3.49}$$

$$
\begin{aligned}
&\sum x^1 m_{(0)} \otimes_{\mathfrak{A}} (m_{(1)^{\mathfrak{A}}}\tilde{X}^1 \otimes x^2 m_{(1)^H_1}\tilde{X}^2) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes x^3 m_{(1)^H_2}\tilde{X}^3) \\
&\qquad = \sum m_{(0)_{(0)}} \otimes_{\mathfrak{A}} (m_{(0)_{(1)^{\mathfrak{A}}}} \otimes m_{(0)_{(1)^H}}) \otimes_{\mathfrak{A}} (m_{(1)^{\mathfrak{A}}} \otimes m_{(1)^H}),
\end{aligned} \tag{3.50}
$$

$$\sum \varepsilon(m_{(1)^H}) m_{(0)} m_{(1)^{\mathfrak{A}}} = m, \tag{3.51}$$

for all $h \in H$, $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$. Applying the canonical isomorphisms, the first three relations are equivalent to

$$\sum (hm)_{(0)}(hm)_{(1)^{\mathfrak{A}}} \otimes (hm)_{(1)^H} = \sum h_1 m_{(0)} m_{(1)^{\mathfrak{A}}} \otimes h_2 m_{(1)^H}, \tag{3.52}$$

$$\sum (m\mathfrak{a})_{(0)}(m\mathfrak{a})_{(1)^{\mathfrak{A}}} \otimes (m\mathfrak{a})_{(1)^H} = \sum m_{(0)} m_{(1)^{\mathfrak{A}}}\mathfrak{a}_{<0>} \otimes m_{(1)^H}\mathfrak{a}_{<1>}, \tag{3.53}$$

$$
\begin{aligned}
&\sum x^1 m_{(0)} m_{(1)^{\mathfrak{A}}}\tilde{X}^1 \otimes x^2 m_{(1)^H_1}\tilde{X}^2 \otimes x^3 m_{(1)^H_2}\tilde{X}^3 \\
&\qquad = \sum m_{(0)_{(0)}} m_{(0)_{(1)^{\mathfrak{A}}}} m_{(1)^{\mathfrak{A}}_{<0>}} \otimes m_{(0)_{(1)^H}} m_{(1)^{\mathfrak{A}}_{<1>}} \otimes m_{(1)^H},
\end{aligned} \tag{3.54}
$$

for all $h \in H$, $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$. Now, if define $\rho_M : M \to M \otimes H$,

$$\rho_M(m) = \sum m_{(0)} m_{(1)^{\mathfrak{A}}} \otimes m_{(1)^H},\ \ \forall\, m \in M$$

then: (3.52) implies that $\rho_M(hm) = \Delta(h)\rho_M(m)$ for all $h \in H$ and $m \in M$, and (3.53) implies that $\rho_M(m\mathfrak{a}) = \rho_M(m)\rho(\mathfrak{a})$ for all $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$, respectively. Moreover, for all $m \in M$ we have that

$$
\begin{aligned}
(\rho_M \otimes id_H)(\rho_M(m)) &= \sum \rho_M(m_{(0)} m_{(1)^{\mathfrak{A}}}) \otimes m_{(1)^H} \\
&= \sum (m_{(0)} m_{(1)^{\mathfrak{A}}})_{(0)} (m_{(0)} m_{(1)^{\mathfrak{A}}})_{(1)^{\mathfrak{A}}} \otimes (m_{(0)} m_{(1)^{\mathfrak{A}}})_{(1)^H} \otimes m_{(1)^H}
\end{aligned}
$$

$$
\begin{aligned}
(3.53) &= \sum m_{(0)_{(0)}} m_{(0)_{(1)^{\mathfrak{A}}}} m_{(1)^{\mathfrak{A}}_{<0>}} \otimes m_{(0)_{(1)^H}} m_{(1)^{\mathfrak{A}}_{<1>}} \otimes m_{(1)^H} \\
(3.54) &= \sum x^1 m_{(0)} m_{(1)^{\mathfrak{A}}} \tilde{X}^1 \otimes x^2 m_{(1)^H_1} \tilde{X}^2 \otimes x^3 m_{(1)^H_2} \tilde{X}^3 \\
&= \Phi^{-1} \cdot \left(\sum m_{(0)} m_{(1)^{\mathfrak{A}}} \otimes \Delta(m_{(1)^H}) \right) \cdot \Phi_\rho \\
&= \Phi^{-1} \cdot (id_M \otimes \Delta)(\rho_M(m)) \cdot \Phi_\rho.
\end{aligned}
$$

By (3.51) it follows that $(id_M \otimes \varepsilon) \circ \rho_M = id_M$, so we have obtained that $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$. In this way we have a functor $\mathbb{F} : \mathcal{D}^{F(\mathfrak{C})} \to {}_H\mathcal{M}_{\mathfrak{A}}^H$ ($\mathbb{F}$ acts as identity on morphisms). We will show that $\mathbb{F}$ provides the desired isomorphism of categories. For, we define the inverse of $\mathbb{F}$ as follows. Let $M \in {}_H\mathcal{M}_{\mathfrak{A}}^H$, and denote by $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ the right coaction of H on M. Then we define

$$
\xi : M \to M \otimes_{\mathfrak{A}} (\mathfrak{A} \otimes H),\ \ \xi(m) = \sum m_{(0)} \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes m_{(1)}),\ \ \forall\, m \in M.
$$

In the same manner as above one can prove that the axioms which define M as a two-sided $(H, \mathfrak{A})$-bimodule imply that ξ satisfies the relations (3.51-3.54). Thus $(M, \xi) \in \mathcal{D}^{F(\mathfrak{C})}$ and we have a well-defined functor $\mathbb{G} :\ {}_H\mathcal{M}_{\mathfrak{A}}^H \to \mathcal{D}^{F(\mathfrak{C})}$ ($\mathbb{G}$ acts as the identity on morphisms). The fact that the functors $\mathbb{F}$ and $\mathbb{G}$ are inverses is obvious, and this finishes our proof. □

Theorem 3.7 enables us to restate the definition of a comodule algebra in terms of monoidal categories.

Proposition 3.8 *Let H be a quasi-bialgebra and $\mathfrak{A}$ an algebra. If $\mathfrak{A} \otimes H$ is viewed in the canonical way as an object in ${}_{\mathfrak{A}\otimes H}\mathcal{M}$ then $\mathfrak{A} \otimes H$ has a coalgebra structure $(\mathfrak{A} \otimes H, \underline{\Delta}, \underline{\varepsilon})$ in the monoidal category $\mathcal{C} = {}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$ such that $\underline{\Delta}(1_{\mathfrak{A}} \otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{A}} \otimes 1_H) = 1_{\mathfrak{A}}$, if and only if $\mathfrak{A}$ is a right H-comodule algebra.*

Proof. One implication follows from the proof of Theorem 3.7. Conversely, suppose that $\mathfrak{A} \otimes H$ is an object of $\mathcal{C}$, and that there exists a coalgebra structure $(\mathfrak{A} \otimes H, \underline{\Delta}, \underline{\varepsilon})$ on $\mathfrak{A} \otimes H$ in the monoidal category $\mathcal{C}$ such that $\underline{\Delta}(1_{\mathfrak{A}} \otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{A}} \otimes 1_H) = 1_{\mathfrak{A}}$. Then we define

$$
\mathfrak{A} \ni \mathfrak{a} \mapsto \rho(\mathfrak{a}) = \sum \mathfrak{a}_{<0>} \otimes \mathfrak{a}_{<1>} := (1_{\mathfrak{A}} \otimes 1_H)\mathfrak{a} \in \mathfrak{A} \otimes H,
$$

and denote

$$
\underline{\Delta}(1_{\mathfrak{A}} \otimes 1_H) := \sum (\tilde{X}^1 \otimes \tilde{X}^2) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes \tilde{X}^3).
$$

Since $\mathfrak{A} \otimes H$ is a right $\mathfrak{A}$-module it is follows that ρ is an algebra map. Also, since $\underline{\Delta}(1_{\mathfrak{A}} \otimes 1_H)$ is invertible we obtain that $\Phi_\rho := \sum \tilde{X}^1 \otimes \tilde{X}^2 \otimes \tilde{X}^3$ is an invertible element in $\mathfrak{A} \otimes H \otimes H$. Now, using the fact that $\underline{\Delta}$ and $\underline{\varepsilon}$ are morphisms in $\mathcal{C}$, and that $\underline{\varepsilon}(1_{\mathfrak{A}} \otimes 1_H) = 1_{\mathfrak{A}}$, it is not hard to see that

$$
\begin{aligned}
&\underline{\Delta}(\mathfrak{a} \otimes h) = \sum (\mathfrak{a}\tilde{X}^1 \otimes h_1\tilde{X}^2) \otimes_{\mathfrak{A}} (1_{\mathfrak{A}} \otimes h_2\tilde{X}^3), \\
&\underline{\varepsilon}(\mathfrak{a} \otimes h) = \varepsilon(h)\mathfrak{a}
\end{aligned}
$$

for all $\mathfrak{a} \in \mathfrak{A}$, $h \in H$. Now, (2.1) and (2.2) follow because of $\underline{\Delta}((1_{\mathfrak{A}} \otimes 1_H)\mathfrak{a}) = \underline{\Delta}(1_{\mathfrak{A}} \otimes 1_H)\mathfrak{a}$ and $\Phi(\underline{\Delta} \otimes id)\underline{\Delta}(\mathfrak{a} \otimes h) = (id \otimes \underline{\Delta})\underline{\Delta}(\mathfrak{a} \otimes h)$ for all $\mathfrak{a} \in \mathfrak{A}$ and $h \in H$, respectively. Finally, it is easy to see that $\underline{\varepsilon}((1_{\mathfrak{A}} \otimes 1_H)\mathfrak{a}) = \mathfrak{a}$ implies (2.3), and the fact that $\underline{\varepsilon}$ is the counit for $\underline{\Delta}$ implies (2.4), respectively. We leave all these details to the reader. □

4 Two-sided crossed products are generalized smash products

Let H be a finite dimensional quasi-bialgebra, and $(\mathfrak{A},\rho,\Phi_\rho)$,$(\mathfrak{B},\lambda,\Phi_\lambda)$ respectively a right and a left H-comodule algebra. As in the case of a Hopf algebra, the right H-coaction (ρ,Φ_ρ) on $\mathfrak{A}$ induces a left H^*-action $\triangleright :\ H^*\otimes\mathfrak{A}\to\mathfrak{A}$ given by

$$\varphi\triangleright\mathfrak{a}=\sum\varphi(\mathfrak{a}_{<1>})\mathfrak{a}_{<0>} \tag{4.1}$$

for all $\varphi\in H^*$ and $\mathfrak{a}\in\mathfrak{A}$, and where $\rho(\mathfrak{a})=\sum a_{<0>}\otimes\mathfrak{a}_{<1>}$ for any $\mathfrak{a}\in\mathfrak{A}$. Similarly, the left H-action (λ,Φ_λ) on $\mathfrak{B}$ provides a right H^*-action $\triangleleft :\mathfrak{B}\otimes H^*\to\mathfrak{B}$ given by

$$\mathfrak{b}\triangleleft\varphi=\sum\varphi(\mathfrak{b}_{[-1]})\mathfrak{b}_{[0]} \tag{4.2}$$

for all $\varphi\in H^*$ and $\mathfrak{b}\in\mathfrak{B}$, where we now denote $\lambda(\mathfrak{b})=\sum\mathfrak{b}_{[-1]}\otimes\mathfrak{b}_{[0]}$ for $\mathfrak{b}\in\mathfrak{B}$. Following [18, Proposition 11.4 (ii)] we can define an algebra structure on the k-vector space $\mathfrak{A}\otimes H^*\otimes\mathfrak{B}$. This algebra is denoted by $\mathfrak{A}\rtimes_\rho H^*\ltimes_\lambda\mathfrak{B}$ and its multiplication is given by

$$\begin{aligned}&(\mathfrak{a}\rtimes\varphi\ltimes\mathfrak{b})(\mathfrak{a}'\rtimes\psi\ltimes\mathfrak{b}')\\ &\quad=\ \sum\mathfrak{a}(\varphi_1\triangleright\mathfrak{a}')\tilde{x}^1_\rho\rtimes(\tilde{x}^1_\lambda\rightharpoonup\varphi_2\leftharpoonup\tilde{x}^2_\rho)(\tilde{x}^2_\lambda\rightharpoonup\psi_1\leftharpoonup\tilde{x}^3_\rho)\ltimes\tilde{x}^3_\lambda(\mathfrak{b}\triangleleft\psi_2)\mathfrak{b}'\end{aligned} \tag{4.3}$$

for all $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$, $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$, and $\varphi,\psi\in H^*$, where we write $\mathfrak{a}\rtimes\varphi\ltimes\mathfrak{b}$ for $\mathfrak{a}\otimes\varphi\otimes\mathfrak{b}$ when viewed as an element of $\mathfrak{A}\rtimes_\rho H^*\ltimes_\lambda\mathfrak{B}$. The comultiplication on H^* is denoted by $\Delta(\varphi)=\sum\varphi_1\otimes\varphi_2$. The unit of the algebra $\mathfrak{A}\rtimes_\rho H^*\ltimes_\lambda\mathfrak{B}$ is $1_\mathfrak{A}\rtimes\varepsilon\ltimes 1_\mathfrak{B}$. Hausser and Nill called this algebra the two-sided crossed product. In this Section we will prove that this two-sided crossed product algebra is a generalized smash product between the quasi-smash product $\mathfrak{A}\,\overline{\#}\,H^*$ and $\mathfrak{B}$.

Proposition 4.1 *Let H be a quasi-bialgebra, A a left H-module algebra and $\mathfrak{B}$ a left H-comodule algebra. Let $A\blacktriangleright\!\!<\mathfrak{B}=A\otimes\mathfrak{B}$ as a k-module, with newly defined multiplication*

$$(a\blacktriangleright\!\!<\mathfrak{b})(a'\blacktriangleright\!\!<\mathfrak{b}')=\sum(\tilde{x}^1\cdot a)(\tilde{x}^2\mathfrak{b}_{[-1]}\cdot a')\blacktriangleright\!\!<\tilde{x}^3\mathfrak{b}_{[0]}\mathfrak{b}' \tag{4.4}$$

for all $a,a'\in A$ and $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$. Then $A\blacktriangleright\!\!<\mathfrak{B}$ is an associative algebra with unit $1_A\blacktriangleright\!\!<1_\mathfrak{B}$.

Proof. For all $a,a',a''\in A$ and $\mathfrak{b},\mathfrak{b}',\mathfrak{b}''\in\mathfrak{B}$ we have:

$$\begin{aligned}
&[(a\blacktriangleright\!\!<b)(a'\blacktriangleright\!\!<\mathfrak{b}')](a''\blacktriangleright\!\!<\mathfrak{b}'')\\
&\quad=\ \sum[(\tilde{x}^1\cdot a)(\tilde{x}^2\mathfrak{b}_{[-1]}\cdot a')\blacktriangleright\!\!<\tilde{x}^3\mathfrak{b}_{[0]}\mathfrak{b}'](a''\blacktriangleright\!\!<\mathfrak{b}'')\\
&\quad=\ \sum[(\tilde{y}^1_1\tilde{x}^1\cdot a)(\tilde{y}^1_2\tilde{x}^2\mathfrak{b}_{[-1]}\cdot a')](\tilde{y}^2\tilde{x}^3_{[-1]}\mathfrak{b}_{[0,-1]}\mathfrak{b}'_{[-1]}\cdot a'')\blacktriangleright\!\!<\tilde{y}^3\tilde{x}^3_{[0]}\mathfrak{b}_{[0,0]}\mathfrak{b}'_{[0]}\mathfrak{b}''\\
(1.27)&\quad=\ \sum(X^1\tilde{y}^1_1\tilde{x}^1\cdot a)[(X^2\tilde{y}^1_2\tilde{x}^2\mathfrak{b}_{[-1]}\cdot a')(X^3\tilde{y}^2\tilde{x}^3_{[-1]}\mathfrak{b}_{[0,-1]}\mathfrak{b}'_{[-1]}\cdot a'')]\\
&\qquad\blacktriangleright\!\!<\tilde{y}^3\tilde{x}^3_{[0]}\mathfrak{b}_{[0,0]}\mathfrak{b}'_{[0]}\mathfrak{b}''\\
(2.6)&\quad=\ \sum(\tilde{x}^1\cdot a)[(\tilde{x}^2_1\tilde{y}^1\mathfrak{b}_{[-1]}\cdot a')(\tilde{x}^2_2\tilde{y}^2\mathfrak{b}_{[0,-1]}\mathfrak{b}'_{[-1]}\cdot a'')]\blacktriangleright\!\!<\tilde{x}^3\tilde{y}^3\mathfrak{b}_{[0,0]}\mathfrak{b}'_{[0]}\mathfrak{b}''\\
(2.5,1.28)&\quad=\ \sum(\tilde{x}^1\cdot a)\{(\tilde{x}^2\mathfrak{b}_{[-1]}\cdot[(\tilde{y}^1\cdot a')(\tilde{y}^2\mathfrak{b}'_{[-1]}\cdot a'')]\}\blacktriangleright\!\!<\tilde{x}^3\mathfrak{b}_{[0]}\tilde{y}^3\mathfrak{b}'_{[0]}\mathfrak{b}''\\
&\quad=\ \sum(a\blacktriangleright\!\!<\mathfrak{b})[(\tilde{y}^1\cdot a')(\tilde{y}^2\mathfrak{b}'_{[-1]}\cdot a'')\blacktriangleright\!\!<\tilde{y}^3\mathfrak{b}'_{[0]}\mathfrak{b}'']\\
&\quad=\ (a\blacktriangleright\!\!<\mathfrak{b})[(a'\blacktriangleright\!\!<\mathfrak{b}')(a''\blacktriangleright\!\!<\mathfrak{b}'')].
\end{aligned}$$

It follows from (2.7), (2.8) and (1.29) that $1_A\blacktriangleright\!\!<1_\mathfrak{B}$ is the unit for $A\blacktriangleright\!\!<\mathfrak{B}$. □

Remark 4.2 Let H be a quasi-bialgebra and A a left H-module algebra. Then H is a left H-comodule algebra so it make sense to consider $A\blacktriangleright\!\!<H$. It is not hard to see that in this case $A\blacktriangleright\!\!<H$ is just the smash product $A\#H$. For this reason we will call the algebra $A\blacktriangleright\!\!<\mathfrak{B}$ in Proposition 4.1 the generalized smash product of A and $\mathfrak{B}$. In fact, our terminology is in agreement with the terminology used over Hopf algebras, see [15] and [10].

Let H be a finite dimensional quasi-bialgebra, $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra and $(\mathfrak{B},\lambda,\Phi_\lambda)$ a left H-comodule algebra. Then the quasi-smash product $\mathfrak{A}\,\overline{\#}\,H^*$ is a left H-module algebra so it makes sense to consider the generalized smash product $(\mathfrak{A}\,\overline{\#}\,H^*)\blacktriangleright\!\!<\mathfrak{B}$. The main result of this Section is now the following:

Proposition 4.3 *With notation as above, the algebras* $(\mathfrak{A}\,\overline{\#}\,H^*)\blacktriangleright\!\!<\mathfrak{B}$ *and* $\mathfrak{A}\rtimes\!\!\!\!\;\lhd_\rho H^*\ltimes_\lambda\mathfrak{B}$ *coincide.*

Proof. Using (4.4), (2.12) and (2.11) we compute that the multiplication on $(\mathfrak{A}\overline{\#}H^*)\blacktriangleright\!\!<\mathfrak{B}$ is given by

$$\begin{aligned}
&[(\mathfrak{a}\,\overline{\#}\,\varphi)\blacktriangleright\!\!<\mathfrak{b}][(\mathfrak{a}'\,\overline{\#}\,\psi)\blacktriangleright\!\!<\mathfrak{b}'] \\
&\quad = \sum[\tilde{x}^1_\lambda\cdot(\mathfrak{a}\,\overline{\#}\,\varphi)][\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\cdot(\mathfrak{a}'\,\overline{\#}\,\psi)]\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}' \\
&\quad = \sum(\mathfrak{a}\,\overline{\#}\,\tilde{x}^1_\lambda\rightharpoonup\varphi)(\mathfrak{a}'\,\overline{\#}\,\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\rightharpoonup\psi)\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}' \\
&\quad = \sum\mathfrak{a}\mathfrak{a}'_{<0>}\tilde{x}^1_\rho\,\overline{\#}\,(\tilde{x}^1_\lambda\rightharpoonup\varphi\leftharpoonup\mathfrak{a}'_{<1>}\tilde{x}^2_\rho)(\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\rightharpoonup\psi\leftharpoonup\tilde{x}^3_\rho)\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}' \qquad (4.5)\\
(4.1,4.2)&\quad = \sum\mathfrak{a}(\varphi_1\triangleright\mathfrak{a}')\tilde{x}^1_\rho\,\overline{\#}\,(\tilde{x}^1_\lambda\rightharpoonup\varphi_2\leftharpoonup\tilde{x}^2_\rho)(\tilde{x}^2_\lambda\rightharpoonup\psi_1\leftharpoonup\tilde{x}^3_\rho)\blacktriangleright\!\!<\tilde{x}^3_\lambda(\mathfrak{b}\triangleleft\psi_2)\mathfrak{b}'
\end{aligned}$$

for $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$, $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$, and $\varphi,\psi\in H^*$. This is just the multiplication rule on the two-sided crossed product $\mathfrak{A}\rtimes\!\!\!\!\;\lhd_\rho H^*\ltimes_\lambda\mathfrak{B}$. □

It follows from (4.5) that the two-sided crossed product can be defined in the situation where H is not finite dimensional. Take $\mathfrak{B}=H$ in Proposition 4.3. From Remark 4.2, we obtain:

Corollary 4.4 *Let H be a quasi-bialgebra and $(\mathfrak{A},\rho,\Phi_\rho)$ a right H-comodule algebra. Then* $(\mathfrak{A}\,\overline{\#}\,H^*)\#H=\mathfrak{A}\rtimes\!\!\!\!\;\lhd_\rho H^*\ltimes_\Delta H$ *as algebras. In particular,* $(H\,\overline{\#}\,H^*)\#H=H\rtimes\!\!\!\!\;\lhd H^*\ltimes H$ *as algebras.*

5 The category of Doi-Hopf modules

5.1 Doi-Hopf modules

Let H be a Hopf algebra over a field k, A an H-comodule algebra and C an H-module coalgebra. A Doi-Hopf module is a k-vector space together with an A-action and a C-coaction satisfying a certain compatibility relation. They were introduced independently by Doi [15] and Koppinen [22], and it turns out that most types of Hopf modules that had been studied before were special cases: Sweedler's Hopf modules [27], Doi's relative Hopf modules [14], Takeuchi's relative Hopf modules [29], Yetter-Drinfeld modules, graded modules and modules graded by a G-set.
Over a quasi-Hopf algebra, the category of relative Hopf modules has been introduced and studied [6], as well as the category of Hopf H-bimodules (see [20]) and the category of Hopf modules ${}_H^H\mathcal{M}_H^H$ (see [26]). We will introduce Doi-Hopf modules, and we will show that, at least in the case where H is finite dimensional, all these categories are isomorphic to certain categories of Doi-Hopf modules. We will also prove that Doi-Hopf modules are special cases of comodules over a coring.
First we recall from [6] the definition of a relative Hopf module. Let H be a quasi-bialgebra and C a right H-module coalgebra. Let N be a k-vector space furnished with the following additional structure:

- N is a right H-module; the right action of $h\in H$ on $n\in N$ is denoted by nh;

- N is a left C-comodule in the monoidal category $\mathcal{M}_H$; we use the following notation for the left C-coaction on N: $\rho_N:\ N\to C\otimes N$, $\rho_N(n)=\sum n_{[-1]}\otimes n_{[0]}$; this means that the following conditions hold, for all $n\in N$:

$$\sum \underline{\varepsilon}(n_{[-1]})n_{[0]}=n$$
$$(\underline{\Delta}\otimes id_N)(\rho_N(n))\Phi^{-1}=(id_C\otimes\rho_N)(\rho_N(n)) \tag{5.1}$$

- we have the following compatibility relation, for all $n\in N$ and $c\in C$:

$$\rho_N(nh)=\sum n_{[-1]}\cdot h_1\otimes n_{[0]}h_2. \tag{5.2}$$

Then N is called a left $[C,H]$-Hopf module. ${}^C\mathcal{M}_H$ is the category of left $[C,H]$-Hopf modules; the morphisms are right H-linear maps which are also left C-comodule maps. We will now generalize this definition.

Definition 5.1 *Let H be a quasi-bialgebra over a field k, C a right H-module coalgebra and $(\mathfrak{B},\lambda,\Phi_\lambda)$ a left H-comodule algebra. A right-left $(H,\mathfrak{B},C)$-Hopf module (or Doi-Hopf module) is a k-module N, with the following additional structure: N is right $\mathfrak{B}$-module (the right action of $\mathfrak{b}$ on n is denoted by $n\mathfrak{b}$), and we have a k-linear map $\rho_N:\ N\to C\otimes N$, such that the following relations hold, for all $n\in N$ and $\mathfrak{b}\in\mathfrak{B}$:*

$$(\underline{\Delta}\otimes id_N)(\rho_N(n))=(id_C\otimes\rho_N)(\rho_N(n))\Phi_\lambda \tag{5.3}$$
$$(\underline{\varepsilon}\otimes id_N)(\rho_N(n))=n \tag{5.4}$$
$$\rho_N(n\mathfrak{b})=\sum n_{[-1]}\cdot\mathfrak{b}_{[-1]}\otimes n_{[0]}\mathfrak{b}_{[0]}. \tag{5.5}$$

As usual, we use the Sweedler-type notation $\rho_N(n)=\sum n_{[-1]}\otimes n_{[0]}$. ${}^C\mathcal{M}(H)_\mathfrak{B}$ is the category of right-left $(H,\mathfrak{B},C)$-Hopf modules and right $\mathfrak{B}$-linear, left C-colinear k-linear maps.

Obviously, if $\mathfrak{B}=H$, $\lambda=\Delta$ and $\Phi_\lambda=\Phi$, then ${}^C\mathcal{M}(H)_\mathfrak{B}={}^C\mathcal{M}_H$.
The main aim of Section 6 will be to define the category of two-sided two-cosided Hopf modules over a quasi-bialgebra, and to prove that it is isomorphic to a module category in the finite dimensional case. To this end, we will need our next result, stating that the category of Doi-Hopf modules is a module category in the case where the coalgebra C is finite dimensional. In fact, for an arbitrary right H-module coalgebra C, the linear dual space of C, C^*, is a left H-module algebra. The multiplication of C^* is the convolution, that is $(c^*d^*)(c)=\sum c^*(c_{\underline{1}})d^*(c_{\underline{2}})$, the unit is $\underline{\varepsilon}$ and the left H-module structure is given by $(h\rightharpoonup c^*)(c)=c^*(c\cdot h)$, for $h\in H$, $c^*,d^*\in C^*$, $c\in C$. Thus C^* is a left H-module algebra and $(\mathfrak{B},\lambda,\Phi_\lambda)$ is a left H-comodule algebra. By Proposition 4.1, it makes sense to consider the generalized smash product algebra $C^*\blacktriangleright\!\!<\mathfrak{B}$.

Proposition 5.2 *Let H be a quasi-bialgebra, C a finite dimensional right H-module coalgebra and $(\mathfrak{B},\lambda,\Phi_\lambda)$ a left H-comodule algebra. Then the category ${}^C\mathcal{M}(H)_\mathfrak{B}$ of right-left $(H,\mathfrak{B},C)$-Hopf modules is isomorphic to the category $\mathcal{M}_{C^*\blacktriangleright\!<\mathfrak{B}}$ of right modules over $C^*\blacktriangleright\!\!<\mathfrak{B}$.*

Proof. We restrict to defining the functors that define the isomorphism of categories, leaving all other details to the reader. Let $\{c_i\}_{i=\overline{1,n}}$ and $\{c^i\}_{i=\overline{1,n}}$ be dual bases in C and C^*.
Let N be a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. Since $\mathbf{i}:\ \mathfrak{B}\to C^*\blacktriangleright\!\!<\mathfrak{B}$, $\mathbf{i}(\mathfrak{b})=\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}$ for $\mathfrak{b}\in\mathfrak{B}$, is an algebra map, it follows that N is a right $\mathfrak{B}$-module via the action $n\mathfrak{b}=n\mathbf{i}(\mathfrak{b})=n(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b})$, $n\in N$, $\mathfrak{b}\in\mathfrak{B}$. The map

$j: C^* \to C^* \blacktriangleright\!\!< \mathfrak{B}$, $j(c^*) = c^* \blacktriangleright\!\!< 1_{\mathfrak{B}}$, $c^* \in C^*$, is not an algebra map (it is not multiplicative) but it can be used to define a left C-coaction on N:

$$\rho_N(n) = \sum n_{[-1]} \otimes n_{[0]} = \sum_{i=1}^{n} c_i \otimes n j(c^i) = \sum_{i=1}^{n} c_i \otimes n(c^i \blacktriangleright\!\!< 1_{\mathfrak{B}}). \tag{5.6}$$

We can easily check that N becomes an object in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$.
Conversely, take $N \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$. Then N is a right $\mathfrak{B}$-module and C^* acts on M from the right as follows: let $nc^* = \sum c^*(n_{[-1]})n_{[0]}$, $n \in N$, $c^* \in C^*$. Now define

$$n(c^* \blacktriangleright\!\!< \mathfrak{b}) = (nc^*)\mathfrak{b} = \sum c^*(n_{[-1]})n_{[0]}\mathfrak{b}. \tag{5.7}$$

Then N becomes a right $C^* \blacktriangleright\!\!< \mathfrak{B}$-module. □

5.2 Doi-Hopf modules and comodules over a coring

Now, we will show that the category of right-left Doi-Hopf modules is isomorphic to a category of right comodules over a certain coring. Let us first recall the definition of a coring.
Let R be a ring (with unit). An R-coring $\mathbf{C}$ is an R-bimodule together with two R-bimodule maps

$$\Delta_{\mathbf{C}} : \mathbf{C} \to \mathbf{C} \otimes_R \mathbf{C} \ \text{ and } \ \varepsilon_{\mathbf{C}} : \mathbf{C} \to R$$

such that the usual coassociativity and counit properties hold, that means:

$$(\Delta_{\mathbf{C}} \otimes_R id_{\mathbf{C}}) \circ \Delta_{\mathbf{C}} = (id_{\mathbf{C}} \otimes_R \Delta_{\mathbf{C}}) \circ \Delta_{\mathbf{C}},$$
$$(\varepsilon_{\mathbf{C}} \otimes_R id_{\mathbf{C}}) \circ \Delta_{\mathbf{C}} = (id_{\mathbf{C}} \otimes_R \varepsilon_{\mathbf{C}}) \circ \Delta_{\mathbf{C}} = id_C.$$

A right $\mathbf{C}$-comodule is a right R-module M together with a right R-linear map $\rho^r : M \to M \otimes_R \mathbf{C}$ such that

$$(\rho^r \otimes_R id_{\mathbf{C}}) \circ \rho^r = (id_M \otimes_R \Delta_{\mathbf{C}}) \circ \rho^r, \tag{5.8}$$
$$(id_M \otimes_R \varepsilon_{\mathbf{C}}) \circ \rho^r = id_M. \tag{5.9}$$

A map $\mathfrak{h} : M \to N$ between two right $\mathbf{C}$-comodules is called a $\mathbf{C}$-comodule map if $\mathfrak{h}$ is a right R-module map and $\rho^r \circ \mathfrak{h} = (\mathfrak{h} \otimes_R id_{\mathbf{C}}) \circ \rho^r$. We denote by $\mathcal{M}^{\mathbf{C}}$ the category of right $\mathbf{C}$-comodules and $\mathbf{C}$-comodule maps. We will use the Sweedler notation for corings and comodules over corings:

$$\Delta_{\mathbf{C}}(c) = \sum c_{(1)} \otimes_R c_{(2)} \ , \ \rho^r(m) = \sum m_{(0)} \otimes_R m_{(1)}.$$

Lemma 5.3 *Let H be a quasi-bialgebra, $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra and C a right H-module coalgebra. Then $\mathbf{C} := \mathfrak{B} \otimes C$ is a $\mathfrak{B}$-coring. First, $\mathbf{C}$ is a $\mathfrak{B}$-bimodule via*

$$\mathfrak{b}(\mathfrak{b}' \otimes c) = \mathfrak{b}\mathfrak{b}' \otimes c \ \textit{ and } \ (\mathfrak{b} \otimes c)\mathfrak{b}' = \sum \mathfrak{b}\mathfrak{b}'_{[0]} \otimes c \cdot \mathfrak{b}'_{[-1]} \tag{5.10}$$

for all $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$ and $c \in C$. Secondly, for all $\mathfrak{b} \in \mathfrak{B}$ and $c \in C$, the two $\mathfrak{B}$-bimodule maps are defined by:

$$\Delta_{\mathbf{C}}(\mathfrak{b} \otimes c) = \sum (\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1), \tag{5.11}$$
$$\varepsilon_{\mathbf{C}}(\mathfrak{b} \otimes c) = \underline{\varepsilon}(c)\mathfrak{b}. \tag{5.12}$$

Proof. Since $\mathfrak{B}$ is an associative unital algebra and $\lambda : \mathfrak{B} \to H \otimes \mathfrak{B}$ is an algebra map, it follows that $\mathfrak{B} \otimes C$ is a $\mathfrak{B}$-bimodule via the actions defined in (5.10). Also, it is not hard to see that $\varepsilon_{\mathbf{C}}$ is a $\mathfrak{B}$-bimodule map. The fact that $\Delta_{\mathbf{C}}$ is left $\mathfrak{B}$-linear is straightforward. It is also right $\mathfrak{B}$-linear since

$$
\begin{aligned}
\Delta_{\mathbf{C}}((\mathfrak{b} \otimes c)\mathfrak{b}') &= \sum \Delta_{\mathbf{C}}(\mathfrak{b}\mathfrak{b}'_{[0]} \otimes c \cdot \mathfrak{b}'_{[-1]}) \\
(1.33) \quad &= \sum (\mathfrak{b}\mathfrak{b}'_{[0]}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \mathfrak{b}'_{[-1]_2}\tilde{x}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \mathfrak{b}'_{[-1]_1}\tilde{x}^1) \\
(2.5, 5.10) \quad &= \sum (\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2)\mathfrak{b}'_{[0]} \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1\mathfrak{b}'_{[-1]}) \\
&= \sum (\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2) \otimes_{\mathfrak{B}} (\mathfrak{b}'_{[0]} \otimes c_{\underline{1}} \cdot \tilde{x}^1\mathfrak{b}'_{[-1]}) \\
(5.10) \quad &= \sum (\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1)\mathfrak{b}' = \Delta_{\mathbf{C}}(\mathfrak{b} \otimes c)\mathfrak{b}'
\end{aligned}
$$

for all $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$ and $c \in C$. Now, for all $\mathfrak{b} \in \mathfrak{B}$ and $c \in C$ we have that:

$$
\begin{aligned}
&(\Delta_{\mathbf{C}} \otimes_{\mathfrak{B}} id_{\mathbf{C}})(\Delta_{\mathbf{C}}(\mathfrak{b} \otimes c)) \\
&= \sum \Delta_{\mathbf{C}}(\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1) \\
(1.33) \quad &= \sum (\mathfrak{b}\tilde{x}^3\tilde{y}^3 \otimes c_{(\underline{2},\underline{2})} \cdot \tilde{x}^2_2\tilde{y}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{(\underline{2},\underline{1})} \cdot \tilde{x}^2_1\tilde{y}^1) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1) \\
(1.32) \quad &= \sum (\mathfrak{b}\tilde{x}^3\tilde{y}^3 \otimes c_{\underline{2}} \cdot x^3\tilde{x}^2_2\tilde{y}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{2})} \cdot x^2\tilde{x}^2_1\tilde{y}^1) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{1})} \cdot x^1\tilde{x}^1) \\
(2.6) \quad &= \sum (\mathfrak{b}\tilde{x}^3\tilde{y}^3_{[0]} \otimes c_{\underline{2}} \cdot \tilde{x}^2\tilde{y}^3_{[-1]}) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{2})} \cdot \tilde{x}^1_2\tilde{y}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{1})} \cdot \tilde{x}^1_1\tilde{y}^1) \\
(5.10) \quad &= \sum (\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2) \otimes_{\mathfrak{B}} (\tilde{y}^3 \otimes c_{(\underline{1},\underline{2})} \cdot \tilde{x}^1_2\tilde{y}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{(\underline{1},\underline{1})} \cdot \tilde{x}^1_1\tilde{y}^1) \\
(1.33, 5.11) \quad &= \sum (\mathfrak{b}\tilde{x}^3 \otimes c_{\underline{2}} \cdot \tilde{x}^2) \otimes_{\mathfrak{B}} \Delta_{\mathbf{C}}(1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1) = (id_{\mathbf{C}} \otimes_{\mathfrak{B}} \Delta_{\mathbf{C}})(\Delta_{\mathbf{C}}(\mathfrak{b} \otimes c)),
\end{aligned}
$$

as needed. It is easy to see that $\varepsilon_{\mathbf{C}}$ is the counit for $\Delta_{\mathbf{C}}$, so the proof is finished. □

We can now prove the following

Theorem 5.4 *Let H be a quasi-bialgebra, $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra and C a right H-module coalgebra. If $\mathbf{C} = \mathfrak{B} \otimes C$ is the $\mathfrak{B}$-coring defined in Lemma 5.3, then the category of right-left Doi-Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category of right $\mathbf{C}$-comodules, $\mathcal{M}^{\mathbf{C}}$.*

Proof. If $M \in \mathcal{M}^{\mathbf{C}}$ then we adopt a similar notation as the one used in the proof of Theorem 3.7. Namely, if $M \in \mathcal{M}^{\mathbf{C}}$ with $\rho^r : M \to M \otimes_{\mathfrak{B}} (\mathfrak{B} \otimes C)$, then we set

$$\rho^r(m) = \sum m_{(0)} \otimes_{\mathfrak{B}} (m_{(1)^{\mathfrak{B}}} \otimes m_{(1)^C}), \ \ \forall \, m \in M.$$

With this notation the fact that ρ^r is right $\mathfrak{B}$-linear means

$$\sum (m\mathfrak{b})_{(0)} \otimes_{\mathfrak{B}} ((m\mathfrak{b})_{(0)^{\mathfrak{B}}} \otimes (m\mathfrak{b})_{(1)^C}) = \sum m_{(0)} \otimes_{\mathfrak{B}} (m_{(1)^{\mathfrak{B}}}\mathfrak{b}_{[0]} \otimes m_{(1)^C} \cdot \mathfrak{b}_{[-1]})$$

for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$, and this is equivalent to

$$\sum (m\mathfrak{b})_{(0)}(m\mathfrak{b})_{(0)^{\mathfrak{B}}} \otimes (m\mathfrak{b})_{(1)^C} = \sum m_{(0)}m_{(1)^{\mathfrak{B}}}\mathfrak{b}_{[0]} \otimes m_{(1)^C} \cdot \mathfrak{b}_{[-1]} \tag{5.13}$$

for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$. Similarly, in this particular case, the relations (5.8) and (5.9) reduce to

$$
\begin{aligned}
&\sum m_{(0)_{(0)}} m_{(0)_{(1)^{\mathfrak{B}}}} m_{(1)^{\mathfrak{B}}_{[0]}} \otimes m_{(0)_{(1)^C}} \cdot m_{(1)^{\mathfrak{B}}_{[-1]}} \otimes m_{(1)^C} \\
&\qquad = \sum m_{(0)}m_{(1)^{\mathfrak{B}}}\tilde{x}^3 \otimes m_{(1)^C_{\underline{2}}} \cdot \tilde{x}^2 \otimes m_{(1)^C_{\underline{1}}} \cdot \tilde{x}^1, \qquad (5.14) \\
&\sum \underline{\varepsilon}(m_{(1)^C})m_{(0)}m_{(1)^{\mathfrak{B}}} = m, \qquad (5.15)
\end{aligned}
$$

for all $\mathfrak{b} \in \mathfrak{B}$ and $m \in M$. Now, if we define

$$\rho_M : M \to C \otimes M, \ \ \rho_M(m) = \sum m_{(1)^C} \otimes m_{(0)} m_{(1)^{\mathfrak{B}}}, \ \ \forall\, m \in M$$

then (5.13) implies that $\rho_M(m\mathfrak{b}) = \rho_M(m)\lambda(\mathfrak{b})$ for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$, and (5.15) implies that $(\underline{\varepsilon} \otimes id_M) \circ \rho_M = id_M$, respectively. Thus, $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$ since

$$\begin{aligned}
(id_C \otimes \rho_M)(\rho_M(m)) &= \sum m_{(1)^C} \otimes \rho_M(m_{(0)} m_{(1)^{\mathfrak{B}}}) \\
(5.13) \quad &= \sum m_{(1)^C} \otimes m_{(0)_{(1)^C}} \cdot m_{(1)^{\mathfrak{B}}_{[-1]}} \otimes m_{(0)_{(0)}} m_{(0)_{(1)^{\mathfrak{B}}}} m_{(1)^{\mathfrak{B}}_{[0]}} \\
(5.14) \quad &= \sum m_{(1)^C_{\underline{1}}} \cdot \tilde{x}^1 \otimes m_{(1)^C_{\underline{2}}} \cdot \tilde{x}^2 \otimes m_{(0)} m_{(1)^{\mathfrak{B}}} \tilde{x}^3 \\
&= (\underline{\Delta} \otimes id_M)(\rho_M(m)) \Phi_\lambda^{-1}
\end{aligned}$$

for all $m \in M$, as needed. Therefore, we have a functor $\mathfrak{F} : \mathcal{M}^{\mathbf{C}} \to {}^C\mathcal{M}(H)_{\mathfrak{B}}$ which acts on objects as above and sends a morphism to itself (the verification of the fact that a morphism in $\mathcal{M}^{\mathbf{C}}$ becomes a morphism in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is left to the reader). Conversely, if $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$ with $\rho_M(m) = \sum m_{[-1]} \otimes m_{[0]}$, $m \in M$, then we define

$$\rho^r : M \to M \otimes_{\mathfrak{B}} (\mathfrak{B} \otimes C), \ \ \rho^r(m) = \sum m_{[0]} \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes m_{[-1]}), \ \ \forall\, m \in M.$$

It is not hard to see that in this way the right $\mathfrak{B}$-module M becomes a right $\mathbf{C}$-comodule, i.e. the relations (5.13-5.15) hold. So we also have a functor $\mathfrak{G} : {}^C\mathcal{M}(H)_{\mathfrak{B}} \to \mathcal{M}^{\mathbf{C}}$ ($\mathfrak{G}$ sends a morphism to itself). Finally, it is routine to check that $\mathfrak{F}$ and $\mathfrak{G}$ are inverses; we leave the details to the reader. □

6 Two-sided two-cosided Hopf modules

Now we define the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$. If H is finite dimensional, then this category is isomorphic to a certain category of right-left Doi-Hopf modules, ${}^C\mathcal{M}(H \otimes H^{\rm op})_{(\mathbb{A} \,\overline{\#}\, H^*)\# H}$. As a consequence, if C is also finite dimensional then this category is isomorphic to the category of right modules over a generalized smash product, by Proposition 5.2.

Definition 6.1 *[18, Definition 8.2]. Let H be a quasi-bialgebra. An H-bicomodule algebra $\mathbb{A}$ is a quintuple $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$, where λ and ρ are left and right H-coactions on $\mathbb{A}$, and where $\Phi_\lambda \in H \otimes H \otimes \mathbb{A}$, $\Phi_\rho \in \mathbb{A} \otimes H \otimes H$ and $\Phi_{\lambda,\rho} \in H \otimes \mathbb{A} \otimes H$ are invertible elements, such that*

- $(\mathbb{A}, \lambda, \Phi_\lambda)$ *is a left H-comodule algebra,*
- $(\mathbb{A}, \rho, \Phi_\rho)$ *is a right H-comodule algebra,*
- *the following compatibility relations hold, for all $a \in \mathbb{A}$:*

$$\Phi_{\lambda,\rho}(\lambda \otimes id)(\rho(a)) = (id \otimes \rho)(\lambda(a))\Phi_{\lambda,\rho} \quad (6.1)$$

$$(1_H \otimes \Phi_{\lambda,\rho})(id \otimes \lambda \otimes id)(\Phi_{\lambda,\rho})(\Phi_\lambda \otimes 1_H) = (id \otimes id \otimes \rho)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_{\lambda,\rho}) \quad (6.2)$$

$$(1_H \otimes \Phi_\rho)(id \otimes \rho \otimes id)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho} \otimes 1_H) = (id \otimes id \otimes \Delta)(\Phi_{\lambda,\rho})(\lambda \otimes id \otimes id)(\Phi_\rho). \quad (6.3)$$

It was pointed out in [18] that the following additional relations hold in an H-bicomodule algebra $\mathbb{A}$:

$$(id_H \otimes id_{\mathbb{A}} \otimes \varepsilon)(\Phi_{\lambda,\rho}) = 1_H \otimes 1_{\mathbb{A}}, \ \ (\varepsilon \otimes id_{\mathbb{A}} \otimes id_H)(\Phi_{\lambda,\rho}) = 1_{\mathbb{A}} \otimes 1_H. \quad (6.4)$$

As a first example, take $\mathbb{A} = H$, $\lambda = \rho = \Delta$ and $\Phi_\lambda = \Phi_\rho = \Phi_{\lambda,\rho} = \Phi$. Related to the left and right comodule algebra structures of $\mathbb{A}$ we will keep the notation of the previous Sections. We will use the following notation:

$$\Phi_{\lambda,\rho} = \sum \Omega^1 \otimes \Omega^2 \otimes \Omega^3 = \sum \overline{\Omega}^1 \otimes \overline{\Omega}^2 \otimes \overline{\Omega}^3 = \text{etc.}$$

and

$$\Phi_{\lambda,\rho}^{-1} = \sum \omega^1 \otimes \omega^2 \otimes \omega^3 = \sum \overline{\omega}^1 \otimes \overline{\omega}^2 \otimes \overline{\omega}^3 = \text{etc.}$$

If H is a quasi-bialgebra, then the opposite algebra $H^{\rm op}$ is also a quasi-bialgebra. The reassociator of $H^{\rm op}$ is $\Phi_{\rm op} = \Phi^{-1}$. $H \otimes H^{\rm op}$ is also a quasi-bialgebra with reassociator

$$\Phi_{H\otimes H^{\rm op}} = \sum (X^1 \otimes x^1) \otimes (X^2 \otimes x^2) \otimes (X^3 \otimes x^3). \tag{6.5}$$

If we identify $H \otimes H^{\rm op}$-modules and (H,H)-bimodules, then the category of (H,H)-bimodules, ${}_H\mathcal{M}_H$, is monoidal. The associativity constraints are given by $\mathbf{a}'_{U,V,W} : (U\otimes V)\otimes W \to U\otimes (V\otimes W)$, where

$$\mathbf{a}'_{U,V,W}((u\otimes v)\otimes w) = \Phi\cdot (u\otimes (v\otimes w))\cdot \Phi^{-1} \tag{6.6}$$

for all $U,V,W \in {}_H\mathcal{M}_H$, $u\in U$, $v \in V$ and $w\in W$. A coalgebra in the category of (H,H)-bimodules will be called an H-bimodule coalgebra. More precisely, an H-bimodule coalgebra C is an (H,H)-bimodule (denote the actions by $h\cdot c$ and $c \cdot h$) with a comultiplication $\underline{\Delta} : \ C \to C\otimes C$ and a counit $\underline{\varepsilon} : \ C \to k$ satisfying the following relations, for all $c\in C$ and $h \in H$:

$$\Phi\cdot (\underline{\Delta}\otimes id_C)(\underline{\Delta}(c))\cdot \Phi^{-1} = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)) \tag{6.7}$$

$$\underline{\Delta}(h\cdot c) = \sum h_1\cdot c_{\underline{1}} \otimes h_2 \cdot c_{\underline{2}},\ \ \underline{\Delta}(c\cdot h) = \sum c_{\underline{1}}\cdot h_1 \otimes c_{\underline{2}}\cdot h_2 \tag{6.8}$$

$$(\underline{\varepsilon}\otimes id_C)\circ \underline{\Delta} = (id_C \otimes \underline{\varepsilon})\circ\underline{\Delta} = id_C \tag{6.9}$$

$$\underline{\varepsilon}(h\cdot c) = \varepsilon(h)\underline{\varepsilon}(c),\ \ \underline{\varepsilon}(c\cdot h) = \underline{\varepsilon}(c)\varepsilon(h) \tag{6.10}$$

where we used the same Sweedler-type notation as before. An H-bimodule coalgebra C becomes a right $H\otimes H^{\rm op}$-module coalgebra via the right $H\otimes H^{\rm op}$-action

$$c\cdot (h\otimes h') = h'\cdot c\cdot h \tag{6.11}$$

for $c\in C$ and $h, h' \in H$. Our next definition extends the definition of two-sided two-cosided Hopf modules from [26].

Definition 6.2 *Let H be a quasi-bialgebra, $(\mathbb{A},\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$ an H-bicomodule algebra and C an H-bimodule coalgebra. A two-sided two-cosided $(H,\mathbb{A},C)$-Hopf module is a k-vector space with the following additional structure:*

- *N is an $(H,\mathbb{A})$-two-sided Hopf module, i.e. $N \in {}_H\mathcal{M}_{\mathbb{A}}^H$; we write $\succ$ for the left H-action, $\prec$ for the right $\mathbb{A}$-action, and $\rho_N^H(n) = \sum n_{(0)}\otimes n_{(1)}$ for the right H-coaction on $n\in N$;*
- *we have k-linear map $\rho_N^C : \ N \to C\otimes N$, $\rho_N^C(n) = \sum n_{[-1]}\otimes n_{[0]}$, called the left C-coaction on N, such that $\sum \underline{\varepsilon}(n_{[-1]})n_{[0]} = n$ and*

$$\Phi(\underline{\Delta}\otimes id_N)(\rho_N^C(n)) = (id_C\otimes \rho_N^C)(\rho_N^C(n))\Phi_\lambda \tag{6.12}$$

  for all $n\in N$;

- *N is a (C,H)-"bicomodule", in the sense that, for all $n\in N$,*

$$\Phi(\rho_N^C\otimes id_H)(\rho_N^H(n)) = (id_C\otimes\rho_N^H)(\rho_N^C(n))\Phi_{\lambda,\rho} \tag{6.13}$$

- *the following compatibility relations hold*

$$\rho_N^C(h\succ n) = \sum h_1\cdot n_{[-1]}\otimes h_2\succ n_{[0]} \tag{6.14}$$
$$\rho_N^C(n\prec a) = \sum n_{[-1]}\cdot a_{[-1]}\otimes n_{[0]}\prec a_{[0]} \tag{6.15}$$

for all $h\in H$, $n\in N$ and $a\in\mathbb{A}$.

${}_H^C\mathcal{M}_{\mathbb{A}}^H$ will be the category of two-sided two-cosided Hopf modules and maps preserving the actions by H and $\mathbb{A}$ and the coactions by H and C.

Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. If H is finite dimensional, then the category ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to a certain category of Doi-Hopf modules. In order to prove this, we first need some lemmas.

Lemma 6.3 *Let H be a quasi-Hopf algebra and $(\mathbb{A},\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$ an H-bicomodule algebra. Consider the map*

$$\wp:\ (\mathbb{A}\,\overline{\#}\,H^*)\#H\to(H\otimes H^{\rm op})\otimes(\mathbb{A}\,\overline{\#}\,H^*)\#H$$

given by

$$\wp((a\,\overline{\#}\,\varphi)\#h) = \sum a_{[-1]}\omega^1\otimes S(y^3h_2)\otimes(a_{[0]}\omega^2\,\overline{\#}\,y^1\rightharpoonup\varphi\leftharpoonup\omega^3)\#y^2h_1 \tag{6.16}$$

for any $a\in\mathbb{A}$, $\varphi\in H^*$ and $h\in H$, where $\Phi_{\lambda,\rho}^{-1}=\sum\omega^1\otimes\omega^2\otimes\omega^3$. Set

$$\Phi_\wp = \sum(\tilde{X}_\lambda^1\otimes g^1S(x^3))\otimes(\tilde{X}_\lambda^2\otimes g^2S(x^2))\otimes(\tilde{X}_\lambda^3\,\overline{\#}\,\varepsilon)\#x^1 \tag{6.17}$$

where $f^{-1}=\sum g^1\otimes g^2$ is the element defined in (1.16). Then $((\mathbb{A}\,\overline{\#}\,H^*)\#H,\wp,\Phi_\wp)$ is a left $H\otimes H^{\rm op}$-comodule algebra.

Proof. We first show that $\wp$ is an algebra map. Using (1.30) and (2.11) we can easily show that the multiplication on $(\mathbb{A}\,\overline{\#}\,H^*)\#H$ is given by

$$\begin{aligned}&((a\,\overline{\#}\,\varphi)\#h)((a'\,\overline{\#}\,\psi)\#h')\\ &\quad= \sum\Big[aa'_{<0>}\tilde{x}_\rho^1\,\overline{\#}\,(x^1\rightharpoonup\varphi\leftharpoonup a'_{<1>}\tilde{x}_\rho^2)(x^2h_1\rightharpoonup\psi\leftharpoonup\tilde{x}_\rho^3)\Big]\#x^3h_2h'\end{aligned} \tag{6.18}$$

for all $a,a'\in\mathbb{A}$, $\varphi,\psi\in H^*$ and $h,h'\in H$. Therefore

$$\begin{aligned}
&\wp(((a\,\overline{\#}\,\varphi)\#h)((a'\,\overline{\#}\,\psi)\#h'))\\
&= \sum a_{[-1]}a'_{<0>_{[-1]}}(\tilde{x}_\rho^1)_{[-1]}\omega^1\otimes S(y^3x_2^3h_{(2,2)}h'_2)\otimes[a_{[0]}a'_{<0>_{[0]}}(\tilde{x}_\rho^1)_{[0]}\omega^2\\
&\qquad\overline{\#}\,(y_1^1x^1\rightharpoonup\varphi\leftharpoonup a'_{<1>}\tilde{x}_\rho^2\omega_1^3)(y_2^1x^2h_1\rightharpoonup\psi\leftharpoonup\tilde{x}_\rho^3\omega_2^3)]\#y^2x_1^3h_{(2,1)}h'_1\\
\stackrel{(6.3,1.3)}{=}& \sum a_{[-1]}a'_{<0>_{[-1]}}\overline{\omega}^1\omega^1\otimes S(y^3x^3h_{(2,2)}h'_2)\otimes[a_{[0]}a'_{<0>_{[0]}}\overline{\omega}^2\omega_{<0>}^2\tilde{x}_\rho^1\\
&\qquad\overline{\#}\,(z^1y^1\rightharpoonup\varphi\leftharpoonup a'_{<1>}\overline{\omega}^3\omega_{<1>}^2\tilde{x}_\rho^2)(z^2y_1^2x^1h_1\rightharpoonup\psi\leftharpoonup\omega^3\tilde{x}_\rho^3)]\#z^3y_2^2x^2h_{(2,1)}h'_1\\
\stackrel{(6.1,1.1)}{=}& \sum a_{[-1]}\overline{\omega}^1a'_{[-1]}\omega^1\otimes S(y^3h_2)\cdot_{op}S(x^3h'_2)\otimes[a_{[0]}\overline{\omega}^2(a'_{[0]}\omega^2)_{<0>}\tilde{x}_\rho^1\,\overline{\#}\,(z^1y^1\\
&\qquad\rightharpoonup\varphi\leftharpoonup\overline{\omega}^3(a'_{[0]}\omega^2)_{<1>}\tilde{x}_\rho^2)(z^2y_1^2h_{(1,1)}x^1\rightharpoonup\psi\leftharpoonup\omega^3\tilde{x}_\rho^3)]\#z^3y_2^2h_{(1,2)}x^2h'_1\\
\stackrel{(2.11)}{=}& \sum a_{[-1]}\overline{\omega}^1a'_{[-1]}\omega^1\otimes S(y^3h_2)\cdot_{op}S(x^3h'_2)\otimes[(a_{[0]}\overline{\omega}^2\,\overline{\#}\,z^1y^1\rightharpoonup\varphi\leftharpoonup\overline{\omega}^3)
\end{aligned}$$

$$\begin{aligned}
& (a'_{[0]}\omega^2 \,\overline{\#}\, z^2 y_1^2 h_{(1,1)} x^1 \rightharpoonup \psi \leftharpoonup \omega^3)]\# z^3 y_2^2 h_{(1,2)} x^2 h'_1 \\
(1.30) \quad = \;& \sum a_{[-1]}\overline{\omega}^1 a'_{[-1]}\omega^1 \otimes S(y^3 h_2) \cdot_{op} S(x^3 h'_2) \otimes [(a_{[0]}\overline{\omega}^2 \,\overline{\#}\, y^1 \rightharpoonup \varphi \leftharpoonup \overline{\omega}^3)\# y^2 h_1] \\
& [(a'_{[0]}\omega^2 \,\overline{\#}\, x^1 \rightharpoonup \psi \leftharpoonup \omega^3)\# x^2 h'_1] \\
= \;& \wp((a \,\overline{\#}\, \varphi)\# h)\wp((a' \,\overline{\#}\, \psi)\# h')
\end{aligned}$$

where $\cdot_{\rm op}$ is the product in $H^{\rm op}$. Obviously $\wp$ respects the unit element and (2.7) and (2.8) hold. (2.5) can be proved using similar computations as above and is left to the reader. Using the notation

$$\Phi_\wp = \sum \tilde{X}^1_\wp \otimes \tilde{X}^2_\wp \otimes \tilde{X}^3_\wp = \text{etc.}$$

we can compute

$$\begin{aligned}
& (id \otimes id \otimes \wp)(\Phi_\wp)(\Delta \otimes id \otimes id)(\Phi\wp) \\
= \;& \sum (\tilde{X}^1_\lambda \otimes g^1 S(x^3))((\tilde{Y}^1_\lambda)_1 \otimes G^1_1 S(y^3)_1) \otimes (\tilde{X}^2_\lambda \otimes g^2 S(x^2)) \\
& ((\tilde{Y}^1_\lambda)_2 \otimes G^1_2 S(y^3)_2) \otimes ((\tilde{X}^3_\lambda)_{[-1]} \otimes S(x^1_2))(\tilde{Y}^2_\lambda \otimes G^2 S(y^2)) \\
& \otimes [((\tilde{X}^3_\lambda)_{[0]} \,\overline{\#}\, \varepsilon)\# x^1_1][(\tilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon)\# y^1] \\
(1.11, 1.3) \quad = \;& \sum (\tilde{X}^1_\lambda (\tilde{Y}^1_\lambda)_1 \otimes G^1_1 g^1 S(y^3 x^3)) \otimes (\tilde{X}^2_\lambda (\tilde{Y}^1_\lambda)_2 \otimes G^1_2 g^2 S(z^3 y^2_2 x^2)) \\
& \otimes ((\tilde{X}^3_\lambda)_{[-1]} \tilde{Y}^2_\lambda \otimes G^2 S(z^2 y^2_1 x^1)) \otimes [((\tilde{X}^3_\lambda)_{[0]} \tilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon)\# z^1 y^1] \\
(2.6, 1.9, 1.18) \quad = \;& \sum (\tilde{Y}^1_\lambda X^1 \otimes x^1 g^1 S(y^3)) \otimes (\tilde{X}^1_\lambda (\tilde{Y}^2_\lambda)_1 X^2 \otimes x^2 g^2_1 G^1 S(z^3 y^2_2)) \\
& \otimes (\tilde{X}^2_\lambda (\tilde{Y}^2_\lambda)_2 X^3 \otimes x^3 g^2_2 G^2 S(z^2 y^2_1)) \otimes [(\tilde{X}^3_\lambda \tilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon)\# z^1 y^1] \\
(1.11) \quad = \;& \sum (\tilde{Y}^1_\lambda \otimes g^1 S(y^3))(X^1 \otimes x^1) \otimes (\tilde{X}^1_\lambda \otimes G^1 S(z^3))((\tilde{Y}^2_\lambda)_1 \otimes g^2_1 S(y^2)_1) \\
& (X^2 \otimes x^2) \otimes (\tilde{X}^2_\lambda \otimes G^2 S(z^2))((\tilde{Y}^2_\lambda)_2 \otimes g^2_2 S(y^2)_2)(X^3 \otimes x^3) \\
& \otimes [(\tilde{X}^3_\lambda \,\overline{\#}\, \varepsilon)\# z^1][(\tilde{Y}^3_\lambda \,\overline{\#}\, \varepsilon)\# y^1] \\
(6.5) \quad = \;& (1_H \otimes \Phi_\wp)(id \otimes \Delta_{H \otimes H^{\rm op}} \otimes id)(\Phi_\wp)(\Phi_{H \otimes H^{\rm op}} \otimes \mathbf{1})
\end{aligned}$$

where $\sum G^1 \otimes G^2$ is another copy of f^{-1} and $\mathbf{1} = (1_{\mathbb{A}} \,\overline{\#}\, \varepsilon)\# 1_H$ is the unit of the algebra $(\mathbb{A} \,\overline{\#}\, H^*)\# H$. $\square$

Let H be a quasi-Hopf algebra, $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ an H-bicomodule algebra and C an H-bimodule coalgebra. By Lemma 6.3, we can consider the category of Doi-Hopf modules ${}^C\mathcal{M}(H \otimes H^{\rm op})_{(\mathbb{A} \,\overline{\#}\, H^*)\# H}$. We will prove that it is isomorphic to the category of two-sided two-cosided Hopf modules ${}^C_H\mathcal{M}^H_{\mathbb{A}}$, in the case where H is finite dimensional.

Lemma 6.4 *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. We have a functor*

$$F : {}^C_H\mathcal{M}^H_{\mathbb{A}} \to {}^C\mathcal{M}(H \otimes H^{\rm op})_{(\mathbb{A} \,\overline{\#}\, H^*)\# H}.$$

$F(N) = N$ as a k-module, with structure maps given by the equations

$$n \leftarrow ((a \,\overline{\#}\, \varphi)\# h) = \sum \langle \varphi, S^{-1}(f^2 n_{(1)} a_{<1>} \tilde{p}^2_\rho) \rangle S(h) f^1 \succ n_{(0)} \prec a_{<0>} \tilde{p}^1_\rho \tag{6.19}$$

$$\tilde{\rho}^C_N(n) = \sum n_{\{-1\}} \otimes n_{\{0\}} = \sum f^1 \cdot n_{[-1]} \otimes f^2 \succ n_{[0]} \tag{6.20}$$

for all $n \in N$, $a \in \mathbb{A}$, $\varphi \in H^*$ and $h \in H$. F sends a morphism to itself.

Proof. Since N is a two-sided $(H,\mathbb{A})$-Hopf module, we know by (3.39) that N is a right $(\mathbb{A}\,\overline{\#}\,H^*)\#H$-module via the action defined by (6.19). Let $\sum F^1\otimes F^2$ be another copy of f. For any $n\in N$, we have that

$$
\begin{aligned}
&(\underline{\Delta}\otimes id_N)(\tilde{\rho}^C_N(n))\Phi^{-1}_{\wp}\\
(6.17)\quad &= \sum n_{\{-1\}_{\underline{1}}}\cdot(\tilde{x}^1_\lambda\otimes S(X^3)F^1)\otimes n_{\{-1\}_{\underline{2}}}\cdot(\tilde{x}^2_\lambda\otimes S(X^2)F^2)\\
&\quad\otimes n_{\{0\}}\leftarrow[(\tilde{x}^3_\lambda\,\overline{\#}\,\varepsilon)\#X^1]\\
(6.11,6.20)\quad &= \sum S(X^3)F^1\cdot(f^1\cdot n_{[-1]})_{\underline{1}}\cdot\tilde{x}^1_\lambda\otimes S(X^2)F^2\cdot(f^1\cdot n_{[-1]})_{\underline{2}}\cdot\tilde{x}^2_\lambda\\
&\quad\otimes S(X^1)f^2\succ n_{[0]}\prec\tilde{x}^3_\lambda\\
(6.8)\quad &= \sum S(X^3)F^1f^1_1\cdot n_{[-1]_{\underline{1}}}\cdot\tilde{x}^1_\lambda\otimes S(X^2)F^2f^1_2\cdot n_{[-1]_{\underline{2}}}\cdot\tilde{x}^2_\lambda\\
&\quad\otimes S(X^1)f^2\succ n_{[0]}\prec\tilde{x}^3_\lambda\\
(6.12,1.9,1.18)\quad &= \sum f^1\cdot n_{[-1]}\otimes F^1f^2_1\cdot n_{[0,-1]}\otimes F^2f^2_2\succ n_{[0,0]}\\
(6.14)\quad &= \sum f^1\cdot n_{[-1]}\otimes F^1\cdot(f^2\succ n_{[0]})_{[-1]}\otimes F^2\succ(f^2\succ n_{[0]})_{[0]}\\
(6.20)\quad &= \sum n_{\{-1\}}\otimes F^1\cdot n_{\{0\}_{[-1]}}\otimes F^2\succ n_{\{0\}_{[0]}}\\
(6.20)\quad &= (id_C\otimes\tilde{\rho}^C_N)(\tilde{\rho}^C_N(n)).
\end{aligned}
$$

We still have to show the compatibility relation (5.5). For, observe that (3.6), (6.3) and (1.5) imply

$$\sum\Omega^1(\tilde{p}^1_\rho)_{[-1]}\otimes\Omega^2(\tilde{p}^1_\rho)_{[0]}\otimes\Omega^3\tilde{p}^2_\rho=\sum\omega^1\otimes\omega^2_{<0>}\tilde{p}^1_\rho\otimes\omega^2_{<1>}\tilde{p}^2_\rho S(\omega^3).\qquad(6.21)$$

Now, for all $n\in N$, $a\in\mathbb{A}$, $\varphi\in H^*$ and $h\in H$ one can show that

$$\tilde{\rho}^C_N(n\leftarrow((a\,\overline{\#}\,\varphi)\#h))=\tilde{\rho}^C_N(n)\wp((a\,\overline{\#}\,\varphi)\#h),$$

completing the proof. □

Lemma 6.5 *Let H be a finite dimensional quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. We have a functor*

$$G:\ {}^C\mathcal{M}(H\otimes H^{\rm op})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}\to{}^C_H\mathcal{M}^H_{\mathbb{A}}.$$

$G(N)=N$ as a k-module, with structure maps given by

$$h\succ n=n\leftarrow[(1_{\mathbb{A}}\,\overline{\#}\,\varepsilon)\#S^{-1}(h)],\ \ n\prec a=n\leftarrow[(a\,\overline{\#}\,\varepsilon)\#1_H],\qquad(6.22)$$

$$\rho^H_N:\ N\to N\otimes H,\ \ \rho^H_N(n)=\sum_{i=1}^n n\leftarrow[(\tilde{q}^1_\rho\,\overline{\#}\,S^{-1}(g^2)\rightharpoonup e^iS\leftharpoonup\tilde{q}^2_\rho)\#S^{-1}(g^1)]\otimes e_i,\qquad(6.23)$$

$$\underline{\rho}^C_N:N\to C\otimes N,\ \ \underline{\rho}^C_N(n)=\sum g^1\cdot n_{[-1]}\otimes g^2\succ n_{[0]}\qquad(6.24)$$

for $n\in N$, $a\in\mathbb{A}$ and $h\in H$. Here $\{e_i\}_{i=\overline{1,n}}$ is a basis of H and $\{e^i\}_{i=\overline{1,n}}$ is the corresponding dual basis of H^*. G sends a morphism to itself.

Proof. Since N is a right $(\mathbb{A}\,\overline{\#}\,H^*)\#H$-module, we already know by (3.36) and (3.38) that H is a two-sided $(H,\mathbb{A})$-Hopf module via (6.22) and (6.23). Thus we only have to check (6.12), (6.13), (6.14) and (6.15). First note that $N\in{}^C\mathcal{M}(H\otimes H^{\rm op})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}$ implies

$$
\begin{aligned}
&\sum n_{[-1]}\otimes n_{[0,-1]}\otimes n_{[0,0]}\\
&\quad=\ \sum S(X^3)f^1\cdot n_{[-1]_{\underline{1}}}\cdot\tilde{x}^1_\lambda\otimes S(X^2)f^2\cdot n_{[-1]_{\underline{2}}}\cdot\tilde{x}^2_\lambda\otimes n_{[0]}\leftarrow[(\tilde{x}^3_\lambda\,\overline{\#}\,\varepsilon)\#X^1]\qquad(6.25)\\
&\sum\{n\leftarrow[(a\,\overline{\#}\,\varphi)\#h]\}_{[-1]}\otimes\{n\leftarrow[(a\,\overline{\#}\,\varepsilon)\#h]\}_{[0]}\\
&\quad=\ \sum S(x^3h_2)\cdot n_{[-1]}\cdot a_{[-1]}\omega^1\otimes n_{[0]}\leftarrow[(a_{[0]}\omega^2\,\overline{\#}\,x^1\rightharpoonup\varphi\leftharpoonup\omega^3)\#x^2h_1]\qquad(6.26)
\end{aligned}
$$

for all $n\in N$, $a\in\mathbb{A}$, $\varphi\in H^*$ and $h\in H$. By the above definitions and (6.26) it is immediate that

$$\underline{\rho}_N^C(h\succ n)=\Delta(h)\underline{\rho}_N^C(n) \text{ and } \underline{\rho}_N^C(n\prec a)=\underline{\rho}_N^C(n)\rho_\lambda(a) \tag{6.27}$$

for all $h\in H$, $n\in N$ and $a\in\mathbb{A}$ (we leave it to the reader to verify the details). Let $\sum G^1\otimes G^2$ be another copy of f^{-1}. We compute that

$$\begin{aligned}
&\Phi(\underline{\Delta}\otimes id_N)(\underline{\rho}_N^C(n))\\
\overset{(6.24)}{=}&\sum X^1\cdot(g^1\cdot n_{[-1]})_{\underline{1}}\otimes X^2\cdot(g^1\cdot n_{[-1]})_{\underline{2}}\otimes X^3g^2\succ n_{[0]}\\
\overset{(6.22,6.8)}{=}&\sum X^1g_1^1\cdot n_{[-1]_{\underline{1}}}\otimes X^2g_2^1\cdot n_{[-1]_{\underline{2}}}\otimes n_{[0]}\leftharpoonup[(1_{\mathbb{A}}\,\overline{\#}\,\varepsilon)\#S^{-1}(X^3g^2)]\\
\overset{(6.25,6.18)}{=}&\sum X^1g_1^1G^1S(x^3)\cdot n_{[-1]}\cdot\tilde{X}_\lambda^1\otimes X^2g_2^1G^2S(x^2)\cdot n_{[0,-1]}\cdot\tilde{X}_\lambda^2\\
&\otimes n_{[0,0]}\leftharpoonup[(\tilde{X}_\lambda^3\,\overline{\#}\,\varepsilon)\#S^{-1}(X^3g^2S(x^1))]\\
\overset{(1.9,1.18)}{=}&\sum g^1\cdot n_{[-1]}\cdot\tilde{X}_\lambda^1\otimes g_1^2G^1\cdot n_{[0,-1]}\cdot\tilde{X}_\lambda^2\\
&\otimes n_{[0,0]}\leftharpoonup[(\tilde{X}_\lambda^3\,\overline{\#}\,\varepsilon)\#S^{-1}(g_2^2G^2)]\\
\overset{(6.22)}{=}&\sum g^1\cdot n_{[-1]}\cdot\tilde{X}_\lambda^1\otimes g_1^2G^1\cdot n_{[0,-1]}\cdot\tilde{X}_\lambda^2\otimes g_2^2G^2\succ n_{[0,0]}\prec\tilde{X}_\lambda^3\\
\overset{(6.24,6.8,6.27)}{=}&(id_C\otimes\underline{\rho}_N^C)(\underline{\rho}_N^C(n))\Phi_\lambda.
\end{aligned}$$

The verification of (6.13) is based on similar computations, and we leave the details to the reader. □

As a consequence of Lemmas 6.4 and 6.5, we have the following description of ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ as a category of Doi-Hopf modules; this description generalizes [4, Proposition 2.3].

Theorem 6.6 *Let H be a finite dimensional quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. Then the categories ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ and ${}^C\mathcal{M}(H\otimes H^{\rm op})_{(\mathbb{A}\,\overline{\#}\,H^*)\#H}$ are isomorphic.*

Proof. We have to verify that the functors F and G defined in Lemmas 6.4 and 6.5 are inverses. For the C-coactions (6.20) and (6.24), this is obvious; for the other structures, it has been already done in Corollary 3.6. □

Propositions 5.2 and 5.4 and Theorem 6.6 immediately imply the following result.

Corollary 6.7 *Let H be a finite dimensional quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. Then ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to the category of right comodules over the coring $\mathbf{C}=((\mathbb{A}\,\overline{\#}\,H^*)\#H)\otimes C$. If C is finite dimensional, then the category ${}_H^C\mathcal{M}_{\mathbb{A}}^H$ is isomorphic to the category of right modules over the generalized smash product $C^*\blacktriangleright\!\!<((\mathbb{A}\,\overline{\#}\,H^*)\#H)$.*

Remark 6.8 Let H be a finite dimensional Hopf algebra. Cibils and Rosso [11] introduced an algebra $X=(H^{\rm op}\otimes H)\underline{\otimes}(H^*\otimes H^{*op})$ having the property that the category of two-sided two-cosided Hopf modules over H^* coincides with the category of left X-modules. Moreover, it was also proved in [11] that X is isomorphic to the direct tensor product of a Heisenberg double and the opposite of a Drinfeld double. Recently, Panaite [25] introduced two other algebras Y and Z with the same property as X. More precisely, Y is the two-sided crossed product $H^*\#(H\otimes H^{\rm op})\#H^{*\rm op}$, and Z is the diagonal crossed product in the sense of [18], $(H^*\otimes H^{*op})\bowtie(H\otimes H^{\rm op})$. Using different methods, we proved that the category of two-sided two-cosided Hopf modules over a finite dimensional quasi-Hopf algebra is isomorphic to the category of right (resp. left) modules over the generalized smash product $\mathcal{A} = H^* \blacktriangleright\!\!< ((H \,\overline{\#}\, H^*)\#H)$ (resp. $\mathcal{A}^{\mathrm{op}}$). Note that, in general, the multiplication on $C^* \blacktriangleright\!\!< ((\mathbb{A} \,\overline{\#}\, H^*)\#H)$ is given by the formula

$$\begin{aligned}
&[c^* \blacktriangleright\!\!< ((a \,\overline{\#}\, \varphi)\#h)][d^* \blacktriangleright\!\!< ((a' \,\overline{\#}\, \psi)\#h')] \\
&\quad = \sum (\tilde{x}^1_\lambda \rightharpoonup c^* \leftharpoonup S(X^3) f^1)(\tilde{x}^2_\lambda a_{[-1]} \omega^1 \rightharpoonup d^* \leftharpoonup S(X^2 x^3 h_2) f^2) \blacktriangleright\!\!< \{[\tilde{x}^3_\lambda a_{[0]} \omega^2 a'_{<0>} \tilde{x}^1_\rho \\
&\qquad \overline{\#}\, (X^1_{(1,1)} y^1 x^1 \rightharpoonup \varphi \leftharpoonup \omega^3 a'_{<1>} \tilde{x}^2_\rho)(X^1_{(1,2)} y^2 x^2_1 h_{(1,1)} \rightharpoonup \psi \leftharpoonup \tilde{x}^3_\rho)] \# X^1_2 y^3 x^2_2 h_{(1,2)} h'\}.
\end{aligned}$$

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