STRONG GROUP COALGEBRAS

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Abstract. We introduce strong group coalgebras, as a generalization of strongly graded coalgebras. We give several characterizations, and study two special types of strong group coalgebras, namely cleft group algebras (or crossed coproduct group coalgebras) and smash coproduct group coalgebras.

Introduction

Graded coalgebras were introduced by Năstăsescu and Torrecillas in [7], and further studied in [3]. At first glance, most results about graded coalgebras seem to be completely similar to corresponding results about graded algebras. However, there are some remarkable differences. For example, in [7], the notion of strongly graded coalgebra is introduced. An interesting property is the fact that strongly graded coalgebras only exist in the case where the grading group G is a finite group; we do not have such a property for strongly graded algebras. In the coalgebra case, this property is basically a consequence of the intrinsic finiteness that is built in the definition of a coalgebra.

Group coalgebras and Hopf group coalgebras were introduced by Turaev in [9]. An algebraic study of Hopf group coalgebras was initiated in [10], and continued in a series of papers by various authors. In [1], it was shown that group coalgebras (resp. Hopf group coalgebras) are in fact coalgebras (resp. Hopf algebras) in a well-chosen symmetric monoidal category.

Group coalgebras are a generalization of graded coalgebras, and the two notions coincide if we work over a finite group G. For an infinite group G, group coalgebras behave better as far as duality is concerned. The dual of a group coalgebra is a graded algebra, and in some situations the category of comodules over the group coalgebra is isomorphic to the category of graded modules over its dual graded algebra. These duality results have been studied in [2], in the slightly more general context of corings rather than coalgebras.

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In this note, we introduce strong group coalgebras. If we work over a finite group G, then strong group coalgebras correspond bijectively to strongly graded coalgebras. However, strong group coalgebras can also exist in the case where G is infinite. The basic example is that of a cofree group coalgebra, playing the role of group algebra in the graded algebra theory. We have several characterizations of strong group coalgebras, see Proposition 1.3 and Theorem 1.7. The most important one is perhaps the following: for any group coalgebra, we have a pair of adjoint functors between the category of comodules over the part of degree e, and the category of group comodules over the group coalgebra; the group coalgebra is strong if and only if this adjunction is a pair of inverse equivalences. Over a group coalgebra, one can define two types of comodules (comparable to the situation in graded ring theory, where one has graded and ungraded modules). In Section 2, we show that there is a relation between these two types: comodules of one type are in fact comodules of the other type, but then over the smash coproduct.
In Hopf algebra theory, cleft comodule algebras have been introduced and studied by Doi (see [4, 5]). In Section 3, we study the corresponding notion of cocleft group coalgebra. We have several equivalent characterizations of cocleft group coalgebras, see Theorem 3.4. In the special situation where the part of degree e of the group coalgebra is cocommutative, we can describe cocleft group coalgebras using group cohomology, this is discussed in detail in Section 4.

1. Strong group coalgebras

In what follows k will denote a fixed field. Unadorned tensor products are meant to be taken over k. By G we will denote a fixed group with identity element e. In any category the identity morphism of an object M will also be denoted by M. For categories $\mathcal{M}_i$, indexed by an arbitrary set I, we will denote the product category by $\prod_{i\in I}\mathcal{M}_i$. If all $\mathcal{M}_i$ are equal to one category $\mathcal{M}$, we will write $\mathcal{M}^I=\prod_{i\in I}\mathcal{M}$. The category of k-vector spaces is denoted by $\mathcal{M}_k$.
Consider a G-group coalgebra (or shorter, G-coalgebra) $\underline{C}$, i.e. a collection $\underline{C}=(C_\alpha)_{\alpha\in G}$ of k-vector spaces, with k-linear maps $\Delta_{\alpha,\beta}:C_{\alpha\beta}\to C_\alpha\otimes C_\beta$ and $\varepsilon:C_e\to k$, for all $\alpha,\beta\in G$, satisfying

$$(\Delta_{\alpha,\beta}\otimes C_\gamma)\circ\Delta_{\alpha\beta,\gamma}=(C_\alpha\otimes\Delta_{\beta,\gamma})\circ\Delta_{\alpha,\beta\gamma},$$
$$(C_\alpha\otimes\varepsilon)\circ\Delta_{\alpha,e}=C_\alpha=(\varepsilon\otimes C_\alpha)\circ\Delta_{e,\alpha},$$

for all $\alpha,\beta,\gamma\in G$. It is clear that C_e is a usual k-coalgebra with comultiplication map $\Delta_{e,e}$ and counit map ε.
Given two G-coalgebras $\underline{C}$ and $\underline{D}$, a morphism $\underline{\varphi}:\underline{C}\to\underline{D}$ of G-coalgebras from $\underline{C}$ to $\underline{D}$ is a collection $\underline{\varphi}=(\varphi_\alpha:C_\alpha\to D_\alpha)_{\alpha\in G}$ of k-linear maps such that, $\varepsilon_{\underline{D}}\circ\varphi_e=\varepsilon_{\underline{C}}$ and $(\varphi_\alpha\otimes\varphi_\beta)\circ\Delta^{\underline{C}}_{\alpha,\beta}=\Delta^{\underline{D}}_{\alpha,\beta}\circ\varphi_{\alpha\beta}$, for all $\alpha,\beta\in G$.
Given a G-coalgebra $\underline{C}$ one can study two different types of modules over $\underline{C}$. Firstly, a k-vector space M together with a family of k-linear maps

$\{\rho_\alpha = \rho_\alpha^M : M \to M \otimes C_\alpha\}_{\alpha \in G}$ such that $(M \otimes \varepsilon) \circ \rho_e = M$ and

$$(M \otimes \Delta_{\alpha,\beta}) \circ \rho_{\alpha\beta} = (\rho_\alpha \otimes C_\beta) \circ \rho_\beta,$$

for all $\alpha, \beta \in G$, is called a right $\underline{C}$-comodule. We will use the Sweedler-type notation $\rho_\alpha(m) = m_{[0]} \otimes m_{[1,\alpha]} \in M \otimes C_\alpha$, for all $m \in M$ and $\alpha \in G$, where summation is implicitly understood. Given two right $\underline{C}$-comodules M and N, a k-linear map $f : M \to N$ is called right $\underline{C}$-colinear if $(f \otimes C_\alpha) \circ \rho_\alpha^M = \rho_\alpha^N \circ f$, for all $\alpha \in G$. The category of right $\underline{C}$-comodules and right $\underline{C}$-colinear maps is denoted by $\mathcal{M}^{\underline{C}}$.

Secondly, a right G-group $\underline{C}$-comodule (or shortly, right G-$\underline{C}$-comodule) is a family of k-vector spaces $\underline{M} = (M_\alpha)_{\alpha \in G}$ equipped with a family of k-linear maps $\{\rho_{\alpha,\beta} = \rho^{\underline{M}}_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes C_\beta\}_{\alpha,\beta \in G}$ such that $(M_\alpha \otimes \varepsilon) \circ \rho_{\alpha,e} = M_\alpha$ and

$$(\rho_{\alpha,\beta} \otimes C_\gamma) \circ \rho_{\alpha\beta,\gamma} = (M_\alpha \otimes \Delta_{\beta,\gamma}) \circ \rho_{\alpha,\beta\gamma},$$

for all $\alpha, \beta, \gamma \in G$. We will use the notation $\rho_{\alpha,\beta}(m) = m_{[0,\alpha]} \otimes m_{[1,\beta]} \in M_\alpha \otimes C_\beta$, for all $m \in M_{\alpha\beta}$ and $\alpha, \beta \in G$. A right G-$\underline{C}$-colinear map between two right G-$\underline{C}$-comodules $\underline{M}$ and $\underline{N}$ is a family of k-linear maps $\underline{\varphi} = (\varphi_\alpha : M_\alpha \to N_\alpha)_{\alpha \in G}$ such that $(f_\alpha \otimes C_\beta) \circ \rho^{\underline{M}}_{\alpha,\beta} = \rho^{\underline{N}}_{\alpha,\beta} \circ f_{\alpha\beta}$, for all $\alpha, \beta \in G$. The category of right G-$\underline{C}$-comodules and right G-$\underline{C}$-colinear maps will be denoted by $\mathcal{M}^{G,\underline{C}}$.

Given a right G-$\underline{C}$-comodule $\underline{M}$, it is clear that $F(\underline{M}) = M_e$ is a right C_e-comodule with coaction map $\rho_{e,e}$. Clearly we have a functor $F : \mathcal{M}^{G,\underline{C}} \to \mathcal{M}^{C_e}$. Given a right C_e-comodule N, we can define a right G-$\underline{C}$-comodule $G(N)$, by $G(N)_\alpha = N \square_{C_e} C_\alpha$ and

$$\rho_{\alpha,\beta} = N \square_{C_e} \Delta_{\alpha,\beta} : N \square_{C_e} C_{\alpha\beta} \to N \square_{C_e} (C_\alpha \otimes C_\beta) \cong (N \square_{C_e} C_\alpha) \otimes C_\beta,$$

for all $\alpha, \beta \in G$.

Proposition 1.1. *(F, G) is a pair of adjoint functors between the categories $\mathcal{M}^{G,\underline{C}}$ and $\mathcal{M}^{C_e}$.*

Proof. The unit of the adjunction is given by $\eta_{\underline{M}} = (\eta_{\underline{M},\alpha})_{\alpha \in G} : \underline{M} \to G(M_e)$, where $\eta_{\underline{M},\alpha} : M_\alpha \to M_e \square_{C_e} C_\alpha$ is the corestriction of $\rho_{e,\alpha} : M_\alpha \to M_e \otimes C_\alpha$ to $M_e \square_{C_e} C_\alpha$, for all $\underline{M} \in \mathcal{M}^{G,\underline{C}}$. The counit is given by the natural isomorphisms $\nu_N : N \square_{C_e} C_e \to N, \nu_N(\sum_i n_i \otimes c_i) = \sum_i \varepsilon(c_i) n_i$, for all $N \in \mathcal{M}^{C_e}$. We still need to verify, for all $\underline{M} \in \mathcal{M}^{G,\underline{C}}$, $N \in \mathcal{M}^{C_e}$ and $\alpha \in G$, that the following diagrams

$$\begin{array}{ccc} M_e & \xrightarrow{\eta_{\underline{M},e}} & M_e \square_{C_e} C_e \\ & \searrow^{=} & \downarrow{\scriptstyle \nu_{M_e}} \\ & & M_e \end{array} \qquad \begin{array}{ccc} N \square_{C_e} C_\alpha & \xrightarrow{\eta_{GN,\alpha}} & (N \square_{C_e} C_e) \square_{C_e} C_\alpha \\ & \searrow^{=} & \downarrow{\scriptstyle \nu_N \square_{C_e} C_\alpha} \\ & & N \square_{C_e} C_\alpha \end{array}$$

commute. Indeed, for all $m \in M_e$ we have that

$$(\nu_{M_e} \circ \eta_{\underline{M},e})(m) = \nu_{M_e}(m_{[0,e]} \otimes m_{[1,e]}) = \varepsilon(m_{[1,e]}) m_{[0,e]} = m;$$

for $\sum_i n_i \otimes c_i \in N\Box_{C_e} C_\alpha$ we have that

$$\big((\nu_N \Box_{C_e} C_\alpha) \circ \eta_{GN,\alpha}\big)\Big(\sum_i n_i \otimes c_i\Big) = \sum_i \nu_N(n_i \otimes c_{i(1,e)}) \otimes c_{i(2,\alpha)}$$
$$= \sum_i n_i \otimes \varepsilon(c_{i(1,e)}) c_{i(2,\alpha)} = \sum_i n_i \otimes c_i.$$

□

Definition 1.2. A G-coalgebra $\underline{C}$ is called *strong* if $\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta$ is a monomorphism, for all $\alpha, \beta \in G$.

Proposition 1.3. *Let $\underline{C}$ be a G-coalgebra. The following assertions are equivalent:*

1. *$\underline{C}$ is a strong G-coalgebra;*
2. *for all $\alpha \in G$, $\Delta_{\alpha,\alpha^{-1}} : C_e \to C_\alpha \otimes C_{\alpha^{-1}}$ is a monomorphism;*
3. *for all $\underline{M} \in \mathcal{M}^{G,\underline{C}}$ and $\alpha, \beta \in G$, $\rho^{\underline{M}}_{\alpha,\beta}$ is a monomorphism.*

Proof. $\underline{(1) \Rightarrow (2)}$ and $\underline{(3) \Rightarrow (1)}$ are trivial. We prove $\underline{(2) \Rightarrow (3)}$. For $\underline{M} \in \mathcal{M}^{G,\underline{C}}$ and $\alpha, \beta \in G$, consider the diagram

$$\begin{array}{ccc} M_{\alpha\beta} & \xrightarrow{\rho_{\alpha\beta,e}} & M_{\alpha\beta} \otimes C_e \\ {\scriptstyle \rho_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle M_{\alpha\beta}\otimes\Delta_{\beta^{-1},\beta}} \\ M_\alpha \otimes C_\beta & \xrightarrow[\rho_{\alpha\beta,\beta^{-1}}\otimes C_\beta]{} & M_{\alpha\beta} \otimes C_{\beta^{-1}} \otimes C_\beta. \end{array}$$

By assumption $\Delta_{\beta^{-1},\beta}$ is monic, hence also $M_{\alpha\beta} \otimes \Delta_{\beta^{-1},\beta}$ (since we work over a field k, $M_{\alpha\beta}$ is k-flat). By the counit property $\rho_{\alpha\beta,e}$ is monic. Indeed, if $\rho_{\alpha\beta,e}(m) = m_{[0,\alpha\beta]} \otimes m_{[1,e]} = 0$, then $m = m_{[0,\alpha\beta]}\varepsilon(m_{[1,e]}) = 0$. Thus $(M_{\alpha\beta} \otimes \Delta_{\beta^{-1},\beta}) \circ \rho_{\alpha\beta,e} = (\rho_{\alpha\beta,\beta^{-1}} \otimes C_\beta) \circ \rho_{\alpha,\beta}$ is monic. It follows that $\rho_{\alpha,\beta}$ is monic, as wanted. □

Proposition 1.4. *Let $\underline{C}$ be a strong G-coalgebra, and $\underline{M} \in \mathcal{M}^{G,\underline{C}}$. The following assertions are equivalent:*

1. *$\underline{M} = 0$, i.e., $M_\alpha = 0$ for all $\alpha \in G$;*
2. *$M_\alpha = 0$ for some $\alpha \in G$.*

Proof. $\underline{(1) \Rightarrow (2)}$ is trivial, we prove $\underline{(2) \Rightarrow (1)}$. Take $\beta \in G$ arbitrary. We have that $\rho_{\alpha,\alpha^{-1}\beta} : M_\beta \to M_\alpha \otimes C_{\alpha^{-1}\beta}$ is monic. Now $M_\alpha = 0$ implies that $M_\beta = 0$, and we find our result. □

Corollary 1.5. *Let $\underline{C}$ be a strong G-coalgebra, and $\underline{\varphi} : \underline{M} \to \underline{N}$ a morphism in $\mathcal{M}^{G,\underline{C}}$. We have that*

1. *φ_α is injective for all $\alpha \in G$ if and only if φ_α is injective for some $\alpha \in G$;*
2. *φ_α is surjective for all $\alpha \in G$ if and only if φ_α is surjective for some $\alpha \in G$;*

(3) *φ_α is bijective for all $\alpha \in G$ if and only if φ_α is bijective for some $\alpha \in G$;*

Proof. This follows from Proposition 1.4. □

Lemma 1.6. *Let $\underline{C}$ be a group coalgebra. $\mathcal{M}^{G,\underline{C}}$ is an abelian category.*

Proof. Let $\underline{\varphi} : \underline{M} \to \underline{N}$ be a morphism in $\mathcal{M}^{G,\underline{C}}$. Write $K_\alpha = \mathrm{Ker}(\varphi_\alpha)$ and $I_\alpha = \mathrm{Im}(\varphi_\alpha)$. For $m \in M_{\alpha\beta}$, we have that

$$(\varphi_\alpha \otimes C_\beta)(\rho^M_{\alpha,\beta}(m)) = \varphi_\alpha(m_{[0,\alpha]}) \otimes m_{[1,\beta]} = \rho^N_{\alpha,\beta}(\varphi_{\alpha\beta}(m)).$$

This implies that $\rho^N_{\alpha,\beta}(I_{\alpha\beta}) \subset I_\alpha \otimes C_\beta$. Also if $m \in K_{\alpha\beta}$, then $\rho^M_{\alpha,\beta}(m) \in \mathrm{Ker}(\varphi_\alpha \otimes C_\beta) = K_\alpha \otimes C_\beta$ (the last equality holds since C_β is k-flat). So $\rho^M_{\alpha,\beta}(K_{\alpha\beta}) \subset K_\alpha \otimes C_\beta$. This shows that $\underline{K} = (K_\alpha)_{\alpha\in G}, \underline{I} = (I_\alpha)_{\alpha\in G} \in \mathcal{M}^{G,\underline{C}}$. □

Theorem 1.7. *Let $\underline{C}$ be a G-coalgebra. The following assertions are equivalent:*

(1) *$\underline{C}$ is a strong G-coalgebra;*
(2) *(F, G) is a pair of inverse equivalences between $\mathcal{M}^{G,\underline{C}}$ and $\mathcal{M}^{C_e}$;*
(3) *$\Delta_{\alpha,\beta}$ corestricts to an isomorphism $C_{\alpha\beta} \to C_\alpha \square_{C_e} C_\beta$, for all $\alpha, \beta \in G$.*

Proof. $\underline{(1) \Rightarrow (2)}$. It suffices to show that, for all $\alpha \in G$ and $\underline{M} \in \mathcal{M}^{G,\underline{C}}$, $\eta_{\underline{M},\alpha} : M_\alpha \to M_e \square_{C_e} C_\alpha$ is bijective. Since $\eta_{\underline{M},e} : M_e \to M_e \square_{C_e} C_e$ is bijective, the result follows from Corollary 1.5.
$\underline{(2) \Rightarrow (3)}$. Consider the σ-suspension $\underline{C}_\sigma = (C_{\sigma\alpha})_{\alpha\in G}$ of $\underline{C}$ (see also [7, Section 3]). The right G-$\underline{C}$-comodule structure on $\underline{C}_\sigma$ is given by the maps $\Delta_{\sigma\alpha,\beta} : C_{\sigma\alpha\beta} \to C_{\sigma\alpha} \otimes C_\beta$. Since (F, G) is a pair of inverse equivalences, $\eta_{\underline{C(\alpha)},\beta} = \Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \square_{C_e} C_\beta$ is an isomorphism, for all $\alpha, \beta \in G$.
$\underline{(3) \Rightarrow (1)}$. It follows from (3) that all maps $\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta$ are monomorphic. □

Let $\underline{C}$ be a G-coalgebra. We can construct a graded k-algebra R as follows (see [2, Sec. 5]). For $\alpha \in G$, let $R_\alpha = C^*_{\alpha^{-1}}$. $R = \oplus_{\alpha\in G} R_\alpha$ is a G-graded k-algebra, with multiplication

$$(f\#g)(c) = f(c_{(2,\alpha^{-1})})g(c_{(1,\beta^{-1})}),$$

for $f \in R_\alpha$, $g \in R_\beta$ and $c \in R_{\beta^{-1}\alpha^{-1}}$. The unit element is $\varepsilon \in R_e$. The multiplication $m_{\alpha,\beta} : R_\alpha \otimes R_\beta \to R_{\alpha\beta}$ is the composition of the dual of the comultiplication map $\Delta_{\beta^{-1},\alpha^{-1}}$ and the canonical inclusion $R_\alpha \otimes R_\beta = C^*_{\alpha^{-1}} \otimes C^*_{\beta^{-1}} \to (C_{\beta^{-1}} \otimes C_{\alpha^{-1}})^*$. If $\underline{C}$ is homogeneously finite, this means that every C_α is finite dimensional, then this canonical inclusion is an isomorphism, and then $m_{\alpha,\beta}$ is surjective if and only if $\Delta_{\beta^{-1},\alpha^{-1}}$ is injective. By definition R is strongly graded if and only if the maps $m_{\alpha,\beta}$ are surjective (see for example [8]). This proves the following result.

Proposition 1.8. *Let $\underline{C}$ be a homogeneously finite G-coalgebra. Then $\underline{C}$ is strong if and only if $R = \oplus_{\alpha\in G} C_\alpha^*$ is a strongly graded k-algebra.*

2. Group comodules and the smash coproduct

Let $\underline{C}$ be a G-coalgebra. We introduce the so-called *smash coproduct G-coalgebra* $\underline{C} \rtimes kG$ of $\underline{C}$ and kG as follows. For $\alpha \in G$ we put $(\underline{C} \rtimes kG)_\alpha = C_\alpha \rtimes kG = C_\alpha \otimes kG$. For $\alpha, \beta \in G$ we define $\Delta_{\alpha,\beta} : C_{\alpha\beta} \rtimes kG \to (C_\alpha \rtimes kG) \otimes (C_\beta \rtimes kG)$ by

$$\Delta_{\alpha,\beta}(c \rtimes \sigma) = (c_{(1,\alpha)} \rtimes \beta\sigma) \otimes (c_{(2,\beta)} \rtimes \sigma),$$

for $c \in C_{\alpha\beta}$ and $\sigma \in G$. We define $\varepsilon : C_e \rtimes kG \to k$ by $\varepsilon(c \rtimes \sigma) = \varepsilon(c)$, for $c \in C_e$ and $\sigma \in G$.

Lemma 2.1. *$\underline{C} \rtimes kG$ is a G-coalgebra.*

Proof. Let $\alpha, \beta, \gamma, \sigma \in G$ and take $c \in C_{\alpha\beta\gamma}$. We then have that

$$\begin{aligned}
&\big((\Delta_{\alpha,\beta} \otimes (C_\gamma \rtimes kG)) \circ \Delta_{\alpha\beta,\gamma}\big)(c \rtimes \sigma) = \Delta_{\alpha,\beta}(c_{(1,\alpha\beta)} \rtimes \gamma\sigma) \otimes (c_{(2,\gamma)} \rtimes \sigma)\\
&= (c_{(1,\alpha)} \rtimes \beta\gamma\sigma) \otimes (c_{(2,\beta)} \rtimes \gamma\sigma) \otimes (c_{(3,\gamma)} \rtimes \sigma)\\
&= (c_{(1,\alpha)} \rtimes \beta\gamma\sigma) \otimes \Delta_{\beta,\gamma}(c_{(2,\beta\gamma)} \rtimes \sigma)\\
&= \big(((C_\alpha \rtimes kG) \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma}\big)(c \rtimes \sigma).
\end{aligned}$$

For $\alpha, \sigma \in G$ and $c \in C_\alpha$ we have

$$\begin{aligned}
&\big(((C_\alpha \rtimes kG) \otimes \varepsilon) \circ \Delta_{\alpha,e}\big)(c \rtimes \sigma) = \varepsilon(c_{(2,e)} \rtimes \sigma)c_{(1,\alpha)} \rtimes \sigma\\
&\qquad = \varepsilon(c_{(2,e)})c_{(1,\alpha)} \rtimes \sigma = c \rtimes \sigma;\\
&\big((\varepsilon \otimes (C_\alpha \rtimes kG)) \circ \Delta_{e,\alpha}\big)(c \rtimes \sigma) = \varepsilon(c_{(1,e)} \rtimes \alpha\sigma)c_{(2,\alpha)} \rtimes \sigma\\
&\qquad = \varepsilon(c_{(1,e)})c_{(2,\alpha)} \rtimes \sigma = c \rtimes \sigma.
\end{aligned}$$

□

Proposition 2.2. *Let $\underline{C}$ be a G-coalgebra. The categories $\mathcal{M}^{\underline{C}\rtimes kG}$ and $\mathcal{M}^{G,\underline{C}}$ are isomorphic.*

Proof. Take $\underline{M} \in \mathcal{M}^{G,\underline{C}}$. Let $M = \bigoplus_{\alpha\in G} M_\alpha$, and define $\rho_\beta : M \to M \otimes (C_\beta \rtimes kG)$ as follows: for $m \in M_\sigma$, let

$$\rho_\beta(m) = m_{[0,\sigma\beta^{-1}]} \otimes (m_{[1,\beta]} \rtimes \sigma^{-1}).$$

Then $M \in \mathcal{M}^{\underline{C}\rtimes kG}$. Indeed, for $m \in M_\sigma$ we have that

$$((M \otimes \varepsilon) \circ \rho_e)(m) = (M \otimes \varepsilon)(m_{[0,\sigma]} \otimes (m_{[1,e]} \rtimes \sigma^{-1})) = \varepsilon(m_{[1,e]})m_{[0,\sigma]} = m.$$

We still need to verify the commutativity of the diagram

$$(1)\qquad \begin{array}{ccc}
M & \xrightarrow{\rho_{\alpha\beta}} & M \otimes (C_{\alpha\beta} \rtimes kG)\\
\downarrow{\scriptstyle \rho_\beta} & & \downarrow{\scriptstyle M\otimes\Delta_{\alpha,\beta}}\\
M \otimes (C_\beta \rtimes kG) & \xrightarrow[\rho_\alpha\otimes(C_\beta\rtimes kG)]{} & M \otimes (C_\alpha \rtimes kG) \otimes (C_\beta \rtimes kG).
\end{array}$$

Take $m \in M_\sigma$. We then have that

$$\begin{aligned}
\big((M\otimes \Delta_{\alpha,\beta})\circ \rho_{\alpha\beta}\big)(m) &= (M\otimes \Delta_{\alpha,\beta})(m_{[0,\sigma\beta^{-1}\alpha^{-1}]}\otimes (m_{[1,\alpha\beta]}\rtimes \sigma^{-1}))\\
&= m_{[0,\sigma\beta^{-1}\alpha^{-1}]}\otimes (m_{[1,\alpha]}\rtimes \beta\sigma^{-1})\otimes (m_{[2,\beta]}\rtimes \sigma^{-1})\\
&= \rho_\alpha(m_{[0,\sigma\beta^{-1}]})\otimes (m_{[1,\beta]}\rtimes \sigma^{-1}) = \big((\rho_\alpha\otimes (C_\beta\rtimes kG))\circ \rho_\beta\big)(m),
\end{aligned}$$

as needed. Conversely, assume that $M \in \mathcal{M}^{\underline{C}\rtimes kG}$. For each $\alpha \in G$ we have a map $\rho_\alpha : M \to M\otimes (C_\alpha \rtimes kG)$. ρ_e turns M into a right $C_e\rtimes kG$-comodule. Since $p_e : C_e \rtimes kG \to kG, p_e(c\rtimes \sigma) = \varepsilon(c)\sigma^{-1}$ is a coalgebra map, we have a right kG-coaction $(M\otimes p_e)\circ\rho_e : M \to M\otimes kG$ on M. This makes M into a G-graded k-vector space $M = \bigoplus_{\sigma\in G} M_\sigma$, where $m \in M_\sigma$ if and only if $\rho_e(m) \in M\otimes (C_e\rtimes \sigma^{-1})$. For $m \in M$, we introduce the notation

$$\rho_\alpha(m) = \sum_{\sigma\in G} m^\sigma_{[0]}\otimes m^\sigma_{[1,\alpha]}\rtimes \sigma^{-1} \in M\otimes (C_\alpha\rtimes kG).$$

With this notation we have that $m \in M_\tau$ if and only if $m^\sigma_{[0]}\otimes m^\sigma_{[1,e]} = 0$, for all $\sigma \neq \tau$. We have that the diagram (1) commutes. Let us write this down explicitely. For all $m \in M$ and $\alpha,\beta \in G$, we have that

$$\begin{aligned}
\big((M\otimes \Delta_{\alpha,\beta})\circ \rho_{\alpha\beta}\big)(m) &= (M\otimes \Delta_{\alpha,\beta})\Big(\sum_\sigma m^\sigma_{[0]}\otimes (m^\sigma_{[1,\alpha\beta]}\rtimes \sigma^{-1})\Big)\\
&= \sum_\sigma m^\sigma_{[0]}\otimes (m^\sigma_{[1,\alpha\beta](1,\alpha)}\rtimes \beta\sigma^{-1})\otimes (m^\sigma_{[1,\alpha\beta](2,\beta)}\rtimes \sigma^{-1})
\end{aligned}$$

equals

$$\begin{aligned}
\big((\rho_\alpha\otimes (C_\beta\rtimes kG))\circ \rho_\beta\big)(m) &= (\rho_\alpha\otimes (C_\beta\rtimes kG))\Big(\sum_\sigma m^\sigma_{[0]}\otimes (m^\sigma_{[1,\beta]}\rtimes \sigma^{-1})\Big)\\
&= \sum_{\sigma,\tau} (m^\sigma_{[0]})^\tau_{[0]}\otimes ((m^\sigma_{[0]})^\tau_{[1,\alpha]}\rtimes \tau^{-1})\otimes (m^\sigma_{[1,\beta]}\rtimes \sigma^{-1}).
\end{aligned}$$

We will refer to this equation as $(\star)$. Before we end the proof of Proposition 2.2, we state and prove two Lemmas.

Lemma 2.3. *Let $\omega \in G$. The following assertions are equivalent:*

(i) $m \in M_\omega$;
(ii) $\rho_\alpha(m) \in M\otimes (C_\alpha\rtimes \omega^{-1})$, *for all* $\alpha \in G$;
(iii) $m^\sigma_{[0]}\otimes m^\sigma_{[1,\alpha]} = 0$, *for all* $\omega \neq \sigma, \alpha \in G$.

Proof. <u>*(ii)* ⇔ *(iii)*</u> and <u>*(iii)* ⇒ *(i)*</u> are obvious. We prove <u>*(i)* ⇒ *(iii)*</u>. Take $\beta = e$ in $(\star)$. We view the equality $(\star)$ as an equality in $\bigoplus_{\sigma,\tau\in G}(M\otimes C_\alpha\otimes C_\beta)e_{\sigma,\tau}$. For $\sigma \neq \omega$, the $(\sigma^{-1},\sigma^{-1})$-component of $(\star)$ comes out as

$$0 = (m^\sigma_{[0]})^\sigma_{[0]}\otimes (m^\sigma_{[0]})^\sigma_{[1,\alpha]}\otimes m^\sigma_{[1,e]} = m^\sigma_{[0]}\otimes m^\sigma_{[1,\alpha](1,\alpha)}\otimes m^\sigma_{[1,\alpha](2,e)}.$$

If we apply ε to the third tensor factor, then we find $0 = m^\sigma_{[0]}\otimes m^\sigma_{[1,\alpha]}$, as needed. □

Lemma 2.4. *Let $\omega \in G$ and $m \in M_\omega$. Then, for all $\beta \in G$,*

$$\rho_\beta(m) \in M_{\omega\beta^{-1}} \otimes C_\beta \rtimes \omega^{-1}.$$

Proof. By Lemma 2.3 ($\star$) takes the form

$$\begin{aligned} & m^\omega_{[0]} \otimes (m^\omega_{[1,\alpha\beta](1,\alpha)} \rtimes \beta\omega^{-1}) \otimes (m^\omega_{[1,\alpha\beta](2,\beta)} \rtimes \omega^{-1}) \\ &\quad = \sum_\tau (m^\omega_{[0]})^\tau_{[0]} \otimes ((m^\omega_{[0]})^\tau_{[1,\alpha]} \rtimes \tau^{-1}) \otimes (m^\omega_{[1,\beta]} \rtimes \omega^{-1}). \end{aligned}$$

Fix $\tau \neq \omega\beta^{-1}$, and take the (τ^{-1}, ω^{-1})-component of both sides. This gives

$$0 = (m^\omega_{[0]})^\tau_{[0]} \otimes (m^\omega_{[0]})^\tau_{[1,\alpha]} \otimes m^\omega_{[1,\beta]},$$

and

$$(\rho_\alpha \otimes (C_\beta \rtimes kG))(\rho_\beta(m)) \in M \otimes (C_\alpha \rtimes \beta\omega^{-1}) \otimes (C_\beta \rtimes \omega^{-1}).$$

Hence $\rho_\beta(m) \in M_{\omega\beta^{-1}} \otimes C_\beta \rtimes \omega^{-1}$. □

Now we complete the proof of Proposition 2.2. For $\alpha, \beta \in G$, and $m \in M_{\alpha\beta}$, we have

$$\rho_\beta(m) = m^{\alpha\beta}_{[0]} \otimes m^{\alpha\beta}_{[1,\beta]} \rtimes \beta^{-1}\alpha^{-1} \in M_\alpha \otimes C_\beta \rtimes \beta^{-1}\alpha^{-1}.$$

We define $\rho_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes C_\beta$ by $\rho_{\alpha,\beta}(m) = m^{\alpha\beta}_{[0]} \otimes m^{\alpha\beta}_{[1,\beta]}$. Let us show that $\underline{M} = (M_\alpha)_{\alpha \in G}$ satisfies the coassociativity condition. If $m \in M_\omega$, then $\rho_\alpha(m) = m^\omega_{[0]} \otimes m^\omega_{[1,\alpha]} \rtimes \omega^{-1}$. Now take $m \in M_{\alpha\beta\gamma}$. Then

$$\begin{aligned} & ((M \otimes \Delta_{\beta,\gamma}) \circ \rho_{\beta\gamma})(m) \\ &\quad = m^{\alpha\beta\gamma}_{[0]} \otimes (m^{\alpha\beta\gamma}_{[1,\beta\gamma](1,\beta)} \rtimes \beta^{-1}\alpha^{-1}) \otimes (m^{\alpha\beta\gamma}_{[1,\beta\gamma](2,\gamma)} \rtimes \gamma^{-1}\beta^{-1}\alpha^{-1}) \end{aligned}$$

is equal to

$$\begin{aligned} & ((\rho_\beta \otimes C_\gamma) \circ \rho_\gamma)(m) \\ &\quad = (m^{\alpha\beta\gamma}_{[0]})^{\alpha\beta}_{[0]} \otimes ((m^{\alpha\beta\gamma}_{[0]})^{\alpha\beta}_{[1,\beta]} \rtimes \beta^{-1}\alpha^{-1}) \otimes (m^{\alpha\beta\gamma}_{[1,\gamma]} \rtimes \gamma^{-1}\beta^{-1}\alpha^{-1}). \end{aligned}$$

Hence

$$\begin{aligned} & ((M_\alpha \otimes \Delta_{\beta,\gamma}) \circ \rho_{\alpha,\beta\gamma})(m) = (M_\alpha \otimes \Delta_{\beta,\gamma})(m^{\alpha\beta\gamma}_{[0]} \otimes m^{\alpha\beta\gamma}_{[1,\beta\gamma]}) \\ &\quad = (m^{\alpha\beta\gamma}_{[0]})^{\alpha\beta}_{[0]} \otimes (m^{\alpha\beta\gamma}_{[0]})^{\alpha\beta}_{[1,\beta]} \otimes m^{\alpha\beta\gamma}_{[1,\gamma]} = ((\rho_{\alpha,\beta} \otimes C_\gamma) \circ \rho_{\alpha\beta,\gamma})(m). \end{aligned}$$

Take $m \in M_\beta$. Then $\rho_e(m) = m^\beta_{[0]} \otimes (m^\beta_{[1,e]} \rtimes \beta^{-1})$, and $\rho_{\beta,e}(m) = m^\beta_{[0]} \otimes m^\beta_{[1,e]}$. It follows that

$$((M \otimes \varepsilon) \circ \rho_{\beta,e})(m) = ((M \otimes \varepsilon) \circ \rho_e)(m) = m,$$

so the counit property is also satisfied. □

Combining Proposition 1.1 and 2.2, we obtain a pair of adjoint functors (F', G') between the categories $\mathcal{M}^{\underline{C} \rtimes kG}$ and $\mathcal{M}^{C_e}$. For $(M, (\rho_\alpha)_{\alpha \in G}) \in \mathcal{M}^{\underline{C} \rtimes kG}$,

$$F'(M) = \{m \in M \mid \rho_e(m) \in M \otimes (C_e \rtimes e)\}.$$

For $N \in \mathcal{M}^{C_e}$, $G'(N) = (\oplus_{\alpha\in G} N\square_{C_e} C_\alpha, (\rho_\alpha)_{\alpha\in G})$, with

$$\rho_\beta :\ \oplus_{\alpha\in G} N\square_{C_e} C_\alpha \to \oplus_{\alpha\in G}(N\square_{C_e} C_\alpha)\otimes (C_\beta \rtimes kG)$$

defined as follows: for $\sum_i n_i \otimes c_i \in N\square_{C_e} C_\alpha$,

$$\rho_\beta(\sum_i n_i \otimes c_i) = \sum_i (n_i \otimes c_{i(1,\alpha\beta^{-1})} \otimes (c_{i(2,\beta)} \rtimes \sigma^{-1}.$$

It follows from Theorem 1.7 and Proposition 2.2 that (F', G') is a pair of inverse equivalences if and only if $\underline{C}$ is a strong G-coalgebra.

3. Crossed coproducts and cocleft G-coalgebras

Let $k\langle G\rangle$ be the G-coalgebra defined by $k\langle G\rangle_\sigma = kp_\sigma$, $\Delta_{\sigma,\tau}(p_{\sigma\tau}) = p_\sigma \otimes p_\tau$ and $\varepsilon(p_e) = 1$, for all $\sigma, \tau \in G$. $k\langle G\rangle$ is even a Hopf G-coalgebra: every kp_σ is a k-algebra (isomorphic to k), and the antipode maps $S_\sigma :\ kp_\sigma \to kp_{\sigma^{-1}}$ are given by $S_\sigma(p_\sigma) = p_{\sigma^{-1}}$.

Let C be a coalgebra. Suppose that we have a weak G-action on C, this is a collection of k-coalgebra maps $\lambda = \{\lambda_\alpha \mid \alpha \in G\}$. Assume moreover that we have a collection of k-linear maps $f = \{f_{\alpha,\beta} \mid \alpha, \beta \in G\} \subset C^*$. We assume that $f_{e,e}$ has a convolution inverse $g_{e,e}$ and that

$$\lambda_e(c) = f_{e,e}(c_{(1)})c_{(2)}g_{e,e}(c_{(3)}). \tag{2}$$

for all $c \in C$. For $\alpha, \beta \in G$, consider the maps

$$\delta_{\alpha,\beta} :\ C \to C \otimes C,\ \delta_{\alpha,\beta}(c) = c_{(1)} \otimes \lambda_\alpha(c_{(2)})f_{\alpha,\beta}(c_{(3)}).$$

Now let $C \rtimes k\langle G\rangle = (C \rtimes p_\alpha)_{\alpha\in G}$, with comultiplication and counit maps given by

$$\Delta_{\alpha,\beta}(c \rtimes p_{\alpha\beta}) = (c_{(1)} \rtimes p_\alpha) \otimes (\lambda_\alpha(c_{(2)})f_{\alpha,\beta}(c_{(3)}) \rtimes p_\beta)$$

and

$$\varepsilon(c \rtimes p_e) = g_{e,e}(c),$$

for all $\alpha, \beta \in G$ and $c \in C$. Straightforward computations now show the following result.

Proposition 3.1. *Let λ be a weak G-action on a coalgebra C, and f a collection of maps satisfying (2). Then $C \rtimes k\langle G\rangle$ is a G-coalgebra if and only if the following conditions are satisfied, for all $c \in C$ and $\alpha, \beta, \gamma \in G$:*

$$f_{e,\alpha} = f_{e,e} \quad ; \quad f_{\alpha,e} = f_{e,e} \circ \lambda_\alpha; \tag{3}$$

$$f_{\beta,\gamma}(\lambda_\alpha(c_{(1)}))f_{\alpha,\beta\gamma}(c_{(2)}) = f_{\alpha,\beta}(c_{(1)})f_{\alpha\beta,\gamma}(c_{(2)}); \tag{4}$$

$$(\lambda_\beta \circ \lambda_\alpha)(c_{(1)})f_{\alpha,\beta}(c_{(2)}) = f_{\alpha,\beta}(c_{(1)})\lambda_{\alpha\beta}(c_{(2)}). \tag{5}$$

If f satisfies the conditions of Proposition 3.1, and every $f_{\alpha,\beta}$ has a convolution inverse $g_{\alpha,\beta}$, then f is called a factor set, and $C \rtimes k\langle G\rangle$ is called a crossed coproduct G-coalgebra. f is called normalized if $f_{\alpha,e} = f_{e,\alpha} = \varepsilon_C$, for all $\alpha \in G$. Then $\lambda_e = C$, and the counit of $C \rtimes k\langle G\rangle$ is given by the formula $\varepsilon(c \rtimes p_e) = \varepsilon_C(c)$.

If $f_{\alpha,\beta} = \varepsilon_C$, for all $\alpha, \beta \in G$, then we call $C \rtimes k\langle G\rangle$ a smash coproduct G-coalgebra. If, in addition, $\lambda_\alpha = C$, then $C \rtimes k\langle G\rangle$ is the cofree G-coalgebra $C\langle G\rangle$ introduced in [2]. In particular, if $C = k$, then we recover the G-coalgebra $k\langle G\rangle$ introduced at the beginning of this Section.

We will now show that the factor set f can be chosen in such a way that it is normalized To this end, we will apply the following construction. Let $\underline{C}$ be a G-coalgebra, and $\underline{\varphi} : \underline{C} \to \underline{D}$ an isomorphism in $(\mathcal{M}_k)^G$. Then we can define a G-coalgebra structure on $\underline{D}$ such that $\underline{\varphi}$ becomes an isomorphism of G-coalgebras: the structure maps on $\underline{D}$ are $\varepsilon' = \varepsilon \circ \varphi_e^{-1} :\ D_e \to k$ and

$$\Delta'_{\alpha,\beta} = (\varphi_\alpha \otimes \varphi_\beta) \circ \Delta_{\alpha,\beta} \circ \varphi_{\alpha\beta}^{-1} :\ D_{\alpha\beta} \to D_\alpha \otimes D_\beta. \tag{6}$$

Proposition 3.2. *Let G be a group acting weakly on a coalgebra C, and f a factor set. Then there exists a set of k-coalgebra maps $\{\lambda'_\alpha \mid \alpha \in G\} \subset$ End(C) and a normalized factor set f' such that $C \rtimes_f k\langle G\rangle \cong C \rtimes_{f'} k\langle G\rangle$ as G-coalgebras.*

Proof. For every $\alpha \in G$, consider the isomorphism $\varphi_\alpha :\ C \rtimes p_\alpha \to C \rtimes p_\alpha$, defined as follows: $\varphi_\alpha = C \rtimes p_\alpha$ if $\alpha \neq e$ and $\varphi_e(c \rtimes p_e) = c_{(1)} g_{e,e}(c_{2)}) \rtimes p_e$. The inverse of φ_e is defined by the formula $\varphi_e^{-1}(c \rtimes p_e) = c_{(1)} f_{e,e}(c_{2)}) \rtimes p_e$. Applying the above construction, we find a new G-coalgebra structure on $C \rtimes k\langle G\rangle$. The new counit ε is defined as follows:

$$\varepsilon'(c \rtimes p_e) = g_{e,e}(c_{(1)} f_{e,e}(c_{(2)})) = \varepsilon(c).$$

We compute the new comultiplication maps $\delta'_{\alpha,\beta} :\ C \to C \otimes C$ using (6). Clearly $\delta'_{\alpha,\beta} = \delta_{\alpha,\beta}$ if $\alpha \neq e$, $\beta \neq e$ and $\alpha\beta \neq e$. For $\alpha \neq e$, we compute

$$\begin{aligned}
\delta'_{e,\alpha}(c) &= (\varphi_e \otimes C)(\delta_{e,\alpha}(c)) = (\varphi_e \otimes C)\big(c_{(1)} \otimes \lambda_e(c_{(2)}) f_{e,\alpha}(c_{(3)})\big)\\
&\overset{(2,3)}{=} c_{(1)} g_{e,e}(c_{(2)}) \otimes f_{e,e}(c_{(3)}) c_{(4)} = c_{(1)} \otimes c_{(2)};\\
\delta'_{\alpha,e}(c) &= (C \otimes \varphi_e)(\delta_{\alpha,e}(c)) = (C \otimes \varphi_e)(c_{(1)} \otimes \lambda_\alpha(c_{(2)}) f_{\alpha,e}(c_{(3)}))\\
&\overset{(3)}{=} c_{(1)} \otimes \lambda_\alpha(c_{(2)}) g_{e,e}(\lambda_\alpha(c_{(3)})) f_{e,e}(\lambda_\alpha(c_{(4)}))\\
&= c_{(1)} \otimes \lambda_\alpha(c_{(2)}) \varepsilon(\lambda_\alpha(c_{(3)})) = c_{(1)} \otimes \lambda_\alpha(c_{(2)});\\
\delta'_{\alpha,\alpha^{-1}}(c) &= \delta_{\alpha,\alpha^{-1}}(\varphi_e^{-1}(c)) = \delta_{\alpha,\alpha^{-1}}(c_{(1)} f_{e,e}(c_{(2)}))\\
&= c_{(1)} \otimes \lambda_\alpha(c_{(2)}) f_{\alpha,\alpha^{-1}}(c_{(3)}) f_{e,e}(c_{(4)});\\
\delta'_{e,e}(c) &= (\varphi_e \otimes \varphi_e)(\delta_{e,e}(c_{(1)}) f_{e,e}(c_{(2)}))\\
&= (\varphi_e \otimes \varphi_e)(c_{(1)} \otimes \lambda_e(c_{(2)}) f_{e,e}(c_{(3)}) f_{e,e}(c_{(4)}))\\
&= c_{(1)} g_{e,e}(c_{(2)}) \otimes \lambda_e(c_{(3)}) g_{e,e}(\lambda_e(c_{(4)})) f_{e,e}(c_{(5)}) f_{e,e}(c_{(6)})\\
&\overset{(2)}{=} c_{(1)} g_{e,e}(c_{(2)}) \otimes f_{e,e}(c_{(3)}) c_{(4)} g_{e,e}(c_{(5)}) f_{e,e}(c_{(6)}) g_{e,e}(c_{(7)})\\
&\quad\ g_{e,e}(c_{(8)}) f_{e,e}(c_{(9)}) f_{e,e}(c_{(10)}) = c_{(1)} \otimes c_{(2)}.
\end{aligned}$$

It follows that $C \rtimes_f k\langle G\rangle \cong C \rtimes_{f'} k\langle G\rangle$ with

$$\lambda'_e = C;\ \lambda'_\alpha = \lambda_\alpha \text{ if } \alpha \neq e;$$

$$f'_{\alpha,e} = f'_{e,\alpha} = \varepsilon_C, \text{ for all } \alpha \in G;$$
$$f'_{\alpha,\alpha^{-1}} = f_{\alpha,\alpha^{-1}} * f_{e,e} \text{ if } \alpha \neq e;$$
$$f'_{\alpha,\beta} = f_{\alpha,\beta} \text{ if } \alpha \neq e,\ \beta \neq e \text{ and } \alpha\beta \neq e.$$

□

Let $\underline{C}$ be a G-coalgebra. In the sequel, we will consider morphisms $\underline{u}:\ \underline{C} \to k\langle G\rangle$ in the category $(\mathcal{M}_k)^G$. Such a morphism is given by a collection of maps $\{u_\alpha \in C_\alpha^* \mid \alpha \in G\}$. The α-component of $\underline{u}$ then sends $c \in C_\alpha$ to $u_\alpha(c)p_\alpha$. We say $\underline{u}$ is convolution invertible if there exists a collection of maps $\{v_\alpha \in C_{\alpha^{-1}}^* \mid \alpha \in G\}$ such that

$$u_\alpha(c_{(1,\alpha)})v_\alpha(c_{(2,\alpha^{-1})}) = v_\alpha(c_{(1,\alpha^{-1})})u_\alpha(c_{(2,\alpha)}) = \varepsilon(c), \tag{7}$$

for all $c \in C_e$. If $\underline{u}$ is a morphism of G-coalgebras, then $\underline{u}$ is convolution invertible: it suffices to take $v_\alpha = u_{\alpha^{-1}}$.

Definition 3.3. A G-coalgebra $\underline{C}$ is called *cocleft* over C_e if there exists a *convolution invertible* morphism $\underline{u} : \underline{C} \to k\langle G\rangle$ in $(\mathcal{M}_k)^G$.

Theorem 3.4. *For a G-coalgebra $\underline{C}$, the following conditions are equivalent.*
(1) *$\underline{C}$ is cocleft;*
(2) *$\underline{C}$ is isomorphic to a crossed coproduct G-coalgebra $C \rtimes_f k\langle G\rangle$;*
(3) *$\underline{C}$ is isomorphic to a crossed coproduct G-coalgebra $C \rtimes_f k\langle G\rangle$, with f normalized;*
(4) *$\underline{C}$ is a strong G-coalgebra, and every C_α is isomorphic to C_e as a left C_e-comodule.*

Proof. $\underline{(1) \Rightarrow (2)}$. For $\alpha, \beta \in G$, we define $\lambda_\alpha :\ C_e \to C_e$ and $f_{\alpha,\beta} :\ C_e \to k$ by the formulas

$$\lambda_\alpha(c) = u_\alpha(c_{(1,\alpha)})c_{(2,e)}v_\alpha(c_{(3,\alpha^{-1})}), \tag{8}$$
$$f_{\alpha,\beta}(c) = u_\alpha(c_{(1,\alpha)})u_\beta(c_{(2,\beta)})v_{\alpha\beta}(c_{(3,\beta^{-1}\alpha^{-1})}). \tag{9}$$

It is easy to verify that the λ_α are coalgebra maps, and that f is a factor set. Clearly the maps $f_{\alpha,\beta}$ are convolution invertible. Now define $\underline{\varphi} : \underline{C} \to C_e \rtimes k\langle G\rangle$ by $\varphi_\alpha(c) = c_{(1,e)}u_\alpha(c_{(2,\alpha)}) \rtimes p_\alpha$. $\underline{\varphi}$ is an isomorphism of G-coalgebras, with inverse given by $\varphi_\alpha^{-1}(c \rtimes p_\alpha) = c_{(1,\alpha)}v_\alpha(c_{(2,\alpha^{-1})})$.
$\underline{(2) \Rightarrow (3)}$ follows immediately from Proposition 3.2.
$\underline{(3) \Rightarrow (4)}$. Let $\underline{C} = C \rtimes k\langle G\rangle$ be a crossed coproduct G-coalgebra. It is easy to see that $\delta_{e,e} = \delta_{e,\alpha}$, and this implies that $C_\alpha = C \rtimes p_\alpha$ is isomorphic to $C_e = C \rtimes p_e$ as left C_e-comodules. In order to show that $\underline{C}$ is strong, it suffices to show that $\delta_{\alpha,\alpha^{-1}}$ is monic, for all $\alpha \in G$, see Proposition 1.3 For all $c \in C$, we have

$$\begin{aligned}(g_{\alpha,\alpha^{-1}} \otimes \lambda_{\alpha^{-1}})(\delta_{\alpha,\alpha^{-1}}(c)) &= g_{\alpha,\alpha^{-1}}(c_{(1)})(\lambda_{\alpha^{-1}} \circ \lambda_\alpha)(c_{(2)})f_{\alpha,\alpha^{-1}}(c_{(3)})\\ &\overset{(5)}{=} g_{\alpha,\alpha^{-1}}(c_{(1)})f_{\alpha,\alpha^{-1}}(c_{(2)})c_{(3)} = c,\end{aligned}$$

where we used the fact that $\lambda_e = C$ (f is normalized).
$\underline{(4) \Rightarrow (1)}$. From Theorem 1.7, we know that the corestriction $\Delta_{\alpha,\beta} :\ C_{\alpha\beta} \to$

$C_\alpha \square_{C_e} C_\beta$ has an inverse $\nabla_{\alpha,\beta}$. Let $\phi_\alpha : C_\alpha \to C_e$ be a left C_e-colinear isomorphism, for all $\alpha \in G$. Then we have the following formulas:

$$\Delta_{e,e}(\phi_\alpha(c)) = c_{(1,e)} \otimes \phi_\alpha(c_{(2,\alpha)}); \tag{10}$$

$$\Delta_{e,\alpha}(\phi_\alpha^{-1}(d)) = d_{(1,e)} \otimes \phi_\alpha^{-1}(d_{(2,e)}), \tag{11}$$

for all $c \in C_\alpha$ and $d \in C_e$. For $c \in C_{\alpha^{-1}}$, we have that $c_{(1,\alpha^{-1})} \otimes \phi_\alpha^{-1}(c_{(2,e)}) \in C_{\alpha^{-1}} \square_{C_e} C_\alpha$. Indeed,

$$\begin{aligned} c_{(1,\alpha^{-1})} \otimes \Delta_{e,\alpha}(\phi_\alpha^{-1}(c_{(2,e)})) &\overset{(11)}{=} c_{(1,\alpha^{-1})} \otimes c_{(2,e)} \otimes \phi_\alpha^{-1}(c_{(3,e)}) \\ &= \Delta_{\alpha^{-1},e}(c_{(1,\alpha^{-1})}) \otimes \phi_\alpha^{-1}(c_{(2,e)}). \end{aligned}$$

This implies that we have a well-defined map

$$v_\alpha = \varepsilon \circ \nabla_{\alpha^{-1},\alpha} \circ (C_{\alpha^{-1}} \otimes \phi_\alpha^{-1}) \circ \Delta_{\alpha^{-1},e} : C_{\alpha^{-1}} \to k.$$

We also consider $u_\alpha = \varepsilon \circ \phi_\alpha : C_\alpha \to k$. For every $c \in C_e$, we now have that

$$\begin{aligned} & u_\alpha(c_{(1,\alpha)}) v_\alpha(c_{(2,\alpha^{-1})}) \\ &= \varepsilon(\phi_\alpha(c_{(1,\alpha)}))(\varepsilon \circ \nabla_{\alpha^{-1},\alpha})(c_{(2,\alpha^{-1})} \otimes \phi_\alpha^{-1}(c_{(3,e)})) \\ &\overset{(11)}{=} \varepsilon(\phi_\alpha(\phi_\alpha^{-1}(c)_{(1,\alpha)}))(\varepsilon \circ \nabla_{\alpha^{-1},\alpha})(\phi_\alpha^{-1}(c)_{(2,\alpha^{-1})} \otimes \phi_\alpha^{-1}(c)_{(3,\alpha)}) \\ &= \varepsilon(\phi_\alpha(\phi_\alpha^{-1}(c)_{(1,\alpha)}))(\varepsilon \circ \nabla_{\alpha^{-1},\alpha} \circ \Delta_{\alpha^{-1},\alpha})(\phi_\alpha^{-1}(c)_{(2,e)}) \\ &= \varepsilon(\phi_\alpha(\phi_\alpha^{-1}(c)_{(1,\alpha)}))\varepsilon(\phi_\alpha^{-1}(c)_{(2,e)}) = \varepsilon(\phi_\alpha(\phi_\alpha^{-1}(c))) = \varepsilon(c); \\ & v_\alpha(c_{(1,\alpha^{-1})}) u_\alpha(c_{(2,\alpha)}) \\ &= (\varepsilon \circ \nabla_{\alpha^{-1},\alpha})(c_{(1,\alpha^{-1})} \otimes \phi_\alpha^{-1}(c_{(2,e)}))(\varepsilon \circ \phi_\alpha)(c_{(3,\alpha)}) \\ &\overset{(10)}{=} (\varepsilon \circ \nabla_{\alpha^{-1},\alpha})(c_{(1,\alpha^{-1})} \otimes \phi_\alpha^{-1}(\phi_\alpha(c_{(2,\alpha)})_{(1,e)})\varepsilon(\phi_\alpha(c_{(2,\alpha)})_{(2,e)})) \\ &= (\varepsilon \circ \nabla_{\alpha^{-1},\alpha})(c_{(1,\alpha^{-1})} \otimes \phi_\alpha^{-1}(\phi_\alpha(c_{(2,\alpha)}))) \\ &= (\varepsilon \circ \nabla_{\alpha^{-1},\alpha} \circ \Delta_{\alpha^{-1},\alpha})(c) = \varepsilon(c), \end{aligned}$$

as needed. □

If there exists a morphism of G-coalgebras $\underline{u} : \underline{C} \to k\langle G\rangle$, then $\underline{C}$ is cocleft. We have the following characterization of this situation.

Theorem 3.5. *For a G-coalgebra $\underline{C}$, the following assertions are equivalent:*

1. *$\underline{C}$ is isomorphic to a smash coproduct G-coalgebra;*
2. *there exists a morphism of G-coalgebras $\underline{u} : \underline{C} \to k\langle G\rangle$.*

Proof. $\underline{(1) \Rightarrow (2)}$. Let $\underline{C} = C \rtimes k\langle G\rangle$ be a smash coproduct G-coalgebra. Then $\delta_{\alpha,\beta}(c) = c_{(1)} \otimes \lambda_\alpha(c_{(2)})$, for all $c \in C$. The map $\underline{u} : C \rtimes k\langle G\rangle \to k\langle G\rangle$, $u_\alpha(c \rtimes p_\alpha) = \varepsilon_C(c)p_\alpha$, is a morphism of G-coalgebras since

$$\begin{aligned} ((u_\alpha \otimes u_\beta) \circ \Delta_{\alpha,\beta})(c \rtimes p_{\alpha\beta}) &= \varepsilon_C(c_{(1)})p_\alpha \otimes \varepsilon_C(\lambda_\alpha(c_{(2)}))p_\beta \\ &= \varepsilon_C(c)p_\alpha \otimes p_\beta = (\Delta_{\alpha,\beta} \circ u_{\alpha\beta}(c \rtimes p_{\alpha\beta}); \\ \varepsilon(c \rtimes p_e) = \varepsilon_C(c) &= \varepsilon(u_e(c \rtimes p_e)). \end{aligned}$$

<u>(2) ⇒ (1)</u>. As we have already mentioned, $\underline{C}$ is cocleft. The convolution inverse of u_α is just $v_\alpha = u_{\alpha^{-1}}$. In the proof of <u>(1) ⇒ (2)</u> in Theorem 3.4, we have seen that $\underline{C} \cong C_e \rtimes_f k\langle G\rangle$, with

$$f_{\alpha,\beta}(c) = u_\alpha(c_{(1,\alpha)})u_\beta(c_{(2,\beta)})u_{\beta^{-1}\alpha^{-1}}(c_{(3,\beta^{-1}\alpha^{-1})}) = u_e(c) = \varepsilon(c).$$

□

In Theorem 3.5, we have characterized crossed coproduct G-coalgebras with trivial factor set. We can also characterize when the maps λ_α are the identity maps.

Theorem 3.6. *For a G-coalgebra $\underline{C}$, the following conditions are equivalent.*

(1) *There exists a convolution invertible $\underline{u} : \underline{C} \to k\langle G\rangle$ such that*

$$u_\alpha(c_{(1,\alpha)})c_{(2,e)} = c_{(1,e)}u_\alpha(c_{(2,\alpha)}), \tag{12}$$

for all $\alpha \in G$ and $c \in C_\alpha$;

(2) *$\underline{C}$ is isomorphic to a crossed coproduct G-coalgebra $C \rtimes_f k\langle G\rangle$, with $\lambda_\alpha = C$, for all $\alpha \in G$;*

(3) *$\underline{C}$ is isomorphic to a crossed coproduct G-coalgebra $C \rtimes_f k\langle G\rangle$, with f normalized and with $\lambda_\alpha = C$, for all $\alpha \in G$;*

(4) *$\underline{C}$ is a strong G-coalgebra, and every C_α is isomorphic to C_e as a C_e-bicomodule.*

Proof. <u>(1) ⇒ (2)</u>. It follows from (12) that the maps λ_α constructed in the proof of <u>(1) ⇒ (2)</u> in Theorem 3.4 are equal to the identity map on C_e.
<u>(2) ⇒ (3)</u>. If the maps λ_α are equal to C in the proof of Proposition 3.2, then we also have that $\lambda'_\alpha = C$, for all $\alpha \in G$.
<u>(3) ⇒ (4)</u>. In the proof of <u>(3) ⇒ (4)</u> in Theorem 3.4, it is shown that $\underline{C}$ is strong, and left C_e-colinear maps $C_\alpha \to C_e$ are given; using the fact that $\lambda_\alpha = C$, we can easily show that these maps are also right C_e-colinear.
<u>(4) ⇒ (1)</u>. We are in the situation of the proof of <u>(4) ⇒ (1)</u> in Theorem 3.4, with the additional hypothesis that ϕ_α is right C_e-colinear, that is,

$$\Delta_{e,e}(\phi_\alpha(c)) = \phi_\alpha(c_{(1,\alpha)}) \otimes c_{(2,e)}.$$

Applying ε to the first tensor factor, we find that $\phi_\alpha(c) = u_\alpha(c_{(1,\alpha)})c_{(2,e)}$. If we apply ε to the second tensor factor of (10), then we obtain that $\phi_\alpha(c) = c_{(1,e)}u_\alpha(c_{(2,\alpha)})$, and (12) follows. □

4. Cohomology

Let C be a cocommutative coalgebra, λ a collection of coalgebra endomorphisms of C and f a factor set (see Proposition 3.1). Then (2) is equivalent to $\lambda_e = C$, and (5) is equivalent to $\lambda_{\alpha\beta} = \lambda_\beta \circ \lambda_\alpha$. This means that C is a right G-module coalgebra, with right G-action $c \cdot \alpha = \lambda_\alpha(c)$. Then C^* is a left G-module algebra, with left G-action $\langle \alpha \cdot f, c\rangle = \langle f, c \cdot \alpha\rangle$. The abelian group $\mathbb{G}_m(C^*)$ consisting of convolution invertible elements of C^* is a left G-module, and we can consider the cohomology groups $H^n(G, \mathbb{G}_m(C^*))$.

Condition (4) can be rewritten as $(\alpha \cdot f_{\beta,\gamma}) * f_{\alpha,\beta\gamma} = f_{\alpha,\beta} * f_{\alpha\beta,\gamma}$, which means precisely that $f \in Z^2(G, \mathbb{G}_m(C^*))$ is a 2-cocycle. (3) follows from (4), after we successively take $\alpha = \beta = e$ and $\beta = \gamma = e$ in (4).

Lemma 4.1. *Let C be a cocommutative right G-module coalgebra, and let $f \in Z^2(G, \mathbb{G}_m(C^*))$. If f' is the cocycle obtained from f using the construction in Proposition 3.2, then $f^{-1} * f' \in B^2(G, \mathbb{G}_m(C^*))$.*

Proof. For every $\alpha \in G$, we define $h_\alpha \in C^*$ as follows $h_\alpha = \varepsilon_C$ if $\alpha \neq e$ and $h_e = f_{e,e}$. We can easily compute $\delta_1(h)_{\alpha,\beta} = h_\alpha * h^{-1}_{\alpha\beta} * (\alpha \cdot h_\beta)$. We find that $\delta_1(h)_{e,\alpha} = f_{e,\alpha}$ and $\delta_1(h)_{\alpha,e} = f_{\alpha,e}$, for all $\alpha \in G$, $\delta_1(h)_{\alpha,\alpha^{-1}} = g_{e,e}$, for all $\alpha \neq e$ and $\delta_1(h)_{\alpha,\beta} = \varepsilon_C$, if $\alpha \neq e$, $\beta \neq e$ and $\alpha\beta \neq e$. Looking at the explicit formula of f' in the proof of Proposition 3.2, we see that $f_{\alpha,\beta} = f'_{\alpha,\beta} * \delta_1(h)_{\alpha,\beta}$. □

Proposition 4.2. *Let C be a cocommutative right G-module coalgebra, and take $f, f' \in Z^2(G, \mathbb{G}_m(C^*))$. The crossed coproduct G-coalgebra $C \rtimes_f k\langle G\rangle$ and $C \rtimes_{f'} k\langle G\rangle$ are isomorphic if and only if there exists a coalgebra automorphism φ of C such that $[f] = [f' \circ \varphi]$ in $H^2(G, \mathbb{G}_m(C^*))$.*

Proof. It follows from Proposition 3.2 and Lemma 4.1 that we may restrict attention to the situation where f and f' are normalized cocycles. First assume that $\{\phi_\alpha \mid \alpha \in G\}$ is a family of k-linear maps $C \to C$ such that $(\phi_\alpha \rtimes p_\alpha)_{\alpha\in G}$ is an isomorphism between the G-coalgebras $C \rtimes_f k\langle G\rangle$ and $C \rtimes_{f'} k\langle G\rangle$. Then we have for al $\alpha, \beta \in G$ and $c \in C$ that

$$\phi_\alpha(c_{(1)}) \otimes \phi_\beta(c_{(2)} \cdot \alpha) f_{\alpha,\beta}(c_{(3)}) = \phi_{\alpha\beta}(c)_{(1)} \otimes (\phi_{\alpha\beta}(c)_{(2)} \cdot \alpha) f'_{\alpha,\beta}(\phi_{\alpha\beta}(c)_{(3)}), \tag{13}$$

and $\varepsilon_C = \varepsilon_C \circ \phi_e$, where we used the fact that f and f' are normalized, so that $\varepsilon_C = g_{e,e} = g'_{e,e}$. In particular, ϕ_e is a coalgebra automorphism of C. Consider the maps $h_\alpha = \varepsilon_C \circ \phi_\alpha \in C^*$. Take $\alpha = e$ in (13) and apply ε_C to the second tensor factor. Again using the normality of f and f', we find that

$$\phi_e(c_{(1)}) h_\beta(c_{(2)}) = \phi_\beta(c). \tag{14}$$

It is then easy to show that $h^{-1}_\alpha = \varepsilon_C \circ \phi^{-1}_\alpha \circ \phi_e$ is the convolution inverse of h_α. Now apply $\varepsilon_C \otimes \varepsilon_C$ to (13), to obtain that

$$h_\alpha(c_{(1)})(\alpha \cdot h_\beta)(c_{(2)}) f_{\alpha,\beta}(c_{(3)}) = f'_{\alpha,\beta}(\phi_{\alpha\beta}(c)) \overset{(14)}{=} f'_{\alpha,\beta}(\phi_e(c_{(1)}) h_{\alpha\beta}(c_{(2)})),$$

or

$$f_{\alpha,\beta} * h_\alpha * (\alpha \cdot h_\beta) * h^{-1}_{\alpha\beta} = f'_{\alpha,\beta} \circ \phi_e, \tag{15}$$

Conversely, assume that $[f] = [f' \circ \varphi]$, that is, there exists a family of convolution invertible maps $\{h_\alpha \mid \alpha \in G\} \subset C^*$ such that (15) holds (with $\phi_e = \varphi$). Then we define ϕ_α by (14). Straightforward computations show that the ϕ_α define an isomorphism of G-coalgebras. □

Now let $\underline{C}$ be a cocleft G-coalgebra, and assume that C_e is cocommutative. Let $u_\alpha:\ C_\alpha \to k$ and $v_\alpha:\ C_{\alpha^{-1}} \to k$ be k-linear maps satisfying (7). Recall from the proof of $\underline{(1) \Rightarrow (2)}$ in Theorem 3.4 that we have maps $\lambda_\alpha:\ C_e \to C_e$ and $f_{\alpha,\beta}:\ C_e \to k$ defined by (8) and (9), and that $\underline{C} \cong C \rtimes_f k\langle G\rangle$. We have seen at the beginning of this Section that C_e is a right G-module coalgebra, with right G-action $c\cdot\alpha = \lambda_\alpha(c)$.

Lemma 4.3. *With notation as above, we have:*

1. *the maps λ_α are independent of the choice of $\underline{u}$;*
2. *for all $\alpha \in G$ and $c \in C_\alpha$, we have $c_{(1,\alpha)} \otimes c_{(2,e)} = c_{(2,\alpha)} \otimes c_{(1,e)} \cdot \alpha$.*

Proof. Applying comultiplication maps to the first tensor factor of the cocommutativity relation $c_{(1,e)} \otimes c_{(2,e)} = c_{(2,e)} \otimes c_{(1,e)}$, we find the following relation

$$c_{(1,\alpha^{-1})} \otimes c_{(2,\alpha)} \otimes c_{(3,e)} = c_{(2,\alpha^{-1})} \otimes c_{(3,\alpha)} \otimes c_{(1,e)}. \tag{16}$$

Let $\underline{u}':\ \underline{C} \to k\langle G\rangle$ be another convolution invertible morphism, with corresponding action $\bullet$. Then

$$\begin{aligned} c\cdot\alpha &= u_\alpha(c_{(1,\alpha)})c_{(2,e)}v_\alpha(c_{(3,\alpha^{-1})})u'_\alpha(c_{(4,\alpha)})v'_\alpha(c_{(5,\alpha^{-1})}) \\ &\overset{(16)}{=} u_\alpha(c_{(1,\alpha)})v_\alpha(c_{(2,\alpha^{-1})})u'_\alpha(c_{(3,\alpha)})c_{(4,e)}v'_\alpha(c_{(5,\alpha^{-1})}) \\ &= u'_\alpha(c_{(1,\alpha)})c_{(2,e)}v'_\alpha(c_{(3,\alpha^{-1})}) = c\bullet\alpha. \end{aligned}$$

and this proves (1). (2) is shown as follows.

$$\begin{aligned} c_{(2,\alpha)} \otimes c_{(1,e)}\cdot\alpha &= c_{(4,\alpha)} \otimes u_\alpha(c_{(1,\alpha)})c_{(2,e)}v_\alpha(c_{(3,\alpha^{-1})}) \\ &\overset{(16)}{=} c_{(3,\alpha)} \otimes u_\alpha(c_{(1,\alpha)})c_{(4,e)}v_\alpha(c_{(2,\alpha^{-1})}) = c_{(1,\alpha)} \otimes c_{(2,e)}. \end{aligned}$$

□

Let $\Omega_{\underline{C}} = \{\underline{u}:\ \underline{C} \to k\langle G\rangle \mid \underline{c} \text{ is a morphism of } G \text{ coalgebras}\}$. $\underline{C}$ is isomorphic to a smash coproduct G-coalgebra if and only if $\Omega_{\underline{C}} \neq \emptyset$. If we know one element of $\Omega_{\underline{C}}$, then we can describe all the others using cohomology.

Proposition 4.4. *If $\Omega_{\underline{C}} \neq \emptyset$, then there is a bijection*

$$\phi:\ \Omega_{\underline{C}} \to Z^1(G, \mathbb{G}_m(C_e^*)).$$

Proof. Fix $\underline{u}^0 \in \Omega_{\underline{C}}$, For $\underline{u} \in \Omega_{\underline{C}}$, we define $\phi(\underline{u}) = \theta:\ G \to \mathbb{G}_m(C_e^*)$ by the formula

$$\theta_\alpha(c) = u_\alpha(c_{(1,\alpha)})u^0_{\alpha^{-1}}(c_{(2,\alpha^{-1})}).$$

We show that θ_α is a 1-cocycle. For all $c \in C_e$, we have

$$\begin{aligned} &((\alpha\cdot\theta_\beta) * \theta_\alpha)(c) = \theta_\beta(c_{(1,e)}\cdot\alpha)\theta_\alpha(c_{(2,e)}) \\ &= \theta_\beta\big(u_\alpha(c_{(1,\alpha)})c_{(2,e)}u_{\alpha^{-1}}(c_{(3,\alpha^{-1})})\big)\theta_\alpha(c_{(4,e)}) \\ &= u_\alpha(c_{(1,\alpha)})u_\beta(c_{(2,\beta)})u^0_{\beta^{-1}}(c_{(3,\beta^{-1})}u_{\alpha^{-1}}(c_{(4,\alpha^{-1})})u_\alpha(c_{(5,\alpha)})u^0_{\alpha^{-1}}(c_{(6,\alpha^{-1})}) \\ &= u_\alpha(c_{(1,\alpha)})u_\beta(c_{(2,\beta)})u^0_{\beta^{-1}}(c_{(3,\beta^{-1})}u^0_{\alpha^{-1}}(c_{(4,\alpha^{-1})}) \\ &= u_{\alpha\beta}(c_{(1,\alpha\beta)})u^0_{\beta^{-1}\alpha^{-1}}(c_{(2,\beta^{-1}\alpha^{-1})}) = \theta_{\alpha\beta}(c). \end{aligned}$$

It follows that $\alpha\cdot\theta_\beta*\theta_\alpha=\theta_{\alpha\beta}$, which is precisely the cocycle relation. ϕ^{-1} is given by the formula $\phi^{-1}(\theta)=\underline{u}$, with $u_\alpha(c)=\theta_\alpha(c_{(1,e)})u^0_\alpha(c_{(2,\alpha)})$.t □

On $\Omega_{\underline{C}}$, we define the following equivalence relation: $\underline{u}\sim\underline{u}'$ if and only if there exists a convolution invertible $f:\ C_e\to k$ such that

$$f(c_{(1,e)})u_\alpha(c_{(2,\alpha)})=u'_\alpha(c_{(1,e)})f(c_{(2,\alpha)}),$$

for all $\alpha\in G$ and $c\in C_\alpha$. We denote

$$\underline{\Omega_{\underline{C}}}=\Omega_{\underline{C}}/\sim.$$

Proposition 4.5. *If $\Omega_{\underline{C}}\neq\emptyset$, then there is a bijection*

$$\underline{\phi}:\ \underline{\Omega_{\underline{C}}}\to H^1(G,\mathbb{G}_m(C_e^*)).$$

Proof. $\underline{u}\sim\underline{u}^0$ if and only if there exists $f\in C_e^*$ with convolution inverse g such that

$$u^0_\alpha(c)=f(c_{(1,e)})u_\alpha(c_{(2,\alpha)})g(c_{(3,e)}),$$

for all $\alpha\in G$, $c\in C_\alpha$. This is equivalent to

$$\begin{aligned}\theta_\alpha(c)&=u_\alpha(c_{(1,\alpha)})f(c_{(2,e)})u_{\alpha^{-1}}(c_{(3,\alpha^{-1})})g(c_{(4,e)})\\&=(a\cdot f)(c_{(1,e)})g(c_{(2,e)})=((a\cdot f)*g)(c).\end{aligned}$$

This is equivalent to the existence of $f\in C_e^*$ with convolution inverse g such that $\theta_\alpha=(\alpha\cdot f)*g$, which means precisely that $\theta_\alpha\in B^1(G,C_e^*)$. □

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