ON GOOD FILTRATION DIMENSIONS FOR STANDARDLY STRATIFIED ALGEBRAS

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To Professor Vlastimil Dlab on the occasion of his 70th birthday.

Abstract. ∇-good filtration dimensions of modules and of algebras are introduced by Parker for quasi-hereditary algebras. These concepts are now generalized to the setting of standardly stratified algebras. Let A be a standardly stratified algebra. The $\overline{\nabla}$-good filtration dimension of A is proved to be the projective dimension of the characteristic module of A. Several characterizations of $\overline{\nabla}$-good filtration dimensions and $\overline{\Delta}$-good filtration dimensions are given for properly stratified algebras. Finally we give an application of these results to the global dimensions of quasi-hereditary algebras with exact Borel subalgebra.

Introduction

As generalizations of quasi-hereditary algebras, properly stratified algebras and standardly stratifed algebras have been introduced by Cline, Parshall, Scott [3] and Dlab [5]. They appear in the work of Futorny, König and Mazorchuk on a generalization of the category O [8]. Analogously to quasi-hereditary algebras [2], [4], [17], given a standardly stratified algebra (A,Λ), of central importance are the modules filted by respectively standard modules, costandard modules, proper standard modules or proper costandard modules (the precise meaning will be given in Section 1) [1], [15].
Recently, in order to calculate the global dimension of the Schur algebra for GL_2 and GL_3, Parker [13] introduced the notion of ∇- (or Δ-)good filtration dimension for a quasi-hereditary algebra. The aim of this note is to calculate these dimensions for standardly stratified algebras and properly stratified algebras.

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In Section 1, we recall some definitions and results which will be needed later on. Various characterizations of $\overline{\nabla}$-good filtration dimensions of standardly stratified algebras and of properly stratified algebra are given in Section 2; the finitistic dimension for a properly stratified algebra is proved to have an upper bound by the sum of projective dimension and injective dimension of its characteristic modules. In Section 3, the results of Section 2 will be applied to quasi-hereditary algebras with a duality and with an exact Borel subalgebra in the sense of König [12]. Their global dimensions have an upper bound by two times of that of their exact Borel subalgebras. This class of algebras includes some algebras arising from Lie theory.

1. Preliminary results

Let R be a commutative Artin ring and A a basic Artin algebra over R. We will consider finitely generated left A-modules; maps between A-modules will be written on the righthand side of the argument, and the composition of maps $f: M_1 \to M_2,\ g: M_2 \to M_3$ will be denoted by fg. The category of left A-modules will be denoted by A-mod. All subcategories considered will be full and closed under isomorphisms.

Given a class Θ of A-modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of all A-modules which have a Θ-filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to an object of Θ for $1 \le i \le t$. The modules in $\mathcal{F}(\Theta)$ are called Θ-good modules, and the category $\mathcal{F}(\Theta)$ is called the Θ-good module category.

In the following, $(A, \le)$ will denote the algebra A together with a fixed ordering on a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents (given by the natural ordering of indices). For $1 \le i \le n$ let $E(i)$ be the simple A-module, which is the simple top of the indecomposable projective $P(i) = Ae_i$. The standard module $\Delta(i)$ is by definition the maximal factor module of $P(i)$ without composition factors $E(j)$ with $j > i$. $\overline{\Delta(i)}$ will be the notation for proper standard module, which is the maximal factor module of $\Delta(i)$ such that the multiplicity condition

$$[\overline{\Delta(i)} : E(i)] = 1$$

holds.

Dually for $1 \le \lambda \le n$, we have costandard modules $\nabla(\lambda)$ and proper costandard modules $\overline{\nabla(\lambda)}$.

Let Δ be the full subcategory consisting of all $\Delta(\lambda),\ \lambda \in \Lambda$, and $\Delta_{<\lambda}$ the full subcategory consisting of all $\Delta(\delta),\ \delta < \lambda$. In a similar way, we introduce ∇ and $\nabla_{<\lambda}$, and so on.

The pair $(A, \leq)$ is called standardly stratified if ${}_AA \in \mathcal{F}(\Delta)$ (compare [1], [15]). $(A, \leq)$ is called properly standardly stratified if ${}_AA \in \mathcal{F}(\Delta)$ and ${}_AA \in \mathcal{F}(\overline{\Delta})$ (cf. [5]). Note that these properties generalize the concept of quasi-hereditary algebras where we require the additional condition that the standard modules are Schur modules.

Let $(A, \leq)$ be a standardly stratified algebra. It was proved in [1], [15] that $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are functorially finite in A-mod, which means that they are at the same time covariantly and contravariantly finite in A-mod. A full subcategory $\mathcal{T}$ of A-mod is called contravariantly finite in A-mod if for any A-module M there is a module $M_1 \in \mathcal{T}$ and a morphism $f: M_1 \to M$ such that the restriction of $\mathrm{Hom_A}(-, \mathrm{f})$ to $\mathcal{T}$ is surjective. Such a morphism f is called a right $\mathcal{T}$-approximation of M. A right $\mathcal{T}$-approximation $f: M_1 \to M$ of M is called minimal if the restriction of f to any non-zero direct summand of M_1 is non-zero. The covariant finiteness of $\mathcal{T}$, left $\mathcal{T}$-approximations of M and the minimal left $\mathcal{T}$-approximation of M can be defined using duality arguments (compare [1], [15]).

It was also proved in [1], [15] that there is a unique basic module ${}_AT$ such that $\mathrm{add}({}_AT) = \mathcal{F}(\Delta) \sqcap \mathcal{F}(\overline{\nabla})$. Such a module ${}_AT$ is a generalized tilting A-module, and is called the characteristic module of A. The endomorphism ring of ${}_AT$ is again a standardly stratified algebra. These properties of standardly stratified algebras are summarized in the next Lemma.

Lemma 1.1. *Let $(A, \leq)$ be a standardly stratified algebra. Then the following statements hold.*

1. *$\mathcal{F}(\Delta)$ is a functorially finite and resolving subcategory*
2. *$\mathcal{F}(\overline{\nabla})$ is a covariantly finite and coresolving subcategory of A-mod.*
3. $\mathcal{F}(\Delta) = \{X \in A\text{-mod} \mid \mathrm{Ext}^1(\mathrm{X}, \mathcal{F}(\overline{\nabla})) = 0\}$.
4. $\mathcal{F}(\overline{\nabla}) = \{Y \in A\text{-mod} \mid \mathrm{Ext}^1(\mathcal{F}(\Delta), \mathrm{Y}) = 0\}$.
5. There exists a tilting module ${}_AT$ with $\mathrm{add}({}_AT) = \mathcal{F}(\Delta) \sqcap \mathcal{F}(\overline{\nabla})$.

It follows from Lemma 1.1 that there exists a finite $\mathcal{F}(\overline{\nabla})$-coresolution

$$0 \to X \to M_0 \to \cdots \to M_d \to 0 \tag{1}$$

with $M_i \in \mathcal{F}(\overline{\nabla})$, for all $X \in A$-mod.

Definition 1.2. [13] Let A be a standardly stratified algebra. The $\overline{\nabla}$-good filtration dimension of X is the smallest number d for which we have an $\mathcal{F}(\overline{\nabla})$-coresolution (1) with $M_i \in \mathcal{F}(\overline{\nabla})$. We then write

$$\overline{\nabla}\text{-gfd}(X) = d$$

Recall the following result from [7], [13]:

Lemma 1.3. *$\overline{\nabla}$-gfd$(X) = d$ if and only if $\mathrm{Ext}^i_A(\Delta(\lambda), X) = 0$ for all $i > d$ and all $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_A(\Delta(\lambda), X) \neq 0$.*

Remark 1.4. Using the duality principle, we can introduce the dual notions of $\overline{\Delta}$-good module filtration and $\overline{\Delta}$-good filtration dimension of an A-module X, denoted by $\overline{\Delta}$-gfd(X).

From [13], we recall the following definition:

Definition 1.5. Let $(A, \leq)$ be a standardly stratified algebra.

$$d = \sup\{\overline{\nabla}\text{-gfd}(X) \mid X \in A\text{-mod}\} =: \overline{\nabla}\text{-gfd}(\mathrm{A})$$

is called the $\overline{\nabla}$-good filtration dimension of A. In a similar way

$$d = \sup\{\overline{\Delta}\text{-gfd}(X) \mid X \in A\text{-mod}\} =: \overline{\Delta}\text{-gfd}(\mathrm{A})$$

is called the $\overline{\Delta}$-good filtration dimension of A.

The $\overline{\nabla}$-good filtration dimension of A considered as a left A-module will be denoted denoted by $\overline{\nabla}$-gfd$({}_AA)$. A similar notation will be used for the $\overline{\Delta}$-good filtration dimension.

2. Main result

Throughout this Section, A will be a standardly stratified algebra with poset $(\Lambda, \leq)$ and $T = \oplus_{\lambda\in\Lambda} T(\lambda)$ is the characteristic module of A.

Proposition 2.1. *Let $X \in A$-mod. Then $\overline{\nabla}$-gfd$(X) = d$ if and only if $\mathrm{Ext}^i_A(T, X) = 0$, for any $i > d$, but there exists $\lambda \in \Lambda$, such that $\mathrm{Ext}^d_A(T(\lambda), X) \neq 0$.*

Proof. Suppose $\overline{\nabla}$-gfd$(X) = d$. It follows from Lemma 1.3 that $\mathrm{Ext}^i_A(T, X) = 0$, for any $i > d$. There exists $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_A(\Delta(\lambda), X) \neq 0$, and let $\lambda_0 \in \Lambda$ be minimal with respect to this property. Applying $\mathrm{Hom}_A(-, X)$ to the exact sequence

$$0 \to \Delta(\lambda_0) \to T(\lambda_0) \to M(\lambda_0) \to 0$$

where $M(\lambda_0) \in \mathcal{F}(\Delta_{<\lambda_0})$, we obtain an exact sequence:

(2)

$$\mathrm{Ext}^d(M(\lambda_0), X) \to \mathrm{Ext}^d(T(\lambda_0), X) \to \mathrm{Ext}^d(\Delta(\lambda_0), X) \to \mathrm{Ext}^{d+1}(M(\lambda_0), X)$$

Therefore $\mathrm{Ext^d(T(\lambda_0), X)} \simeq \mathrm{Ext^d(\Delta(\lambda_0), X)} \neq 0$.
To prove the converse, we use induction on λ to prove that $\mathrm{Ext^i(\Delta(\lambda), X)} = 0$ for all λ and $i > d$. If λ is minimal in Λ, then $\Delta(\lambda) = T(\lambda)$, and we are done. Take a non-minimal $\lambda \in \Lambda$; applying $\mathrm{Hom_A(-, X)}$ to the exact sequence

$$0 \to \Delta(\lambda) \to T(\lambda) \to M(\lambda) \to 0$$

where $M(\lambda) \in \mathcal{F}(\Delta_{<\lambda})$, we obtain an exact sequence
(3)

$$\mathrm{Ext^i(M(\lambda), X)} \to \mathrm{Ext^i(T(\lambda), X)} \to \mathrm{Ext^i(\Delta(\lambda), X)} \to \mathrm{Ext^{i+1}(M(\lambda), X)}$$

It follows that $\mathrm{Ext^i(\Delta(\lambda), X)} = 0$ for all $i > d$, and we have that $\overline{\nabla}\text{-gfd}(X) \leq d$.
Let λ be the minimal weight such that $\mathrm{Ext^d(T(\lambda), X)} \neq 0$. It follows from the exact sequence (3) (with i=d) that

$$\mathrm{Ext^d(\Delta(\lambda), X)} \simeq \mathrm{Ext^d(T(\lambda), X)} \neq 0$$

Then $\overline{\nabla}\text{-gfd}(X) = d$, finishing our proof. □

As a consequence, we find

Proposition 2.2. $\overline{\nabla}\text{-gfd}(A) = s$ *if and only if* $\text{proj.dim}_A T = s$.

Proof. $\overline{\nabla}\text{-gfd}(A) = s$ if and only if $\overline{\nabla}\text{-gfd}(X) \leq s$ for all $X \in A\text{-mod}$, and there is an A-module M such that $\overline{\nabla}\text{-gfd}(M) = s$. By Proposition 2.1, this is equivalent to $\mathrm{Ext^j(T, X)} = 0$, for all $j > s$ and the existence of an A-module N such that $\mathrm{Ext^s(T, X)} = 0$. The last condition is equivalent to $\text{proj.dim}_A T = s$, finishing the proof. □

Proposition 2.3. $\text{proj.dim}_A T = d$ *if and only if* $\mathrm{Ext^i(T, A)} = 0$ *for all* $i > d$, *and* $\mathrm{Ext^d(T, A)} \neq 0$.

Proof. Assume that $\text{proj.dim}_A T = d$. It is easy to check that $\mathrm{Ext^d(T, A)} \neq 0$.
The proof of the converse consists of two steps. We first prove that for any λ and any $j > d$, $\mathrm{Ext^j(T, \Delta(\lambda))} = 0$. If λ is maximal in Λ, then $\Delta(\lambda) = T(\lambda)$, and then $\mathrm{Ext^j(T, \Delta(\lambda))} = 0$ for any $j > d$. If λ is not maximal, then we have an exact sequence

$$0 \to U(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$$

where $U(\lambda) \in \mathcal{F}(\Delta_{>\lambda})$. Applying $\mathrm{Hom(T, -)}$, we find the exact sequence:

$$\mathrm{Ext^j(T, P(\lambda))} \to \mathrm{Ext^j(T, \Delta(\lambda))} \to \mathrm{Ext^{j+1}(T, U(\lambda))}$$

and it follows that

$$\mathrm{Ext^j(T, \Delta(\lambda))} = 0$$

for any $j > d$.
Secondly, for any $M \in A$-mod we have that $\mathrm{Ext^j(T, M)} = 0$, for any $j > d$. Let $f : X \to M$ be the minimal $\mathcal{F}(\Delta)-$approximation of M. Then f is surjective and $K = \mathrm{Ker\, f} \in \mathcal{F}(\nabla)$, and we have the exact sequence

$$0 \to K \to X \to M \to 0$$

Applying $\mathrm{Hom(T, -)}$, we find an exact sequence

$$\mathrm{Ext^j(T, X))} \to \mathrm{Ext^j(T, M)} \to \mathrm{Ext^{j+1}(T, K)}$$

From the fact that $\mathrm{Ext^{j+1}(T, K)} = 0$, it follows that $\mathrm{Ext^j(T, M)} = 0$ for all $j > d$, and therefore proj.dim$_A T = d$. □

Recall from [10], we recall the notion of T-codimension of an A-module X: it is the smallest number s such that we have an exact sequence

$$0 \to X \to T_0 \to \cdots \to T_s \to 0$$

with $T_i \in \mathrm{add}T$ for all $0 \leq i \leq s$. We then write

$$T\text{-codim}(\mathrm{X}) = \mathrm{s}$$

In a similar way, we introduce $T\text{-}\dim(X)$, see [19].

Proposition 2.4. $\mathrm{proj.dim}(T) = T\text{-codim}(_{\mathrm{A}}\mathrm{A})$.

Proof. It follows from [10, Lemma 2.2] that $T\text{-codim}(_{\mathrm{A}}\mathrm{A}) \leq \mathrm{proj.dim(T)}$. Let $T\text{-codim}(_{\mathrm{A}}\mathrm{A}) = \mathrm{d}$. Then we have a T-coresolution of A of length d, namely

$$0 \to A \to T_0 \to \cdots \to T_d \to 0$$

with $T_i \in \mathrm{add}T$ for $0 \leq i \leq d$. It follows from [10] that any indecomposable direct summand of T appears in the coresolution, and therefore we have that $\mathrm{Ext^j(T, A)} = 0$ for any $j > d$. It follows from Proposition 2.3. that $\mathrm{proj.dim}(T) \leq d$, as needed. □

Combining everything together, we find the following descriptions of the $\overline{\nabla}$-good filtration dimension of a standard stratified algebra A.

Theorem 2.5. *Let $(A, \leq)$ be a standard stratified algebra, T the characteristic (full tilting) module of A and d a non-negative integer. Then the following are equivalent:*

1. $\overline{\nabla}$-gfd$(A) = d$,
2. $\mathrm{proj.dim}(T) = d$,
3. $\overline{\nabla}$-gfd$(_A A) = d$,
4. $T\text{-codim}(_{\mathrm{A}}\mathrm{A}) = \mathrm{d}$.

Proof. The equivalence of 1) and 2) is just Proposition 2.2, the equivalence of 2) and 4) is Proposition 2.3, and the equivalence of 2) and 3) is Proposition 2.4. □

Remark 2.6. Let A be a quasi-hereditary algebra. According to the Definitions in Section 1, the good filtration dimension of A as an algebra, and A considered as an A-module, could be different (see also the remark following Definition 2.3 in [13]). The equivalence of 1) and 4) in Theorem 2.5 tells us that they are equal:

$$\overline{\nabla}\text{-gfd}(A) = \overline{\nabla}\text{-gfd}({}_AA) \tag{4}$$

In the following we deal with properly stratified algebras; for basic properties of these algebras we refer to [5]. We are indebted to the referee for pointing out to us that Theorem 2.7 also follows from [1], together with Dlab's result (see [5]) that the oppposite algebra of a properly stratified algebra is standardly stratified. We kept a short proof for the sake of completeness.

Theorem 2.7. *Let $(A, \leq)$ be a properly stratified algebra. Then there exist a tilting module T and a cotilting module S such that* add$T$ = $\mathcal{F}(\Delta) \sqcap \mathcal{F}(\overline{\nabla})$ *and* add$S = \mathcal{F}(\nabla) \sqcap \mathcal{F}(\overline{\Delta})$.

Proof. Let $(A, \leq)$ be a properly stratified algebra. Then $(A, \leq)$ is standardly stratified, and then there exists a tilting module T with add$T = \mathcal{F}(\Delta) \sqcap \mathcal{F}(\overline{\nabla})$. Since ${}_AA \in \mathcal{F}(\overline{\Delta})$, we have $D(A_A) \in \mathcal{F}(\nabla)$. The dual version of Lemma 1.1 or [1, Theorem 2.1] tells us that

$$\mathcal{F}(\overline{\Delta}) = {}^{\perp}\mathcal{F}(\nabla) \text{ and } \mathcal{F}(\nabla) = {}^{\perp}\mathcal{F}(\overline{\Delta})$$

By the dual version of [1, Theorem 2.1], we have a basic cotilting A-module S with add$S = F(\nabla) \sqcap \mathcal{F}(\overline{\Delta})$. □

Corollary 2.8. *Let $(A, \leq)$ be a properly stratified algebra and take T and S as in Theorem 2.7. Then*

$$\begin{aligned} &(5) \quad \overline{\nabla}\text{-gfd}(A) &= \quad &\text{proj.dim}(T) = T\text{-codim}({}_AA) = \overline{\nabla}\text{-gfd}({}_AA) \\ &(6) \quad \overline{\Delta}\text{-gfd}(A) &= \quad &\text{inj.dim}(S) = S\text{-dim}(D(A_A)) = \overline{\Delta}\text{-gfd}(D(A_A)) \end{aligned}$$

Proof. (5) follows from Theorem 2.5, and (6) is the dual version of it. □

Applying Corollary 2.8 to quasi-hereditary algebras, we recover the various descriptions of good filtration dimensions introduced in [13].

Corollary 2.9. *Let $(A, \leq)$ be a quasi-hereditary algebra and T the characteristic module of A. Then*

$$\begin{aligned} &(7) \quad \nabla\text{-gfd}(A) &= \quad &\text{proj.dim}(T) = T\text{-codim}({}_AA) = \nabla\text{-gfd}({}_AA) \\ &(8) \quad \Delta\text{-gfd}(A) &= \quad &\text{inj.dim}(T) = T\text{-dim}(D(A_A)) = \Delta\text{-gfd}(D(A_A)) \end{aligned}$$

Proof. If $(A, \leq)$ be quasi-hereditary, then $\overline{\Delta}(i) = \Delta(i)$ and $\overline{\nabla}(i) = \nabla(i)$, and therefore $S = T$. All the assertions follow. □

Theorem 2.10. *Let $(A, \leq)$ be a properly stratified algebra with $T = S$, where T and S are as in Theorem 2.7. Then the sum of the projective and injective dimension of T is finite, and is an upper bound for the finitistic dimension of A.*

Proof. Let $(A, \leq)$ be a properly stratified algebra with $T = S$. Then $\text{proj.dim}(T) = t < \infty$ and $\text{inj.dim}(T) = s < \infty$. Let X be any left A-module with finite projective dimension and $\Omega^s(X)$ the s-th syzygy module of X. It follows that $\mathrm{Ext^{s+i}(X, T)} = 0$ for any $i > 0$. Therefore $\mathrm{Ext^i(\Omega^s(X), T)} = 0$ for any $i > 0$. One has that $\Omega^s(X) \in {}^{\perp}T$. It follows from $\text{proj.dim}(\Omega^s(X)) < \infty$ that $\Omega^s(X) \in \mathcal{F}(\Delta)$ (see [13]). Therefore we have that $\text{proj.dim}(\Omega^s(X)) \leq t$, and $\text{proj.dim}(X) \leq t + s$. It follows that the finitistic dimension of A is at most $s + t$. □

Note that quasi-hereditary algebras are examples of properly stratified algebras satisfying the condition in Theorem 2.10. A non-quasi-hereditary example is the following: let A be the quiver algebra of the quiver consisting of one vertex and one arrow α, with relation $\alpha^2 = 0$. Then $\mathcal{F}(\Delta) = \mathcal{F}(\nabla) = \text{add}(A)$. It is easy to see that A is a properly stratified algebra with $T = S$.

Applying Theorem 2.10 to a quasi-hereditary algebra A, we find that the global dimension of a quasi-hereditary algebra is bounded by the sum of the projective and injective dimension of their characteristic module T. We remark at this point that the same formula holds for the global dimension of the endomorphism algebra of a tilting module T over an Artin algebra, see [9].

As a direct consequence of Theorem 2.10, we have

Corollary 2.11. *Let A be a quasi-hereditary algebra and T the characteristic module of A. Then*

$$\max\{\text{proj.dim}(T), \text{inj.dim}(T)\} \leq \text{gl.dim}(A) \leq \text{proj.dim}(T) + \text{inj.dim}(T)$$

Example 2.12. There exist quasi-hereditary algebras such that the right inequality in Corollary 2.11 is strict. Let A be the path algebra given by

$$\overset{2}{\circ} \longrightarrow \overset{1}{\circ} \longrightarrow \overset{3}{\circ}$$

For the ordering $\Lambda = \{ 1 < 2 < 3 \}$, (A, Λ) is a quasi-hereditary algebra with characteristic module $T = E(1) \oplus P(1) \oplus P(2)$. Then $\text{gl.dim}(A) < \text{proj.dim}(T) + \text{inj.dim}(T)$.

Let $A' = \text{End}_A T$. A' is also a quasi-hereditary algebra with respect to the opposite ordering of Λ, and is called the Ringel dual of A. $T' = \text{Hom}_A(T, D(A))$ is the characteristic module of A', and the endomorphism ring of ${}_{A'}T'$ is Morita equivalent to A as a quasi-hereditary algebra. Applying Theorem 2.10 to the Ringel dual of A, we obtain a description of its global dimension.

Corollary 2.13. *Let A be a quasi-hereditary algebra, T its characteristic module, and A' its Ringel dual. Then*

$$\begin{aligned} \max\{\text{proj.dim}({}_AT), \text{inj.dim}({}_AT)\} &\leq \text{gl.dim}(A') \\ &\leq \text{proj.dim}({}_AT) + \text{inj.dim}({}_AT) \end{aligned}$$

Proof. We assume that T is the basic characteristic module, that is, it contains exactly $|\Lambda|$ indecomposables. Let $\text{proj.dim}({}_AT) = s$, $\text{inj.dim}({}_AT)$ $= t$. Then from (8) in Corollary 2.9 and the fact that the endomorphism ring $\text{End}_{A'}\text{T}'$ is isomorphic to A as a quasi-hereditary algebra [17], we have that $\text{proj.dim}({}_{A'}T') = t$ and $\text{proj.dim}({}_{A'}T') = s$. By Corollary 2.11, we have that

$$\text{gl.dim}(A') \leq \text{proj.dim}({}_AT) + \text{inj.dim}({}_AT)$$

□

Remark 2.14. It would be interesting to know for which quasi-hereditary algebra A we have the equality

$$\text{gl.dim}(A) = \text{proj.dim}({}_AT) + \text{inj.dim}({}_AT) \tag{9}$$

It was proved in [18] and [6, Sec. 4.8] (9) holds for Schur algebras $S(n, r)$, with p arbitrary and $n \geq r$. Parker [14] recently proved that (9) holds for Schur algebras with $n = 2, 3$ and p arbitrary, or $p > n$ and r arbitrary. In Section 3, we will show that this equality holds for quasi-hereditary algebras with simple modules as standard modules.

3. Applications and Examples

In this section we will apply the results of Section 2 to some special classes of quasi-hereditary algebras, especially to algebras with exact Borel subalgebras. For the definition of an exact Borel subalgebra of a quasi-hereditary algebra, we refer to [12] or [16]. Let A be a quasi-hereditary algebra. We say that A has a duality if there exists an involutory, contravariant functor $\phi :$ A-mod $\rightarrow$ A-mod preserving simple modules, i.e. $\phi(E(i)) = E(i)$ for any i. If A is a quasi-hereditary algebra A with a duality ϕ, then

$$\phi(\Delta(i)) = \nabla(i) \text{ and } \phi(T(i)) = T(i)$$

for all i.

Lemma 3.1. *Let (A, Λ) be a quasi-hereditary algebra with simple modules as $\Delta-$modules. Then*

$$\text{gl.dim}(A) = \text{proj.dim}(D(A)) = \text{inj.dim}({}_AA)$$

Proof. If A is a quasi-hereditary algebra with simple standard modules, then $\mathcal{F}(\Delta) = A$-mod. Therefore the injective module is the characteristic module of A. By Corollary 2.9 or Theorem 2.10, we have that gl.dim$(A) =$ proj.dim$(D(A))$. A is also a quasi-hereditary algebra with respect to the opposite ordering of Λ. For this quasi-hereditary algebra (A, Λ^{op}), Δ-modules are indecomposable projective modules, and its characteristic module is ${}_AA$. Then gl.dim$(A) =$ inj.dim(A). □

Lemma 3.2. *Let B be an exact Borel subalgebra of a quasi-hereditary algebra A. Then for any $B-$module ${}_BM$, the $A-$module $A \otimes_B M$ has a Δ_A-filtation. Moreover*

$$\text{proj.dim}(A \otimes_B M) \leq \text{proj.dim}({}_BM)$$

Proof. First, we prove that $A \otimes_B M \in \mathcal{F}({}_A\Delta)$. Since B is a Borel subalgebra of A, the functor

$$F = A \otimes - : \ B\text{-mod} \to \text{A-mod}$$

is exact and $F(\Delta_B(\lambda)) \simeq \Delta_A(\lambda)$ (compare [12], [16]). Let $G: A$-mod $\to$ B-mod be the restriction of scalars functor. We first prove that $G(\nabla_A(j)) \in \mathcal{F}({}_B\nabla)$ for all j. This follows from the isomorphisms

$$\begin{aligned} \text{Ext}^1_B(\Delta_B(i), G(\nabla_A(j))) &\simeq \text{Ext}^1_A(F(\Delta_B(i)), \nabla_A(j))) \\ &\simeq \text{Ext}^1_A(\Delta_A(i), \nabla_A(j)) = 0 \end{aligned}$$

for any $i, j \in \Lambda$. Then from the isomorphisms

$$\text{Ext}^1_A(F({}_BM), \nabla_A(j)) \simeq \text{Ext}^1_B({}_BM, G(\nabla_A(j))) = 0$$

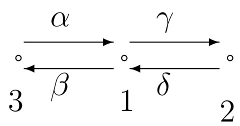
for any $j \in \Lambda$, we have that $F({}_BM) \in \mathcal{F}({}_A\Delta)$. Since $A \otimes -$ is an exact functor, applying $A \otimes -$ to a minimal projective resolution of ${}_BM$ gives a projective resolution of the characteristic module $A \otimes_B M$. This means that proj.dim$(A \otimes_B M) \leq$ proj.dim$({}_BM)$. This finishes the proof. □

Theorem 3.3. *Let A be a quasi-hereditary algebra with a duality and with an exact Borel subalgebra B. Then* gl.dim$(A) \leq 2$gl.dim(B).

Proof. For a quasi-hereditary algebra A with a duality ϕ, the characteristic module T is self-dual, i.e. $\phi(T) \simeq T$. Then $\text{proj.dim}(T) = \text{inj.dim}(T)$, and it follows from Lemmas 3.1 and 3.2 and Corollary 2.11 that $\text{gl.dim}(A) \leq 2\text{proj.dim}(T) = 2\sup\{\text{proj.dim}\Delta_A(\lambda) | \lambda \in \Lambda\} \leq 2\sup\{\text{proj.dim}\Delta_B(i) | \lambda \in \Lambda\} = 2\text{gl.dim}(B)$. □

Example 3.4. This example, taken from [12], shows that it is possible that $\text{gl.dim}(A) = 2\text{gl.dim}(B)$ for a Borel subalgebra B of a quasi-hereditary algebra A.

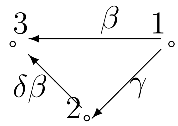
Let A be the algebra given by



with relations

$$\delta \cdot \gamma \cdot \alpha = 0, \ \beta \cdot \delta \cdot \gamma = 0, \ \beta \cdot \alpha = 0, \ \gamma \cdot \delta = 0$$

Ordering the weights by $1 < 2 < 3$, A is a quasi-hereditary algebra which has an exact Borel subalgebra B given by



with relation

$$\beta \cdot \delta \cdot \gamma = 0$$

It is easy to see that $\text{gl.dim}(B) = 2$ and $\text{gl.dim}(A) = 2\text{gl.dim}(B) = 4$.

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