SEMISIMPLICITY OF THE CATEGORIES OF YETTER-DRINFELD MODULES AND LONG DIMODULES

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Abstract. Let k be a field, and H a Hopf algebra with bijective antipode. If H is commutative, noetherian, semisimple and cosemisimple, then the category ${}_H\mathcal{YD}^H$ of Yetter-Drinfeld modules is semisimple. We also prove a similar statement for the category of Long dimodules, without the assumption that H is commutative.

Introduction

Let H be a Hopf algebra at the same time acting and coacting on a vector space M. We can impose various compatibility relations between the action and coaction, leading to different notions of Hopf modules. Hopf modules are already considered by Sweedler [15], and they have to satisfy the relation

$$\rho(hm) = \Delta(h)\rho(m) = h_1m_0 \otimes h_2m_1$$

One can also require that the H-coaction is H-linear:

$$\rho(hm) = h\rho(m) = hm_0 \otimes m_1$$

A module satisfying this condition is called a Long dimodule. Long dimodules are the building stones of the Brauer-Long group, in the case where the Hopf algebra H is commutative, cocommutative and faithfully projective (see [8], and [1] for a detailed discussion). Long dimodules are also connected to a non-linear equation (see [10]).
Another - at first sight complicated and artificial - compatibility relation is the following:

$$h_1m_0 \otimes h_2m_1 = (h_2m)_0 \otimes (h_2m)_1h_1$$

A module that satisfies it is called a Yetter-Drinfeld module. There is a close connection between Yetter-Drinfeld modules and the Drinfeld double (see [5]): if H is finitely generated projective, then the

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category of Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld double. Yetter-Drinfeld modules have been studied intensively by several authors over the passed fifteen years, see for example [3], [7], [9], [13], this list is far from exhaustive. One of the important features is the fact that the category of Yetter-Drinfeld modules is braided monoidal. As Long dimodules, Yetter-Drinfeld modules are related to a non-linear equation, the quantum Yang-Baxter equation (see e.g. [6]). If H is commutative and cocommutative, then Yetter-Drinfeld modules coincide with Long dimodules.
In this note, we give sufficient conditions for the categories of Yetter-Drinfeld modules and Long dimodules to be semisimple (Section 3) and we study projective and injective dimension in these categories. Our main result is that the category of Yetter-Drinfeld modules is semisimple if H is a commutative, noetherian, semisimple and cosemisimple Hopf algebra over a field k. The same is true for the category of Long dimodules, without the assumption that H is commutative. In the case where H is finite dimensional there is a partial overlap with [12, Prop. 7], where necessary and sufficient conditions for the semiplicity of the Drinfeld double $D(H)$ are given (see Remark 3.5).
For generalities on Hopf algebras, we refer the reader to [4], [11], [15]. For a detailed study of Hopf modules and their generalizations, we refer to [3].

1. Preliminary Results

Let k be a commutative ring, and H a faithfully flat Hopf algebra with bijective antipode S. Unadorned $\otimes$ and Hom will be over k. $\mathcal{M}$ will be the category of k-modules. We will use the Sweedler-Heyneman notation for comultiplication and coaction: for $h \in H$, we write

$$\Delta(h) = h_1 \otimes h_2$$

(summation implicitly understood), and for a right H-comodule (M, ρ_M) and $m \in M$, we write

$$\rho_M(m) = m_0 \otimes m_1$$

${}_H\mathcal{M}$ and $\mathcal{M}^H$ will be the categories of respectively left H-modules and left H-linear maps, and right H-comodules and right H-colinear maps. If M and N are right H-comodules, then we denote the k-module consisting of right H-colinear maps from M to N by $\mathrm{Hom}^H(M, N)$.

$$M^{\mathrm{co}H} = \{m \in M \mid \rho_M(m) = m \otimes 1\}$$

is called the k-submodule of coinvariants of M. Observe that $H^{\mathrm{co}H} = k$. Suppose that a k-vector space M is at the same time a left H-module and a right H-comodule. Recall that M is called a left-right Yetter-Drinfeld module if

$$h_1m_0 \otimes h_2m_1 = (h_2m)_0 \otimes (h_2m)_1h_1$$

or, equivalently,

$$\rho(hm) = h_2m_0 \otimes h_3m_1S^{-1}(h_1)$$

for all $m \in M$ and $h \in H$. M is called a left-right Long dimodule if

$$\rho(hm) = hm_0 \otimes m_1$$

for all $m \in M$ and $h \in H$. If H is commutative and cocommutative, then a Long dimodule is the same as a Yetter-Drinfeld module. ${}_H\mathcal{YD}^H$ and ${}_H\mathcal{L}^H$ will be the categories of respectively Yetter-Drinfeld modules and Long dimodules, and H-linear H-colinear maps. The k-module consisting of all H-linear H-colinear maps between two Yetter-Drinfeld modules or two Long dimodules M and N will be denoted by ${}_H\mathrm{Hom}^H(M,N)$. If H is finitely generated and projective, then the category ${}_H\mathcal{YD}^H$ is isomorphic to the category ${}_{D(H)}\mathcal{M}$, where $D(H)$ is the Drinfeld double of H, and ${}_H\mathcal{L}^H$ is isomorphic to ${}_{H\otimes H^*}\mathcal{M}$.
The functors

$$(-)^{\mathrm{co}H}:\ {}_H\mathcal{YD}^H \to \mathcal{M} \text{ and } (-)^{\mathrm{co}H}:\ {}_H\mathcal{L}^H \to \mathcal{M}$$

are exact if

$$(-)^{\mathrm{co}H}:\ \mathcal{M}^H \to \mathcal{M}$$

is exact. This is the case if H is cosemisimple and k is a field.

Lemma 1.1. (1) *Let M and N be objects of ${}_H\mathcal{YD}^H$. Then $M\otimes N$ is an object of ${}_H\mathcal{YD}^H$; the H-action and H-coaction are given by*

$$h(m\otimes n) = h_1m \otimes h_2n \quad \textit{and} \quad \rho(m\otimes n) = m_0\otimes n_0 \otimes n_1m_1$$

(2) *Let M and N be objects of ${}_H\mathcal{L}^H$. Then $M\otimes N$ is an object of ${}_H\mathcal{L}^H$; the H-action and H-coaction are given by*

$$h(m\otimes n) = h_1m \otimes h_2n \quad \textit{and} \quad \rho(m\otimes n) = m_0\otimes n_0 \otimes m_1n_1$$

(3) *For any H-comodule N, $H\otimes N$ is an object of ${}_H\mathcal{YD}^H$ via the following structures*

$$h(h'\otimes n) = hh'\otimes n \quad \textit{and} \quad \rho(h\otimes n) = h_2\otimes n_0\otimes h_3n_1S^{-1}(h_1)$$

(4) *For any H-comodule N, $H\otimes N$ is an object of ${}_H\mathcal{L}^H$ via the following structures*

$$h(h'\otimes n) = hh'\otimes n \quad \textit{and} \quad \rho(h\otimes n) = h\otimes n_0\otimes n_1$$

Proof. This result is well-known, and the proof is a straightforward computation. It may be found in [1, p. 440], [1, Prop. 12.1.2], [3, Prop. 123], and [3, Sec. 7.2]. □

Lemma 1.2. (1) *Let M and N be in ${}_H\mathcal{YD}^H$. If H is commutative, then $M \otimes_H N$ is an object of ${}_H\mathcal{YD}^H$. The H-action and H-coaction are given by*

$$h(m \otimes n) = hm \otimes n = m \otimes hn$$

and

$$\rho_{M\otimes_H N}(m \otimes n) = m_0 \otimes n_0 \otimes n_1 m_1$$

(2) *Let H be commutative. Let M and N be in ${}_H\mathcal{YD}^H$ with M finitely generated projective in ${}_H\mathcal{M}$. Then*

(a) ${}_H\mathrm{Hom}(M, N) \in \mathcal{M}^H$ *and*

$${}_H\mathrm{Hom}^H(M, N) = {}_H\mathrm{Hom}(M, N)^{coH}$$

The coaction is defined by

$$\rho(f) = f_0 \otimes f_1 \in {}_H\mathrm{Hom}(M, N) \otimes H$$

if and only if

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S(m_1) \tag{1}$$

for all $m \in M$.

(b) ${}_H\mathrm{Hom}(M, N) \in {}_H\mathcal{YD}^H$*; the H-action is defined by $(hf)(m) = hf(m) = f(hm)$.*

Proof. 1) It is clear that $M \otimes_H N$ is an H-module. An easy verification shows that the H-coaction is well-defined on the tensor product over H and that the necessary associativity and counit properties are satisfied, so that $M \otimes_H N$ is also an H-comodule. $M \otimes_H N$ is a Yetter-Drinfeld module, since we have for every $h \in H$ that

$$\begin{aligned}
\rho_{M\otimes_H N}(hm \otimes n) &= (hm)_0 \otimes n_0 \otimes n_1 (hm)_1 \\
&= h_2 m_0 \otimes n_0 \otimes n_1 h_3 m_1 S^{-1}(h_1) \\
&= h_2(m_0 \otimes n_0) \otimes h_3 n_1 m_1 S^{-1}(h_1) \\
&= h_2(m \otimes n)_0 \otimes h_3 (m \otimes n)_1 S^{-1}(h_1)
\end{aligned}$$

2a) Let us define a map

$$\pi : \ \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N \otimes H)$$

by

$$\pi(f)(m) = f(m_0)_0 \otimes f(m_0)_1 S(m_1)$$

Let f be H-linear. Using the commutativity of H, we obtain

$$\begin{aligned}\pi(f)(hm) &= f((hm)_0)_0 \otimes f((hm)_0)_1 S((hm)_1)\\ &= (h_2 f(m_0))_0 \otimes (h_2 f(m_0))_1 S(h_3 m_1 S^{-1}(h_1))\\ &= h_3 f(m_0)_0 \otimes h_4 f(m_0)_1 S^{-1}(h_2) h_1 S(m_1) S(h_5)\\ &= h f(m_0)_0 \otimes f(m_0)_1 S(m_1) = h\pi(f)(m)\end{aligned}$$

so $\pi(f)$ is H-linear, and π restricts to a map

$$\pi : {}_H\mathrm{Hom}(M,N) \to {}_H\mathrm{Hom}(M, N \otimes H)$$

Now M is finitely generated and projective as an H-module, so we have a natural isomorphism ${}_H\mathrm{Hom}(M, N\otimes H) \cong {}_H\mathrm{Hom}(M,N) \otimes H$, and we obtain a map

$$\pi : {}_H\mathrm{Hom}(M,N) \to {}_H\mathrm{Hom}(M,N) \otimes H$$

with $\pi(f) = f_0 \otimes f_1$ if and only if

$$(\pi(f))(m) = f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S(m_1)$$

It is straightforward to show that π makes ${}_H\mathrm{Hom}(M,N)$ a right H-comodule. Now take $f \in {}_H\mathrm{Hom}^H(M,N)$ and $m \in M$. Then

$$\begin{aligned}\pi(f)(m) = f_0(m) \otimes f_1 &= f(m_0)_0 \otimes f(m_0)_1 S(m_1)\\ &= f(m_0) \otimes m_1 S(m_2) = f(m) \otimes 1 = (f \otimes 1)(m)\end{aligned}$$

so f is coinvariant. Conversely, take $f \in {}_H\mathrm{Hom}(M,N)^{\mathrm{co}H}$. Then for every $m \in M$

$$f(m_0)_0 \otimes f(m_0)_1 S(m_1) = f_0(m) \otimes f_1 = f(m) \otimes 1$$

and

$$f(m_0)_0 \otimes f(m_0)_1 S(m_1) m_2 = f(m_0) \otimes m_1$$

and it follows that

$$\rho_N(f(m)) = \rho_N(f(m_0))\varepsilon(m_1) = f(m_0) \otimes m_1$$

and f is H-colinear.

2b) Clearly ${}_H\mathrm{Hom}(M,N)$ is an H-module and, by a), it is an H-comodule. On the other hand, we have

$$\begin{aligned}((hf)_0 \otimes (hf)_1)(m) &= ((hf)(m_0))_0 \otimes ((hf)(m_0))_1 S(m_1)\\ &= (h(f(m_0))_0 \otimes (h(f(m_0))_1 S(m_1)\\ &= h_2(f(m_0)_0) \otimes h_3(f(m_0)_1) S^{-1}(h_1) S(m_1)\\ &= h_2(f(m_0)_0) \otimes h_3(f(m_0)_1) S(m_1) S^{-1}(h_1)\\ &= h_2(f_0(m)) \otimes h_3 f_1 S^{-1}(h_1)\\ &= (h_2 f_0 \otimes h_3 f_1 S^{-1}(h_1))(m)\end{aligned}$$

so ${}_H\mathrm{Hom}(M,N) \in {}_H\mathcal{YD}^H$. □

Remark 1.3. The results in Lemma 1.2 remain true after we replace ${}_H\mathcal{Y}D^H$ by ${}_H\mathcal{L}^H$. The H-coaction on $M \otimes_H N$ is given by

$$\rho_{M\otimes_H N}(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1$$

The H-coaction on ${}_H\mathrm{Hom}(M,N)$ is also defined by (1). Part 2a) of Lemma 1.2 then also holds if H is noncommutative.

Lemma 1.4. *Let V be a k-module and N an H-module.*

1. ${}_H\mathrm{Hom}(H\otimes V,N)$ *and* $\mathrm{Hom}(V,N)$ *are isomorphic as k-modules.*
2. *If V is projective as k-module, then $H\otimes V$ is projective in ${}_H\mathcal{M}$.*

Proof. 1) is well-known: the k-isomorphism

$${}_H\mathrm{Hom}(H \otimes V, N) \to \mathrm{Hom}(V, N)$$

is defined by $\phi(f)(v) = f(1 \otimes v)$.
2) follows immediately from (1). □

Let V be a right H-comodule which is finitely generated and projective as a k-module. For any right H-comodule N, $\mathrm{Hom}(V,N) \in \mathcal{M}^H$: let $\{(v_i, v_i^*) \mid i = 1,\cdots,n\} \subset V \otimes V^*$ be a finite dual basis of V, and define, for $g \in \mathrm{Hom}(V,N)$:

$$\rho(g) = \sum_i v_i^* \otimes g(v_{i0})_0 \otimes g(v_{i0})_1 S(v_{i1}) \in V^* \otimes N \otimes H \cong \mathrm{Hom}(V,N) \otimes H$$

This means that $\rho(g) = g_0 \otimes g_1$ if and only if

$$g_0(v) \otimes g_1 = g(v_0)_0 \otimes g(v_0)_1 S(v_1)$$

for all $v \in V$.
By Lemmas 1.1 and 1.4, $H\otimes V$ is an object in ${}_H\mathcal{YD}^H$ and in ${}_H\mathcal{L}^H$, and is finitely generated projective as an H-module. So if N is an object of ${}_H\mathcal{YD}^H$ and if H is commutative, then, by Lemma 1.2, ${}_H\mathrm{Hom}(H\otimes V,N)$ is an object in ${}_H\mathcal{YD}^H$. If N is an object of ${}_H\mathcal{L}^H$, then by Remark 1.3, ${}_H\mathrm{Hom}(H\otimes V,N)$ is an object of $\mathcal{M}^H$; if furthermore H is commutative, then ${}_H\mathrm{Hom}(H \otimes V,N)$ is an object of ${}_H\mathcal{L}^H$.

Lemma 1.5. *Let H be commutative and $N \in {}_H\mathcal{YD}^H$.*

1. *If V is an H-comodule which is finitely generated and projective as a k-module, then the H-comodules ${}_H\mathrm{Hom}(H \otimes V,N)$ and $\mathrm{Hom}(V,N)$ are isomorphic.*
2. *Let k be a field and V a finite-dimensional H-comodule that is projective as an H-comodule. Then $H\otimes V$ is a projective object of ${}_H\mathcal{YD}^H$.*

Proof. 1) Consider the canonical k-isomorphism

$$\phi:\ {}_H\mathrm{Hom}(H\otimes V,N)\to \mathrm{Hom}(V,N),\ \phi(f)(v)=f(1\otimes v)$$

ϕ is H-colinear since

$$\begin{aligned}\phi(f)_0(v)\otimes\phi(f)_1 &= (\phi(f)(v_0))_0\otimes(\phi(f)(v_0))_1S(v_1)\\ &= f(1\otimes v_0)_0\otimes f(1\otimes v_0)_1S(v_1)\\ &= f((1\otimes v)_0)_0\otimes f((1\otimes v)_0)_1S((1\otimes v)_1)\\ &= f_0(1\otimes v)\otimes f_1\\ &= (\phi(f_0))(v)\otimes f_1\end{aligned}$$

2) By 1) and Lemma 1.2, we have

$$\begin{aligned}{}_H\mathrm{Hom}^H(H\otimes V,N) &\cong {}_H\mathrm{Hom}(H\otimes V,N)^{\mathrm{co}H}\\ &\cong \mathrm{Hom}(V,N)^{\mathrm{co}H}\cong \mathrm{Hom}^H(V,N)\end{aligned}$$

□

Lemma 1.5 also holds with ${}_H\mathcal{YD}^H$ replaced by ${}_H\mathcal{L}^H$, and without the assumption that H is commutative.

Proposition 1.6. *Let k be a field. An object M of ${}_H\mathcal{YD}^H$ or ${}_H\mathcal{L}^H$ is finitely generated as an H-module if and only if there exists a finite dimensional H-comodule V and an H-linear H-colinear epimorphism $\pi:\ H\otimes V\longrightarrow M$.*

Proof. If there exist a finite dimensional H-comodule V and an epimorphism of H-modules $\pi:\ H\otimes V\longrightarrow M$, then $H\otimes V$ is finitely generated as an H-module and M is a quotient of $H\otimes V$ in ${}_H\mathcal{M}$, so M is finitely generated in ${}_H\mathcal{M}$.
Suppose that M is finitely generated as an H-module, with generators $\{m_1,\cdots,m_n\}$. By [4, 5.1.1], there exists a finite dimensional H-subcomodule V of M containing $\{m_1,\cdots,m_n\}$ and the k-linear map

$$\pi:\ H\otimes V\to M,\ \pi(h\otimes v)=hv$$

is an H-linear H-colinear epimorphism. □

Let H^* be the linear dual of H. If M and N are H-comodules, then $\mathrm{Hom}_k(M,N)$ is a left H^*-module, with H^*-action

$$(h^*f)(m)=h^*(f(m_0)_1S(m_1))f(m_0)_0$$

(adapt the proof of [14, Proposition 1.1]).

Lemma 1.7. *Let H be commutative. For $M,N\in{}_H\mathcal{YD}^H$, ${}_H\mathrm{Hom}(M,N)$ is a left H^*-submodule of $\mathrm{Hom}_k(M,N)$.*

Proof. For all $\alpha \in H^*$, $f \in \mathrm{Hom}_H(M,N)$, $h \in H$ and $m \in M$, we have

$$\begin{aligned}(\alpha f)(hm) &= \alpha\Big(f((hm)_0)_1 S((hm)_1)\Big) f((hm)_0)_0\\ &= \alpha\Big(f(h_2m_0)_1 S(h_3m_1S^{-1}(h_1))\Big) f(h_2m_0)_0\\ &= \alpha\Big((h_2(f(m_0)))_1 h_1 S(m_1)S(h_3)\Big)(h_2f(m_0))_0\\ &= \alpha\Big(h_4 f(m_0)_1 S^{-1}(h_2)h_1S(m_1)S(h_5)\Big) h_3 f(m_0)_0\\ &= \alpha\big(f(m_0)_1 S(m_1)\big) h f(m_0)_0\\ &= h((\alpha f)(m))\end{aligned}$$

and it follows that αf is H-linear. Observe that we used the commutativity of H. □

Recall that a left H^*-module M is called rational if there exists a right H-coaction on M inducing the left H^*-action.

Proposition 1.8. *Let H be a commutative Hopf algebra over a field k. If $M, N \in {}_H\mathcal{YD}^H$ with M finitely generated as H-module, then ${}_H\mathrm{Hom}(M,N) \in {}_H\mathcal{YD}^H$.*

Proof. By Proposition 1.6, there exist a finite dimensional H-subcomodule V of M and an H-linear H-colinear epimorphism $\pi : H \otimes V \longrightarrow M$. So we obtain an injective k-linear map

$${}_H\mathrm{Hom}(\pi, N) : {}_H\mathrm{Hom}(M,N) \to {}_H\mathrm{Hom}(H\otimes V, N)$$

For all $\alpha \in H^*$, $f \in {}_H\mathrm{Hom}(M,N)$, $h \in H$ and $v \in V$, we have $\pi(h\otimes v) = hv$, $(1\otimes v)_0 \otimes (1\otimes v)_1 = 1 \otimes v_0 \otimes v_1$ and

$$\begin{aligned}((\alpha f)\circ\pi)(1\otimes v) &= (\alpha f)(v) = \alpha(f(v_0)_1 S(v_1))f(v_0)_0\\ &= \alpha(f(\pi(1\otimes v_0))_1 S(v_1)) f(\pi(1\otimes v_0))_0\\ &= \alpha(f(\pi(1\otimes v)_0))_1 S((1\otimes v)_1) f(\pi((1\otimes v)_0))_0\\ &= (\alpha(f\circ\pi))(1\otimes v)\end{aligned}$$

This relation and the fact that $(\alpha f)\circ\pi$ and $\alpha(f\circ\pi)$ are H-linear imply that $((\alpha f)\circ\pi)(h\otimes v) = (\alpha(f\circ\pi))(h\otimes v)$, and it follows that the map ${}_H\mathrm{Hom}(\pi,N)$ is H^*-linear. By Lemma 1.2, ${}_H\mathrm{Hom}(H\otimes V, N)$ is an H-comodule, and therefore a rational H^*-module. It follows that ${}_H\mathrm{Hom}(M,N)$ is a rational H^*-module, being an H^*-submodule of the rational H^*-module ${}_H\mathrm{Hom}(H\otimes V,N)$. This shows that ${}_H\mathrm{Hom}(M,N)$ is an H-comodule. By Lemma 1.2, ${}_H\mathrm{Hom}(M,N) \in {}_H\mathcal{YD}^H$. □

Remark 1.9. 1) Lemma 1.7 is still true if we replace ${}_H\mathcal{YD}^H$ by ${}_H\mathcal{L}^H$, without the assumption that H is commutative.

2) We have the following Long dimodule version of Proposition 1.8: for a (not necessarily commutative) Hopf algebra over a field k, and $M, N \in {}_H\mathcal{L}^H$, with M finitely generated as an H-module, ${}_H\mathrm{Hom}(M, N) \in \mathcal{M}^H$.

2. Projective and injective dimension in the category of Yetter-Drinfeld modules

Lemma 2.1. *Let H be commutative, and $M, N, P \in {}_H\mathcal{YD}^H$, with N finitely generated projective as an H-module.*

(1) *We have a k-isomorphism*

$$ {}_H\mathrm{Hom}^H(M, {}_H\mathrm{Hom}(N, P)) \cong {}_H\mathrm{Hom}^H(M \otimes_H N, P) $$

(2) *The functor*

$$ {}_H\mathrm{Hom}(N, -) : \ {}_H\mathcal{YD}^H \to {}_H\mathcal{YD}^H $$

preserves injective objects.

Proof. 1) We have a natural isomorphism

$$ \phi : \ {}_H\mathrm{Hom}(M, {}_H\mathrm{Hom}(N, P)) \to {}_H\mathrm{Hom}(M \otimes {}_H N, P) $$

given by $\phi(f)(m \otimes n) = f(m)(n)$. We will show that ϕ restricts to an isomorphism between ${}_H\mathrm{Hom}^H(M, {}_H\mathrm{Hom}(N, P))$ and ${}_H\mathrm{Hom}^H(M \otimes_H N, P)$. Take $f \in {}_H\mathrm{Hom}(M, {}_H\mathrm{Hom}(N, P))$ and $\phi(f) = g$. Then f is H-colinear if and only if

$$ f(m_0) \otimes m_1 = f(m)_0 \otimes f(m)_1 $$

for all $m \in M$. Using (1), we find that this is equivalent to

$$ f(m_0)(n) \otimes m_1 = f(m)_0(n) \otimes f(m)_1 = f(m)(n_0)_0 \otimes f(m)(n_0)_1 S(n_1) $$

for all $m \in M$ and $n \in N$, or

$$ g(m_0 \otimes n) \otimes m_1 = g(m \otimes n_0)_0 \otimes g(m \otimes n_0)_1 S(n_1) $$

which is equivalent to

$$ g(m_0 \otimes n_0) \otimes m_1 n_1 = g(m \otimes n)_0 \otimes g(m \otimes n)_1 $$

and this equation means that g is H-colinear.

2) If I is an injective object of ${}_H\mathcal{YD}^H$, then the functor

$$ {}_H\mathrm{Hom}^H(-, I) : \ {}_H\mathcal{YD}^H \to \mathcal{M} $$

is exact. On the other hand, N is H-projective, hence the functor

$$ (-) \otimes_H N : \ {}_H\mathcal{YD}^H \to {}_H\mathcal{YD}^H $$

is exact, and it follows from (1) that

$$ {}_H\mathrm{Hom}^H(-, {}_H\mathrm{Hom}(N, I)) : \ {}_H\mathcal{YD}^H \to \mathcal{M} $$

is exact. □

If k is a field, then the category of Yetter-Drinfeld modules ${}_H\mathcal{YD}^H$ is Grothendieck, and every object has an injective resolution. For every Yetter-Drinfeld module M, we can define the right derived functors ${}_H\mathrm{Ext}^{H^i}(M,-)$ of the covariant left exact functor

$${}_H\mathrm{Hom}^H(M,-):\ {}_H\mathcal{YD}^H \to \mathcal{M}$$

Proposition 2.2. *Let H be a commutative Hopf algebra over a field k, and $M,N,P \in {}_H\mathcal{YD}^H$ with N finitely generated projective as an H-module. Then*

$${}_H\mathrm{Ext}^{H^i}(M, {}_H\mathrm{Hom}(N,P)) \cong {}_H\mathrm{Ext}^{H^i}(M \otimes_H N, P)$$

Proof. By the first part of Lemma 2.1, the functors

$${}_H\mathrm{Hom}^H(M, {}_H\mathrm{Hom}(N,-)) \quad \text{and} \quad {}_H\mathrm{Hom}^H(M \otimes_H N, -)$$

coincide on ${}_H\mathcal{YD}^H$. By the projectivity and the finiteness assumptions on N, the ${}_H\mathrm{Hom}(N,-)$ is an exact endofunctor of ${}_H\mathcal{YD}^H$. By the second part of Lemma 2.1, it preserves the injective objects of ${}_H\mathcal{YD}^H$. Thus the functor ${}_H\mathrm{Hom}(N,-)$ preserves injective resolutions in ${}_H\mathcal{YD}^H$. □

In the following corollary, ${}_H\mathrm{pdim}^H(-)$ and ${}_H\mathrm{injdim}^H(-)$ denote respectively the projective and injective dimension in the category ${}_H\mathcal{YD}^H$.

Corollary 2.3. *Let H be a commutative Hopf algebra over a field k, and $M,N,P \in {}_H\mathcal{YD}^H$ with N finitely generated projective as an H-module. Then*

(1) ${}_H\mathrm{pdim}^H(M \otimes_H N) \leq {}_H\mathrm{pdim}^H(M)$.
(2) ${}_H\mathrm{injdim}^H({}_H\mathrm{Hom}(N,P)) \leq {}_H\mathrm{injdim}^H(P)$.

Remarks 2.4. 1) Let H be semisimple. Then the projectivity assumption in Lemma 2.1, Proposition 2.2 and Corollary 2.3 is no longer needed.
2) If k is a field, then ${}_H\mathcal{L}^H$ is a Grothendieck category with enough injective objects, and every Long dimodule has an injective resolution. For every $M \in {}_H\mathcal{L}^H$, we can then define the right derived functors ${}_H\mathrm{Ext}^{H^i}(M,-)$ of the covariant left exact functor

$${}_H\mathrm{Hom}^H(M,-):\ {}_H\mathcal{L}^H \to \mathcal{M}$$

All the results of this Section remain valid for ${}_H\mathcal{L}^H$. If H is semisimple, then the projectivity assumptions are not needed.

3. Semisimplicity of the category of Yetter-Drinfeld modules

Throughout this Section, k will be a field, and H a commutative Hopf algebra. Recall that $M \in {}_H\mathcal{YD}^H$ is called simple if it has no proper subobjects; a direct sum of simples is called semisimple. If every $M \in {}_H\mathcal{YD}^H$ is semisimple, then we call the category ${}_H\mathcal{YD}^H$ semisimple. We say that ${}_H\mathcal{YD}^H$ satisfies condition (†) if the following holds:
if $M \in {}_H\mathcal{YD}^H$ is finitely generated as a left H-module, then ${}_H\mathrm{Hom}(M,-) : {}_H\mathcal{YD}^H \to {}_H\mathcal{YD}^H$ is exact.
By Proposition 1.8, ${}_H\mathrm{Hom}(M,N) \in {}_H\mathcal{YD}^H$ if H is commutative and M is finitely generated as an H-module. Also observe that ${}_H\mathcal{YD}^H$ satisfies condition (†) if H is semisimple.

Proposition 3.1. *Let H be commutative. Assume that ${}_H\mathcal{YD}^H$ satisfies condition (†) and that the functor*

$$(-)^{\mathrm{co}H} : \ {}_H\mathcal{YD}^H \to \mathcal{M}$$

is exact. If $M \in {}_H\mathcal{YD}^H$ is finitely generated as an H-module, then M is a projective object in ${}_H\mathcal{YD}^H$.

Proof. We know that

$${}_H\mathrm{Hom}^H(M,-) \cong {}_H\mathrm{Hom}(M,-)^{\mathrm{co}H}$$

so ${}_H\mathrm{Hom}^H(M,-)$ is exact since it is isomorphic to the composition of two exact functors. □

Corollary 3.2. *With the same assumptions as in Proposition 3.1, and with H noetherian, we have that every object $M \in {}_H\mathcal{YD}^H$ which is finitely generated as an H-module is a direct sum in ${}_H\mathcal{YD}^H$ of a family of simple subobjects that are finitely generated as H-modules.*

Proof. Let N be a subobject of M in ${}_H\mathcal{YD}^H$. Then M/N is finitely generated as an H-module and we have an exact sequence

$$0 \to N \to M \to M/N \to 0 \tag{2}$$

in ${}_H\mathcal{YD}^H$. N is finitely generated as H-module, since H is noetherian, so it follows from Proposition 3.1 that M/N and N are projective in ${}_H\mathcal{YD}^H$, hence the sequence (2) splits in ${}_H\mathcal{YD}^H$. □

Take $M \in {}_H\mathcal{YD}^H$ and V a right H-subcomodule of M. We will set

$$HV = \{\sum_{i\in I} a_i v_i \mid a_i \in H, v_i \in V, \text{ where } I \text{ is a finite set}\}$$

HV is a subobject of M in ${}_H\mathcal{YD}^H$; the H-action and H-coaction on HV are given by

$$\begin{aligned} h(\sum_{i\in I} a_iv_i) &= \sum_{i\in I} ha_iv_i \\ \rho(\sum_{i\in I} a_iv_i) &= \sum_{i\in I} (a_i)_2(v_i)_0 \otimes (a_i)_3(v_i)_1 S^{-1}((a_i)_1) \end{aligned}$$

Corollary 3.3. *Let H be commutative and noetherian. Assume that ${}_H\mathcal{YD}^H$ satisfies condition (†), and that the functor $(-)^{\mathrm{co}H}$ from ${}_H\mathcal{YD}^H$ to $\mathcal{M}$ is exact. Then $M \in {}_H\mathcal{YD}^H$ is a direct sum in ${}_H\mathcal{YD}^H$ of a family of simple subobjects that are finitely generated as H-modules. Therefore M is a semisimple object in ${}_H\mathcal{YD}^H$ and ${}_H\mathcal{YD}^H$ is a semisimple category.*

Proof. Every $m \in M$ is contained in a finite-dimensional H-subcomodule V_m of M, see e.g. [4, 5.1.1]. Then HV_m is finitely generated as H-module, and, by Corollary 3.2, each HV_m is a direct sum of a family of simple subobjects of HV_m (and of M) in ${}_H\mathcal{YD}^H$, which are finitely generated as an H-module.
Let $\mathcal{N}$ be the collection of all direct sums $N = \oplus_{i\in I} N_i$, with N_i simple in ${}_H\mathcal{YD}^H$ and finitely generated as an H-module. Then every $m \in M$ is contained in some $N \in \mathcal{N}$. The sum of two elements of $\mathcal{N}$ is again in $\mathcal{N}$, since the intersection of two simple objects is trivial. $\mathcal{N}$ is partially ordered by the inclusion, so it contains a maximal element N', by Zorn's Lemma. Every $m \in M$ is contained in some $N \in \mathcal{N}$, and therefore in $N + N' \in \mathcal{N}$. Since N' is maximal, $N + N' = N'$, and $m \in N'$. It follows that $M = N' \in \mathcal{N}$, and M is a sum of simple objects finitely generated as an H-module. □

Corollary 3.4. *Let H be commutative, noetherian (in particular: finite dimensional), semisimple and cosemisimple. Then each $M \in {}_H\mathcal{YD}^H$ is a direct sum in ${}_H\mathcal{YD}^H$ of a family of simple subobjects of M finitely generated as H-modules. Hence M is semisimple in ${}_H\mathcal{YD}^H$ and ${}_H\mathcal{YD}^H$ is a semisimple category.*

Proof. The cosemisimplicity of H implies that the functor

$$(-)^{coH} : \mathcal{M}^H \to \mathcal{M}$$

is exact, and, a fortiori

$$(-)^{coH} : {}_H\mathcal{Y}D^H \to \mathcal{M}$$

is exact. □

Remark 3.5. If H is finite dimensional, then the category of Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld double $D(H)$, so ${}_H\mathcal{YD}^H$ is semisimple if and only if $D(H)$ is a semisimple algebra. Semisimplicity of $D(H)$ has been examined by Radford, see [12, Prop. 7]: he proves that $D(H)$ is semisimple if and only if H and H^* are semisimple. Observe that Radford does not need the commutativity of H.
Another related result in the finite dimensional case is [2, Corollary 3.5]: if H is finite dimensional, cosemisimple and unimodular, then the extension $H \to D(H)$ is right semisimple . By this we mean that every submodule of a right $D(H)$-module, which is a right H-direct summand, is also a $D(H)$-direct summand.

Take $M, N \in {}_H\mathcal{L}^H$, with M finitely generated as an H-module. By Proposition 1.8 and Remark 1.9, ${}_H\mathrm{Hom}(M, N) \in \mathcal{M}^H$, and we can study the semisimplicity of ${}_H\mathcal{L}^H$. We will say that ${}_H\mathcal{L}^H$ satisfies condition (†) if the functor

$$ {}_H\mathrm{Hom}(M, -): \ {}_H\mathcal{L}^H \to \mathcal{M}^H $$

is exact for every H-finitely generated $M \in {}_H\mathcal{L}^H$. The previous results of this Section then remain true after we replace the category of Yetter-Drinfeld modules by Long dimodules, and without the assumption that H is commutative. We state the results without proof.

Proposition 3.6. *Assume that ${}_H\mathcal{L}^H$ satisfies condition (†) and that the functor*

$$ (-)^{\mathrm{co}H}: \ \mathcal{M}^H \to \mathcal{M} $$

is exact. Then every H-finitely generated $M \in {}_H\mathcal{L}^H$ is a projective object in ${}_H\mathcal{L}^H$.

Corollary 3.7. *Let H be left noetherian, and assume that the conditions of Proposition 3.6 are satisfied. Then every H-finitely generated $M \in {}_H\mathcal{L}^H$ is a direct sum in ${}_H\mathcal{L}^H$ of a family of simple subobjects of M that are finitely generated as H-modules. ${}_H\mathcal{L}^H$ is a semisimple category.*

Corollary 3.8. *Let H be left noetherian (in particular: finite dimensional), semisimple and cosemisimple. Then each $M \in {}_H\mathcal{L}^H$ is a direct sum in ${}_H\mathcal{L}^H$ of a family of simple subobjects of M that are finitely generated as H-modules. Hence $M \in {}_H\mathcal{L}^H$ is semisimple and ${}_H\mathcal{L}^H$ is a semisimple category.*

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