PARTIAL ENTWINING STRUCTURES

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Abstract. We introduce partial (co)actions of a Hopf algebra on an algebra A. To this end, we introduce first the notion of lax coring, generalizing Wisbauer's notion of weak coring. We also have the dual notion of lax ring. We then introduce partial and lax entwining structures. Several duality results are given, and we develop Galois theory for partial entwining structures.

Introduction

Partial group actions were considered first by Exel [15], in the context of operator algebras. A treatment from a purely algebraic point of view was given recently in [11, 12, 13, 14]. In particular, Galois theory over commutative rings can be generalized to partial group actions, see [13] (at least under the additional assumption that the associated ideals are generated by idempotents).

The following questions arise naturally: can we develop a theory of partial (co)actions of Hopf algebras? Is it possible to generalize Hopf-Galois theory to the partial situation? The aim of this paper is to give a positive answer to these questions.

Partial group actions were studied from the point of view of corings by the first author and De Groot in [6]. Namely, a partial group action in the sense of [13] gives rise to a coring. The Galois theory of [13] can then be considered as a special case of the Galois theory of corings (see [2, 3, 4, 18]). There is a remarkable analogy with the Galois theory that can be developed for weak Hopf algebras (see [7]): in both cases, the associated coring is a direct factor of the tensor product of the Galois extension A, and a coalgebra. In the partial group action case, the coalgebra is the dual of the group algebra, in the other case it is the weak Hopf algebra that we started with. The right A-module structure of the coring is induced by a kind of entwining map. In the weak Hopf algebra case, it is a weak entwining map, as introduced in [5]. The map in the partial group action case, however, does not satisfy the axioms of a weak entwining structure.

Wisbauer [17] introduces weak entwining structures from the point of view of *weak* corings; these are corings with a bimodule structure that is not necessarily unital. If $\mathcal{C}$ is a left-unital weak A-coring, then $\mathcal{C}1_A$ is an A-coring

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that is a direct summand of $\mathcal{C}$. Weak entwining structures are then in one-to-one correspondence with left-unital weak A-coring structures on $A \otimes C$, where A is an algebra, and C is a coalgebra.
If a finite group G acts partially on an algebra A, then we can define a left-unital A-bimodule structure on $A \otimes (kG)^*$, such that $(A \otimes (kG)^*)1_A$ is an A-coring, and a direct factor of $A \otimes (kG)^*$. But $A \otimes (kG)^*$ does not satisfy Wisbauer's axioms of a weak coring. This observation has lead us to the introduction of *lax corings*. The counit property of a lax coring is weaker than that of a weak coring, but it is still designed in such a way that $\mathcal{C}1_A$ is a coring.
Our next step is then to examine lax coring structures on $A \otimes C$. A subtlety that appears is that we have two possible choices for the counit: we can consider $A \otimes \epsilon$ and $(A \otimes \epsilon) \circ \pi$, where ϵ is the counit on C, and π is the projection of $A \otimes C$ onto $(A \otimes C)1_A$. This leads to the introduction of lax and partial entwining structures. The notion of lax entwining structure is the most general, and includes partial and weak entwining structures as special cases. If (A, C, ψ) is at the same time a partial and weak entwining structure, then it is an entwining structure.
Now let $C = H$ be a Hopf algebra, and consider a map $\rho : A \to A \otimes H$. We call A a right partial (resp. lax) H-comodule algebra if (A, C, ψ), with ψ given by the formula $\psi(h \otimes a) = a_{[0]} \otimes ha_{[1]}$ is a partial (resp. lax) entwining structure.
We have a dual theory: we can introduce lax A-rings, and lax and partial smash product structures. We then obtain the definition of partial (resp. lax) H-module algebra. In the case where H is a group algebra, we recover the definition of partial group action. We also discuss duality results. For example, if C is a finitely generated projective coalgebra, then we have a bijective correspondence between lax (resp. partial) entwining structures of the form (A, C, ψ) and lax (resp. partial) smash product structures of the form $(A^{\rm op}, C^*, R)$ (see Theorem 5.7). In the final Section 7, we applied the theory of Galois corings to corings arising from partial entwining structures.

1. Lax rings and corings

In this Section, A is a ring with unit. A-modules will not necessarily be unital.

Proposition 1.1. *Let P be an unital left A-module. There is a bijective correspondence between*

1. *(non-unital) right A-module structures on P making P an A-bimodule;*
2. *unital right A-module structures on left A-linear direct factors $\underline{P}$ of P, making $\underline{P}$ a unital A-bimodule.*

Proof. Let P be an A-bimodule as in (1). The map $\pi : P \to P$, $\pi(p) = p1_A$ is a left A-linear projection. The right A-action on P restricts to a unital right A-action on $\underline{P} = {\rm Im}\,(\pi)$.

Conversely, let $\pi: P \to \underline{P}$ be a left A-linear projection, and let $\underline{P}$ be an unital A-bimodule. We extend the right A-action from $\underline{P}$ to P as follows: $pa = \pi(p)a \in \underline{P}$. This action is associative, since $(pa)b = (\pi(p)a)b = \pi(p)(ab) = p(ab)$. $\square$

We observe that π is then also right A-linear, so $\underline{P}$ is an A-bimodule direct factor of P. The inclusion $\iota: \underline{P} \to P$ is a right inverse of π.

Recall that an A-coring $(\mathcal{C}, \Delta, \varepsilon)$ is a coalgebra in the monoidal category ${}_A\mathcal{M}_A$ of unital A-bimodules. This means that $\mathcal{C}$ is an unital A-bimodule, and that $\Delta: \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon: \mathcal{C} \to A$ are A-bimodule maps such that

$$(\Delta \otimes_A \mathcal{C}) \circ \Delta = (\mathcal{C} \otimes_A \Delta) \circ \Delta \tag{1}$$

and

$$(\varepsilon \otimes_A \mathcal{C}) \circ \Delta = (\mathcal{C} \otimes_A \varepsilon) \circ \Delta = \mathcal{C}. \tag{2}$$

We will often use the Sweedler-Heyneman notation

$$\Delta(c) = c_{(1)} \otimes_A c_{(2)},$$

where summation is implicitly understood.

Now take a left unital A-bimodule $\mathcal{C}$ and two A-bimodule maps $\Delta: \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon: \mathcal{C} \to A$ satisfying (1). We consider the projection $\pi: \mathcal{C} \to \underline{\mathcal{C}} = \mathcal{C}1$ and its right inverse ι. For all $c \in \mathcal{C}$, we have that

$$\Delta(c1) = \Delta(c)1 = c_{(1)} \otimes_A 1c_{(2)}1 = c_{(1)}1 \otimes_A c_{(2)}1 \in \underline{\mathcal{C}} \otimes_A \underline{\mathcal{C}},$$

so Δ restricts to a map $\underline{\Delta}: \underline{\mathcal{C}} \to \underline{\mathcal{C}} \otimes_A \underline{\mathcal{C}}$. $\varepsilon \circ \iota$ is then the restriction of ε to $\underline{\mathcal{C}}$.

- We call $(\mathcal{C}, \Delta, \varepsilon)$ a left unital *lax A-coring* if $(\underline{\mathcal{C}}, \underline{\Delta}, \varepsilon \circ \iota)$ is an A-coring; this is equivalent to

$$c = \varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)}), \tag{3}$$

  for all $c \in \underline{\mathcal{C}}$.
- Following [17], we call $(\mathcal{C}, \Delta, \varepsilon)$ a left unital *weak A-coring* if

$$c1_A = \varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)}), \tag{4}$$

  for all $c \in \mathcal{C}$.

It is clear that weak corings are lax corings.

Recall that an A-ring $(\mathcal{R}, \mu, \eta)$ is an algebra in the category ${}_A\mathcal{M}_A$ of unital A-bimodules. This means that $\mu: \mathcal{R} \otimes_A \mathcal{R} \to \mathcal{R}$ and $\eta: A \to \mathcal{R}$ are A-bimodule maps such that

$$\mu \circ (\mu \otimes_A \mathcal{R}) = \mu \circ (\mathcal{R} \otimes_A \mu) \tag{5}$$

and

$$\mu \circ (\eta \otimes_A \mathcal{R}) = \mu \circ (\mathcal{R} \otimes_A \eta) = \mathcal{R}. \tag{6}$$

Then $\mathcal{R}$ is a ring with unit $\eta(1_A)$, and $\eta:\ A \to \mathcal{R}$ is a ring morphism. It follows from (6) that the A-bimodule structure on $\mathcal{R}$ is induced by η. So an A-ring is a ring $\mathcal{R}$ together with a ring morphism $\eta:\ A \to \mathcal{R}$.

Let $\mathcal{R}$ be a left unital A-bimodule, and consider the projection $\pi:\ \mathcal{R} \to \underline{\mathcal{R}} = \mathcal{R}1$, and an A-bimodule map $\mu:\ \mathcal{R}\otimes_A \mathcal{R} \to \mathcal{R}$ satisfying (5). μ restricts to a map $\underline{\mu}:\ \underline{\mathcal{R}}\otimes_A \underline{\mathcal{R}} \to \underline{\mathcal{R}}$, since $\mu(r1_A\otimes_A s1_A) = \mu(r1_A\otimes_A s)1_A \in \underline{\mathcal{R}}$, for all $r,s\in\mathcal{R}$. We will write $\mu(r\otimes_A s) = rs$, as usual. Let $\eta:\ A \to \mathcal{R}$ be an A-bimodule map, and write $\eta(1_A) = 1_{\mathcal{R}}$. Then $\pi(1_{\mathcal{R}}) = 1_{\mathcal{R}}1_A = \eta(1_A)1_A = \eta(1_A) = 1_{\mathcal{R}}$, so that $\pi\circ\eta$ is the corestriction of η to $\underline{\mathcal{R}}$.

- $(\mathcal{R},\mu,\eta)$ is called a left unital *lax A-ring* if $(\underline{\mathcal{R}},\underline{\mu},\pi\circ\eta)$ is an A-ring, that is,

$$r = 1_{\mathcal{R}}r = r1_{\mathcal{R}}, \tag{7}$$

  for all $r\in\underline{\mathcal{R}}$;
- $(\mathcal{R},\mu,\eta)$ is called a left unital *weak A-ring* if

$$r1_A = 1_{\mathcal{R}}r = r1_{\mathcal{R}}, \tag{8}$$

  for all $r\in\mathcal{R}$.

In what follows we will also need a right unital version of lax and weak A-ring.
Let $\mathcal{R}$ be a right unital A-bimodule, put $\underline{\mathcal{R}} = 1\mathcal{R}$, and consider an A-bimodule map $\mu:\ \mathcal{R}\otimes_A \mathcal{R} \to \mathcal{R}$ satisfying (5). As before μ restricts to a map $\underline{\mu}:\ \underline{\mathcal{R}}\otimes_A \underline{\mathcal{R}} \to \underline{\mathcal{R}}$, and an A-bimodule map $\eta:\ A\to\mathcal{R}$ corestricts to the map $\pi\circ\eta:\ A \to \underline{\mathcal{R}}$.

- $(\mathcal{R},\mu,\eta)$ is called a right unital *lax A-ring* if $(\underline{\mathcal{R}},\underline{\mu},\pi\circ\eta)$ is an A-ring, that is, condition (7) is fulfilled, for all $r\in\underline{\mathcal{R}}$;
- $(\mathcal{R},\mu,\eta)$ is called a right unital *weak A-ring* if

$$1_A r = 1_{\mathcal{R}}r = r1_{\mathcal{R}}, \tag{9}$$

  for all $r\in\mathcal{R}$.

Duality. Let $\mathcal{C}$ be a left unital A-bimodule. Then ${}^*\mathcal{C} = {}_A\mathrm{Hom}(\mathcal{C},A)$ is a right unital A-bimodule, with A-action

$$(afb)(c) = f(ca)b,$$

for all $a,b\in A$, $c\in\mathcal{C}$ and $f\in{}^*\mathcal{C}$. If $\Delta:\ \mathcal{C}\to\mathcal{C}\otimes_A\mathcal{C}$ is a coassociative A-bimodule map, then $\mu:\ {}^*\mathcal{C}\otimes_A{}^*\mathcal{C}\to{}^*\mathcal{C}$, $\mu(f\otimes_A g) = f\#g$ given by

$$(f\#g)(c) = g(c_{(1)}f(c_{(2)})),$$

for all $c\in\mathcal{C}$, is an associative A-bimodule map: we compute easily that

$$(f\#g\#h)(c) = h(c_{(1)}g(c_{(2)}f(c_{(3)}))).$$

Let $\varepsilon: \mathcal{C} \to A$ be an A-bimodule map. For all $a \in A$ and $c \in \mathcal{C}$, we have that

$$(a\varepsilon)(c) = (\varepsilon a)(c) = \varepsilon(c)a,$$

so $\eta: A \to {}^*\mathcal{C}$, $\eta(a) = a\varepsilon = \varepsilon a$, is an A-bimodule map.
For all $f \in {}^*\mathcal{C}$ and $c \in \mathcal{C}$, we compute that

$$(\varepsilon \# f)(c) = f(c_{(1)}\varepsilon(c_{(2)}))$$

and

$$(f\#\varepsilon)(c) = \varepsilon(c_{(1)}f(c_{(2)})) = \varepsilon(c_{(1)})f(c_{(2)}) = f(\varepsilon(c_{(1)})c_{(2)}).$$

Proposition 1.2. *If $(\mathcal{C}, \Delta, \varepsilon)$ is a left unital weak (resp. lax) A-coring, then $({}^*\mathcal{C}, \mu, \eta)$ is a right unital weak (resp. lax) A-ring. The A-rings $1_A{}^*\mathcal{C}$ and ${}^*(\mathcal{C}1_A)$ are isomorphic.*

Proof. Let $(\mathcal{C}, \Delta, \varepsilon)$ be a left unital weak A-coring. For all $f \in {}^*\mathcal{C}$ and $c \in \mathcal{C}$, we have

$$\begin{aligned}(\varepsilon\# f)(c) &= f(c_{(1)}\varepsilon(c_{(2)})) \overset{(4)}{=} f(c1_A) = (1_A f)(c);\\ (f\#\varepsilon)(c) &= f(\varepsilon(c_{(1)})c_{(2)}) \overset{(4)}{=} f(c1_A) = (1_A f)(c).\end{aligned}$$

Thus (9) holds, and $({}^*\mathcal{C}, \mu, \eta)$ is a right unital weak A-ring.
Now assume that $(\mathcal{C}, \Delta, \varepsilon)$ is a left unital lax A-coring. For all $\underline{f} = 1_A f \in 1_A{}^*\mathcal{C} = \underline{{}^*\mathcal{C}}$ and $c \in \mathcal{C}$, we have

$$\begin{aligned}(\varepsilon\#\underline{f})(c) &= (\varepsilon\# 1_A f)(c) = (1_A f)(c_{(1)}\varepsilon(c_{(2)}))\\ &= f(c_{(1)}\varepsilon(c_{(2)})1_A) = f(c_{(1)}\varepsilon(c_{(2)})) \overset{(3)}{=} f(c1_A) = (1_A f)(c)\\ (\underline{f}\#\varepsilon)(c) &= \varepsilon(c_{(1)}(1_A f)(c_{(2)})) = \varepsilon(c_{(1)}f(c_{(2)}1_A))\\ &= \varepsilon(c_{(1)})f(c_{(2)}1_A) = f(\varepsilon(c_{(1)})c_{(2)}1_A) \overset{(3)}{=} f(c1_A) = (1_A f)(c).\end{aligned}$$

So $\varepsilon\#\underline{f} = \underline{f}\#\varepsilon = \underline{f}$, and (7) holds, and it follows that $({}^*\mathcal{C}, \mu, \eta)$ is a right unital lax A-ring.
To prove the final statement, we observe first that $f \in 1_A{}^*\mathcal{C}$ if and only if $1 \cdot f = f$, or $f(c \cdot 1) = f(c)$, for all $c \in \mathcal{C}$. The map

$$\alpha: 1_A{}^*\mathcal{C} \to {}^*(\mathcal{C}1_A),\ \alpha(f) = f_{|\mathcal{C}1_A}$$

has an inverse β, defined by the formula

$$\beta(g)(c) = g(c \cdot 1_A),$$

for all $g \in {}^*(\mathcal{C}1_A)$. It is clear that $\beta(g) \in 1_A{}^*\mathcal{C}$, and

$$\beta(\alpha(f))(c) = \alpha(f)(c \cdot 1_A) = f(c \cdot 1_A) = f(c),$$

for all $f \in 1_A{}^*\mathcal{C}$, and

$$\alpha(\beta(g))(c \cdot 1) = \beta(g)(c \cdot 1) = g(c \cdot 1),$$

for all $g \in {}^*(\mathcal{C}1_A)$. □

2. Partial entwining structures

Let k be a commutative ring, A a k-algebra, and M a k-module. $A\otimes M$ is then an unital left A-module, with left A-action induced by multiplication on A. For a k-linear map $\psi:\ M\otimes A\to A\otimes M$, we will adopt the notation

$$\psi(m\otimes a)=a_\psi\otimes m^\psi=a_\Psi\otimes m^\Psi,$$

for all $a\in A$ and $m\in M$. Summation is implicitly understood.

Lemma 2.1. *There is a bijective correspondence between*

- *(non-unital) right A-actions on $A\otimes M$, compatible with the left A-action;*
- *k-linear maps $\psi:\ M\otimes A\to A\otimes M$ satisfying the condition*

$$(ab)_\psi\otimes m^\psi=a_\psi b_\Psi\otimes m^{\psi\Psi}, \tag{10}$$

for all $a,b\in A$ and $m\in M$.

Then $\underline{A\otimes M}=(A\otimes M)1_A$ is an unital A-bimodule, and we have a projection

$$\pi:\ A\otimes M\to \underline{A\otimes M},\ \pi(a\otimes m)=(a\otimes m)1=a1_\psi\otimes m^\psi.$$

$A\otimes M$ is right A-unital if

$$1_\psi\otimes m^\psi=1\otimes m, \tag{11}$$

for all $m\in M$.

Proof. Given a right A-action, we define $\psi:\ M\otimes A\to A\otimes M$ by the formula

$$\psi(m\otimes a)=(1\otimes m)a.$$

(10) follows from the associativity of the right A-action. Conversely, given ψ, we define a right A-action by $(a\otimes m)b=ab_\psi\otimes m^\psi$. □

Now let (C,δ,ϵ) be a k-coalgebra, and $\psi:\ C\otimes A\to A\otimes C$ a k-linear map satisfying (10). Consider the maps

$$\Delta=(\pi\otimes C)\circ(A\otimes\delta):\ A\otimes C\to A\otimes C\otimes_A A\otimes C\cong \underline{A\otimes C}\otimes C,$$

$$\Delta(a\otimes c)=(a\otimes c_{(1)})\otimes_A(1\otimes c_{(2)})\cong a1_\psi\otimes c_{(1)}^\psi\otimes c_{(2)},$$

$$\varepsilon=A\otimes\epsilon:\ A\otimes C\to A,\ \varepsilon(a\otimes c)=\epsilon(c)a,$$

$$\underline{\varepsilon}=(A\otimes\epsilon)\circ\pi:\ A\otimes C\to A,\ \underline{\varepsilon}(a\otimes c)=\epsilon(c^\psi)a1_\psi.$$

We will now investigate when $(A\otimes C,\Delta,\varepsilon)$ and $(A\otimes C,\Delta,\underline{\varepsilon})$ are left unital weak, resp. lax A-corings. Then Δ and ε (or $\underline{\varepsilon}$) have to be right A-linear.

Lemma 2.2. *Let A be a k-algebra, C a k-coalgebra, and $\psi:\ C\otimes A\to A\otimes C$ a k-linear map satisfying (10).*

1. *Δ is right A-linear if and only if*

$$a_\psi 1_\Psi\otimes(c^\psi)_{(1)}^\Psi\otimes(c^\psi)_{(2)}=a_{\psi\Psi}\otimes c_{(1)}^\Psi\otimes c_{(2)}^\psi, \tag{12}$$

for all $a\in A$ and $c\in C$.

(2) $\underline{\varepsilon}$ *is right* A*-linear if and only if*

$$\epsilon(c^{\psi})1_{\psi}a = \epsilon(c^{\psi})a_{\psi}, \tag{13}$$

for all $a \in A$ *and* $c \in C$.

(3) ε *is right* A*-linear if and only if* $\varepsilon = \underline{\varepsilon}$, *that is*

$$\epsilon(c)a = \epsilon(c^{\psi})a_{\psi}, \tag{14}$$

for all $a \in A$ *and* $c \in C$.

Proof. 1) Δ is right A-linear if and only if

$$\Delta(1 \otimes c)a = (1 \otimes c_{(1)}) \otimes_A (a_{\psi} \otimes c^{\psi}_{(2)}) \cong a_{\psi\Psi} \otimes c^{\Psi}_{(1)} \otimes c^{\psi}_{(2)}$$

equals

$$\begin{aligned}\Delta((1 \otimes c)a) &= \Delta(a_{\psi} \otimes c^{\psi}) \\ &= (a_{\psi} \otimes (c^{\psi})_{(1)}) \otimes_A (1 \otimes (c^{\psi})_{(2)}) \cong a_{\psi}1_{\Psi} \otimes (c^{\psi})^{\Psi}_{(1)} \otimes (c^{\psi})_{(2)}.\end{aligned}$$

2) $\underline{\varepsilon}$ is right A-linear if and only if $\underline{\varepsilon}(b\otimes c)a = \epsilon(c^{\psi})b1_{\psi}a$ equals $\underline{\varepsilon}((b\otimes c)a) = \epsilon(c^{\psi})ba_{\psi}$, for all $a, b \in A$ and $c \in C$, and this is equivalent to (13).
3) ε is right A-linear if and only if $\varepsilon(b \otimes c)a = \epsilon(c)ba$ equals $\varepsilon((b \otimes c)a) = \epsilon(c^{\psi})ba_{\psi}$, and this is equivalent to (14). □

If Δ is right A-linear, then Δ restricts to a map $\underline{\Delta} : \underline{A \otimes C} \to \underline{A \otimes C} \otimes_A \underline{A \otimes C}$.

Proposition 2.3. *Let* A *be a* k*-algebra,* C *a* k*-coalgebra, and* $\psi : C \otimes A \to A \otimes C$ *a* k*-linear map. The following assertions are equivalent:*

(1) $(A \otimes C, \Delta, \underline{\varepsilon})$ *is a left unital weak* A*-coring;*

(2) *the conditions (10,12,13) and*

$$1_{\psi} \otimes c^{\psi} = \epsilon(c^{\psi}_{(1)})1_{\psi} \otimes c_{(2)} \tag{15}$$

are satisfied for all $a, b \in A$ *and* $c \in C$;

(3) *the conditions (10,13,15) and*

$$a_{\psi} \otimes \delta(c^{\psi}) = a_{\psi\Psi} \otimes c^{\Psi}_{(1)} \otimes c^{\psi}_{(2)}, \tag{16}$$

are satisfied for all $a, b \in A$ *and* $c \in C$.

We then say that (A, C, ψ) *is a weak entwining structure.*

Proof. If $(A \otimes C, \Delta, \underline{\varepsilon})$ is a left unital weak A-coring, then (10,12,13) hold, by Lemmas 2.1 and 2.2.
Assume that (10,12,13) are satisfied. The left counit property $(\underline{\varepsilon}\otimes_A \mathcal{C}) \circ \Delta = \pi$ (cf. (4)) holds if and only if

$$\underline{\varepsilon}(a \otimes c_{(1)})(1 \otimes c_{(2)}) = \epsilon(c^{\psi}_{(1)})a1_{\psi} \otimes c_{(2)}$$

equals

$$(a \otimes c)1 = a1_{\psi} \otimes c^{\psi},$$

for all $a \in A$ and $c \in C$, if and only if (15) holds for all $c \in C$. This proves that $\underline{(1) \Rightarrow (2)}$. It also proves that $\underline{(2) \Rightarrow (1)}$, if we can show that the right counit condition is satisfied. Indeed,

$$\begin{aligned}(a \otimes c_{(1)})\underline{\varepsilon}(1 \otimes c_{(2)}) &= \epsilon(c_{(2)}^{\psi})(a \otimes c_{(1)})1_{\psi}\\ &= \epsilon(c_{(2)}^{\psi})(a1_{\psi\Psi} \otimes c_{(1)}^{\Psi})\\ &\overset{(12)}{=} \epsilon((c^{\psi})_{(2)})a1_{\psi}1_{\Psi} \otimes (c^{\psi})_{(1)}^{\Psi}\\ &= a1_{\psi}1_{\Psi} \otimes c^{\psi\Psi}\\ &\overset{(10)}{=} a1_{\psi} \otimes c^{\psi} = (a \otimes c)1.\end{aligned}$$

$\underline{(2) \Leftrightarrow (3)}$. We will prove that the left hand sides of the formulas (12) and (16) are equal if ψ satisfies (10) and (15). Indeed,

$$\begin{aligned}a_{\psi}1_{\Psi} \otimes (c^{\psi})_{(1)}^{\Psi} \otimes (c^{\psi})_{(2)} &\overset{(15)}{=} a_{\psi}\epsilon\big((c^{\psi})_{(1)}^{\Psi}\big)1_{\Psi} \otimes (c^{\psi})_{(2)} \otimes (c^{\psi})_{(3)}\\ &\overset{(15)}{=} a_{\psi}1_{\Psi} \otimes (c^{\psi\Psi})_{(1)} \otimes (c^{\psi\Psi})_{(2)}\\ &\overset{(10)}{=} a_{\psi} \otimes (c^{\psi})_{(1)} \otimes (c^{\psi})_{(2)} = a_{\psi} \otimes \delta(c^{\psi}).\end{aligned}$$

□

Weak entwining structures were first introduced by the first author and De Groot [5], and the defining axioms are (10,13,15) and (16). Wisbauer [17] introduced weak corings, and proved the equivalence of (1) and (2) in Proposition 2.3 (see [17, 4.1]). So his axioms characterizing weak entwining structures are (10,12,13) and (15). In a remark following 4.1 in [17], it is observed that the defining axioms in [5] and [17] are not the same. It follows from Proposition 2.3 that the two sets of axioms are equivalent.

Proposition 2.4. *Let A be a k-algebra, C a k-coalgebra, and $\psi:\ C \otimes A \to A \otimes C$ a k-linear map. The following assertions are equivalent:*

1. *$(A \otimes C, \Delta, \varepsilon)$ is an A-coring;*
2. *$(A \otimes C, \Delta, \varepsilon)$ is a left unital weak A-coring;*
3. *(A, C, ψ) is an entwining structure; this means that the conditions (10,11,14) and (16) are satisfied.*

Proof. $\underline{(1) \Rightarrow (2)}$ is trivial.
$\underline{(2) \Rightarrow (3)}$. It follows from Lemmas 2.1 and 2.2 that (10,12,14) hold. Using (4), we find

$$a1_{\psi} \otimes c^{\psi} = (a \otimes c)1 = \varepsilon(a \otimes c_{(1)})(1 \otimes c_{(2)}) = \epsilon(c_{(1)})a \otimes c_{(2)} = a \otimes c,$$

and (11) follows after taking $a = 1$. (16) follows easily from (11) and (12).
$\underline{(3) \Rightarrow (1)}$ is well-known (see e.g. [2]). □

Proposition 2.5. *Let A be a k-algebra, C a k-coalgebra, and $\psi:\ C \otimes A \to A \otimes C$ a k-linear map. The following assertions are equivalent:*

1. *$(A \otimes C, \Delta, \underline{\varepsilon})$ is a left unital lax A-coring;*

(2) *the conditions (10,12,13) and*

$$1_\psi \otimes c^\psi = \epsilon(c^\Psi_{(1)})1_{\psi\Psi} \otimes c^\psi_{(2)}, \tag{17}$$

are satisfied for all $a, b \in A$ *and* $c \in C$*;*

(3) *the conditions (10,12,13) and*

$$1_\psi \otimes c^\psi = \epsilon(c^\Psi_{(1)})1_\Psi 1_\psi \otimes c^\psi_{(2)}, \tag{18}$$

are satisfied for all $a, b \in A$ *and* $c \in C$*.*

We then say that (A, C, ψ) *is a lax entwining structure.*

Proof. $\underline{(1) \Rightarrow (2)}$. It follows from Lemmas 2.1 and 2.2 that (10,12,13) hold. Take $1_\psi \otimes c^\psi \in \underline{A \otimes C}$. Then

$$\begin{aligned} \underline{\Delta}(1_\psi \otimes c^\psi) &= (1_\psi \otimes (c^\psi)_{(1)}) \otimes_A (1 \otimes (c^\psi)_{(2)}) \\ &= (1_\psi 1_\Psi \otimes (c^\psi)^\Psi_{(1)}) \otimes_A (1 \otimes (c^\psi)_{(2)}). \end{aligned}$$

From the left counit property in (3), it follows that

$$\begin{aligned} 1_\psi \otimes c^\psi &= ((\underline{\varepsilon} \circ \iota) \otimes_A \underline{A \otimes C})\underline{\Delta}(1_\psi \otimes c^\psi) \\ &= (A \otimes \epsilon)(1_\psi 1_\Psi \otimes (c^\psi)^\Psi_{(1)})(1 \otimes (c^\psi)_{(2)}) \\ &= \epsilon((c^\psi)^\Psi_{(1)})1_\psi 1_\Psi \otimes (c^\psi)_{(2)} \\ &\overset{(12)}{=} \epsilon(c^\Psi_{(1)})1_{\psi\Psi} \otimes c^\psi_{(2)}, \end{aligned}$$

so (17) holds.

$\underline{(2) \Rightarrow (1)}$. If (A, C, ψ) satisfies (10,12,13,17), then Δ is a coassociative comultiplication on $A \otimes C$. One equality in (3) is equivalent to (17), and the other one can be proved as follows: we have shown in the proof of Proposition 2.3 that (10) and (12) imply that

$$((A \otimes C) \otimes_A \underline{\varepsilon}) \circ \Delta = \pi.$$

This entails that

$$((\underline{A \otimes C}) \otimes_A (\underline{\varepsilon} \circ \iota)) \circ \underline{\Delta} = \underline{A \otimes C}.$$

$\underline{(2) \Leftrightarrow (3)}$. Using (13), we find that (17) is equivalent to (18) (take $a = 1_\psi$ in (13)). □

Proposition 2.6. *Let* A *be a* k*-algebra,* C *a* k*-coalgebra, and* $\psi : C \otimes A \to A \otimes C$ *a* k*-linear map. The following assertions are equivalent:*

(1) $(A \otimes C, \Delta, \varepsilon)$ *is a left unital lax* A*-coring;*

(2) (A, C, ψ) *is a lax entwining structure and*

$$\epsilon(c^\psi)1_\psi = \epsilon(c)1, \tag{19}$$

for all $c \in C$*;*

(3) *The conditions (10,12) and (14) are satisfied, for all* $a \in A$ *and* $c \in C$*.*

We then say that (A, C, ψ) *is a partial entwining structure.*

Proof. (1) ⇒ (2). It follows from Lemma 2.2 that (14) holds; taking $a = 1$, we find (19). It is clear that (A, C, ψ) is a lax entwining structure.
(2) ⇒ (1). From (19) it follows that $\varepsilon = \underline{\varepsilon}$, and thus $(A \otimes C, \Delta, \varepsilon = \underline{\varepsilon})$ is a left-unital lax A-coring.
(2) ⇒ (3). Combining (19) and (13), we find (14).
(3) ⇒ (2). Taking $a = 1$ in (14), we find (19). (13) then follows from (19) and (14). Also (18) follows easily:

$$\epsilon(c^{\Psi}_{(1)})1_{\Psi}1_{\psi} \otimes c^{\psi}_{(2)} \overset{(19)}{=} \epsilon(c_{(1)})11_{\psi} \otimes c^{\psi}_{(2)} = 1_{\psi} \otimes c^{\psi}.$$

□

Proposition 2.7. *(A, C, ψ) is an entwining structure if and only if it is at the same time a partial and weak entwining structure.*

Proof. One implication is obvious. Conversely, if (A, C, ψ) is a weak entwining structure, then, by Proposition 2.3, (10,15,16) are satisfied. If (A, C, ψ) is a partial entwining structure, then (14) holds, by Proposition 2.6. Then (11) can be shown as follows:

$$1_{\psi} \otimes c^{\psi} \overset{(15)}{=} \epsilon(c^{\psi}_{(1)})1_{\psi} \otimes c_{(2)} \overset{(14)}{=} \epsilon(c_{(1)})1 \otimes c_{(2)} = 1 \otimes c.$$

□

The lax Koppinen smash product. Let (A, C, ψ) be a weak (or lax) entwining structure, i.e. $(\mathcal{C} = A \otimes C, \Delta, \underline{\varepsilon})$ is a (left unital) weak (or lax) A-coring. Given the k-module isomorphism

$${}^*\mathcal{C} = {}_A\mathrm{Hom}(A \otimes C, A) \cong \mathrm{Hom}(C, A), f \mapsto f \circ (\eta_A \otimes C),$$

the (right unital) weak (or lax) A-ring structure on ${}^*\mathcal{C}$ (see Proposition 1.2) induces a (right unital) weak (or lax) A-ring structure on $\mathrm{Hom}(C, A)$. It is given by the following formulas, for all $a, b \in A, c \in C$ and $f, g \in \mathrm{Hom}(C, A)$:

$$(afb)(c) = a_{\psi}f(c^{\psi})b,$$

$$(f\#g)(c) = f(c_{(2)})_{\psi}g(c^{\psi}_{(1)}),$$

and

$$\eta : A \to \mathrm{Hom}(C, A), \eta(a)(c) = \epsilon(c^{\psi})a_{\psi}.$$

$\mathrm{Hom}(C, A)$ with this weak A-ring structure (but with slightly modified multiplication) is called in [5] the weak Koppinen smash product; when it is equipped with the lax A-ring structure, we will call it the lax Koppinen smash product. It is usually denoted by $\#(C, A)$.
The left dual of the corresponding A-coring $\underline{\mathcal{C}} = \mathcal{C}1_A$ is then isomorphic to

$$\underline{\#}(C, A) = 1_A\#(C, A) = \{f \in \#(C, A) \mid f(c) = 1_{\psi}f(c^{\psi}), \text{ for all } c \in C\}.$$

3. Lax entwined modules

Let (A, C, ψ) be a lax entwining structure, $\mathcal{C} = A \otimes C$ the associated lax A-coring, and $\underline{\mathcal{C}} = (A \otimes C)1_A$ the associated A-coring. For a k-linear map $\rho: M \to M \otimes C$, we will adopt the notation $\rho(m) = m_{[0]} \otimes m_{[1]}$, $(\rho \otimes C)(\rho(m)) = \rho^2(m) = m_{[0]} \otimes m_{[1]} \otimes m_{[2]}$, etc. We do not assume that ρ is coassociative.

A lax entwined module is a right A-module M, together with a k-linear map $\rho: M \to M \otimes C$ such that the following conditions are satisfied, for all $m \in M$:

$$m_{[0]}\epsilon(m_{[1]}) = m; \tag{20}$$

$$\rho^2(m) = m_{[0]}1_{\psi\Psi} \otimes (m_{[1](1)})^{\Psi} \otimes (m_{[1](2)})^{\psi}; \tag{21}$$

$$\rho(ma) = m_{[0]}a_{\psi} \otimes m_{[1]}^{\psi}. \tag{22}$$

A morphism between two lax entwined modules M and N is a right A-linear map $f: M \to N$ such that $f(m_{[0]}) \otimes m_{[1]} = f(m)_{[0]} \otimes f(m)_{[1]}$, for all $m \in M$. $\mathcal{M}(\psi)_A^C$ will denote the category of lax entwined modules.

Proposition 3.1. *For a lax entwining structure (A, C, ψ), the categories $\mathcal{M}^{\underline{\mathcal{C}}}$ and $\mathcal{M}(\psi)_A^C$ are isomorphic.*

Proof. Let M be a right A-module. We will first show that $\mathrm{Hom}_A(M, M \otimes_A \underline{\mathcal{C}})$ is isomorphic to the submodule of $\mathrm{Hom}(M, M \otimes C)$ consisting of maps ρ satisfying (22). Take $\rho: M \to M \otimes C$ satisfying (22), and define $\alpha(\rho): M \to M \otimes_A \underline{\mathcal{C}}$ as follows:

$$\alpha(\rho)(m) = m_{[0]} \otimes_A (1_{\psi} \otimes m_{[1]}^{\psi}).$$

$\alpha(\rho)$ is right A-linear since

$$\begin{aligned} \alpha(\rho)(ma) &= m_{[0]}a_{\Psi} \otimes_A (1_{\psi} \otimes m_{[1]}^{\Psi\psi}) \\ &= m_{[0]} \otimes_A (a_{\Psi}1_{\psi} \otimes m_{[1]}^{\Psi\psi}) \\ &\overset{(10)}{=} m_{[0]} \otimes_A (a_{\psi} \otimes m_{[1]}^{\psi}) \\ &= m_{[0]} \otimes_A (1_{\psi} \otimes m_{[1]}^{\psi})a = (\alpha(\rho)(m))a. \end{aligned}$$

Conversely, take $\tilde{\rho} \in \mathrm{Hom}_A(M, M \otimes_A \underline{\mathcal{C}})$, and define $\beta(\tilde{\rho}) \in \mathrm{Hom}(M, M \otimes C)$ as follows: for $m \in M$, there exist (a finite number of) $m_i \in M$ and $c_i \in C$ such that $\tilde{\rho}(m) = \sum_i m_i \otimes_A (1_{\psi} \otimes c_i^{\psi})$; then we define

$$\beta(\tilde{\rho})(m) = \sum_i m_i 1_{\psi} \otimes c_i^{\psi}.$$

$\beta(\tilde{\rho})$ satisfies (22): $\tilde{\rho}(ma) = \sum_i m_i \otimes_A (a_{\psi} \otimes c_i^{\psi}) = \sum_i m_i a_{\psi} \otimes_A (1_{\Psi} \otimes c_i^{\psi\Psi})$, hence

$$\beta(\tilde{\rho})(ma) = \sum_i m_i a_{\psi} 1_{\Psi} \otimes c_i^{\psi\Psi} = \sum_i m_i 1_{\psi} a_{\Psi} \otimes c_i^{\psi\Psi}.$$

α and β are inverses:

$$\alpha(\beta(\tilde{\rho}))(m) = \sum_i m_i 1_\psi \otimes_A (1_\Psi \otimes c_i^{\psi\Psi})$$
$$= \sum_i m_i \otimes_A (1_\psi \otimes c_i^{\psi}) = \tilde{\rho}(m)$$
$$\beta(\alpha(\rho))(m) = m_{[0]}1_\psi \otimes m_{[1]}^\psi \overset{(22)}{=} \rho(m).$$

Now take $\rho: M \to M \otimes C$ satisfying (22) and the corresponding right A-linear map $\tilde{\rho}$. We claim that $\tilde{\rho}$ is coassociative if and only if ρ satisfies (21). First compute

$$\tilde{\rho}^2(m) = m_{[0]} \otimes_A (1_{\Psi_1} \otimes m_{[1]}^{\Psi_1}) \otimes_A (1_\psi \otimes m_{[2]}^\psi);$$

$$((M \otimes_A \underline{\Delta}) \circ \tilde{\rho})(m) = m_{[0]} \otimes_A (1_\psi \otimes (m_{[1]}^\psi)_{(1)}) \otimes_A (1 \otimes (m_{[1]}^\psi)_{(2)}).$$

If $\tilde{\rho}$ is coassociative, then it follows that

$$m_{[0]}1_{\Psi_1}1_{\psi\Psi} \otimes m_{[1]}^{\Psi_1\Psi} \otimes m_{[2]}^\psi$$
$$\overset{(10)}{=} m_{[0]}1_{\psi\Psi} \otimes m_{[1]}^\Psi \otimes m_{[2]}^\psi$$
$$\overset{(22)}{=} \rho(m_{[0]}1_\psi) \otimes m_{[1]}^\psi \overset{(22)}{=} \rho(m_{[0]}) \otimes m_{[1]}$$

equals

$$m_{[0]}1_\psi 1_\Psi \otimes (m_{[1]}^\psi)_{(1)}^\Psi \otimes (m_{[1]}^\psi)_{(2)}$$
$$\overset{(12)}{=} m_{[0]}1_{\psi\Psi} \otimes (m_{[1](1)})^\Psi \otimes (m_{[1](2)})^\psi,$$

and (21) follows. Conversely, if (21) holds, then

$$\begin{aligned}
&((M \otimes_A \underline{\Delta}) \circ \tilde{\rho})(m)\\
&= m_{[0]} \otimes_A (1_\psi 1_\Psi \otimes (m_{[1]}^\psi)_{(1)}^\Psi) \otimes_A (1 \otimes (m_{[1]}^\psi)_{(2)}) \cdot 1\\
&\overset{(12)}{=} m_{[0]} \otimes_A (1_{\psi\Psi} \otimes (m_{[1](1)})^\Psi) \otimes_A (1 \otimes (m_{[1](2)})^\psi) \cdot 1\\
&= m_{[0]}1_{\psi\Psi} \otimes_A (1 \otimes (m_{[1](1)})^\Psi) \cdot 1 \otimes_A (1 \otimes (m_{[1](2)})^\psi) \cdot 1\\
&\overset{(21)}{=} m_{[0]} \otimes_A (1 \otimes m_{[1]}) \cdot 1 \otimes_A (1 \otimes m_{[2]}) \cdot 1 = \tilde{\rho}^2(m),
\end{aligned}$$

so $\tilde{\rho}$ is coassociative. Finally, $\tilde{\rho}$ satisfies the counit property if and only if

$$m = m_{[0]}\epsilon(m_{[1]}^\psi)1_\psi \overset{(22)}{=} m_{[0]}\epsilon(m_{[1]}).$$

□

Remark 3.2. Let (A, C, ψ) be a weak entwining structure. Using (16), we find that (21) is equivalent to

$$\rho^2(m) = m_{[0]}1_\psi \otimes \delta(m_{[1]}^\psi) \overset{(22)}{=} m_{[0]} \otimes \delta(m_{[1]}),$$

so (20,21) are equivalent to saying that M is a right C-comodule. We then recover [2, Prop. 2.3(3)].

4. Partial coactions

Let k be a commutative ring, A a k-algebra and H a k-bialgebra. Consider a map

$$\rho:\ A \to A \otimes H,\ \rho(a) = a_{[0]} \otimes a_{[1]}.$$

To ρ, we associate a map

$$\psi:\ H \otimes A \to A \otimes H,\ \psi(h \otimes a) = a_{[0]} \otimes ha_{[1]} = a_\psi \otimes h^\psi. \tag{23}$$

Lemma 4.1. *ψ satisfies (10) if and only if*

$$\rho(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}; \tag{24}$$

ψ satisfies (11) if and only if

$$\rho(1_A) = 1_A \otimes 1_H; \tag{25}$$

ψ satisfies (12) if and only if

$$\rho(a_{[0]}) \otimes a_{[1]} = a_{[0]}1_{[0]} \otimes a_{[1](1)}1_{[1]} \otimes a_{[1](2)}; \tag{26}$$

ψ satisfies (13) if and only if

$$\epsilon(a_{[1]})a_{[0]} = \epsilon(1_{[1]})1_{[0]}a; \tag{27}$$

ψ satisfies (14) if and only if

$$\epsilon(a_{[1]})a_{[0]} = a; \tag{28}$$

ψ satisfies (15) if and only if

$$\rho(1_A) = \epsilon(1_{[1]})1_{[0]} \otimes 1_H; \tag{29}$$

ψ satisfies (16) if and only if

$$\rho(a_{[0]}) \otimes a_{[1]} = a_{[0]} \otimes \delta(a_{[1]}); \tag{30}$$

ψ satisfies (17) if and only if

$$\rho(1_A) = \epsilon(1_{[0][1]})1_{[0][0]} \otimes 1_{[1]}; \tag{31}$$

ψ satisfies (18) if and only if

$$\rho(1_A) = \epsilon(1_{[1']})1_{[0']}1_{[0]} \otimes 1_{[1]}; \tag{32}$$

ψ satisfies (19) if and only if

$$\epsilon(1_{[1]})1_{[0]} = 1_A. \tag{33}$$

Proof. Let us show the equivalence between (12) and (26); the proof of the other equivalences is similar. (12) holds if and only if

$$a_{[0][0]} \otimes h_{(1)}a_{[0][1]} \otimes h_{(2)}a_{[1]} = a_{[0]}1_{[0]} \otimes h_{(1)}a_{[1](1)}1_{[1]} \otimes h_{(2)}a_{[1](2)},$$

and this is equivalent with (26). □

It follows immediately from Lemma 4.1 that (A, H, ψ) is an entwining structure if and only if A is a right H-comodule algebra. It follows also that (A, H, ψ) is a partial entwining structure if and only if (24,26,28) hold. We will then say that H *coacts partially on* A, or that A is a *right partial H-comodule algebra*. Similarly, we say that H is a *right lax H-comodule algebra* if (A, H, ψ) is a lax entwining structure, i.e. (24,26,27), and (31) or (32) are satisfied.

Example 4.2. Let $e \in H$ be an idempotent such that $e \otimes e = \Delta(e)(e \otimes 1)$ and $\epsilon(e) = 1$. Then we can define the following partial H-coaction on $A = k$:

$$\rho(x) = x \otimes e \in k \otimes_k H.$$

It is straightforward to verify that (24,26,28) hold:

$$\begin{aligned} \rho(x)\rho(y) &= xy \otimes e^2 = xy \otimes e = \rho(xy); \\ \rho(x_{[0]}) \otimes x_{[1]} &= x \otimes e \otimes e = x \otimes e_{(1)}e \otimes e_{(2)} = x_{[0]}1_{[0]} \otimes x_{[1](1)}1_{[1]} \otimes x_{[1](2)}; \\ x\epsilon(e) &= x. \end{aligned}$$

Such an idempotent e exists in a finite dimensional semisimple Hopf algebra. There exists a left integral t such that $\epsilon(t) = 1$. t is an idempotent, since $t^2 = \epsilon(t)t = t$, and $\Delta(t)(t \otimes 1) = t_{(1)}t \otimes t_{(2)} = \epsilon(t_{(1)})t \otimes t_{(2)} = t \otimes t$.

The proof of our next result is left to the reader.

Proposition 4.3. *Let H be a k-bialgebra, C a k-coalgebra, and consider a map*

$$\kappa : \ C \otimes H \to C, \ \ \kappa(c \otimes h) = c \cdot h,$$

and define $\psi : \ C \otimes H \to H \otimes C$ by the formula

$$\psi(c \otimes h) = h_{(1)} \otimes c \cdot h_{(2)}.$$

Then (H, C, ψ) is a partial entwining structure if and only if

(34) $\quad (c \cdot h) \cdot k = c \cdot (hk);$

(35) $\quad (c \cdot h)_{(1)} \cdot 1_H \otimes (c \cdot h)_{(2)} = c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)};$

(36) $\quad \epsilon_C(c \cdot h) = \epsilon_C(c)\epsilon_H(h),$

for all $c \in C$, $h, k \in H$. We then call C a right partial H-module coalgebra.

We are now able to define partial Doi-Hopf data.

Proposition 4.4. *Let H be a k-bialgebra, A a right partial H-comodule algebra, and C a right partial H-module coalgebra. Consider the map*

$$\psi : \ C \otimes A \to A \otimes C, \ \ \psi(c \otimes a) = a_{[0]} \otimes c \cdot a_{[1]}.$$

Then (A, C, ψ) is a partial entwining structure. We will say that (H, A, C) is a (right-right) partial Doi-Hopf structure.

Proof. Clearly the conditions (10,14) are satisfied. Let us show that (12) holds.

$$\begin{aligned} a_\psi 1_\Psi \otimes (c^\psi)^\Psi_{(1)} \otimes (c^\psi)_{(2)} &= a_{[0]}1_{[0]} \otimes (c\cdot a_{[1]})_{(1)} \cdot 1_{[1]} \otimes (c\cdot a_{[1]})_{(2)} \\ &\overset{(34)}{=} a_{[0]}1_{[0]} \otimes ((c\cdot a_{[1]})_{(1)} \cdot 1_H)\cdot 1_{[1]} \otimes (c\cdot a_{[1]})_{(2)} \\ &\overset{(35)}{=} a_{[0]}1_{[0]} \otimes (c_{(1)} \cdot a_{[1](1)})\cdot 1_{[1]} \otimes c_{(2)} \cdot a_{[1](2)} \\ &\overset{(34)}{=} a_{[0]}1_{[0]} \otimes c_{(1)} \cdot (a_{[1](1)}1_{[1]}) \otimes c_{(2)} \cdot a_{[1](2)} \\ &\overset{(26)}{=} a_{[0][0]} \otimes c_{(1)} \cdot a_{[0][1]} \otimes c_{(2)} \cdot a_{[1]} = a_{\psi\Psi} \otimes c^\Psi_{(1)} \otimes c^\psi_{(2)}. \end{aligned}$$

□

Remark 4.5. In a similar way, we can define lax H-module coalgebras and lax Doi-Hopf data.

5. Partial smash products

Let A and B be k-algebras, and $R:\ B\otimes A \to A\otimes B$ a k-linear map. We use the notation

$$R(b\otimes a) = a_R \otimes b^R = a_r \otimes b^r.$$

We assume that $A\otimes B$ is a left unital A-bimodule under the action $c'(a\otimes b)c = c'ac_R \otimes b^R$. Then the following condition is satisfied, for all $a, c \in A$ and $b \in B$ (see (10)):

$$(ac)_R \otimes b^R = a_R c_r \otimes b^{Rr}. \tag{37}$$

The map $\mu:\ (A\otimes B)\otimes_A (A\otimes B) \to A\otimes B$,

$$\mu((a\otimes b)\otimes_A (c\otimes d)) = ac_R \otimes b^R d$$

is well-defined since

$$\begin{aligned} \mu((a\otimes b)a' \otimes_A (c\otimes d)) &= \mu((aa'_R \otimes b^R)\otimes_A (c\otimes d)) = aa'_R c_r \otimes b^{Rr} d \\ &\overset{(37)}{=} a(a'c)_R \otimes b^R d = \mu((a\otimes b)\otimes_A (a'c\otimes d)). \end{aligned}$$

$A\#B$ will be our notation for $A\otimes B$ together with the multiplication μ. We then write

$$\mu((a\otimes b)\otimes_A (c\otimes d)) = (a\#b)(c\#d).$$

We also consider the maps

$$\begin{aligned} \eta &= A\otimes \eta_B:\ A \to A\otimes B,\ \eta(a) = a\otimes 1_B; \\ \underline{\eta} &= \pi\circ\eta:\ A \to \underline{A\otimes B},\ \underline{\eta}(a) = a1_R \otimes 1^R. \end{aligned}$$

Lemma 5.1. *Assume that $R:\ B\otimes A \to A\otimes B$ satisfies (37).*

(1) *μ is right A-linear if and only if μ is associative if and only if*

$$(ac_R)_r \otimes b^r d^R = a_R c_r \otimes (b^R d)^r, \tag{38}$$

for all $a, c \in A$ and $b, d \in B$.

(2) *η is right A-linear if and only if*

$$a\otimes 1 = a_R\otimes 1^R, \tag{39}$$

for all $a\in A$.

(3) *$\underline{\eta}$ is right A-linear if and only if*

$$a1_R\otimes 1^R = a_R\otimes 1^R, \tag{40}$$

for all $a\in A$.

Proof. 1) μ is right A-linear if and only if

$$(a'\#b)((a\#d)c) = (a'\#b)(ac_R\#d^R) = a'(ac_R)_r\#b^r d^R$$

equals

$$((a'\#b)(a\#d))c = (a'a_R\#b^R d)c = a'a_R c_r\#(b^R d)^r,$$

for all $a',a,c\in A$ and $b,d\in B$. This is equivalent to (38). It is obvious that μ is associative if and only if μ is right A-linear.
2) η is right A-linear if and only if $\eta(a)=a\otimes 1$ equals $\eta(1)a=(1\otimes 1)a=a_R\otimes 1^R$.
3) $\underline{\eta}$ is right A-linear if and only if $\underline{\eta}(a)=a1_R\otimes 1^R$ equals

$$\underline{\eta}(1)a = (1_R\otimes 1^R)a = 1_R a_r\otimes 1^{Rr} \overset{(37)}{=} a_R\otimes 1^R.$$

□

Proposition 5.2. *Let A and B be k-algebras, and $R:\ B\otimes A\to A\otimes B$ a k-linear map. The following assertions are equivalent:*

(1) *$(A\#B,\mu,\underline{\eta})$ is a left unital weak A-ring;*
(2) *the conditions (37,38,40) and*

$$1_R\otimes b^R = 1_R\otimes 1^R b \tag{41}$$

are satisfied, for all $a,c\in A$ and $b,d\in B$;

(3) *(A,B,R) is a weak smash product structure in the sense of* [5], *that is, the conditions (37,40,41) and*

$$a_{Rr}\otimes b^r d^R = a_R\otimes (bd)^R \tag{42}$$

are satisfied, for all $a,c\in A$ and $b,d\in B$.

Proof. <u>(1) ⇒ (2)</u>. (37,38,40) follow from Lemma 5.1. From (8), it follows that

$$(1\#b)1 = 1_R\#b^R$$

equals

$$(1_R\#1^R)(1\#b) = 1_R1_r\#1^{Rr}b \overset{(37)}{=} 1_R\#1^R b,$$

and (41) follows.
<u>(2) ⇒ (1)</u>. If (37,38,40) hold, then we only need to verify (8), by Lemma 5.1.

We compute that

$$\begin{aligned}(1_R\#1^R)(a\#b) &= 1_Ra_r\#1^{Rr}b \overset{(37)}{=} a_R\#1^Rb\\ &\overset{(40)}{=} a1_R\#1^Rb \overset{(41)}{=} a1_R\#b^R = (a\#b)1;\\ (a\#b)(1_R\#1^R) &= a1_{Rr}\#b^r1^R = a(11_R)_r\#b^r1^R\\ &\overset{(38)}{=} a1_R1_r\#(b^R1)^r = a1_R1_r\#b^{Rr} \overset{(37)}{=} a1_R\#b^R = (a\#b)1.\end{aligned}$$

(2) ⇒ (3). Replacing b by bd in (41), we obtain

$$1_R\otimes (bd)^R = 1_R\otimes 1^Rbd. \tag{43}$$

Taking $a=1$ in (38), we obtain

$$\begin{aligned}c_{Rr}\otimes b^rd^R &\overset{(38)}{=} 1_Rc_r\otimes (b^Rd)^r \overset{(41)}{=} 1_Rc_r\otimes (1^Rbd)^r\\ &\overset{(43)}{=} 1_Rc_r\otimes (bd)^{Rr} \overset{(37)}{=} c_R\otimes (bd)^R,\end{aligned}$$

and (42) follows.
(3) ⇒ (2). We have to show that (38) holds. Indeed,

$$(ac_R)_r\otimes b^rd^R \overset{(37)}{=} a_rc_{Rr'}\otimes b^{rr'}d^R \overset{(42)}{=} a_rc_R\otimes (b^rd)^R.$$

□

Proposition 5.3. *Let A and B be k-algebras, and $R:\ B\otimes A\to A\otimes B$ a k-linear map. The following assertions are equivalent:*

1. *$(A\#B,\mu,\eta)$ is an A-ring;*
2. *$(A\#B,\mu,\eta)$ is a left unital weak A-ring;*
3. *(A,B,R) is a smash product structure; this means that the following conditions hold: (37,39,42) and*

$$1\otimes b = 1_R\otimes b^R, \tag{44}$$

for all $b\in B$.

Proof. (1) ⇒ (2) is trivial.
(2) ⇒ (3). It follows from Lemma 5.1 that (37,38,39) hold. Using (8), we find that

$$1_R\#b^R = (1\#b)1 = (1\#1)(1\#b) = 1_R\#1^Rb \overset{(39)}{=} 1\#b,$$

proving (44). Taking $a=1$ in (38) and using (44), we find (42).
(3) ⇒ (1) is well-known (and easy to prove). □

Proposition 5.4. *Let A and B be k-algebras, and $R:\ B\otimes A\to A\otimes B$ a k-linear map. The following assertions are equivalent:*

1. *$(A\#B,\mu,\underline{\eta})$ is a left unital lax A-ring;*
2. *the conditions (37,38,40) and*

$$a1_R\otimes b^R = (a1_r)_R\otimes 1^Rb^r \tag{45}$$

are fulfilled, for all $a,c\in A$ and $b,d\in B$;

(3) *the conditions (37,38,40) and*

$$a1_R \otimes b^R = a_R 1_r \otimes (1^R b)^r \tag{46}$$

are fulfilled, for all $a, c \in A$ and $b, d \in B$.

We then say that (A, B, R) is a lax smash product structure.

Proof. (1) ⇒ (2). (37,38,40) follow from Lemma 5.1. Using (7), we find

$$a1_R\#b^R = (1_r\#1^r)(a1_R\#b^R) = 1_r(a1_R)_{r'}\#1^{rr'}b^R \overset{(37)}{=} (a1_R)_r\#1^r b^R,$$

so (45) is satisfied.

(2) ⇒ (3). We compute that

$$a1_R\#b^R = (a1_R)_r\#1^r b^R \overset{(38)}{=} a_R 1_r\#(1^R b)^r.$$

(3) ⇒ (1). It follows from the above computations that

$$a1_R\#b^R = (1_r\#1^r)(a1_R\#b^R).$$

We also have

$$\begin{aligned}(a1_R\#b^R)(1_r\#1^r) &= a1_R 1_{rr'}\#b^{Rr'}1^r \\ &\overset{(37)}{=} a(11_r)_R\#b^R 1^r \overset{(38)}{=} a1_R 1_r\#b^{Rr} \overset{(37)}{=} a1_R\#b^R,\end{aligned}$$

as needed. □

Proposition 5.5. *Let A and B be k-algebras, and $R:\ B \otimes A \to A \otimes B$ a k-linear map. The following assertions are equivalent:*

(1) *$(A\#B, \mu, \eta)$ is a left unital lax A-ring;*

(2) *(A, B, R) is a lax smash product structure and*

$$1 \otimes 1 = 1_R \otimes 1^R; \tag{47}$$

(3) *the conditions (37,38,39) are satisfied.*

We then say that (A, B, R) is a partial smash product structure.

Proof. (1) ⇒ (2). Let $(A\#B, \mu, \eta)$ be a left unital lax A-ring. It follows from Lemma 5.1 that (39) holds. (47) follows after we take $a = 1$ in (39). (47) implies that $\eta = \underline{\eta}$, so $(A\#B, \mu, \underline{\eta})$ is a left unital lax A-ring.

(2) ⇒ (1) follows also from the fact that (47) implies that $\eta = \underline{\eta}$.

(1) ⇒ (3) follows immediately from Lemma 5.1.

(3) ⇒ (1). We have to show that (7) holds:

$$(a1_R\#b^R)(1\#1) = a1_R 1_r\#b^{Rr} \overset{(37)}{=} a1_R\#b^R;$$

$$(1\#1)(a1_R\#b^R) = (a1_R)_r\#1^r b^R \overset{(39)}{=} a1_R\#b^R.$$

□

Proposition 5.6. *(A, B, R) is a smash product structure if and only if it is at the same time a weak and partial smash product structure.*

Proof. One implication is obvious. Conversely, if (A, B, R) is a weak smash product structure, then (37,40,41,42) are satisfied (cf. Proposition 5.2). If (A, C, ψ) is a partial smash product structure, then (39) holds (cf. Proposition 5.5) and (44) can be computed as follows:

$$1_R\#b^R \overset{(41)}{=} 1_R\#1^R b \overset{(47)}{=} 1\#b.$$

□

Theorem 5.7. *Let A be a k-algebra, and C a finitely generated projective k-coalgebra. Then there is a one-to-one correspondence between lax (resp. partial) entwining structures of the form (A, C, ψ) and lax (resp. partial) smash product structures of the form $(A^{\rm op}, C^*, R)$.*

Proof. Let $\{c_i, c_i^* \mid i = 1, \dots, n\}$ be a dual basis for C. Then it is well-known (see for example [8, (1.5)]) that

$$\sum_i \Delta(c_i) \otimes c_i^* = \sum_{i,j} c_i \otimes c_j \otimes c_i^* * c_j^*. \tag{48}$$

We also have an isomorphism of k-modules

$$\mathrm{Hom}(C \otimes A, A \otimes C) \cong \mathrm{Hom}(C^* \otimes A, A \otimes C^*).$$

The isomorphism can be described as follows. If $R:\ C^* \otimes A \to A \otimes C^*$ corresponds to $\psi:\ C \otimes A \to A \otimes C$, then

$$R(c^* \otimes a) = a_R \otimes (c^*)^R = \sum_i \langle c^*, c_i^\psi \rangle a_\psi \otimes c_i^*; \tag{49}$$

$$\psi(c \otimes a) = a_\psi \otimes c^\psi = \sum_i \langle (c_i^*)^R, c \rangle a_R \otimes c_i. \tag{50}$$

Assume that (A, C, ψ) is a lax entwining structure; we will show that $(A^{\rm op}, C^*, R)$ is a lax smash product structure. The multiplication in $A^{\rm op}$ will be denoted by a dot: $a \cdot b = ba$. We have to show that R satisfies (37,38,40) and (45).

$$\begin{aligned}
a_R \cdot b_r \otimes (c^*)^{Rr} &= b_r a_R \otimes (c^*)^{Rr} = \sum_i \langle (c^*)^R, c_i^\psi \rangle b_\psi a_R \otimes c_i^* \\
&= \sum_{i,j} \langle c_j^*, c_i^\psi \rangle \langle c^*, c_j^\Psi \rangle b_\psi a_\Psi \otimes c_i^* = \sum_i \langle c^*, c_i^{\psi\Psi} \rangle b_\psi a_\Psi \otimes c_i^* \\
&\overset{(10)}{=} \sum_i \langle c^*, c_i^\psi \rangle (ba)_\psi \otimes c_i^* = (ba)_R \otimes (c^*)^R = (a \cdot b)_R \otimes (c^*)^R;
\end{aligned}$$

$$\begin{aligned}
&(a \cdot b_R)_r \otimes (c^*)^r * (d^*)^R = (b_R a)_r \otimes (c^*)^r * (d^*)^R \\
&\quad= \sum_i \langle d^*, c_i^\psi \rangle (b_\psi a)_r \otimes (c^*)^r * c_i^* \\
&\quad= \sum_{i,j} \langle d^*, c_i^\psi \rangle \langle c^*, c_j^\Psi \rangle (b_\psi a)_\Psi \otimes c_j^* * c_i^*
\end{aligned}$$

$$
\begin{aligned}
&\overset{(48)}{=} \sum_i \langle d^*, c_{i(2)}^{\psi}\rangle\langle c^*, c_{i(1)}^{\Psi}\rangle (b_\psi a)_\Psi \otimes c_i^*\\
&\overset{(10)}{=} \sum_i \langle d^*, c_{i(2)}^{\psi}\rangle\langle c^*, c_{i(1)}^{\Psi\Psi'}\rangle b_{\psi\Psi} a_{\Psi'} \otimes c_i^*\\
&\overset{(12)}{=} \sum_i \langle d^*, (c_i^{\psi})_{(2)}\rangle\langle c^*, (c_i^{\psi})_{(1)}^{\Psi\Psi'}\rangle b_\psi 1_\Psi a_{\Psi'} \otimes c_i^*\\
&\overset{(10)}{=} \sum_i \langle d^*, (c_i^{\psi})_{(2)}\rangle\langle c^*, (c_i^{\psi})_{(1)}^{\Psi}\rangle b_\psi a_\Psi \otimes c_i^*\\
&= \sum_{i,j} \langle d^*, (c_i^{\psi})_{(2)}\rangle\langle c^*, c_j^{\Psi}\rangle\langle c_j^*, (c_i^{\psi})_{(1)}\rangle b_\psi a_\Psi \otimes c_i^*\\
&= \sum_{i,j} \langle c_j^* * d^*, c_i^{\psi}\rangle\langle c^*, c_j^{\Psi}\rangle b_\psi a_\Psi \otimes c_i^*\\
&= b_r a_R \otimes ((c^*)^R * d^*)^r = a_R \cdot b_r \otimes ((c^*)^R * d^*)^r;
\end{aligned}
$$

$$
\begin{aligned}
a \cdot 1_R \otimes \epsilon^R = 1_R a \otimes \epsilon^R &= \sum_i \langle \epsilon, c_i^{\psi}\rangle 1_\psi a \otimes c_i^*\\
&\overset{(13)}{=} \sum_i \langle \epsilon, c_i^{\psi}\rangle a_\psi \otimes c_i^* = a_R \otimes \epsilon^R;
\end{aligned}
$$

$$
\begin{aligned}
a \cdot 1_R \otimes (c^*)^R = 1_R a \otimes (c^*)^R &= \sum_i \langle c^*, c_i^{\psi}\rangle 1_\psi a \otimes c_i^*\\
&\overset{(18)}{=} \sum_i \langle \epsilon, c_{i(1)}^{\Psi}\rangle\langle c^*, c_{i(2)}^{\psi}\rangle 1_\Psi 1_\psi a \otimes c_i^*\\
&\overset{(13)}{=} \sum_i \langle \epsilon, c_{i(1)}^{\Psi}\rangle\langle c^*, c_{i(2)}^{\psi}\rangle (1_\psi a)_\Psi \otimes c_i^*\\
&\overset{(48)}{=} \sum_{i,j} \langle \epsilon, c_i^{\Psi}\rangle\langle c^*, c_j^{\psi}\rangle (1_\psi a)_\Psi \otimes c_i^* * c_j^*\\
&= \sum_i \langle \epsilon, c_i^{\Psi}\rangle (1_r a)_\Psi \otimes c_i^* * (c^*)^r\\
&= (1_r a)_R \otimes \epsilon^R * (c^*)^r = (a \cdot 1_r)_R \otimes \epsilon^R * (c^*)^r.
\end{aligned}
$$

If (A, C, ψ) is a partial entwining structure, then $(A^{\rm op}, C^*, R)$ is a partial smash product structure. It suffices to show that (47) holds.

$$
\begin{aligned}
1_R \otimes \epsilon^R &= \sum_i \langle \epsilon, c_i^{\psi}\rangle 1_\psi \otimes c_i^*\\
&\overset{(19)}{=} \sum_i \langle \epsilon, c_i\rangle 1 \otimes c_i^* = 1 \otimes \epsilon.
\end{aligned}
$$

Conversely, if $(A^{\rm op}, C^*, R)$ is a lax, resp. partial smash product structure, then (A, C, ψ) is a lax, resp. partial entwining structure. The computations are similar to the ones above, and are left to the reader. □

Remark 5.8. Theorem 5.7 also holds for (weak) entwining structures versus (weak) smash product structures. We refer to [8, Theorem 8] and to [5].

Proposition 5.9. *Assume that C is finitely generated and projective as a k-module. Let (A, C, ψ) be a lax entwining structure, and $(A^{\rm op}, C^*, R)$ the corresponding lax smash product structure. Then ${}^*(A\otimes C)^{\rm op}$ is isomorphic to $A^{\rm op}\#C^*$ as left unital lax $A^{\rm op}$-rings, and ${}^*(\underline{A\otimes C})^{\rm op}$ is isomorphic to $\underline{A^{\rm op}\#C^*}$ as $A^{\rm op}$-rings. Consequently the categories ${}_{\underline{A^{\rm op}\#C^*}}\mathcal{M}$ and $\mathcal{M}(\psi)_A^C$ are isomorphic.*

Proof. We know from Section 2 that ${}^*(A\otimes C)^{\rm op}\cong \#(C, A)^{\rm op}$, with multiplication

$$(f\bullet g)(c) = g(c_{(2)})_\psi f(c_{(1)}^\psi)$$

in $\#(A, C)^{\rm op}$. The map

$$\alpha:\ A^{\rm op}\otimes C^*\to {\rm Hom}(C, A),\ \alpha(a\#c^*)(c) = a\langle c^*, c\rangle$$

is an isomorphism of k-modules since C is finitely generated and projective. The first statement follows after we show that α preserves the multiplication.

$$\begin{aligned}
\alpha((a\#c^*)(b\#d^*))(c) &= \alpha(b_Ra\#(c^*)^R * d^*)(c)\\
&= b_Ra\langle (c^*)^R * d^*, c\rangle = \sum_i \langle c^*, c_i^\psi\rangle b_\psi a\langle c_i^* * d^*, c\rangle\\
&= \sum_i \langle c^*, c_i^\psi\rangle b_\psi a\langle c_i^*, c_{(1)}\rangle\langle d^*, c_{(2)}\rangle = b_\psi a\langle c^*, c_{(1)}^\psi\rangle\langle d^*, c_{(2)}\rangle\\
&= (\alpha(b\#d^*)(c_{(2)}))_\psi \alpha(a\#c^*)(c_{(1)}^\psi) = (\alpha(a\#c^*)\bullet\alpha(b\#d^*))(c).
\end{aligned}$$

Applying Proposition 1.2 we see that

$${}^*(\underline{A\otimes C})^{\rm op}\cong (1_A\cdot {}^*(A\otimes C))^{\rm op} = {}^*(A\otimes C)^{\rm op}\cdot 1_A \cong (A^{\rm op}\#C^*)\cdot 1_A = \underline{A^{\rm op}\#C^*}.$$

□

6. Partial actions

Let k be a commutative ring, A a k-algebra and H a k-bialgebra. Consider a map

$$\kappa:\ H\otimes A\to A,\ \kappa(h\otimes a) = h\cdot a.$$

To κ, we associate a map

(51) $$R:\ H\otimes A\to A\otimes H,\ R(h\otimes a) = h_{(1)}\cdot a\otimes h_{(2)} = a_R\otimes h^R.$$

Lemma 6.1. *R satisfies (37) if and only if*

(52) $$h\cdot(ac) = (h_{(1)}\cdot a)(h_{(2)}\cdot c);$$

R satisfies (38) if and only if

(53) $$h\cdot(a(k\cdot c)) = (h_{(1)}\cdot a)((h_{(2)}k)\cdot c);$$

R satisfies (39) if and only if

(54) $$1_H\cdot a = a;$$

R satisfies (40) if and only if

$$a(1_H \cdot 1_A) = 1_H \cdot a; \tag{55}$$

R satisfies (41) if and only if

$$h \cdot 1_A = \epsilon(h)1_H \cdot 1_A; \tag{56}$$

R satisfies (42) if and only if

$$h \cdot (k \cdot a) = (hk) \cdot a; \tag{57}$$

R satisfies (44) if and only if

$$h \cdot 1_A = \epsilon(h)1_A; \tag{58}$$

R satisfies (45) if and only if

$$a(h \cdot 1_A) = 1_H \cdot (a(h \cdot 1_A)); \tag{59}$$

R satisfies (46) if and only if

$$a(h \cdot 1_A) = (1_H \cdot a)(h \cdot 1_A); \tag{60}$$

R satisfies (47) if and only if

$$1_H \cdot 1_A = 1_A. \tag{61}$$

Proof. Let us show the equivalence between (38) and (53); the proof of the other equivalences is similar. (38) holds if and only if

$$h_{(1)} \cdot (a(k_{(1)} \cdot c)) \otimes h_{(2)}k_{(2)} = (h_{(1)} \cdot a)((h_{(2)}k_{(1)}) \cdot c) \otimes h_{(3)}k_{(2)},$$

and this is equivalent with (53). □

Remark 6.2. If (52) holds, then (53) is equivalent to

$$h \cdot (k \cdot a) = (h_{(1)} \cdot 1_A)((h_{(2)}k) \cdot a). \tag{62}$$

It follows immediately from Lemma 6.1 that (A, H, R) is a smash product structure if and only if A is a left H-module algebra. It follows also that (A, H, R) is a partial smash product structure if and only if (52,62,54) hold. We will then say that H *acts partially on* A, or that A is a *left partial H-module algebra*. Likewise we call A a *left lax H-module algebra* if (A, H, R) is a lax smash product structure, i.e. (52,62,55), and (59) or (60) are satisfied. We will investigate the former notion in the particular situation where $H = kG$ is a group algebra. We will recover the partial group actions introduced in [13] and generalized in [6].

Partial group actions. Let G be a group, and A a k-algebra. A *partial action* of G on A consists of a set of idempotents $\{e_\sigma \mid \sigma \in G\} \subset A$, and a set of isomorphisms $\alpha_\sigma : e_{\sigma^{-1}}A \to e_\sigma A$ such that

$$e_1 = 1_A \ ; \ \alpha_1 = A,$$

and

$$e_\sigma \alpha_{\sigma\tau}(e_{\tau^{-1}\sigma^{-1}}a) = \alpha_\sigma(e_{\sigma^{-1}}\alpha_\tau(e_{\tau^{-1}}a)); \tag{63}$$

$$\alpha_\sigma(e_{\sigma^{-1}}ab) = \alpha_\sigma(e_{\sigma^{-1}}a)\alpha_\sigma(e_{\sigma^{-1}}b); \tag{64}$$

$$\alpha_\sigma(e_{\sigma^{-1}}) = e_\sigma, \tag{65}$$

for all $\sigma, \tau \in G$ and $a, b \in A$. This slightly generalizes the definitions in [13] and [6]: in [13], it is assumed that A is commutative and that the isomorphisms α_σ are multiplicative; in [6], it is assumed that, for all $\sigma \in G$, e_σ is central and α_σ is multiplicative. In both cases (64) and (65) are automatically satisfied.

Proposition 6.3. *Let A be a k-algebra, and G a group. Then there is a bijective correspondence between partial G-actions and partial kG-actions on A.*

Proof. Assume first that kG acts partially on A. For each $\sigma \in G$, let

$$e_\sigma = \sigma \cdot 1_A.$$

Take $a = c = 1_A$ in (52); then it follows that $e_\sigma^2 = e_\sigma$. It follows from (54) that $e_1 = 1 \cdot 1_A = 1_A$. From (62), it follows that

$$\sigma \cdot (\tau \cdot a) = e_\sigma((\sigma\tau) \cdot a). \tag{66}$$

We then compute

$$\sigma \cdot e_{\sigma^{-1}} = \sigma \cdot (\sigma^{-1} \cdot 1_A) = e_\sigma 1_A = e_\sigma, \tag{67}$$

and

$$\sigma \cdot (e_{\sigma^{-1}}a) \overset{(52)}{=} (\sigma \cdot e_{\sigma^{-1}})(\sigma \cdot a) = e_\sigma(\sigma \cdot a). \tag{68}$$

It follows that the map $A \to A$, $a \mapsto \sigma \cdot a$ restricts to a map

$$\alpha_\sigma : \ e_{\sigma^{-1}}A \to e_\sigma A.$$

Observe that

$$\sigma \cdot (e_{\sigma^{-1}}a) \overset{(68)}{=} e_\sigma(\sigma \cdot a) = (\sigma \cdot 1_A)(\sigma \cdot a) \overset{(52)}{=} \sigma \cdot a. \tag{69}$$

Now

$$\begin{aligned} \alpha_{\sigma^{-1}}(\alpha_\sigma(e_{\sigma^{-1}}a) &\overset{(68)}{=} \alpha_{\sigma^{-1}}(e_\sigma(\sigma \cdot a)) \overset{(69)}{=} \sigma^{-1} \cdot (\sigma \cdot a) \\ &\overset{(66)}{=} e_{\sigma^{-1}}((\sigma^{-1}\sigma) \cdot a) \overset{(54)}{=} e_{\sigma^{-1}}a. \end{aligned}$$

In a similar way, we find that $\alpha_\sigma(\alpha_{\sigma^{-1}}(e_\sigma a)) = e_\sigma a$, and it follows that $\alpha_\sigma : \ e_{\sigma^{-1}}A \to e_\sigma A$ is an isomorphism. It is also clear that

$$\alpha_1(a) = 1 \cdot a \overset{(54)}{=} a,$$

and

$$\alpha_\sigma(e_{\sigma^{-1}}) = \sigma\cdot(e_{\sigma^{-1}}) \overset{(67)}{=} e_\sigma.$$

(64) can be shown as follows:

$$\begin{aligned}\alpha_\sigma(e_{\sigma^{-1}}a)\alpha_\sigma(e_{\sigma^{-1}}b) &= (\sigma\cdot(e_{\sigma^{-1}}a))(\sigma\cdot(e_{\sigma^{-1}}b))\\ &\overset{(52)}{=} (\sigma\cdot e_{\sigma^{-1}})(\sigma\cdot a)(\sigma\cdot e_{\sigma^{-1}})(\sigma\cdot b) = e_\sigma(\sigma\cdot a)e_\sigma(\sigma\cdot b)\\ &= (\sigma\cdot 1_A)(\sigma\cdot a)(\sigma\cdot 1_A)(\sigma\cdot b) \overset{(52)}{=} \sigma\cdot(ab) \overset{(69)}{=} \alpha_\sigma(e_{\sigma^{-1}}ab).\end{aligned}$$

We are left to prove that (63) holds:

$$\begin{aligned}\alpha_\sigma(e_{\sigma^{-1}}\alpha_\tau(e_{\tau^{-1}}a)) &\overset{(69)}{=} \sigma\cdot(\tau\cdot a)\\ &\overset{(66)}{=} e_\sigma((\sigma\tau)\cdot a) \overset{(69)}{=} e_\sigma\alpha_{\sigma\tau}(e_{\tau^{-1}\sigma^{-1}}a).\end{aligned}$$

Conversely, assume that G acts partially on A, and define an action of kG on A by extending

$$\sigma\cdot a = \alpha_\sigma(e_{\sigma^{-1}}a) \in e_\sigma A$$

linearly to kG. This defines a partial action of kG on A, since

$$\begin{aligned}&\sigma\cdot(ab) = \alpha_\sigma(e_{\sigma^{-1}}ab) \overset{(64)}{=} \alpha_\sigma(e_{\sigma^{-1}}a)\alpha_\sigma(e_{\sigma^{-1}}b) = (\sigma\cdot a)(\sigma\cdot b);\\ &\sigma\cdot(\tau\cdot a) = \alpha_\sigma(e_{\sigma^{-1}}\alpha_\tau(e_{\tau^{-1}}a)) \overset{(63)}{=} e_\sigma\alpha_{\sigma\tau}(e_{\tau^{-1}\sigma^{-1}}a) = e_\sigma((\sigma\tau)\cdot a);\\ &1\cdot a = \alpha_1(e_1 a) = \alpha_1(a) = a.\end{aligned}$$

It is easy to check that condition (65) establishes the bijectivity of the correspondence. □

A Frobenius property. Let $i:\ R\to S$ be a ring homomorphism. Recall that i is called Frobenius (or we say that S/R is Frobenius) if there exists a Frobenius system (ν, e). This consists of an R-bimodule map $\nu:\ S\to R$ and an element $e=\sum e^1\otimes_R e^2 \in S\otimes_R S$ such that $se=es$, for all $s\in S$, and $\sum \nu(e^1)e^2 = \sum e^1\nu(e^2) = 1$.
A Hopf algebra H over a commutative ring k is Frobenius if and only if it is finitely generated projective, and the space of integrals is free of rank one. If H is Frobenius, then there exists a left integral $t\in H$ and a left integral $\varphi\in H^*$ such that

$$\langle\varphi, t\rangle = 1. \tag{70}$$

The Frobenius system is $(\varphi, t_{(2)}\otimes \overline{S}(t_{(1)}))$. In particular, we have

$$\langle\varphi, t_{(2)}\rangle\overline{S}(t_{(1)}) = t_{(2)}\langle\varphi, \overline{S}(t_{(1)})\rangle = 1_H. \tag{71}$$

For a detailed discussion, we refer to the literature, see for example [8, Sec. 3.2].
If $t\in H$ is a left integral, then it is easy to prove that

$$t_{(2)}\otimes\overline{S}(t_{(1)})h = ht_{(2)}\otimes\overline{S}(t_{(1)}), \tag{72}$$

for all $h\in H$ (see [8, Prop. 58] for a similar statement).
Assume that A is a left H-module algebra, and that H is Frobenius. Then the ring homomorphism $A\to A\#H$ is also Frobenius (see [8, Prop. 5.1]). Similar properties hold for a module algebra over a weak Hopf algebra and

for an algebra with a partial group action (see [6, 7]). Our aim is now to prove such a statement for a partial module algebra over a Frobenius Hopf algebra H. Assume that we have an action of H on an algebra A satisfying (52,62,54). The smash product $A\#H$ has multiplication rule

$$(a\#h)(b\#k) = a(h_{(1)}\cdot b)\#h_{(2)}k, \tag{73}$$

and $\underline{A\#H}$ is the subalgebra generated by the elements of the form $(a\#h)1_A = a(h_{(1)}\cdot 1_A)\#h_{(2)}$.

Proposition 6.4. *Let H be a Frobenius Hopf algebra, let t and φ be as above, and take a left partial H-module algebra A. Suppose that $h\cdot 1_A$ is central in A, for every $h\in H$, and that t satisfies the following cocommutativity property:*

$$t_{(1)}\otimes t_{(2)}\otimes t_{(3)}\otimes t_{(4)} = t_{(1)}\otimes t_{(3)}\otimes t_{(2)}\otimes t_{(4)}. \tag{74}$$

Then $\underline{A\#H}/A$ is Frobenius, with Frobenius system

$$(\underline{\nu} = (A\#\varphi)\circ\iota, \underline{e} = (1_A\#t_{(2)})1_A\otimes_A (1_A\#\overline{S}(t_{(1)}))1_A),$$

where $\iota:\ \underline{A\#H}\to A\#H$ is the inclusion map.

Proof. Applying Δ to the first tensor factor of (74), we see that

$$t_{(1)}\otimes t_{(2)}\otimes t_{(3)}\otimes t_{(4)}\otimes t_{(5)} = t_{(1)}\otimes t_{(2)}\otimes t_{(4)}\otimes t_{(3)}\otimes t_{(5)}. \tag{75}$$

For all $a\in A$ and $h\in H$, we have

$$\begin{aligned}
&(1_A\#t_{(2)})1_A\otimes_A (1_A\#\overline{S}(t_{(1)}))1_A(a\#h)1_A\\
=\;& (1_A\#t_{(2)})\otimes_A (1_A\#\overline{S}(t_{(1)}))(a\#h)1_A\\
=\;& (1_A\#t_{(2)})\otimes_A ((1_A\#\overline{S}(t_{(1)}))(a(h_{(1)}\cdot 1_A)\#h_{(2)}))1_A\\
=\;& (1_A\#t_{(3)})\otimes_A (\overline{S}(t_{(2)})\cdot(a(h_{(1)}\cdot 1_A))\#\overline{S}(t_{(1)})h_{(2)})1_A\\
=\;& \big(t_{(3)}\cdot(\overline{S}(t_{(2)})\cdot(a(h_{(1)}\cdot 1_A)))\#t_{(4)}\big)\otimes_A (1_A\#\overline{S}(t_{(1)})h_{(2)})1_A\\
\overset{(62)}{=}\;& \big((t_{(3)}\cdot 1_A)((t_{(4)}\overline{S}(t_{(2)}))\cdot(a(h_{(1)}\cdot 1_A)))\#t_{(5)}\big)\otimes_A (1_A\#\overline{S}(t_{(1)})h_{(2)})1_A\\
\overset{(75)}{=}\;& \big((t_{(4)}\cdot 1_A)((t_{(3)}\overline{S}(t_{(2)}))\cdot(a(h_{(1)}\cdot 1_A)))\#t_{(5)}\big)\otimes_A (1_A\#\overline{S}(t_{(1)})h_{(2)})1_A\\
=\;& \big((t_{(2)}\cdot 1_A)a(h_{(1)}\cdot 1_A)\#t_{(3)}\big)\otimes_A (1_A\#\overline{S}(t_{(1)})h_{(2)})1_A\\
=\;& \big(a(h_{(1)}\cdot 1_A)(t_{(2)}\cdot 1_A)\#t_{(3)}\big)\otimes_A (1_A\#\overline{S}(t_{(1)})h_{(2)})1_A\\
\overset{(72)}{=}\;& \big(a(h_{(1)}\cdot 1_A)((h_{(2)}t_{(2)})\cdot 1_A)\#h_{(3)}t_{(3)}\big)\otimes_A (1_A\#\overline{S}(t_{(1)}))1_A\\
=\;& (a(h_{(1)}\cdot 1_A)\#h_{(2)}t_{(2)})1_A\otimes_A (1_A\#\overline{S}(t_{(1)}))1_A\\
=\;& (a\#h)(1_A\#t_{(2)})\otimes_A (1_A\#\overline{S}(t_{(1)}))1_A\\
=\;& (a\#h)1_A(1_A\#t_{(2)})1_A\otimes_A (1_A\#\overline{S}(t_{(1)}))1_A.
\end{aligned}$$

Using the fact that φ is a left integral, we easily find that

$$\underline{\nu}((a\#h)1_A) = \langle\varphi, h_{(2)}\rangle a(h_{(1)}\cdot 1_A) = \langle\varphi, h\rangle a(1_H\cdot 1_A) \overset{(54)}{=} \langle\varphi, h\rangle a.$$

The left A-linearity of $\underline{\nu}$ is obvious, and the right A-linearity can be established as follows:

$$\begin{aligned}\underline{\nu}((a\#h)1_Ab) = \underline{\nu}((a\#h)b1_A) &= \underline{\nu}((a(h_{(1)}\cdot b)\#h_{(2)})1_A)\\ &= \langle\varphi,h_{(2)}\rangle a(h_{(1)}\cdot b) = \langle\varphi,h\rangle a(1_H\cdot b)\\ &\overset{(54)}{=} \langle\varphi,h\rangle ab = \underline{\nu}((a\#h)1_A)b.\end{aligned}$$

Finally,

$$\begin{aligned}&\underline{\nu}(1_A\#t_{(2)})1_A)((1_A\#\overline{S}(t_{(1)}))1_A) = (\langle\varphi,t_{(2)}\rangle 1_A\#\overline{S}(t_{(1)}))1_A\\ &\overset{(71)}{=} (1_A\#1_H)1_A = 1_A\#1_H;\\ &(1_A\#t_{(2)})\underline{\nu}((1_A\#\overline{S}(t_{(1)}))1_A) = (1_A\#t_{(2)})\langle\varphi,\overline{S}(t_{(1)})\rangle 1_A\\ &\overset{(71)}{=} (1_A\#1_H)1_A = 1_A\#1_H.\end{aligned}$$

□

Remark 6.5. It follows from (74) that t is cocommutative. Obviously (74) is satisfied if H is cocommutative.

7. Galois theory

Let (A,C,ψ) be a lax entwining structure, and consider the corresponding A-coring $\underline{\mathcal{C}} = \underline{A\otimes C}$. The aim is to give a structure theorem for lax entwined modules, based on the Galois theory for corings. To this end, we first study the grouplike elements of $\underline{\mathcal{C}}$.

Proposition 7.1. *Let (A,C,ψ) be a lax entwining structure, and $g\in G(C)$. The element $x = 1_\psi\otimes g^\psi$ satisfies the equation $\Delta(x) = x\otimes_A x$. x is grouplike in the following situations:*

- *(A,C,ψ) is a partial entwining structure;*
- *$C = H$ is a weak bialgebra, A is a right H-comodule algebra in the sense of* [1] *or* [5]*, and $\psi:\ H\otimes A\to A\otimes H$ is given by the formula $\psi(h\otimes a) = a_{[0]}\otimes ha_{[1]}$, and $g = 1_H$.*

Proof.

$$\begin{aligned}\Delta(x) &= (1_\psi\otimes(g^\psi)_{(1)})\otimes_A(1\otimes(g^\psi)_{(2)}) = (1_\psi 1_\Psi\otimes(g^\psi)^\Psi_{(1)})\otimes_A(1\otimes(g^\psi)_{(2)})\\ &\overset{(12)}{=} (1_{\psi\Psi}\otimes g^\Psi)\otimes_A(1\otimes g^\psi)\overset{(10)}{=}(1_\Psi 1_{\psi\Psi'}\otimes g^{\Psi\Psi'})\otimes_A(1\otimes g^\psi)\\ &= (1_\Psi\otimes g^\Psi)\otimes_A(1_\psi\otimes g^\psi) = x\otimes_A x.\end{aligned}$$

If (A,C,ψ) is a partial entwining structure, then

$$\varepsilon(x) = \epsilon(g^\psi)1_\psi \overset{(19)}{=} \epsilon(g)1 = 1.$$

If A is a comodule algebra over a weak bialgebra H, then A is, in particular, an H-comodule (see [1] or or [5]). Then $x = 1_{[0]}\otimes 1_{[1]}$, and $\varepsilon(x) = \epsilon(1_{[1]})1_{[0]} = 1$ (see also [7, Lemma 2.1]). □

The situation where $C = H$ is a weak bialgebra has been studied in [7]. Let us here focus attention to the situation where (A, C, ψ) is a partial entwining structure. We keep the notation from Proposition 7.1. Then $A \in \mathcal{M}(\psi)_A^C$, with coaction

$$\rho(a) = a_\psi \otimes g^\psi.$$

Let

$$T = A^{\mathrm{co}C} = \{b \in A \mid b1_\psi \otimes g^\psi = b_\psi \otimes g^\psi\}.$$

We have a morphism of corings

$$\mathrm{can} :\ A \otimes_T A \to \underline{A \otimes C},\ \mathrm{can}(a \otimes b) = ab_\psi \otimes g^\psi.$$

From [4, Sec. 1 and Prop. 3.8] and [18, Sec. 3] we obtain immediately the following.

Proposition 7.2. *Let (A, C, ψ) be a partial entwining structure, and $g \in G(C)$. We have a pair of adjoint functors (F, G) between the categories $\mathcal{M}_T$ and $\mathcal{M}(\psi)_A^C$. $G = (-)^{\mathrm{co}C}$, and $F(N) = N \otimes_T A$ with coaction $\rho(n \otimes_T a) = (n \otimes_T a_\psi) \otimes g^\psi$. The following conditions are equivalent:*

1. can *is an isomorphism and A is faithfully flat as a left T-module;*
2. *(F, G) is an equivalence of categories and A is flat as a left T-module;*
3. *$\underline{A \otimes C}$ is flat as a left A-module, and A is a projective generator of $\mathcal{M}(\psi)_A^C$.*

In this situation, we will say that A is a faithfully flat partial coalgebra-Galois extension of T.

From now on, we assume that C is finitely generated and projective as a k-module, with finite dual basis $\{(c_i, c_i^*) \mid i = 1, \ldots, n\}$. If there exists $g \in G(C)$, then C is a k-progenerator: C is a generator, because $\epsilon(g) = 1$ (see for example [10, I.1] for a discussion of (pro)generator modules). Suppose in addition that A is finitely generated and projective as a left T-module. Then can is an isomorphism if and only if its left dual

$${}^*\mathrm{can} :\ {}^*(\underline{A \otimes C}) \cong \underline{\#}(C, A) \to {}^*(A \otimes_T A) \cong {}_T\mathrm{End}(A)^{\mathrm{op}}$$

is an isomorphism. Viewed as a map $\underline{\#}(C, A) \to {}_T\mathrm{End}(A)^{\mathrm{op}}$, ${}^*\mathrm{can}$ is given by the formula ${}^*\mathrm{can}(f)(a) = a_\psi f(g^\psi)$. Composing ${}^*\mathrm{can}$ with the isomorphism ${}^*(\underline{A \otimes C})^{\mathrm{op}} \cong \underline{A^{\mathrm{op}}\#C^*}$ (see Proposition 5.9), we obtain an A^{op}-ring isomorphism $\theta :\ \underline{A^{\mathrm{op}}\#C^*} \to {}_T\mathrm{End}(A)$. We compute the map θ explicitly:

$$\begin{aligned}\theta((a\#c^*) \cdot 1)(b) &= \theta(1_R a\#(c^*)^R)(b) = \theta(\sum_i 1_\psi a\#\langle c^*, c_i^\psi\rangle c_i^*)(b)\\ &= \sum_i b_\Psi 1_\psi a\langle c^*, c_i^\psi\rangle\langle c_i^*, g^\Psi\rangle = b_\Psi 1_\psi a\langle c^*, g^{\Psi\psi}\rangle \overset{(10)}{=} b_\psi a\langle c^*, g^\psi\rangle.\end{aligned}$$

Recall (see [9, Sec. 3]) that we can associate a Morita context $(T, {}^*\underline{\mathcal{C}}, A, Q, \tau, \mu)$ to an A-coring $\underline{\mathcal{C}}$ with a fixed grouplike element x. We will now compute this Morita context for $\underline{\mathcal{C}} = \underline{A \otimes C}$ and $x = 1_\psi \otimes g^\psi$, in the case where (A, C, ψ) is a partial entwining structure. First recall that $Q = \{q \in$

${}^*\underline{\mathcal{C}} \mid c_{(1)}q(c_{(2)}) = q(c)x,\ \text{for all}\ c \in \underline{\mathcal{C}}\}$. We first compute Q as a submodule of $\underline{\#}(C,A)$. $q \in \underline{\#}(C,A)$ satisfies the equation

$$q(c) = 1_\psi q(c^\psi), \tag{76}$$

for all $c \in C$. Let φ be the map in ${}^*(\underline{A\otimes C})$ corresponding to q. For $\gamma = a1_\psi \otimes c^\psi \in \underline{A\otimes C}$, we have that

$$\Delta(\gamma) = (a1_{\psi\Psi} \otimes c^\Psi_{(1)}) \otimes_A (1 \otimes c^\psi_{(2)}),$$

hence $q \in Q$ if and only if

$$\begin{aligned}\gamma_{(1)}\varphi(\gamma_{(2)}) &= (a1_{\psi\Psi} \otimes c^\Psi_{(1)})q(c^\psi_{(2)}) = a1_{\psi\Psi}q(c^\psi_{(2)})_{\Psi'} \otimes c^{\Psi\Psi'}_{(1)}\\ &\overset{(10)}{=} a(1_\psi q(c^\psi_{(2)}))_\Psi \otimes c^\Psi_{(1)} \overset{(76)}{=} aq(c_{(2)})_\Psi \otimes c^\Psi_{(1)}\end{aligned}$$

equals

$$\varphi(\gamma)x = a1_\psi q(c^\psi)(1_\Psi \otimes g^\Psi) = aq(c)1_\psi \otimes g^\psi,$$

for all $a \in A$ and $c \in C$. We conclude that Q is the submodule of $\#(C,A)$ consisting of the maps q that satisfy (76) and

$$q(c_{(2)})_\psi \otimes c^\psi_{(1)} = q(c)1_\psi \otimes g^\psi, \tag{77}$$

for all $c \in C$.

Now we want to describe Q as a submodule of $\underline{A^{\text{op}}\#C^*} \cong \underline{\#}(C,A)^{\text{op}}$. Take $k = \sum_j a_j\#d^*_j \in \underline{A^{\text{op}}\#C^*}$ corresponding to $q \in \underline{\#}(C,A)^{\text{op}}$. Then

$$\sum_j a_j\#d^*_j = \sum_j 1_R a_j\#(d^*_j)^R, \tag{78}$$

where R is the map from the partial smash product structure corresponding to (A,C,ψ), cf. Theorem 5.7. Then $k \in Q$ if and only if

$$\sum_j a_{j\psi}\langle d^*_j, c_{(2)}\rangle \otimes c^\psi_{(1)} = \sum_j a_j\langle d^*_j, c\rangle 1_\psi \otimes g^\psi,$$

for all $c \in C$. This is equivalent to stating that

$$\begin{aligned}\sum_j a_{j\psi}\langle d^*_j, c_{(2)}\rangle\langle c^*, c^\psi_{(1)}\rangle &= \sum_{i,j} a_{j\psi}\langle d^*_j, c_{(2)}\rangle\langle c^*, c^\psi_i\rangle\langle c^*_i, c_{(1)}\rangle\\ &\overset{(49)}{=} \sum_j a_{jR}\langle d^*_j, c_{(2)}\rangle\langle (c^*)^R, c_{(1)}\rangle = \sum_j a_{jR}\langle (c^*)^R * d^*_j, c\rangle\end{aligned}$$

equals

$$\begin{aligned}\sum_j a_j 1_\psi\langle d^*_j, c\rangle\langle c^*, g^\psi\rangle &= \sum_{i,j} a_j 1_\psi\langle d^*_j, c\rangle\langle c^*, c^\psi_i\rangle\langle c^*_i, g\rangle\\ &\overset{(49)}{=} \sum_j a_j 1_R\langle d^*_j, c\rangle\langle (c^*)^R, g\rangle,\end{aligned}$$

for all $c \in C$ and $c^* \in C^*$. We conclude that Q consists of the elements $k = \sum_j a_j \# d_j^* \in A^{\rm op} \# C^*$ satisfying (78) and

$$\sum_j a_{jR} \#((c^*)^R * d_j^*) = \sum_j a_j 1_R \# \langle (c^*)^R, g\rangle d_j^*, \tag{79}$$

for all $c^* \in C^*$.
It follows from [9, Lemma 3.1] that Q is a $(\underline{\#}(C, A), T)$-bimodule. The left action is given by the multiplication in $\underline{\#}(C, A)$. We also know (cf. [9, Prop. 2.2]) that A is a $(T, \underline{\#}(C, A))$-bimodule, with right $\underline{\#}(C, A)$-action given by the formula $a \cdot f = a_\psi f(g^\psi)$. We have well-defined maps

$$\tau : \ A \otimes_{\underline{\#}(C,A)} Q \to T, \ \tau(a \otimes q) = a \cdot q = a_\psi q(g^\psi);$$

$$\mu : \ Q \otimes_T A \to \underline{\#}(C, A), \ \mu(q \otimes a)(c) = q(c)a.$$

$(T, \underline{\#}(C, A), A, Q, \tau, \mu)$ is a Morita context. The map τ is surjective if and only if there exists $q \in Q$ such that $q(g) = 1$.

Theorem 7.3. *Let (A, C, ψ) be a partial entwining structure, $g \in G(C)$, and assume that C is finitely generated and projective as a k-module. Then the following assertions are equivalent.*

1. *A is a faithfully flat partial coalgebra-Galois extension of T;*
2. *θ is an isomorphism and A is a left T-progenerator;*
3. *the Morita context $(T, \underline{\#}(C, A), A, Q, \tau, \mu)$ is strict;*
4. *(F, G) is an equivalence of categories.*

Proof. Since C is finitely generated projective as a k-module, $A \otimes C$ is finitely generated and projective as a left A-module. Being a direct factor of $A \otimes C$, $\underline{A \otimes C}$ is also finitely generated and projective as a left A-module. $\underline{A \otimes C}$ is a left A-generator since

$$\varepsilon(1_\psi \otimes g^\psi) = 1_\psi \epsilon(g^\psi) \overset{(14)}{=} 1\epsilon(g) = 1.$$

It follows that $\underline{A \otimes C}$ is a left A-progenerator, and the result then follows immediately from [4, Theorem 4.7]. □

We now consider the situation where $C = H$ is a finitely generated projective Hopf algebra, $g = 1_H$, A is a right partial H-comodule algebra, and ψ is given by the formula (23). We will compute the corresponding partial smash product structure $(A^{\rm op}, H^*, R)$ (see Theorem 5.7), and show that it comes from a left partial $H^{*{\rm cop}}$-action on $A^{\rm op}$, given by the formula

$$h^* \rightharpoonup a = \langle h^*, a_{[1]}\rangle a_{[0]}.$$

Proposition 7.4. *With notation and assumptions as above, $A^{\rm op}$ is a left partial $H^{*{\rm cop}}$-module algebra, and $(A^{\rm op}, H^*, R)$ is the corresponding smash product structure, as discussed in Section 6.*

Proof. We compute R using (49):

$$\begin{aligned} R(h^*\otimes a) &= \sum_i \langle h^*, h_i a_{[1]}\rangle a_{[0]}\otimes h_i^* \\ &= \sum_i \langle h^*_{(1)}, h_i\rangle\langle h^*_{(2)}, a_{[1]}\rangle a_{[0]}\otimes h_i^* = h^*_{(2)}\rightharpoonup a\otimes h^*_{(1)}. \end{aligned}$$

It follows that R is given by formula (51), starting from the Hopf algebra $H^{*\rm cop}$. From Theorem 5.7, it follows that $(A^{\rm op}, H^*, R)$ is a partial smash product structure. Hence, by definition, $A^{\rm op}$ is a left partial $H^{*\rm cop}$-module algebra. It is also possible to verify (52,62,54) directly. □

Assume that H is Frobenius. Then $H^{*\rm cop}$ is also Frobenius. Assume, moreover, that

$$a1_{[0]}\otimes 1_{[1]} = 1_{[0]}a\otimes 1_{[1]} \text{ and } \langle\varphi, hklm\rangle = \langle\varphi, hlkm\rangle,$$

for all $a\in A$ and $h,k,l,m\in H$. Then it follows from Proposition 6.4 that $\underline{A^{\rm op}\# H^{*\rm cop}}/A^{\rm op}$ is Frobenius. The Morita context $(T, \underline{\#}(H,A)\cong \underline{A^{\rm op}\# H^{*\rm cop}}, A, Q, \tau, \mu)$ is the Morita context associated to the $A^{\rm op}$-ring $\underline{A^{\rm op}\# H^{*\rm cop}}$, see [9, Theorem 3.5]. It follows from [9, Theorem 2.7] that $Q\cong A$ as k-modules. So we conclude that the Morita context is of the form $(T, \underline{A^{\rm op}\# H^{*\rm cop}}, A, A, \tau, \mu)$.

References

[1] G. Böhm, Doi-Hopf modules over weak Hopf algebras, *Comm. Algebra* **28** (2000), 4687–4698.

[2] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois properties, *Algebr. Representat. Theory* **5** (2002), 389–410.

[3] T. Brzeziński and R. Wisbauer, "Corings and comodules", *London Math. Soc. Lect. Note Ser.* **309**, Cambridge University Press, Cambridge, 2003.

[4] S. Caenepeel, *Galois corings from the descent theory point of view*, *Fields Inst. Comm* **43** (2004), 163–186.

[5] S. Caenepeel and E. De Groot, Modules over weak entwining structures, *Contemp. Math.* **267** (2000), 31–54.

[6] S. Caenepeel, E. De Groot, Corings applied to partial Galois theory, in "Proceedings of the International Conference on Mathematics and its Applications, ICMA 2004", S.L. Kalla and M.M. Chawla (Eds.), Kuwait University, Kuwait, 2005, 117–134.

[7] S. Caenepeel, E. De Groot, Galois theory for weak Hopf algebras, *Rev. Roumaine Math. Pures Appl.*, to appear.

[8] S. Caenepeel, G. Militaru and Shenglin Zhu, "Frobenius and separable functors for generalized module categories and nonlinear equations", *Lect. Notes in Math.* **1787**, Springer Verlag, Berlin, 2002.

[9] S. Caenepeel, J. Vercruysse and Shuanhong Wang, Morita theory for corings and cleft entwining structures, *J. Algebra* **276** (2004), 210–235.

[10] F. DeMeyer, E. Ingraham, Separable algebras over commutative rings, *Lecture Notes in Math.* **181**, Springer Verlag, Berlin, 1971.

[11] M. Dokuchaev, R. Exel, and P. Piccione, Partial representations and partial group algebras, *J. Algebra* **226** (2000), 251–268.

[12] M. Dokuchaev and R. Exel, Associativity of crossed products by partial actions, enveloping actions and partial representations, *Trans. Amer. Math. Soc.* **357** (2005), 1931–1952.
[13] M. Dokuchaev, M. Ferrero, and A. Pacques, Partial Galois theory of commutative rings, *J. Pure Appl. Algebra*, to appear.
[14] M. Dokuchaev, N. Zhukavets, On finite degree partial representations of groups, *J. Algebra* **274** (2004), 309–334.
[15] R. Exel, Twisted partial actions: a classification of regular C^*-algebraic bundels, *Proc. London Math. Soc.* **74** (1997), 417–443.
[16] M. E. Sweedler, The predual Theorem to the Jacobson-Bourbaki Theorem, *Trans. Amer. Math. Soc.* **213** (1975), 391–406.
[17] R. Wisbauer, Weak corings, *J. Algebra* **245** (2001), 123–160.
[18] R. Wisbauer, On Galois corings, in “Hopf algebras in non-commutative geometry and physics”, Caenepeel S. and Van Oystaeyen, F. (eds.), *Lect. Notes Pure Appl. Math.* **239**, Dekker, New York, 2004.

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