

DOI-HOPF MODULES AND YETTER-DRINFELD MODULES FOR QUASI-HOPF ALGEBRAS

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ABSTRACT. For a quasi-Hopf algebra H , a left H -comodule algebra \mathfrak{B} and a right H -module coalgebra C we will characterize the category of Doi-Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ in terms of modules. We will also show that for an H -bicomodule algebra \mathfrak{A} and an H -bimodule coalgebra C the category of generalized Yetter-Drinfeld modules ${}_{\mathfrak{A}}\mathcal{YD}(H)^C$ is isomorphic to a certain category of Doi-Hopf modules. Using this isomorphism we will transport the properties from the category of Doi-Hopf modules to the category of generalized Yetter-Drinfeld modules.

INTRODUCTION

Recall that the defining axioms for a quasi-bialgebra H are the same as for a bialgebra, with the coassociativity of the comultiplication replaced by a weaker property, called quasi-coassociativity: the comultiplication is coassociative up to conjugation by an invertible element $\Phi \in H \otimes H \otimes H$, called the reassociator. There are important differences with ordinary quasi-bialgebras: the definition of a quasi-bialgebra is not selfdual, and we cannot consider comodules over quasi-bialgebras, since they are not coassociative coalgebras. However, the category of (left or right) modules over a quasi-bialgebra is a monoidal category.

Using this categorical point of view, the category of relative Hopf modules has been introduced and studied in [6]. A right H -module coalgebra C is a coalgebra in the monoidal category \mathcal{M}_H , and a left $[C, H]$ -Hopf module is a left C -comodule in the monoidal category \mathcal{M}_H . A generalization of this concept was presented in [3]: replacing the right H -action by an action of a left H -comodule algebra, we can define the notion of Doi-Hopf module over a quasi-bialgebra. At this point, we have to mention that there is a philosophical problem with the introduction of H -comodule algebras: we cannot introduce them as algebras in the category of H -comodules, since this category does not exist, as we mentioned above. However, a formal definition of H -comodule (and H -bicomodule) algebra was given by Hausser and Nill in [14]. A more conceptual definition has been proposed in [3]. If \mathfrak{A} is an associative algebra then the category of $(\mathfrak{A} \otimes H, \mathfrak{A})$ -bimodules is monoidal. Moreover, $\mathfrak{A} \otimes H$ has a coalgebra structure within this monoidal category “compatible” with the unit element $1_{\mathfrak{A}} \otimes 1_H$ if and only if \mathfrak{A} is a right H -comodule algebra (for the complete statement see Proposition 1.2 below). Of course, a similar result holds

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for a left H -comodule algebra (see Proposition 1.3). Moreover, if H is a quasi-Hopf algebra then any H -bicomodule algebra can be viewed in two different (but twist equivalent) ways as a left (right) H -comodule algebra.

The aim of this paper is to study the category of Doi-Hopf modules over a quasi-Hopf algebra H , and its connections to the category of Yetter-Drinfeld modules. If H is a quasi-bialgebra, \mathfrak{B} a left H -comodule algebra and C a right H -module coalgebra then $\mathfrak{B} \otimes C$ is a \mathfrak{B} -coring (this means a coalgebra in the monoidal category of \mathfrak{B} -bimodules) and the category of right-left (H, \mathfrak{B}, C) -Hopf modules, denoted by ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, is isomorphic to the category of right comodules over the coring $\mathfrak{B} \otimes C$, conform [3, Theorem 5.4]. In particular, since $\mathfrak{B} \otimes C$ is flat as a left \mathfrak{B} -module we obtain that ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a Grothendieck category.

It was shown in [3, Proposition 5.2] that the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category of right modules over the generalized smash product $C^* \blacktriangleright \mathfrak{B}$, if C is finite dimensional. In Section 2 we look at the case where C is infinite dimensional. Following the methods developed in [9, 11, 19], we will present two characterizations of the category of Doi-Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$. We first introduce the notion of rational (right) $C^* \blacktriangleright \mathfrak{B}$ -module, and then we will show that the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to $\text{Rat}(\mathcal{M}_{C^* \blacktriangleright \mathfrak{B}})$, the category of rational (right) $C^* \blacktriangleright \mathfrak{B}$ -modules. We notice that, in the coassociative case, the notion of rational (right) $C^* \blacktriangleright \mathfrak{B}$ -module reduces to the notion of right $C^* \blacktriangleright \mathfrak{B}$ -module which is rational as a right C^* -module. Secondly, we will show that $\text{Rat}(\mathcal{M}_{C^* \blacktriangleright \mathfrak{B}})$ is equal to $\sigma_{C^* \blacktriangleright \mathfrak{B}}[C \otimes \mathfrak{B}]$, the smallest closed subcategory of $\mathcal{M}_{C^* \blacktriangleright \mathfrak{B}}$ containing $C \otimes \mathfrak{B}$ (see Theorem 2.6). In this way we recover that ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a Grothendieck category, a fortiori with enough injective objects. We will also introduce a generalized version of Koppinen's smash product [16], relate it to the generalized smash product, and characterize the category of Doi-Hopf modules as the full subcategory of modules over the Koppinen smash product, consisting of rational modules.

In Section 3, we will generalize a result from [8]. If H is a quasi-Hopf algebra, then an H -bicomodule algebra \mathbb{A} can be viewed in two different, but twist equivalent, ways as a right $H^{\text{op}} \otimes H$ -comodule algebra. To this end, we first prove that any left H -comodule algebra \mathfrak{B} can be turned into a right H^{op} -comodule algebra. So, by this correspondence, the two (twist equivalent) left $H \otimes H^{\text{op}}$ -comodule algebra structures on \mathbb{A} obtained in [7] provide two (twist equivalent) right $H^{\text{op}} \otimes H$ -comodule algebra structures on \mathbb{A} , which we will denote by \mathbb{A}^1 and \mathbb{A}^2 . If C is an H -bimodule coalgebra (that is, a coalgebra in the monoidal category of H -bimodules), then C becomes in a natural way a left $H^{\text{op}} \otimes H$ -module coalgebra, thus it makes sense to consider the Hopf module category ${}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C$. The main result of Section 3 asserts that the category of generalized left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to ${}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C$, and also to ${}_{\mathbb{A}^1}\mathcal{M}(H^{\text{op}} \otimes H)^C$. Using the first isomorphism, we will characterize ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ as a category of comodules over a coring. In Section 3.3, we will characterize the category of Yetter-Drinfeld modules as a category of modules.

1. PRELIMINARY RESULTS

1.1. Quasi-Hopf algebras. We work over a commutative field k . All algebras, linear spaces etc. will be over k ; unadorned \otimes means \otimes_k . Following Drinfeld [12], a quasi-bialgebra is a fourtuple $(H, \Delta, \varepsilon, \Phi)$, where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \rightarrow H \otimes H$ and $\varepsilon : H \rightarrow k$

are algebra homomorphisms satisfying the identities

$$(1.1) \quad (id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1},$$

$$(1.2) \quad (id \otimes \varepsilon)(\Delta(h)) = h, \quad (\varepsilon \otimes id)(\Delta(h)) = h,$$

for all $h \in H$; Φ has to be a normalized 3-cocycle, in the sense that

$$(1.3) \quad (1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi),$$

$$(1.4) \quad (id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1.$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras [20] we denote $\Delta(h) = h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention (summation implicitly understood):

$$(\Delta \otimes id)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and those of Φ^{-1} by small letters, namely

$$\Phi = X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \dots$$

$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \dots$$

A quasi-Hopf algebra is a quasi-bialgebra H equipped with an anti-automorphism S of the algebra H and elements $\alpha, \beta \in H$ such that

$$(1.5) \quad S(h_1)\alpha h_2 = \varepsilon(h)\alpha \text{ and } h_1\beta S(h_2) = \varepsilon(h)\beta,$$

$$(1.6) \quad X^1\beta S(X^2)\alpha X^3 = 1 \text{ and } S(x^1)\alpha x^2\beta S(x^3) = 1.,$$

for all $h \in H$.

The antipode of a quasi-Hopf algebra is determined uniquely up to a transformation $\alpha \mapsto U\alpha$, $\beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, with $U \in H$ invertible. The axioms imply that $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β , we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$ and $\varepsilon \circ S = \varepsilon$. The identities (1.2-1.4) also imply that

$$(\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1.$$

If $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ is a quasi-bialgebra or a quasi-Hopf algebra then H^{op} , H^{cop} and $H^{\text{op,cop}}$ are also quasi-bialgebras (respectively quasi-Hopf algebras), where the superscript ‘‘op’’ means opposite multiplication and ‘‘cop’’ means opposite comultiplication. The structure maps are obtained by putting $\Phi_{\text{op}} = \Phi^{-1}$, $\Phi_{\text{cop}} = (\Phi^{-1})^{321}$, $\Phi_{\text{op,cop}} = \Phi^{321}$, $S_{\text{op}} = S_{\text{cop}} = (S_{\text{op,cop}})^{-1} = S^{-1}$, $\alpha_{\text{op}} = S^{-1}(\beta)$, $\beta_{\text{op}} = S^{-1}(\alpha)$, $\alpha_{\text{cop}} = S^{-1}(\alpha)$, $\beta_{\text{cop}} = S^{-1}(\beta)$, $\alpha_{\text{op,cop}} = \beta$ and $\beta_{\text{op,cop}} = \alpha$.

The definition of a quasi-bialgebra H is designed in such a way that the categories of left and right representations over H are monoidal (see [15, 18] for the terminology). Let $(H, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. For are left (resp. right) H -modules U, V, W , the associativity constraints $a_{U,V,W}$ (resp. $\mathbf{a}_{U,V,W}$) : $(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ are given by the formulas

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w));$$

$$\mathbf{a}_{U,V,W}((u \otimes v) \otimes w) = (u \otimes (v \otimes w)) \cdot \Phi^{-1}.$$

Next we recall that the definition of a quasi-bialgebra or quasi-Hopf algebra is ‘‘twist covariant’’ in the following sense. An invertible element $F \in H \otimes H$ is called a gauge transformation or twist if $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$. If H is

a quasi-bialgebra or a quasi-Hopf algebra and $F = F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = G^1 \otimes G^2$, then we can define a new quasi-bialgebra (respectively quasi-Hopf algebra) H_F by keeping the multiplication, unit, counit (and antipode in the case of a quasi-Hopf algebra) of H and replacing the comultiplication, reassociator and the elements α and β by

$$(1.7) \quad \Delta_F(h) = F\Delta(h)F^{-1},$$

$$(1.8) \quad \Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1),$$

$$(1.9) \quad \alpha_F = S(G^1)\alpha G^2, \quad \beta_F = F^1\beta S(F^2).$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f \in H \otimes H$ such that

$$(1.10) \quad f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\text{cop}}(h)), \text{ for all } h \in H.$$

The element f can be computed explicitly. First set

$$\begin{aligned} A^1 \otimes A^2 \otimes A^3 \otimes A^4 &= (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}), \\ B^1 \otimes B^2 \otimes B^3 \otimes B^4 &= (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1), \end{aligned}$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \text{ and } \delta = B^1\beta S(B^4) \otimes B^2\beta S(B^3).$$

Then f and f^{-1} are given by the formulae

$$(1.11) \quad f = (S \otimes S)(\Delta^{\text{cop}}(x^1))\gamma\Delta(x^2\beta S(x^3)),$$

$$(1.12) \quad f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{cop}}(x^3)).$$

Furthermore the corresponding twisted reassociator (see (1.8)) is given by

$$(1.13) \quad \Phi_f = (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1).$$

1.2. Comodule and bicomodule algebras. A formal definition of comodule algebras over a quasi-bialgebra was given by Hausser and Nill [14].

Definition 1.1. Let H be a quasi-bialgebra. A right H -comodule algebra is a unital associative algebra \mathfrak{A} together with an algebra morphism $\rho : \mathfrak{A} \rightarrow \mathfrak{A} \otimes H$ and an invertible element $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ such that:

$$(1.14) \quad \begin{aligned} \Phi_\rho(\rho \otimes id)(\rho(\mathfrak{a})) &= (id \otimes \Delta)(\rho(\mathfrak{a}))\Phi_\rho, \text{ for all } \mathfrak{a} \in \mathfrak{A}, \\ (1_{\mathfrak{A}} \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi_\rho)(\Phi_\rho \otimes 1_H) & \end{aligned}$$

$$(1.15) \quad = (id \otimes id \otimes \Delta)(\Phi_\rho)(\rho \otimes id \otimes id)(\Phi_\rho),$$

$$(1.16) \quad (id \otimes \varepsilon) \circ \rho = id,$$

$$(1.17) \quad (id \otimes \varepsilon \otimes id)(\Phi_\rho) = (id \otimes id \otimes \varepsilon)(\Phi_\rho) = 1_{\mathfrak{A}} \otimes 1_H.$$

In a similar way, a left H -comodule algebra is a unital associative algebra \mathfrak{B} together with an algebra morphism $\lambda : \mathfrak{B} \rightarrow H \otimes \mathfrak{B}$ and an invertible element $\Phi_\lambda \in H \otimes H \otimes \mathfrak{B}$

such that the following relations hold:

$$(1.18) \quad (id \otimes \lambda)(\lambda(\mathbf{b}))\Phi_\lambda = \Phi_\lambda(\Delta \otimes id)(\lambda(\mathbf{b})), \quad \forall \mathbf{b} \in \mathfrak{B},$$

$$(1_H \otimes \Phi_\lambda)(id \otimes \Delta \otimes id)(\Phi_\lambda)(\Phi \otimes 1_{\mathfrak{B}})$$

$$(1.19) \quad = (id \otimes id \otimes \lambda)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_\lambda),$$

$$(1.20) \quad (\varepsilon \otimes id) \circ \lambda = id,$$

$$(1.21) \quad (id \otimes \varepsilon \otimes id)(\Phi_\lambda) = (\varepsilon \otimes id \otimes id)(\Phi_\lambda) = 1_H \otimes 1_{\mathfrak{B}}.$$

Observe that H is a left and right H -comodule algebra: take $\rho = \lambda = \Delta$, $\Phi_\rho = \Phi_\lambda = \Phi$.

If $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H -comodule algebra then

- $(\mathfrak{B}, \lambda \circ \tau_{H, \mathfrak{B}}, (\Phi_\lambda^{-1})^{321})$ is a right H^{cop} -comodule algebra;
- $(\mathfrak{B}^{\text{op}}, \lambda \circ \tau_{H, \mathfrak{B}}, \Phi_\lambda^{321})$ is a right $H^{\text{op, cop}}$ -comodule algebra;
- $(\mathfrak{B}^{\text{op}}, \lambda, \Phi_\lambda^{-1})$ is a left H^{op} -comodule algebra,

and vice versa. $\tau_{X, Y} : X \otimes Y \rightarrow Y \otimes X$ is the switch map mapping $x \otimes y$ to $y \otimes x$. From [14], we recall the notion of twist equivalence for the coaction on a right H -comodule algebra $(\mathfrak{A}, \rho, \Phi_\rho)$: if $\mathbb{V} \in \mathfrak{A} \otimes H$ is an invertible element such that

$$(1.22) \quad (id_{\mathfrak{A}} \otimes \varepsilon)(\mathbb{V}) = 1_{\mathfrak{A}}$$

then we can construct a new right H -comodule algebra $(\mathfrak{A}, \rho', \Phi_{\rho'})$ with

$$(1.23) \quad \rho'(\mathbf{a}) = \mathbb{V}\rho(\mathbf{a})\mathbb{V}^{-1}$$

and

$$(1.24) \quad \Phi_{\rho'} = (id_{\mathfrak{A}} \otimes \Delta)(\mathbb{V})\Phi_\rho(\rho \otimes id_H)(\mathbb{V}^{-1})(\mathbb{V}^{-1} \otimes 1_H).$$

We say that $(\mathfrak{A}, \rho, \Phi_\rho)$ and $(\mathfrak{A}, \rho', \Phi_{\rho'})$ are twist equivalent right H -comodule algebras.

We obviously have a similar notion for left H -comodule algebras. More precisely, if $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H -comodule algebra and $\mathbb{U} \in H \otimes \mathfrak{B}$ is an invertible element such that $(\varepsilon \otimes id_{\mathfrak{B}})(\mathbb{U}) = 1_{\mathfrak{B}}$ then we have a new left H -comodule algebra $(\mathfrak{B}, \lambda', \Phi_{\lambda'})$ with $\lambda'(\mathbf{b}) = \mathbb{U}\lambda(\mathbf{b})\mathbb{U}^{-1}$, for all $\mathbf{b} \in \mathfrak{B}$, and

$$\Phi_{\lambda'} = (1_H \otimes \mathbb{U})(id_H \otimes \lambda)(\mathbb{U})\Phi_\lambda(\Delta \otimes id_{\mathfrak{B}})\mathbb{U}^{-1}.$$

$(\mathfrak{B}, \lambda, \Phi_\lambda)$ and $(\mathfrak{B}, \lambda', \Phi_{\lambda'})$ are called twist equivalent.

For a right H -comodule algebra $(\mathfrak{A}, \rho, \Phi_\rho)$ we will use the following Sweedler-type notation, for any $\mathbf{a} \in \mathfrak{A}$.

$$\rho(\mathbf{a}) = \mathbf{a}_{(0)} \otimes \mathbf{a}_{(1)}, \quad (\rho \otimes id)(\rho(\mathbf{a})) = \mathbf{a}_{(0,0)} \otimes \mathbf{a}_{(0,1)} \otimes \mathbf{a}_{(1)} \text{ etc.}$$

Similarly, for a left H -comodule algebra $(\mathfrak{B}, \lambda, \Phi_\lambda)$, if $\mathbf{b} \in \mathfrak{B}$, we adopt the following notation:

$$\lambda(\mathbf{b}) = \mathbf{b}_{[-1]} \otimes \mathbf{b}_{[0]}, \quad (id \otimes \lambda)(\lambda(\mathbf{b})) = \mathbf{b}_{[-1]} \otimes \mathbf{b}_{[0,-1]} \otimes \mathbf{b}_{[0,0]} \text{ etc.}$$

In analogy with the notation for the reassociator Φ of H , we will write

$$\Phi_\rho = \tilde{X}_\rho^1 \otimes \tilde{X}_\rho^2 \otimes \tilde{X}_\rho^3 = \tilde{Y}_\rho^1 \otimes \tilde{Y}_\rho^2 \otimes \tilde{Y}_\rho^3 = \dots$$

and

$$\Phi_\rho^{-1} = \tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 \otimes \tilde{x}_\rho^3 = \tilde{y}_\rho^1 \otimes \tilde{y}_\rho^2 \otimes \tilde{y}_\rho^3 = \dots$$

We use a similar notation for the element Φ_λ of a left H -comodule algebra \mathfrak{B} .

If H is an associative bialgebra and \mathfrak{A} is an ordinary right H -comodule algebra, then $\mathfrak{A} \otimes H$ is a coalgebra in the monoidal category of \mathfrak{A} -bimodules. The quasi-bialgebra analog of this property was given in [3]. Let H be a quasi-bialgebra and \mathfrak{A} a unital associative algebra. We define by ${}_{\mathfrak{A} \otimes H} \mathcal{M}_{\mathfrak{A}}$ the category of \mathfrak{A} -bimodules and (H, \mathfrak{A}) -bimodules M such that $h \cdot (\mathfrak{a} \cdot m) = \mathfrak{a} \cdot (h \cdot m)$, for all $\mathfrak{a} \in \mathfrak{A}$, $h \in H$ and $m \in M$. Morphisms are left H -linear maps which are also \mathfrak{A} -bimodule maps. It is not hard to see that ${}_{\mathfrak{A} \otimes H} \mathcal{M}_{\mathfrak{A}}$ is a monoidal category. The tensor product is $\otimes_{\mathfrak{A}}$ and for any two objects $M, N \in {}_{\mathfrak{A} \otimes H} \mathcal{M}_{\mathfrak{A}}$, $M \otimes_{\mathfrak{A}} N$ is an object of ${}_{\mathfrak{A} \otimes H} \mathcal{M}_{\mathfrak{A}}$ via

$$(\mathfrak{a} \otimes h) \cdot (m \otimes_{\mathfrak{A}} n) \cdot \mathfrak{a}' = \mathfrak{a} \cdot (h_1 \cdot m) \otimes_{\mathfrak{A}} h_2 \cdot n \cdot \mathfrak{a}'$$

for all $m \in M$, $n \in N$, $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, and $h \in H$. The associativity constraints are given by

$$\begin{aligned} \underline{a}_{M,N,P} &: (M \otimes_{\mathfrak{A}} N) \otimes_{\mathfrak{A}} P \rightarrow M \otimes_{\mathfrak{A}} (N \otimes_{\mathfrak{A}} P), \\ \underline{a}_{M,N,P} &((m \otimes_{\mathfrak{A}} n) \otimes_{\mathfrak{A}} p) = X^1 \cdot m \otimes_{\mathfrak{A}} (X^2 \cdot n \otimes_{\mathfrak{A}} X^3 \cdot p); \end{aligned}$$

the unit object is \mathfrak{A} viewed as a trivial left H -module, and the left and right unit constraints are the usual ones. Now, the definition of a comodule algebra in terms of monoidal categories can be restated as follows.

Proposition 1.2. ([3, Proposition 3.8]) *Let H be a quasi-bialgebra and \mathfrak{A} an algebra, and view $\mathfrak{A} \otimes H$ in the canonical way as an object in ${}_{\mathfrak{A} \otimes H} \mathcal{M}$. There is a bijective correspondence between coalgebra structures $(\mathfrak{A} \otimes H, \underline{\Delta}, \underline{\varepsilon})$ in the monoidal category ${}_{\mathfrak{A} \otimes H} \mathcal{M}_{\mathfrak{A}}$ such that $\underline{\Delta}(1_{\mathfrak{A}} \otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{A}} \otimes 1_H) = 1_{\mathfrak{A}}$, and right H -comodule algebra structures on \mathfrak{A} .*

A similar result holds for a left H -comodule algebra \mathfrak{B} . Denote by ${}_{\mathfrak{B}} \mathcal{M}_{\mathfrak{B} \otimes H}$ the category whose objects are \mathfrak{B} -bimodules and (\mathfrak{B}, H) -bimodules M such that $(m \cdot \mathfrak{b}) \cdot h = (m \cdot h) \cdot \mathfrak{b}$ for all $m \in M$, $\mathfrak{b} \in \mathfrak{B}$ and $h \in H$. Morphisms are right H -linear maps which are also \mathfrak{B} -bimodule maps. We can easily check that ${}_{\mathfrak{B}} \mathcal{M}_{\mathfrak{B} \otimes H}$ is a monoidal category with tensor product $\otimes_{\mathfrak{B}}$ given via Δ , this means

$$\mathfrak{b} \cdot (m \otimes_{\mathfrak{B}} n) \cdot (\mathfrak{b}' \otimes h) := \mathfrak{b} \cdot m \cdot h_1 \otimes_{\mathfrak{B}} (n \cdot h_2) \cdot \mathfrak{b}'$$

for all $M, N \in {}_{\mathfrak{B}} \mathcal{M}_{\mathfrak{B} \otimes H}$, $m \in M$, $n \in N$, $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$ and $h \in H$. The associativity constraints are given by

$$\begin{aligned} \underline{a}'_{M,N,P} &: (M \otimes_{\mathfrak{B}} N) \otimes_{\mathfrak{B}} P \rightarrow M \otimes_{\mathfrak{B}} (N \otimes_{\mathfrak{B}} P), \\ \underline{a}'_{M,N,P} &((m \otimes_{\mathfrak{B}} n) \otimes_{\mathfrak{B}} p) = m \cdot x^1 \otimes_{\mathfrak{B}} (n \cdot x^2 \otimes_{\mathfrak{B}} p \cdot x^3), \end{aligned}$$

the unit object is \mathfrak{B} viewed as a right H -module via ε , and the left and right unit constraints are the usual ones.

Proposition 1.3. *Let H be a quasi-bialgebra and \mathfrak{B} an algebra, and view $\mathfrak{B} \otimes H$ in the canonical way as an object in ${}_{\mathfrak{B} \otimes H} \mathcal{M}$. There is a bijective correspondence between coalgebra structures $(\mathfrak{B} \otimes H, \underline{\Delta}, \underline{\varepsilon})$ in the monoidal category ${}_{\mathfrak{B} \otimes H} \mathcal{M}_{\mathfrak{B}}$ such that $\underline{\Delta}(1_{\mathfrak{B}} \otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{B}} \otimes 1_H) = 1_{\mathfrak{B}}$, and left H -comodule algebra structures on \mathfrak{B} .*

Proof. Since the proof is similar to the one given in [3, Proposition 3.8], we restrict ourselves to a description of the correspondence.

Suppose that $(\mathfrak{B} \otimes H, \underline{\Delta}, \underline{\varepsilon})$ is a coalgebra in ${}_{\mathfrak{B} \otimes H} \mathcal{M}_{\mathfrak{B}}$ such that $\underline{\Delta}(1_{\mathfrak{B}} \otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{B}} \otimes 1_H) = 1_{\mathfrak{B}}$. Write

$$\underline{\Delta}(1_{\mathfrak{B}} \otimes 1_H) = (1_{\mathfrak{B}} \otimes \tilde{X}_{\lambda}^1) \otimes_{\mathfrak{B}} (\tilde{X}_{\lambda}^3 \otimes \tilde{X}_{\lambda}^2),$$

and consider $\Phi_\lambda = \tilde{X}_\lambda^1 \otimes \tilde{X}_\lambda^2 \otimes \tilde{X}_\lambda^3$. Define $\lambda : \mathfrak{B} \rightarrow H \otimes \mathfrak{B}$ by

$$\lambda(b) = \tau_{\mathfrak{B}, H}(b \cdot (1_{\mathfrak{B}} \otimes 1_H)),$$

for all $b \in \mathfrak{B}$. Then $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H -comodule algebra.

Conversely, if $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H -comodule algebra then $\mathfrak{B} \otimes H \in {}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B} \otimes H}$ via

$$b \cdot (b' \otimes h) \cdot (b'' \otimes h') = b_{[0]}b'b'' \otimes b_{[-1]}hh'.$$

Moreover, $\mathfrak{B} \otimes H$ is a coalgebra in ${}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B} \otimes H}$, with comultiplication and counit given by the formulas

$$\begin{aligned} \underline{\Delta}(b \otimes h) &= (1_{\mathfrak{B}} \otimes \tilde{X}_\lambda^1 h_1) \otimes_{\mathfrak{B}} (\tilde{X}_\lambda^3 b \otimes \tilde{X}_\lambda^2 h_2), \\ \underline{\varepsilon}(b \otimes h) &= \varepsilon(h)b, \end{aligned}$$

for all $b \in \mathfrak{B}$ and $h \in H$. □

Let \mathfrak{B} be a left H -comodule algebra, and consider the elements \tilde{p}_λ and \tilde{q}_λ in $H \otimes \mathfrak{B}$ given by the following formulas:

$$(1.25) \quad \tilde{p}_\lambda = \tilde{p}_\lambda^1 \otimes \tilde{p}_\lambda^2 = \tilde{X}_\lambda^2 S^{-1}(\tilde{X}_\lambda^1 \beta) \otimes \tilde{X}_\lambda^3, \quad \tilde{q}_\lambda = S(\tilde{x}_\lambda^1) \alpha \tilde{x}_\lambda^2 \otimes \tilde{x}_\lambda^3.$$

Then we have the following formulas, for all $b \in \mathfrak{B}$ (see [14]):

$$(1.26) \quad \lambda(b_{[0]})\tilde{p}_\lambda[S^{-1}(b_{[-1]}) \otimes 1_{\mathfrak{B}}] = \tilde{p}_\lambda[1_H \otimes b],$$

$$(1.27) \quad [S(b_{[-1]}) \otimes 1_H]\tilde{q}_\lambda\lambda(b_{[0]}) = [1 \otimes b]\tilde{q}_\lambda,$$

$$(1.28) \quad \lambda(\tilde{q}_\lambda^2)\tilde{p}_\lambda[S^{-1}(\tilde{q}_\lambda^1) \otimes 1_{\mathfrak{B}}] = 1_H \otimes 1_{\mathfrak{B}},$$

$$(1.29) \quad [S(\tilde{p}_\lambda^1) \otimes 1_{\mathfrak{B}}]\tilde{q}_\lambda\lambda(\tilde{p}_\lambda^2) = 1_H \otimes 1_{\mathfrak{B}},$$

$$(1.30) \quad \begin{aligned} &\Phi_\lambda^{-1}(id_{\mathfrak{B}} \otimes \lambda)(\tilde{p}_\lambda)(1_{\mathfrak{B}} \otimes \tilde{p}_\lambda) \\ &= (\Delta \otimes id_{\mathfrak{B}})(\lambda(\tilde{X}_\lambda^3)\tilde{p}_\lambda)[S^{-1}(\tilde{X}_\lambda^2 g^2) \otimes S^{-1}(\tilde{X}_\lambda^1 g^1) \otimes 1_{\mathfrak{B}}], \end{aligned}$$

$$(1.31) \quad \begin{aligned} &(1_H \otimes \tilde{q}_\lambda)(id_H \otimes \lambda)(\tilde{q}_\lambda)\Phi_\lambda \\ &= [S(\tilde{x}_\lambda^2) \otimes S(\tilde{x}_\lambda^1) \otimes 1_{\mathfrak{B}}][f \otimes 1_{\mathfrak{B}}](\Delta \otimes id_{\mathfrak{B}})(\tilde{q}_\lambda\lambda(\tilde{x}_\lambda^3)). \end{aligned}$$

Bicomodule algebras were introduced by Hausser and Nill in [14], under the name “quasi-commuting pair of H -coactions”.

Definition 1.4. Let H be a quasi-bialgebra. An H -bicomodule algebra \mathbb{A} is a quintuple $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda, \rho})$, where λ and ρ are left and right H -coactions on \mathbb{A} , and where $\Phi_\lambda \in H \otimes H \otimes \mathbb{A}$, $\Phi_\rho \in \mathbb{A} \otimes H \otimes H$ and $\Phi_{\lambda, \rho} \in H \otimes \mathbb{A} \otimes H$ are invertible elements, such that

- $(\mathbb{A}, \lambda, \Phi_\lambda)$ is a left H -comodule algebra,
- $(\mathbb{A}, \rho, \Phi_\rho)$ is a right H -comodule algebra,
- the following compatibility relations hold, for all $u \in \mathbb{A}$:

$$(1.32) \quad \begin{aligned} \Phi_{\lambda, \rho}(\lambda \otimes id)(\rho(u)) &= (id \otimes \rho)(\lambda(u))\Phi_{\lambda, \rho} \\ (1_H \otimes \Phi_{\lambda, \rho})(id \otimes \lambda \otimes id)(\Phi_{\lambda, \rho})(\Phi_\lambda \otimes 1_H) \end{aligned}$$

$$(1.33) \quad \begin{aligned} &= (id \otimes id \otimes \rho)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_{\lambda, \rho}) \\ (1_H \otimes \Phi_\rho)(id \otimes \rho \otimes id)(\Phi_{\lambda, \rho})(\Phi_{\lambda, \rho} \otimes 1_H) \end{aligned}$$

$$(1.34) \quad = (id \otimes id \otimes \Delta)(\Phi_{\lambda, \rho})(\lambda \otimes id \otimes id)(\Phi_\rho).$$

It was pointed out in [14] that the following additional relations hold in an H -bicomodule algebra \mathbb{A} :

$$(id_H \otimes id_{\mathbb{A}} \otimes \varepsilon)(\Phi_{\lambda, \rho}) = 1_H \otimes 1_{\mathbb{A}}, \quad (\varepsilon \otimes id_{\mathbb{A}} \otimes id_H)(\Phi_{\lambda, \rho}) = 1_{\mathbb{A}} \otimes 1_H.$$

As a first example, take $\mathbb{A} = H$, $\lambda = \rho = \Delta$ and $\Phi_\lambda = \Phi_\rho = \Phi_{\lambda,\rho} = \Phi$.

Let $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ be an H -bicomodule algebra; it is not hard to show that

- $(\mathbb{A}, \rho \circ \tau, \lambda \circ \tau, (\Phi_\rho^{-1})^{321}, (\Phi_\lambda^{-1})^{321}, (\Phi_{\lambda,\rho}^{-1})^{321})$ is an H^{cop} -bicomodule algebra,
- $(\mathbb{A}^{\text{op}}, \rho \circ \tau, \lambda \circ \tau, \Phi_\rho^{321}, \Phi_\lambda^{321}, \Phi_{\lambda,\rho}^{321})$ is an $H^{\text{op,cop}}$ -bicomodule algebra,
- $(\mathbb{A}^{\text{op}}, \lambda, \rho, \Phi_\lambda^{-1}, \Phi_\rho^{-1}, \Phi_{\lambda,\rho}^{-1})$ is an H^{op} -bicomodule algebra.

We will use the following notation:

$$\Phi_{\lambda,\rho} = \Theta^1 \otimes \Theta^2 \otimes \Theta^3 = \bar{\Theta}^1 \otimes \bar{\Theta}^2 \otimes \bar{\Theta}^3;$$

$$\Phi_{\lambda,\rho}^{-1} = \theta^1 \otimes \theta^2 \otimes \theta^3 = \bar{\theta}^1 \otimes \bar{\theta}^2 \otimes \bar{\theta}^3.$$

Let H be a quasi-Hopf algebra and \mathbb{A} an H -bicomodule algebra. We define two left $H \otimes H^{\text{op}}$ -coactions $\lambda_1, \lambda_2 : \mathbb{A} \rightarrow (H \otimes H^{\text{op}}) \otimes \mathbb{A}$ on \mathbb{A} , as follows:

$$\begin{aligned} \lambda_1(u) &= (u_{<0>[-1]} \otimes S^{-1}(u_{<1>})) \otimes u_{<0>[0]}, \\ \lambda_2(u) &= (u_{[-1]} \otimes S^{-1}(u_{[0]<1>})) \otimes u_{[0]<0>}, \end{aligned}$$

for all $u \in \mathbb{A}$. We also consider the following elements $\Phi_{\lambda_1}, \Phi_{\lambda_2} \in (H \otimes H^{\text{op}}) \otimes (H \otimes H^{\text{op}}) \otimes \mathbb{A}$:

$$\begin{aligned} \Phi_{\lambda_1} &= \left(\Theta^1 \tilde{X}_\lambda^1(\tilde{x}_\rho^1)_{[-1]_1} \otimes S^{-1}(\tilde{x}_\rho^2 g^2) \right) \\ &\quad \otimes \left(\Theta_{[-1]}^2 \tilde{X}_\lambda^2(\tilde{x}_\rho^1)_{[-1]_2} \otimes S^{-1}(\Theta^3 \tilde{x}_\rho^2 g^1) \right) \otimes \Theta_{[0]}^2 \tilde{X}_\lambda^3(\tilde{x}_\rho^1)_{[0]}, \\ \Phi_{\lambda_2} &= \left(\tilde{Y}_\lambda^1 \otimes S^{-1}(\theta^3 \tilde{y}_\rho^3(\tilde{Y}_\lambda^3)_{<1>_2} g^2) \right) \\ &\quad \otimes \left(\theta^1 \tilde{Y}_\lambda^2 \otimes S^{-1}(\theta_{<1>}^2 \tilde{y}_\rho^2(\tilde{Y}_\lambda^3)_{<1>_1} g^1) \right) \otimes \theta_{<0>}^2 \tilde{y}_\rho^1(\tilde{Y}_\lambda^3)_{<0>}. \end{aligned}$$

It was proved in [7] that $(\mathbb{A}, \lambda_1, \Phi_{\lambda_1})$ and $(\mathbb{A}, \lambda_2, \Phi_{\lambda_2})$ are twist equivalent left $H \otimes H^{\text{op}}$ -comodule algebras. In particular, if H is a quasi-Hopf algebra then the notion of H -bicomodule algebra can be restated in terms of monoidal categories. In Section 3 we will see that \mathbb{A} can be also viewed in two twist equivalent ways as a right $H^{\text{op}} \otimes H$ -comodule algebra.

2. DOI-HOPF MODULES AND RATIONALITY PROPERTIES

2.1. Doi-Hopf modules. The category of left (right) modules over a quasi-bialgebra is monoidal. A coalgebra in ${}_H\mathcal{M}$ (resp. \mathcal{M}_H) is called a left (right) H -module coalgebra. Thus a left H -module coalgebra is a left H -module C together with a comultiplication $\underline{\Delta} : C \rightarrow C \otimes C$ and a counit $\underline{\varepsilon} : C \rightarrow k$ such that

$$(2.1) \quad \Phi(\underline{\Delta} \otimes id_C)(\underline{\Delta}(c)) = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)),$$

$$(2.2) \quad \underline{\Delta}(h \cdot c) = h_1 \cdot c_1 \otimes h_2 \cdot c_2,$$

$$(2.3) \quad \underline{\varepsilon}(h \cdot c) = \varepsilon(h)\underline{\varepsilon}(c),$$

for all $c \in C$ and $h \in H$. Similarly, a right H -module coalgebra C is a right H -module together with a comultiplication $\underline{\Delta} : C \rightarrow C \otimes C$ and a counit $\underline{\varepsilon} : C \rightarrow k$, satisfying the following relations

$$(2.4) \quad (\underline{\Delta} \otimes id_C)(\underline{\Delta}(c))\Phi^{-1} = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)),$$

$$(2.5) \quad \underline{\Delta}(c \cdot h) = c_1 \cdot h_1 \otimes c_2 \cdot h_2,$$

$$(2.6) \quad \underline{\varepsilon}(c \cdot h) = \underline{\varepsilon}(c)\varepsilon(h),$$

for all $c \in C$ and $h \in H$. Here we used the Sweedler-type notation

$$\underline{\Delta}(c) = c_{\underline{1}} \otimes c_{\underline{2}}, \quad (\underline{\Delta} \otimes id_C)(\underline{\Delta}(c)) = c_{(\underline{1},\underline{1})} \otimes c_{(\underline{1},\underline{2})} \otimes c_{\underline{2}} \quad \text{etc.}$$

It is easy to see that a left H -module coalgebra C is in a natural way a right H^{op} -module coalgebra (and vice versa).

Let H be a quasi-bialgebra and C a right H -module coalgebra. A left $[C, H]$ -Hopf module is a left C -comodule in the monoidal category \mathcal{M}_H . This definition was generalized in [3].

Definition 2.1. Let H be a quasi-bialgebra over a field k , C a right H -module coalgebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H -comodule algebra. A right-left (H, \mathfrak{B}, C) -Hopf module (or Doi-Hopf module) is a k -module M , with the following additional structure: M is right \mathfrak{B} -module (the right action of \mathfrak{b} on m is denoted by $m \cdot \mathfrak{b}$), and we have a k -linear map $\lambda_M : M \rightarrow C \otimes M$, such that the following relations hold, for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$:

$$(2.7) \quad (\underline{\Delta} \otimes id_M)(\lambda_M(m)) = (id_C \otimes \lambda_M)(\lambda_M(m))\Phi_\lambda,$$

$$(2.8) \quad (\underline{\varepsilon} \otimes id_M)(\lambda_M(m)) = m,$$

$$(2.9) \quad \lambda_M(m \cdot \mathfrak{b}) = m_{\{-1\}} \cdot \mathfrak{b}_{[-1]} \otimes m_{\{0\}} \cdot \mathfrak{b}_{[0]}.$$

As usual, we use the Sweedler-type notation $\lambda_M(m) = m_{\{-1\}} \otimes m_{\{0\}}$. ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is the category of right-left (H, \mathfrak{B}, C) -Hopf modules and right \mathfrak{B} -linear, left C -colinear k -linear maps.

Let M be a right \mathfrak{B} -module; then $C \otimes M$ is a right-left (H, \mathfrak{B}, C) -Hopf module, with structure maps given by the following formulas

$$(2.10) \quad (c \otimes m) \cdot \mathfrak{b} = c \cdot \mathfrak{b}_{[-1]} \otimes m \cdot \mathfrak{b}_{[0]},$$

$$(2.11) \quad \lambda_{C \otimes M}(c \otimes m) = c_{\underline{1}} \cdot \tilde{x}_\lambda^1 \otimes c_{\underline{2}} \cdot \tilde{x}_\lambda^2 \otimes m \cdot \tilde{x}_\lambda^3,$$

for all $c \in C$, $\mathfrak{b} \in \mathfrak{B}$ and $m \in M$. We obtain a functor $\mathcal{F} = C \otimes \bullet : \mathcal{M}_{\mathfrak{B}} \rightarrow {}^C\mathcal{M}(H)_{\mathfrak{B}}$. The functor \mathcal{F} sends a morphism ϑ to $id_C \otimes \vartheta$. In particular, $C \otimes \mathfrak{B} \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$, via the structure maps

$$(c \otimes \mathfrak{b}) \cdot \mathfrak{b}' = c \cdot \mathfrak{b}'_{[-1]} \otimes \mathfrak{b}\mathfrak{b}'_{[0]},$$

$$\lambda_{C \otimes \mathfrak{B}}(c \otimes \mathfrak{b}) = c_{\underline{1}} \cdot \tilde{x}_\lambda^1 \otimes c_{\underline{2}} \cdot \tilde{x}_\lambda^2 \otimes \mathfrak{b}\tilde{x}_\lambda^3,$$

for all $c \in C$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$.

The functor \mathcal{F} has a left and a right adjoint, so it is an exact functor.

Proposition 2.2. *Let H be a quasi-bialgebra, \mathfrak{B} a left H -comodule algebra and C a right H -module coalgebra. Then the functor $\mathcal{F} = C \otimes \bullet$ is a right adjoint of the forgetful functor*

$${}^C\mathcal{U} : {}^C\mathcal{M}(H)_{\mathfrak{B}} \rightarrow \mathcal{M}_{\mathfrak{B}},$$

and a left adjoint of the functor

$$\text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, \bullet) : {}^C\mathcal{M}(H)_{\mathfrak{B}} \rightarrow \mathcal{M}_{\mathfrak{B}}$$

defined as follows. For $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$, $\text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, M)$ is a right \mathfrak{B} -module via the formula

$$(\eta \cdot \mathfrak{b})(c \otimes \mathfrak{b}') = \eta(c \otimes \mathfrak{b}\mathfrak{b}'),$$

for all $\eta \in \text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, M)$, $c \in C$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. For a morphism $\kappa : M \rightarrow N$ in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, we let

$$\text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, \kappa)(v) = \kappa \circ v,$$

for all $v \in \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(C \otimes \mathfrak{B}, M)$.

Proof. Let M be a right-left (H, \mathfrak{B}, C) -Hopf module and N a right \mathfrak{B} -module. Define

$$\xi_{M,N} : \text{Hom}_{\mathfrak{B}}(M, N) \rightarrow \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(M, C \otimes N), \quad \xi_{M,N}(\varsigma)(m) = m_{\{-1\}} \otimes \varsigma(m_{\{0\}}),$$

for all $\varsigma \in \text{Hom}_{\mathfrak{B}}(M, N)$ and $m \in M$, and

$$\zeta_{M,N} : \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(M, C \otimes N) \rightarrow \text{Hom}_{\mathfrak{B}}(M, N), \quad \zeta_{M,N}(\chi)(m) = (\varepsilon \otimes id_N)(\chi(m)),$$

for all $\chi \in \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(M, C \otimes N)$ and $m \in M$. It is not hard to see that $\xi_{M,N}$ and $\zeta_{M,N}$ are well-defined natural transformations that are inverse to each other.

For $M \in \mathcal{M}_{\mathfrak{B}}$ and $N \in {}^{\mathcal{C}}\mathcal{M}(H)_{\mathfrak{B}}$, we define

$$\xi'_{M,N} : \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(C \otimes M, N) \rightarrow \text{Hom}_{\mathfrak{B}}(M, \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(C \otimes \mathfrak{B}, N))$$

by

$$\xi'_{M,N}(\varsigma')(m)(c \otimes \mathfrak{b}) = \varsigma'(c \otimes m \cdot \mathfrak{b})$$

and

$$\zeta'_{M,N} : \text{Hom}_{\mathfrak{B}}(M, \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(C \otimes \mathfrak{B}, N)) \rightarrow \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(C \otimes M, N)$$

by

$$\zeta'_{M,N}(\chi')(c \otimes m) = \chi'(m)(c \otimes 1_{\mathfrak{B}}),$$

for all $\varsigma' \in \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(C \otimes M, N)$, $\chi' \in \text{Hom}_{\mathfrak{B}}(M, \text{Hom}_{\mathfrak{B}}^{\mathcal{C}}(C \otimes \mathfrak{B}, N))$, $m \in M$, $c \in C$ and $\mathfrak{b} \in \mathfrak{B}$. Then ξ' and ζ' are well-defined natural transformations that are inverse to each other. \square

2.2. Doi-Hopf modules and comodules over a coring. It was proved in [3] that the category of right-left Doi-Hopf modules is isomorphic to the category of right comodules over a suitable \mathfrak{B} -coring. For a general treatment of the theory of corings, we refer to [1, 2].

Let R be a ring with unit. An R -coring \mathcal{C} is an R -bimodule together with two R -bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_R \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow R$ such that the usual coassociativity and counit properties hold. A right \mathcal{C} -comodule is a right R -module together with a right R -linear map $\rho_M^r : M \rightarrow M \otimes_R \mathcal{C}$ such that

$$\begin{aligned} (\rho_M^r \otimes_R id_{\mathcal{C}}) \circ \rho_M^r &= (id_M \otimes_R \Delta_{\mathcal{C}}) \circ \rho_M^r, \\ (id_M \otimes_R \varepsilon_{\mathcal{C}}) \circ \rho_M^r &= id_M. \end{aligned}$$

A map $h : M \rightarrow N$ between two right \mathcal{C} -comodules is called right \mathcal{C} -colinear if h is right R -linear and $\rho_N^r \circ h = (h \otimes_R id_{\mathcal{C}}) \circ \rho_M^r$. $\mathcal{M}^{\mathcal{C}}$ will be the category of right \mathcal{C} -comodules and right \mathcal{C} -comodule maps. It is well-known that $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category (in particular with enough injective objects) if \mathcal{C} is flat as a left R -module. The category ${}^{\mathcal{C}}\mathcal{M}$ of left \mathcal{C} -comodules and left \mathcal{C} -comodule maps can be introduced in a similar way.

Let H be a quasi-bialgebra, $(\mathfrak{B}, \lambda, \Phi_{\lambda})$ a left H -comodule algebra and C a right H -module coalgebra. It was proved in [3] that $\mathcal{C} = \mathfrak{B} \otimes C$ is a \mathfrak{B} -coring. More precisely, \mathcal{C} is a \mathfrak{B} -bimodule via

$$\mathfrak{b} \cdot (\mathfrak{b}' \otimes c) = \mathfrak{b}\mathfrak{b}' \otimes c \text{ and } (\mathfrak{b} \otimes c) \cdot \mathfrak{b}' = \mathfrak{b}\mathfrak{b}'_{[0]} \otimes c \cdot \mathfrak{b}'_{[-1]},$$

and the comultiplication and counit are given by

$$\Delta_{\mathcal{C}}(\mathfrak{b} \otimes c) = (\mathfrak{b}\tilde{x}_{\lambda}^3 \otimes c_2 \cdot \tilde{x}_{\lambda}^2) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_1 \cdot \tilde{x}_{\lambda}^1)$$

and

$$\varepsilon_C(c \otimes \mathfrak{b}) = \underline{\varepsilon}(c)\mathfrak{b},$$

for all $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$ and $c \in C$. Then we have an isomorphism of categories (see [3, Theorem 5.4])

$${}^C\mathcal{M}(H)_{\mathfrak{B}} \cong \mathcal{M}^C.$$

C is free, and therefore flat, as a left \mathfrak{B} -module, and we conclude that ${}^C\mathcal{M}(H)_{\mathfrak{B}} \cong \mathcal{M}^C$ is a Grothendieck category, and has therefore enough injectives (see [13, Prop. 1.2] or [2, 18.14]).

2.3. Doi-Hopf modules and the generalized smash product. We now want to discuss when the category of Doi-Hopf modules is isomorphic to a module category. In the case where C is finite dimensional, this was already done in [3, Proposition 5.2]. Let us explain this more precisely.

Let H be a quasi-bialgebra. A left H -module algebra A is an algebra in the monoidal category ${}_H\mathcal{M}$. This means that A is a left H -module with a multiplication $A \otimes A \rightarrow A$ and a unit element 1_A satisfying the following conditions:

$$(2.12) \quad (aa')a'' = (X^1 \cdot a)[(X^2 \cdot a')(X^3 \cdot a'')],$$

$$(2.13) \quad h \cdot (aa') = (h_1 \cdot a)(h_2 \cdot a'),$$

$$(2.14) \quad h \cdot 1_A = \varepsilon(h)1_A,$$

for all $a, a', a'' \in A$ and $h \in H$. Following [3], we can define the generalized smash product of a left H -module algebra A and a left H -comodule algebra \mathfrak{B} : $A \blacktriangleright \blacktriangleleft \mathfrak{B} = A \otimes B$ as a vector space, with multiplication

$$(a \blacktriangleright \blacktriangleleft \mathfrak{b})(a' \blacktriangleright \blacktriangleleft \mathfrak{b}') = (\tilde{x}_\lambda^1 \cdot a)(\tilde{x}_\lambda^2 \mathfrak{b}_{[-1]} \cdot a') \blacktriangleright \blacktriangleleft \tilde{x}_\lambda^3 \mathfrak{b}_{[0]} \mathfrak{b}',$$

for all $a, a' \in A$, $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. $A \blacktriangleright \blacktriangleleft \mathfrak{B}$ is an associative algebra with unit $1_A \blacktriangleright \blacktriangleleft 1_B$.

The linear dual C^* of a right H -module coalgebra C is a left H -module algebra. The multiplication is the convolution, the unit is $\underline{\varepsilon}$, and the left H -action is given by the formula $(h \cdot c^*)(c) = c^*(c \cdot h)$, for all $h \in H$, $c^* \in C^*$ and $c \in C$. So we can consider the generalized smash product algebra $C^* \blacktriangleright \blacktriangleleft \mathfrak{B}$. Moreover, we have a functor

$$\mathfrak{G} : {}^C\mathcal{M}(H)_{\mathfrak{B}} \rightarrow \mathcal{M}_{C^* \blacktriangleright \blacktriangleleft \mathfrak{B}}, \quad \mathfrak{G}(M) = M,$$

with right $C^* \blacktriangleright \blacktriangleleft \mathfrak{B}$ -action given by

$$m \cdot (c^* \blacktriangleright \blacktriangleleft \mathfrak{b}) = c^*(m_{\{-1\}})m_{\{0\}} \cdot \mathfrak{b},$$

for all $m \in M$, $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. If C is finite dimensional, then \mathfrak{G} is an isomorphism of categories (see [3]).

Now let C be infinite dimensional. We will show that the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category $\sigma_{C^* \blacktriangleright \blacktriangleleft \mathfrak{B}}[C \otimes \mathfrak{B}]$. Recall that if \mathcal{A} is a Grothendieck category and M is an object of \mathcal{A} then $\sigma_{\mathcal{A}}[M]$ is the class of all objects $N \in \mathcal{A}$ which are subgenerated by M , that is, N is a subobject of a quotient of direct sums of copies of M . It is well-known that $\sigma_{\mathcal{A}}[M]$ is the smallest closed subcategory of \mathcal{A} containing M , and that for any closed subcategory \mathcal{D} of \mathcal{A} there exists an object M of \mathcal{A} such that $\mathcal{D} = \sigma_{\mathcal{A}}[M]$. In particular, $\sigma_{\mathcal{A}}[M]$ is a Grothendieck category, and has a fortiori enough injective objects. For more detail, the reader is invited to consult [10, 21].

Let H be a quasi-bialgebra, C a right H -module coalgebra and \mathfrak{B} a left H -comodule coalgebra. For a right $C^* \blacktriangleleft \mathfrak{B}$ -module M we define the linear maps

$$\mu_M : M \rightarrow \text{Hom}(C^* \otimes \mathfrak{B}, M), \quad \mu_M(m)(c^* \otimes \mathfrak{b}) = m \cdot (c^* \blacktriangleleft \mathfrak{b}),$$

$$\nu_M : C \otimes M \rightarrow \text{Hom}(C^* \otimes \mathfrak{B}, M), \quad \nu_M(c \otimes m)(c^* \otimes \mathfrak{b}) = c^*(c)m \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}),$$

for all $m \in M$, $c \in C$, $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. It is easily verified that μ_M and ν_M are injective linear maps.

Inspired by [9, 11] we propose the following

Definition 2.3. Let M be a right $C^* \blacktriangleleft \mathfrak{B}$ -module. We say that M is rational if $\text{Im}(\mu_M) \subseteq \text{Im}(\nu_M)$. $\text{Rat}(\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}})$ will be the full subcategory of $\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}}$ consisting of rational $C^* \blacktriangleleft \mathfrak{B}$ -modules.

Remark 2.4. If H is a coassociative bialgebra, \mathfrak{B} a left H -comodule algebra and C a right H -module coalgebra in the usual sense, then a rational $C^* \blacktriangleleft \mathfrak{B}$ -module M is nothing else than a $C^* \blacktriangleleft \mathfrak{B}$ -module which is rational as a C^* -module. Here M is viewed as a right C^* -module via the canonical algebra map $C^* \hookrightarrow C^* \blacktriangleleft \mathfrak{B}$.

It follows easily from Definition 2.3 that a right $C^* \blacktriangleleft \mathfrak{B}$ -module is rational if and only if for every $m \in M$ there exist two finite sets $\{c_i\}_i \subseteq C$ and $\{m_i\}_i \subseteq M$ such that

$$(2.15) \quad m \cdot (c^* \blacktriangleleft \mathfrak{b}) = \sum_i c^*(c_i)m_i \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. If $\{c'_j\}_j \subseteq C$ and $\{m'_j\}_j \subseteq M$ are two other finite sets satisfying (2.15), then $\sum_i c_i \otimes m_i = \sum_j c'_j \otimes m'_j$, because of the injectivity of the map ν_M . So we have a well-defined map

$$\lambda_M : M \rightarrow C \otimes M, \quad \lambda_M(m) = \sum_i c_i \otimes m_i,$$

for all $m \in M$. If C is finite dimensional, then any right $C^* \blacktriangleleft \mathfrak{B}$ -module is rational. Indeed, we take a finite dual basis $\{(c_i, c^i)\}_i$ of C and then consider for each $m \in M$ the finite sets $\{c_i\}_i \subseteq C$ and $\{m_i = m \cdot (c^i \blacktriangleleft 1_{\mathfrak{B}})\}_i \subseteq M$.

We now summarize the properties of rational $C^* \blacktriangleleft \mathfrak{B}$ -modules.

Proposition 2.5. *Let H be a quasi-bialgebra, C a right H -module coalgebra and \mathfrak{B} a left H -comodule algebra. Then:*

- i) *A cyclic submodule of a rational $C^* \blacktriangleleft \mathfrak{B}$ -module is finite dimensional.*
- ii) *If M is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module and N is a $C^* \blacktriangleleft \mathfrak{B}$ -submodule of M , then N and M/N are rational $C^* \blacktriangleleft \mathfrak{B}$ -modules.*
- iii) *If $(M_i)_{i \in I}$ is a family of rational $C^* \blacktriangleleft \mathfrak{B}$ -modules, then the direct sum $M = \bigoplus_{i \in I} M_i$ in $\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}}$ is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module.*
- iv) *To any right $C^* \blacktriangleleft \mathfrak{B}$ -module M we can associate a unique maximal rational submodule M^{rat} , which is equal to $\mu_M^{-1}(\text{Im}(\nu_M))$. It is also equal to the sum of all rational $C^* \blacktriangleleft \mathfrak{B}$ -submodule of M . We have a left exact functor*

$$\text{Rat} : \mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}} \rightarrow \mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}}, \quad \text{Rat}(M) = M^{\text{rat}}.$$

Proof. In fact this is a straightforward generalization of [20, Theorem 2.1.3] and [10, Theorem 2.2.6]. Consider an element m of a rational $C^* \blacktriangleleft \mathfrak{B}$ -module M . In what follows, $\{c_i\}_i \subseteq C$ and $\{m_i\}_i \subseteq M$ will then be two finite sets satisfying (2.15).

- i) Let $m \cdot (C^* \blacktriangleleft \mathfrak{B})$ be a cyclic submodule of a rational $C^* \blacktriangleleft \mathfrak{B}$ -module M . $m \cdot (C^* \blacktriangleleft \mathfrak{B})$ is generated by the finite set $\{m_i\}_i$, so it is finite dimensional.
- ii) Take $m \in N \subset M$. Choose the c_i in such a way that they are linearly independent. Fix j , and take $c^* \in C^*$ such that $c^*(c_i) = \delta_{i,j}$. Then N contains $m \cdot (c^* \blacktriangleleft 1_{\mathfrak{B}}) = \sum_i c^*(c_i) m_i \cdot (\underline{\varepsilon} \blacktriangleleft 1_{\mathfrak{B}}) = m_j$, as needed.

Let \bar{m} be the class in M/N represented by $m \in M$. For all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$, we have that $\bar{m} \cdot (c^* \blacktriangleleft \mathfrak{b}) = \sum_i c^*(c_i) \bar{m}_i \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b})$, and it follows that M/N is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module.

- iii) Every $m \in M$ can be written in a unique way as

$$m = \sum_{j \in J} m_j,$$

with $J \subset I$ finite, and $m_j \in M_j$. Since M_j is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module, there exist two finite sets $\{c_k^j\}_k \subseteq C$ and $\{m_k^j\}_k \subseteq M_j$ such that

$$m_j \cdot (c^* \blacktriangleleft \mathfrak{b}) = \sum_k c^*(c_k^j) m_k^j \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. We therefore have that

$$m \cdot (c^* \blacktriangleleft \mathfrak{b}) = \sum_{j \in J} m_j \cdot (c^* \blacktriangleleft \mathfrak{b}) = \sum_{j \in J, k} c^*(c_k^j) m_k^j \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$, and it follows that M is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module.

- iv) Let M be a right $C^* \blacktriangleleft \mathfrak{B}$ -module. We define $M^{\text{rat}} = \mu_M^{-1}(\text{Im}(\nu_M))$.

We first prove that M^{rat} is a right $C^* \blacktriangleleft \mathfrak{B}$ -module. Take $m \in M^{\text{rat}}$. Then there exist two finite sets $\{c_i\}_i \subseteq C$ and $\{m_i\}_i \subseteq M$ such that $m \cdot (c^* \blacktriangleleft \mathfrak{b}) = c^*(c_i) m_i \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b})$, for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. Therefore:

$$\begin{aligned} (m \cdot (c^* \blacktriangleleft \mathfrak{b})) \cdot (d^* \blacktriangleleft \mathfrak{b}') &= m \cdot ((c^* \blacktriangleleft \mathfrak{b})(d^* \blacktriangleleft \mathfrak{b}')) \\ &= m \cdot ((\tilde{x}_\lambda^1 \cdot c^*)(\tilde{x}_\lambda^2 \mathfrak{b}_{[-1]} \cdot d^*) \blacktriangleleft \tilde{x}_\lambda^3 \mathfrak{b}_{[0]} \mathfrak{b}') \\ &= c^*((c_i)_1 \cdot \tilde{x}_\lambda^1) d^*((c_i)_2 \cdot \tilde{x}_\lambda^2 \mathfrak{b}_{[-1]})(m_i \cdot (\underline{\varepsilon} \blacktriangleleft \tilde{x}_\lambda^3 \mathfrak{b}_{[0]})) \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}'), \end{aligned}$$

for all $m \in M$, $c^*, d^* \in C^*$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. Thus $m \cdot (c^* \blacktriangleleft \mathfrak{b}) \in M^{\text{rat}}$, hence M^{rat} is a $C^* \blacktriangleleft \mathfrak{B}$ -submodule of M . Using an argument similar to the one in the first part of the proof of assertion ii), we can easily check that M^{rat} is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module.

Let N be a rational $C^* \blacktriangleleft \mathfrak{B}$ -submodule of M , this means $\text{Im}(\mu_N) \subseteq \text{Im}(\nu_N)$. Then

$$\mu_M(N) = \mu_N(N) \subseteq \text{Im}(\nu_N) \subseteq \text{Im}(\nu_M),$$

hence $N \subseteq \mu_M^{-1}(\text{Im}(\nu_M)) = M^{\text{rat}}$, and we conclude that M^{rat} is the unique maximal rational submodule of M . Assertions ii) and iii) show that M^{rat} is also equal to the sum of all rational $C^* \blacktriangleleft \mathfrak{B}$ -submodules of M . The proof of the final assertion is identical to the proof of [10, Theorem 2.2.6 iv)]. \square

We are now able to prove the main result of this Section.

Theorem 2.6. *Let H be a quasi-bialgebra, \mathfrak{B} a left H -comodule algebra and C a right H -module coalgebra. The categories ${}^C \mathcal{M}(H)_{\mathfrak{B}}$ and $\text{Rat}(\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}})$ are isomorphic, and $\text{Rat}(\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}})$ is equal to $\sigma_{C^* \blacktriangleleft \mathfrak{B}}[C \otimes \mathfrak{B}]$.*

Proof. Recall that we have a functor

$$\mathfrak{G} : {}^C\mathcal{M}(H)_{\mathfrak{B}} \rightarrow \mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}}, \quad \mathfrak{G}(M) = M,$$

with right $C^* \blacktriangleleft \mathfrak{B}$ -action given by the formula

$$m \cdot (c^* \blacktriangleleft \mathfrak{b}) = c^*(m_{\{-1\}})m_{\{0\}} \cdot \mathfrak{b}.$$

It is clear that $\mathfrak{G}(M)$ is rational as a $C^* \blacktriangleleft \mathfrak{B}$ -module.

Consider a rational $C^* \blacktriangleleft \mathfrak{B}$ -module M . Then M is a right \mathfrak{B} -module through $m \cdot \mathfrak{b} = m \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b})$. We define a linear map $\lambda_M : M \rightarrow C \otimes M$ as follows:

$$\lambda_M(m) = \sum_i c_i \otimes m_i$$

if and only if

$$m \cdot (c^* \blacktriangleleft \mathfrak{b}) = \sum_i c^*(c_i)m_i \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. It is clear that λ_M is well-defined.

Fix i , and assume that

$$\lambda_M(m_i) = \sum_j c_j^i \otimes m_j^i,$$

or, equivalently,

$$m_i \cdot (c^* \blacktriangleleft \mathfrak{b}) = c^*(c_j^i)m_j^i \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. Therefore we have, for all $c^*, d^* \in C^*$ and $m \in M$, that

$$\begin{aligned} \langle c^* \otimes d^* \otimes id_M, (id_C \otimes \lambda_M)(\lambda_M(m)) \cdot \Phi_\lambda \rangle &= c^*(c_i \cdot \tilde{X}_\lambda^1) d^*(c_j^i \cdot \tilde{X}_\lambda^2) m_j^i \cdot \tilde{X}_\lambda^3 \\ &= c^*(c_i \cdot \tilde{X}_\lambda^1) m_i \cdot (\tilde{X}_\lambda^2 \cdot d^* \blacktriangleleft \tilde{X}_\lambda^3) = (m \cdot (\tilde{X}_\lambda^1 \cdot c^* \blacktriangleleft 1_{\mathfrak{B}})) \cdot (\tilde{X}_\lambda^2 \cdot d^* \blacktriangleleft \tilde{X}_\lambda^3) \\ &= m \cdot ((\tilde{X}_\lambda^1 \cdot c^* \blacktriangleleft 1_{\mathfrak{B}})(\tilde{X}_\lambda^2 \cdot d^* \blacktriangleleft \tilde{X}_\lambda^3)) = m \cdot (c^* d^* \blacktriangleleft 1_{\mathfrak{B}}) \\ &= c^* d^*(c_i)m_i = \langle c^* \otimes d^* \otimes id_M, (\underline{\Delta} \otimes id_M)(\lambda_M(m)) \rangle, \end{aligned}$$

proving that (2.7) is satisfied. (2.8) is trivial since $m = m \cdot (\underline{\varepsilon} \blacktriangleleft 1_{\mathfrak{B}})$ for all $m \in M$.

We will next prove that (2.9) holds. First observe that

$$\begin{aligned} (m \cdot \mathfrak{b}) \cdot (c^* \blacktriangleleft \mathfrak{b}') &= (m \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b})) \cdot (c^* \blacktriangleleft \mathfrak{b}') = m \cdot ((\underline{\varepsilon} \blacktriangleleft \mathfrak{b})(c^* \blacktriangleleft \mathfrak{b}')) \\ &= m \cdot (\mathfrak{b}_{[-1]} \cdot c^* \blacktriangleleft \mathfrak{b}_{[0]} \mathfrak{b}') = c^*(c_i \cdot \mathfrak{b}_{[-1]}) (m_i \cdot \mathfrak{b}_{[0]}) \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}'), \end{aligned}$$

for all $m \in M$, $c^* \in C^*$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. This shows that $\lambda_M(m \cdot \mathfrak{b}) = c_i \cdot \mathfrak{b}_{[-1]} \otimes m_i \cdot \mathfrak{b}_{[0]}$, as needed, and it follows that $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$.

Let $\eta : M \rightarrow N$ be a morphism of rational $C^* \blacktriangleleft \mathfrak{B}$ -modules. Take $m \in M$, and assume that $\lambda_M(m) = \sum_i c_i \otimes m_i$. We compute that

$$\eta(m) \cdot (c^* \blacktriangleleft \mathfrak{b}) = \eta(m \cdot (c^* \blacktriangleleft \mathfrak{b})) = \sum_i c^*(c_i) \eta(m_i) \cdot (\underline{\varepsilon} \blacktriangleleft \mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. This is equivalent to $\lambda_N(\eta(m)) = c_i \otimes \eta(m_i) = (id_C \otimes \eta)(\lambda_M(m))$, hence η is left C -colinear. It is clear that η is right \mathfrak{B} -linear, so η is a morphism in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, and we have a functor $\mathfrak{F} : \text{Rat}(\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}}) \rightarrow {}^C\mathcal{M}(H)_{\mathfrak{B}}$, which is inverse to \mathfrak{G} .

The proof of the fact that $\text{Rat}(\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}})$ and $\sigma_{C^* \blacktriangleleft \mathfrak{B}}[C \otimes \mathfrak{B}]$ are equal is similar to the proof of [19, Lemma 3.9].

We first show that $M \in \sigma_{C^* \blacktriangleleft \mathfrak{B}}[C \otimes \mathfrak{B}]$ for every $M \in \text{Rat}(\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}})$. We recall first that a right $C^* \blacktriangleleft \mathfrak{B}$ -module belongs to $\sigma_{C^* \blacktriangleleft \mathfrak{B}}[C \otimes \mathfrak{B}]$ if and only if there

exists a set I , a right $C^* \blacktriangleleft \mathfrak{B}$ -module N , and two $C^* \blacktriangleleft \mathfrak{B}$ -linear maps $\iota : M \rightarrow N$ and $\pi : (C \otimes \mathfrak{B})^{(I)} \rightarrow N$ such that ι is injective and π is surjective.

Let M be a rational $C^* \blacktriangleleft \mathfrak{B}$ -module. Thus $M \in {}^C \mathcal{M}(H)_{\mathfrak{B}}$ and $\mathfrak{G}(M) = M$ as right $C^* \blacktriangleleft \mathfrak{B}$ -modules. It is easy to check that $\iota = \lambda_M : M \rightarrow C \otimes M$ is an injective morphism in ${}^C \mathcal{M}(H)_{\mathfrak{B}}$; here $C \otimes M$ has the right-left Doi-Hopf module structure given by (2.10, 2.11). The map

$$\pi_M : \mathfrak{B}^{(M)} \rightarrow M, \pi_M((\mathfrak{b}_m)_m) = m \cdot \mathfrak{b}_m,$$

is a surjective right \mathfrak{B} -linear map. By Proposition 2.2, it provides a surjective morphism $id_C \otimes \pi_M : C \otimes \mathfrak{B}^{(M)} \rightarrow C \otimes M$ in ${}^C \mathcal{M}(H)_{\mathfrak{B}}$, and since $C \otimes \mathfrak{B}^{(M)} \cong (C \otimes \mathfrak{B})^{(M)}$ in ${}^C \mathcal{M}(H)_{\mathfrak{B}}$, we conclude that there exists a surjective morphism $\pi : (C \otimes \mathfrak{B})^{(M)} \rightarrow C \otimes M$ in ${}^C \mathcal{M}(H)_{\mathfrak{B}}$, so $M \in \sigma_{C^* \blacktriangleleft \mathfrak{B}}[C \otimes \mathfrak{B}]$.

Take $M \in \sigma_{C^* \blacktriangleleft \mathfrak{B}}[C \otimes \mathfrak{B}]$. Then we have right $C^* \blacktriangleleft \mathfrak{B}$ -morphisms $\iota : M \rightarrow N$ and $\pi : (C \otimes \mathfrak{B})^{(I)} \rightarrow N$ such that ι is injective and π is surjective. Since the right $C^* \blacktriangleleft \mathfrak{B}$ -module $C \otimes \mathfrak{B}$ lies in the image of \mathfrak{G} , it follows that $C \otimes \mathfrak{B}$ is a rational module. By Proposition 2.5 iii), $(C \otimes \mathfrak{B})^{(I)}$ is a rational module too and since π is surjective we deduce from Proposition 2.5 ii) that N is rational. Finally, from Proposition 2.5 ii) and the fact that ι is injective, it follows that M is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module. \square

Remark 2.7. It is well-known that the category $\sigma_{\mathcal{A}}[M]$ is a Grothendieck category, so it follows from Theorem 2.6 that ${}^C \mathcal{M}(H)_{\mathfrak{B}}$ is a Grothendieck category, an observation that we already made before.

Corollary 2.8. [3, Proposition 5.2] *Let H be a quasi-bialgebra, \mathfrak{B} a left H -comodule algebra and C a finite dimensional right H -module coalgebra. Then the categories ${}^C \mathcal{M}(H)_{\mathfrak{B}}$ and $\mathcal{M}_{C^* \blacktriangleleft \mathfrak{B}}$ are isomorphic.*

Corollary 2.9. *Let M be a right-left (H, \mathfrak{B}, C) -Hopf module. Then the following assertions hold:*

- i) *The right-left (H, \mathfrak{B}, C) -Hopf submodule generated by an element of M is finite dimensional.*
- ii) *M is the sum of its finite dimensional (H, \mathfrak{B}, C) -Hopf submodules.*

Proof. i) M is a rational $C^* \blacktriangleleft \mathfrak{B}$ -module, so the right-left (H, \mathfrak{B}, C) -Hopf submodule generated by an element $m \in M$ coincides with the cyclic $C^* \blacktriangleleft \mathfrak{B}$ -submodule generated by m , by Theorem 2.6. We know from Proposition 2.5 i) that it is finite dimensional.

ii) View M as a rational $C^* \blacktriangleleft \mathfrak{B}$ -module. Obviously, M is the sum of its cyclic $C^* \blacktriangleleft \mathfrak{B}$ -submodules and all of these are finite dimensional right-left (H, \mathfrak{B}, C) -Hopf submodules. \square

2.4. Doi-Hopf modules and Koppinen's smash product. We begin this Section with some general results about corings, taken from [2, Sec. 19 and 20]. Let R be a ring, and \mathcal{C} an R -coring. Then ${}^* \mathcal{C} = {}_R \text{Hom}(\mathcal{C}, R)$ is an R -ring, with multiplication

$$(\varphi \# \psi)(c) = \psi(c_{(1)})\varphi(c_{(2)}),$$

for all $\varphi, \psi \in {}^* \mathcal{C}$. We have a functor $F : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_{{}^* \mathcal{C}}$, $F(M) = M$, with $m \cdot \varphi = m_{[0]}\varphi(m_{[1]})$, for all $m \in M$ and $\varphi \in {}^* \mathcal{C}$. If \mathcal{C} is finitely generated and projective as a left R -module, then F is an isomorphism of categories.

Assume now that \mathcal{C} is locally projective as a left R -module. Then $\mathcal{M}^{\mathcal{C}}$ is isomorphic to the category $\sigma_{*\mathcal{C}}[\mathcal{C}]$, see [2, 19.3]; observe that our multiplication on ${}^*\mathcal{C}$ is opposite to the one from [2], so that left ${}^*\mathcal{C}$ -modules in [2] are our right ${}^*\mathcal{C}$ -modules. Take a right ${}^*\mathcal{C}$ -module M . $m \in M$ is called rational if there exists $\sum_i m_i \otimes c_i \in M \otimes_R \mathcal{C}$ such that $m \cdot \varphi = \sum_i m_i \varphi(c_i)$, for all $\varphi \in {}^*\mathcal{C}$. Then $\text{Rat}^{\mathcal{C}}(M) = \{m \in M \mid m \text{ is rational}\}$ is a right \mathcal{C} -comodule, and we obtain a functor $\text{Rat}^{\mathcal{C}} : \mathcal{M}_{*\mathcal{C}} \rightarrow \mathcal{M}^{\mathcal{C}}$, which is right adjoint to F . M is called rational if $\text{Rat}^{\mathcal{C}}(M) = M$. $\mathcal{M}^{\mathcal{C}}$ is isomorphic to the full category of $\mathcal{M}_{*\mathcal{C}}$ consisting of rational right ${}^*\mathcal{C}$ -modules.

Now let H be a quasi-bialgebra, \mathfrak{B} a left H -comodule algebra, and C a right H -comodule algebra. We consider the \mathfrak{B} -coring $\mathcal{C} = \mathfrak{B} \otimes C$ from Section 2.2. Since we work over a field k , C is projective as a k -module, hence \mathcal{C} is projective (and a fortiori locally projective) as a left \mathfrak{B} -module. Hence we can apply the above results to this situation. We have an isomorphism of vector spaces

$${}^*\mathcal{C} = {}_{\mathfrak{B}}\text{Hom}(\mathfrak{B} \otimes C, \mathfrak{B}) \cong \text{Hom}(C, \mathfrak{B}).$$

The multiplication on ${}^*\mathcal{C}$ can be transported to a multiplication on $\text{Hom}(C, \mathfrak{B})$. This multiplication makes $\text{Hom}(C, \mathfrak{B})$ into a B -ring $\#(C, \mathfrak{B})$, which we will call the Koppinen smash product. The multiplication is given by the following formula:

$$(2.16) \quad (f\#g)(c) = \tilde{x}_{\lambda}^3 f(c_{\perp} \cdot \tilde{x}_{\lambda}^1)_{[0]} g(c_{\perp} \cdot \tilde{x}_{\lambda}^2 f(c_{\perp} \cdot \tilde{x}_{\lambda}^1)_{[1]}).$$

In the situation where H is an associative bialgebra, we recover the smash product introduced first by Koppinen in [16]. The relation to the generalized smash product introduced in Section 2.3 is discussed in Proposition 2.10.

Proposition 2.10. *The k -linear map*

$$\alpha : C^* \blacktriangleleft \mathfrak{B} \rightarrow \#(C, B), \quad \alpha(c^* \blacktriangleleft \mathfrak{b}) = f,$$

with $f(c) = \langle c^*, c \rangle b$, for all $c \in C$, is a morphism of \mathfrak{B} -rings. It is an isomorphism if C is finite dimensional.

Proof. We have to show that α is multiplicative. Take $c^* \blacktriangleleft \mathfrak{b}, d^* \blacktriangleleft \mathfrak{b}' \in C^* \blacktriangleleft \mathfrak{B}$, and write $\alpha(c^* \blacktriangleleft \mathfrak{b}) = f$, $\alpha(d^* \blacktriangleleft \mathfrak{b}') = g$. Using (2.16), we compute that

$$(f\#g)(c) = \tilde{x}_{\lambda}^3 \langle c^*, c_{\perp} \cdot \tilde{x}_{\lambda}^1 \rangle \mathfrak{b}_{[0]} \langle d^*, c_{\perp} \cdot \tilde{x}_{\lambda}^2 \rangle \mathfrak{b}'_{[1]} \mathfrak{b}'.$$

We also have that

$$(c^* \blacktriangleleft \mathfrak{b})(d^* \blacktriangleleft \mathfrak{b}') = (\tilde{x}_{\lambda}^1 \cdot c^*)(\tilde{x}_{\lambda}^2 \mathfrak{b}'_{[-1]} \cdot d^*) \blacktriangleleft \tilde{x}_{\lambda}^3 \mathfrak{b}_{[0]} \mathfrak{b}',$$

so

$$\alpha((c^* \blacktriangleleft \mathfrak{b})(d^* \blacktriangleleft \mathfrak{b}'))(c) = \langle c^*, c_{\perp} \cdot \tilde{x}_{\lambda}^1 \rangle \langle d^*, c_{\perp} \cdot \tilde{x}_{\lambda}^2 \rangle \tilde{x}_{\lambda}^3 \mathfrak{b}_{[0]} \mathfrak{b}',$$

as needed. \square

Take a right $\#(C, B)$ -module M . $m \in M$ is rational if there exists $\sum_i m_i \otimes c_i \in M \otimes C$ such that

$$m \cdot f = \sum_i m_i f(c_i),$$

for all $f : C \rightarrow B$. M is called rational if every $m \in M$ is rational.

Corollary 2.11. *Now let H be a quasi-bialgebra, \mathfrak{B} a left H -comodule algebra, and C a right H -comodule algebra. Then the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the full subcategory of $\mathcal{M}_{\#(C,B)}$, which is also equal to $\sigma_{\#(C,\mathfrak{B})}(\mathfrak{B} \otimes C)$.*

2.5. Left, right and right-left Doi-Hopf modules. For the sake of completeness, we also define the other Doi-Hopf module categories. In fact, we have four different types of Doi-Hopf modules. The first one was already studied, namely the right-left version. We also have the left-right, right-right and left-left versions.

Definition 2.12. Let H be a quasi-bialgebra, \mathfrak{A} a right H -comodule algebra and \mathfrak{B} a left H -comodule algebra. In the statements below we assume in i) and iii) that C is a left H -module coalgebra, and in ii) that C is a right H -module coalgebra, respectively.

i) A left-right (H, \mathfrak{A}, C) -Hopf module (or Doi-Hopf module) is a left \mathfrak{A} -module M together with a k -linear map $\rho_M : M \rightarrow M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that the following relations hold, for all $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$:

$$(2.17) \quad \Phi_\rho \cdot (\rho_M \otimes id_C)(\rho_M(m)) = (id_M \otimes \underline{\Delta})(\rho_M(m)),$$

$$(2.18) \quad (id_M \otimes \underline{\varepsilon})(\rho_M(m)) = m,$$

$$(2.19) \quad \rho_M(\mathfrak{a} \cdot m) = \mathfrak{a}_{(0)} \cdot m_{(0)} \otimes \mathfrak{a}_{(1)} \cdot m_{(1)}.$$

${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is the category of left-right (H, \mathfrak{A}, C) -Hopf modules and left \mathfrak{A} -linear, right C -colinear maps.

ii) A right-right (H, \mathfrak{A}, C) -Hopf module (or Doi-Hopf module) is a right \mathfrak{A} -module M together with a k -linear map $\rho_M : M \rightarrow M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that the following relations hold, for all $m \in M$ and $\mathfrak{a} \in \mathfrak{A}$:

$$(2.20) \quad (\rho_M \otimes id_M)(\rho_M(m)) = (id_M \otimes \underline{\Delta})(\rho_M(m)) \cdot \Phi_\rho,$$

$$(2.21) \quad (id_M \otimes \underline{\varepsilon})(\rho_M(m)) = m,$$

$$(2.22) \quad \rho_M(m \cdot \mathfrak{a}) = m_{(0)} \cdot \mathfrak{a}_{(0)} \otimes m_{(1)} \cdot \mathfrak{a}_{(1)}.$$

$\mathcal{M}(H)_{\mathfrak{A}}^C$ is the category of right-right (H, \mathfrak{A}, C) -Hopf modules and left \mathfrak{A} -linear, right C -colinear maps.

iii) A left-left (H, \mathfrak{B}, C) -Hopf module (or Doi-Hopf module) is a left \mathfrak{B} -module M together with a left k -linear map $\lambda_M : M \rightarrow C \otimes M$, $\lambda_M(m) = m_{\{-1\}} \otimes m_{\{0\}}$, such that the following relations hold, for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$:

$$(2.23) \quad \Phi_\lambda \cdot (\underline{\Delta} \otimes id_M)(\lambda_M(m)) = (id_C \otimes \lambda_M)(\lambda_M(m)),$$

$$(2.24) \quad (\underline{\varepsilon} \otimes id_M)(\lambda_M(m)) = m,$$

$$(2.25) \quad \lambda_M(\mathfrak{b} \cdot m) = \mathfrak{b}_{\{-1\}} \cdot m_{\{-1\}} \otimes \mathfrak{b}_{\{0\}} \cdot m_{\{0\}}.$$

${}_{\mathfrak{B}}\mathcal{M}(H)$ is the category of left-left (H, \mathfrak{A}, C) -Hopf modules and left \mathfrak{A} -linear, left C -colinear maps.

We remind that if $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right H -comodule algebra then $\mathfrak{A}^{\text{op}} = (\mathfrak{A}^{\text{op}}, \rho \circ \tau_{\mathfrak{A}, H}, (\Phi_\rho)^{321})$ is a left $H^{\text{op}, \text{cop}}$ -comodule algebra and $\underline{\mathfrak{A}} = (\mathfrak{A}, \rho \circ \tau_{\mathfrak{A}, H}, (\Phi_\rho^{-1})^{321})$ is a left H^{cop} -comodule algebra, where $\tau_{\mathfrak{A}, H} : \mathfrak{A} \otimes H \rightarrow H \otimes \mathfrak{A}$ is the switch map. Also, if \mathfrak{B} is a left H -comodule algebra then $\mathfrak{B}^{\text{op}} = (\mathfrak{B}^{\text{op}}, \lambda, \Phi_\lambda^{-1})$ is a left H^{op} -comodule algebra.

On the other hand, if C is a left H -module coalgebra then C is a right H^{op} -module coalgebra and C^{cop} is a right $H^{\text{op,cop}}$ -module coalgebra (and vice versa). So if C is a right H -module coalgebra then C^{cop} is a right H^{cop} -module coalgebra.

Having these correspondences one can easily see that

$$\begin{aligned} \mathfrak{A}\mathcal{M}(H)^C &\cong {}^{C^{\text{cop}}}\mathcal{M}(H^{\text{op,cop}})_{\mathfrak{A}^{\text{op}}}, \quad \mathcal{M}(H)_{\mathfrak{A}}^C \cong {}^{C^{\text{cop}}}\mathcal{M}(H^{\text{cop}})_{\mathfrak{A}}, \\ {}_{\mathfrak{B}}^C\mathcal{M}(H) &\cong {}^C\mathcal{M}(H^{\text{op}})_{\mathfrak{B}^{\text{op}}}. \end{aligned}$$

It follows that the four different types of Doi-Hopf modules are isomorphic to categories of comodules over suitable corings, and they are Grothendieck categories with enough injective objects. On the other hand, if C is finite dimensional then the above categories are isomorphic to categories of modules over certain generalized smash product algebras. More precisely, we have:

Remarks 2.13. i) The category ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to the category of right modules over the generalized smash product $(C^{\text{cop}})^* \blacktriangleright \mathfrak{A}^{\text{op}}$ (over $H^{\text{op,cop}}$), and therefore also to the category of left modules over $((C^{\text{cop}})^* \blacktriangleright \mathfrak{A}^{\text{op}})^{\text{op}}$. Is not hard to see that the multiplication rule in $((C^{\text{cop}})^* \blacktriangleright \mathfrak{A}^{\text{op}})^{\text{op}}$ is

$$(c^* \blacktriangleright \mathfrak{a})(d^* \blacktriangleright \mathfrak{a}') = (c^* \cdot \mathfrak{a}'_{(1)} \tilde{x}_\rho^2)(d^* \cdot \tilde{x}_\rho^3) \blacktriangleright \mathfrak{a}\mathfrak{a}'_{(0)} \tilde{x}_\rho^1,$$

for all $c^*, d^* \in C^*$ and $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$, where $(c^* \cdot h)(c) = c^*(h \cdot c)$ for all $c^* \in C^*$, $h \in H$ and $c \in C$. Therefore, under the trivial permutation of tensor factors we have that $((C^{\text{cop}})^* \blacktriangleright \mathfrak{A}^{\text{op}})^{\text{op}} = \mathfrak{A} \blacktriangleright C^*$, the right generalized smash product between the right H -comodule algebra \mathfrak{A} and the right H -module algebra C^* (see [7] for more details). We conclude that ${}_{\mathfrak{A}}\mathcal{M}(H)^C \cong {}_{\mathfrak{A} \blacktriangleright C^*}\mathcal{M}$ if C is finite dimensional.

ii) The above arguments entail that

$$\mathcal{M}(H)_{\mathfrak{A}}^C \cong {}^{C^{\text{cop}}}\mathcal{M}(H^{\text{op}})_{\mathfrak{A}} \cong \mathcal{M}_{(C^{\text{cop}})^* \blacktriangleright \mathfrak{A}} \cong ((C^{\text{cop}})^* \blacktriangleright \mathfrak{A})^{\text{op}} \mathcal{M},$$

where the generalized smash product is over H^{cop} . The explicit formula for the multiplication \odot on $((C^{\text{cop}})^* \blacktriangleright \mathfrak{A})^{\text{op}}$ is given by

$$(2.26) \quad (c^* \blacktriangleright \mathfrak{a}) \odot (d^* \blacktriangleright \mathfrak{a}') = (\tilde{X}_\rho^2 \mathfrak{a}'_{(1)} \cdot c^*)(\tilde{X}_\rho^3 \cdot d^*) \blacktriangleright \tilde{X}_\rho^1 \mathfrak{a}'_{(0)} \mathfrak{a},$$

for all $c^*, d^* \in C^*$ and $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$.

iii) Obviously, ${}_{\mathfrak{B}}^C\mathcal{M}(H) \cong \mathcal{M}_{C^* \blacktriangleright \mathfrak{B}^{\text{op}}}$, where the generalized smash product is taken over H^{op} .

Let H be a quasi-Hopf algebra and C a finite dimensional right H -module coalgebra. It was proved in [6, Proposition 3.2] that the category $\mathcal{M}_H^C = \mathcal{M}(H)_H^C$ is isomorphic to the category of left modules over the smash product algebra $C^* \# H = C^* \blacktriangleright H$. Now, by Remark 2.13 ii) the category \mathcal{M}_H^C is isomorphic to the category of left modules over $((C^{\text{cop}})^* \# \underline{H})^{\text{op}}$. The next result shows that the smash product algebras $C^* \# H$ and $((C^{\text{cop}})^* \# \underline{H})^{\text{op}}$ are isomorphic.

Proposition 2.14. *Let H be a quasi-Hopf algebra and C a right H -module coalgebra. Then the map*

$$\varphi : C^* \# H \rightarrow ((C^{\text{cop}})^* \# \underline{H})^{\text{op}}, \quad \varphi(c^* \# h) = S^{-1}(\mathfrak{q}^1 h_1 g^1) \cdot c^* \# S^{-1}(\mathfrak{q}^2 h_2 g^2)$$

is an algebra isomorphism. Here $f^{-1} = g^1 \otimes g^2$ is the element defined by (1.12) and $\tilde{\mathfrak{q}}_\Delta = \mathfrak{q}^1 \otimes \mathfrak{q}^2$ is the element $\tilde{\mathfrak{q}}_\lambda$ defined in (1.25), in the special case where $\mathfrak{B} = H$.

Proof. We first show that φ is an algebra map. Let $f = f^1 \otimes f^2$, $f^{-1} = g^1 \otimes g^2 = G^1 \otimes G^2$ and $\tilde{q}_\Delta = \mathfrak{q}^1 \otimes \mathfrak{q}^2 = \Omega^1 \otimes \Omega^2$ be the elements defined by (1.11), (1.12) and (1.25), respectively. We compute

$$\begin{aligned}
& \varphi((c^* \# h)(d^* \# h')) = \varphi((x^1 \cdot c^*)(x^2 h_1 \cdot d^*) \# x^3 h_2 h') \\
& \stackrel{(2.13, 1.10)}{=} [S^{-1}(f^2 \mathfrak{q}_2^1 x_{(1,2)}^3 (h_2 h')_{(1,2)} g_2^1 G^2) x^1 \cdot c^*] \\
& \quad [S^{-1}(f^1 \mathfrak{q}_1^1 x_{(1,1)}^3 (h_2 h')_{(1,1)} g_1^1 G^1) x^2 h_1 \cdot d^*] \# S^{-1}(\mathfrak{q}^2 x_2^3 (h_2 h')_2 g^2) \\
& \stackrel{(1.31, 1.1)}{=} [S^{-1}(\mathfrak{q}^1 \Omega_1^2 (h_2 h')_{(2,1)} X^2 g_2^1 G^2) \cdot c^*] [S^{-1}(\Omega^1 (h_2 h')_1 X^1 g_1^1 G^1) h_1 \cdot d^*] \\
& \quad \# S^{-1}(\mathfrak{q}^2 \Omega_2^2 (h_2 h')_{(2,2)} X^3 g^2) \\
& \stackrel{(1.27, 1.8, 1.13)}{=} [X^2 S^{-1}(\mathfrak{q}^1 h_1 \Omega_1^2 h'_{(2,1)} g_1^2 G^1) \cdot c^*] [X^3 S^{-1}(\Omega^1 h'_1 g^1) \cdot d^*] \\
& \quad \# X^1 S^{-1}(\mathfrak{q}^2 h_2 \Omega_2^2 h'_{(2,2)} g_2^2 G^2) \\
& \stackrel{(1.10)}{=} [X^2 S^{-1}(\Omega^2 h'_2 g^2)_2 S^{-1}(\mathfrak{q}^1 h_1 G^1) \cdot c^*] [X^3 S^{-1}(\Omega^1 h'_1 g^1) \cdot d^*] \\
& \quad \# X^1 S^{-1}(\Omega^2 h'_2 g^2)_1 S^{-1}(\mathfrak{q}^2 h_2 G^2) \\
& \stackrel{(2.26)}{=} \varphi(c^* \# h) \odot \varphi(d^* \# h'),
\end{aligned}$$

as needed. It is clear that $\varphi(\varepsilon \# 1_H) = \varepsilon \# 1_H$, so it remains to be shown that φ is bijective.

First we introduce some notation. Let \mathfrak{A} be a right H -comodule algebra, and define the element $\tilde{q}_\rho \in \mathfrak{A} \otimes H$ as follows:

$$(2.27) \quad \tilde{q}_\rho = \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2 = \tilde{X}_\rho^1 \otimes S^{-1}(\alpha \tilde{X}_\rho^3) \tilde{X}_\rho^2$$

In the special situation where $\mathfrak{A} = H$, the element \tilde{q}_Δ will be denoted by $\tilde{q}_\Delta = q^1 \otimes q^2$.

We now claim that $\varphi^{-1} : ((C^{\text{cop}})^* \# H)^{\text{op}} \rightarrow C^* \# H$ is given by the formula

$$\varphi^{-1}(c^* \# h) = g^1 S(q^2 h_2) \cdot c^* \# g^2 S(q^1 h_1),$$

for all $c^* \in C^*$ and $h \in H$.

From [6, Lemma 2.6], we recall the following formula:

$$(2.28) \quad g^2 \alpha S^{-1}(g^1) = S^{-1}(\beta).$$

Let $\tilde{p}_\Delta = \mathfrak{p}^1 \otimes \mathfrak{p}^2$ be the element \tilde{p}_λ defined in (1.25), in the special case where $\mathfrak{B} = H$. We then compute, for all $c^* \in C^*$ and $h \in H$:

$$\begin{aligned}
& \varphi^{-1} \circ \varphi(c^* \# h) = \varphi^{-1}(S^{-1}(\mathfrak{q}^1 h_1 g^1) \cdot c^* \# S^{-1}(\mathfrak{q}^2 h_2 g^2)) \\
& \stackrel{(1.10)}{=} \mathfrak{q}_1^2 h_{(2,1)} g_1^2 G^1 S(q^2) S^{-1}(\mathfrak{q}^1 h_1 g^1) \cdot c^* \# \mathfrak{q}_2^2 h_{(2,2)} g_2^2 G^2 S(q^1) \\
& \stackrel{(2.27, 1.8, 1.13)}{=} \mathfrak{q}_1^2 h_{(2,1)} X^2 g_2^1 G^2 \alpha S^{-1}(\mathfrak{q}^1 h_1 X^1 g_1^1 G^1) \cdot c^* \# \mathfrak{q}_2^2 h_{(2,2)} X^3 g^2 \\
& \stackrel{(2.28, 1.5, 1.25)}{=} S^{-1}(\mathfrak{q}^1 h_1 S(\mathfrak{q}_1^2 h_{(2,1)} \mathfrak{p}^1)) \cdot c^* \# \mathfrak{q}_2^2 h_{(2,2)} \mathfrak{p}^2 \\
& \stackrel{(1.26, 1.28)}{=} c^* \# h.
\end{aligned}$$

The proof of the fact that $\varphi \circ \varphi^{-1} = id_{((C^{\text{cop}})^* \# H)^{\text{op}}}$ is based on similar computations. \square

Remark 2.15. The isomorphism φ in Proposition 2.14 can be defined more generally for a left H -module algebra A instead of a right H -module coalgebra C .

Observe that H cannot be replaced by an H -bicomodule algebra \mathbb{A} , because of the appearance of the antipode S of H on the second position of the tensor product.

3. YETTER-DRINFELD MODULES ARE DOI-HOPF MODULES

In this Section, we will show that Yetter-Drinfeld modules are special case of Doi-Hopf modules. We will then apply the properties of Doi-Hopf modules to Yetter-Drinfeld modules.

3.1. Yetter-Drinfeld modules over quasi-bialgebras. The category of Yetter-Drinfeld modules over a quasi-Hopf algebra H was introduced by Majid, as the center of the monoidal category ${}_H\mathcal{M}$. His aim was to define the quantum double by an implicit Tannaka-Krein reconstruction procedure, see [17]. From [7], we recall the following more general definition of Yetter-Drinfeld modules.

The category of (H, H) -bimodules, ${}_H\mathcal{M}_H$, is monoidal. The associativity constraints $\mathbf{a}'_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ are given by

$$(3.1) \quad \mathbf{a}'_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)) \cdot \Phi^{-1}$$

for all $U, V, W \in {}_H\mathcal{M}_H$, $u \in U$, $v \in V$ and $w \in W$. A coalgebra in the category of (H, H) -bimodules will be called an H -bimodule coalgebra. More precisely, an H -bimodule coalgebra C is an (H, H) -bimodule (denote the actions by $h \cdot c$ and $c \cdot h$) with a comultiplication $\underline{\Delta} : C \rightarrow C \otimes C$ and a counit $\underline{\varepsilon} : C \rightarrow k$ satisfying the following relations, for all $c \in C$ and $h \in H$:

$$(3.2) \quad \Phi \cdot (\underline{\Delta} \otimes id_C)(\underline{\Delta}(c)) \cdot \Phi^{-1} = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)),$$

$$(3.3) \quad \underline{\Delta}(h \cdot c) = h_1 \cdot c_{\underline{1}} \otimes h_2 \cdot c_{\underline{2}}, \quad \underline{\Delta}(c \cdot h) = c_{\underline{1}} \cdot h_1 \otimes c_{\underline{2}} \cdot h_2,$$

$$(3.4) \quad (\underline{\varepsilon} \otimes id_C) \circ \underline{\Delta} = (id_C \otimes \underline{\varepsilon}) \circ \underline{\Delta} = id_C,$$

$$(3.5) \quad \underline{\varepsilon}(h \cdot c) = \varepsilon(h)\underline{\varepsilon}(c), \quad \underline{\varepsilon}(c \cdot h) = \underline{\varepsilon}(c)\varepsilon(h),$$

where we used the same Sweedler-type notation as introduced before.

For further use we note that an H -bimodule coalgebra C can be always viewed as a left $H^{\text{op}} \otimes H$ -module coalgebra via the left $H^{\text{op}} \otimes H$ -action given for all $c \in C$ and $h, h' \in H$ by

$$(3.6) \quad (h \otimes h') \cdot c = h' \cdot c \cdot h.$$

Definition 3.1. ([7]) Let H be a quasi-bialgebra, C an H -bimodule coalgebra and \mathbb{A} an H -bicomodule algebra. A left-right (H, \mathbb{A}, C) -Yetter-Drinfeld module is a k -vector space M with the following additional structure:

- M is a left \mathbb{A} -module; we write \cdot for the left \mathbb{A} -action;
- we have a k -linear map $\rho_M : M \rightarrow M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, called the right C -coaction on M , such that for all $m \in M$, $\underline{\varepsilon}(m_{(1)})m_{(0)} = m$ and

$$(3.7) \quad \begin{aligned} & (\theta^2 \cdot m_{(0)})_{(0)} \otimes (\theta^2 \cdot m_{(0)})_{(1)} \cdot \theta^1 \otimes \theta^3 \cdot m_{(1)} \\ & = \tilde{x}_\rho^1 \cdot (\tilde{x}_\lambda^3 \cdot m)_{(0)} \otimes \tilde{x}_\rho^2 \cdot (\tilde{x}_\lambda^3 \cdot m)_{(1)\underline{1}} \cdot \tilde{x}_\lambda^1 \otimes \tilde{x}_\rho^3 \cdot (\tilde{x}_\lambda^3 \cdot m)_{(1)\underline{2}} \cdot \tilde{x}_\lambda^2, \end{aligned}$$

- for all $u \in \mathbb{A}$ and $m \in M$ the following compatibility relation holds:

$$(3.8) \quad u_{\langle 0 \rangle} \cdot m_{(0)} \otimes u_{\langle 1 \rangle} \cdot m_{(1)} = (u_{[0]} \cdot m)_{(0)} \otimes (u_{[0]} \cdot m)_{(1)} \cdot u_{[-1]}.$$

${}_{\mathbb{A}}\mathcal{YD}(H)^C$ will be the category of left-right (H, \mathbb{A}, C) -Yetter-Drinfeld modules and maps preserving the \mathbb{A} -action C -coaction.

We have seen in Section 1.2 that H is an H -bicomodule algebra; it is also clear that H is an H -bimodule coalgebra (take $\underline{\Delta} = \Delta$ and $\underline{\varepsilon} = \varepsilon$). If we take $H = \mathbb{A} = C$ in Definition 3.1, then we recover the category of Yetter-Drinfeld modules introduced by Majid in [17], and studied in [4, 5]. It is remarkable that a quasi-bialgebra H is a coalgebra in the category of H -bimodules, but not in the category of vector spaces, or in the category of left (or right) H -modules.

Let H be a quasi-bialgebra, \mathbb{A} an H -bicomodule algebra and C a left H -module coalgebra. It is straightforward to check that C with the right H -module structure given by ε is an H -bimodule coalgebra. Then (3.7) and (3.8) reduce respectively to (2.17) and (2.19), in which only the right H -coaction on \mathbb{A} appears. So in this particular case the category ${}_{\mathbb{A}}\mathcal{M}(H)^C$ is just ${}_{\mathbb{A}}\mathcal{YD}(H)^C$.

In order to show that the Yetter-Drinfeld modules are special case of Doi-Hopf modules we need a Doi-Hopf datum. As we have already seen, an H -bimodule coalgebra C can be viewed as a left $H^{\text{op}} \otimes H$ -module via the structure defined in (3.6). In order to provide a right $H^{\text{op}} \otimes H$ -comodule algebra structure on an H -bicomodule algebra \mathbb{A} , we need the following result.

Lemma 3.2. *Let H be a quasi-Hopf algebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H -comodule algebra. Then \mathfrak{B} is a right H^{op} -comodule algebra via the structure*

$$(3.9) \quad \rho : \mathfrak{B} \rightarrow \mathfrak{B} \otimes H, \quad \rho(\mathfrak{b}) = \mathfrak{b}_{[0]} \otimes S^{-1}(\mathfrak{b}_{[-1]}),$$

$$(3.10) \quad \Phi_\rho = \tilde{x}_\lambda^3 \otimes S^{-1}(f^2 \tilde{x}_\lambda^2) \otimes S^{-1}(f^1 \tilde{x}_\lambda^1) \in \mathfrak{B} \otimes H \otimes H,$$

where $f^1 \otimes f^2$ is the Drinfeld twist defined in (1.11). Moreover, if $(\mathfrak{B}, \lambda, \Phi_\lambda)$ and $(\mathfrak{B}, \lambda', \Phi_{\lambda'})$ are twist equivalent left H -comodule algebras then the corresponding right H^{op} -comodule algebras are also twist equivalent.

Proof. The relation (1.14) follows easily by applying (1.18) and (1.10), and the relations (1.16, 1.17) are trivial. We prove now (1.15). Since $\Phi_{\text{op}} = \Phi^{-1}$ we have

$$\begin{aligned} & \tilde{X}_\rho^1 \tilde{Y}_\rho^1 \otimes x^1 \cdot_{\text{op}} (\tilde{X}_\rho^2)_1 \cdot_{\text{op}} \tilde{Y}_\rho^3 \otimes x^2 \cdot_{\text{op}} (\tilde{X}_\rho^2)_2 \cdot_{\text{op}} \tilde{Y}_\rho^3 \otimes x^3 \cdot_{\text{op}} \tilde{X}_\rho^3 \\ &= \tilde{x}_\lambda^3 \tilde{y}_\lambda^3 \otimes S^{-1}(F^2 \tilde{y}_\lambda^2) S^{-1}(f^2 \tilde{x}_\lambda^2)_1 x^1 \\ & \quad \otimes S^{-1}(F^1 \tilde{y}_\lambda^1) S^{-1}(f^2 \tilde{x}_\lambda^2)_2 x^2 \otimes S^{-1}(f^1 \tilde{x}_\lambda^1) x^3 \\ (1.10) \quad &= \tilde{x}_\lambda^3 \tilde{y}_\lambda^3 \otimes S^{-1}(S(x^1) F^2 f_2^2(\tilde{x}_\lambda^2)_2 \tilde{y}_\lambda^2) \\ & \quad \otimes S^{-1}(S(x^2) F^1 f_1^2(\tilde{x}_\lambda^2)_1 \tilde{y}_\lambda^1) \otimes S^{-1}(S(x^3) f^1 \tilde{x}_\lambda^1) \\ (1.8, 1.13) \quad &= \tilde{x}_\lambda^3 \tilde{y}_\lambda^3 \otimes S^{-1}(F^2 x^3(\tilde{x}_\lambda^2)_2 \tilde{y}_\lambda^2) \\ & \quad \otimes S^{-1}(f^2 F_2^1 x^2(\tilde{x}_\lambda^2)_1 \tilde{y}_\lambda^1) \otimes S^{-1}(f^1 F_1^1 x^1 \tilde{x}_\lambda^1) \\ (1.19, 1.10) \quad &= \tilde{y}_\lambda^3(\tilde{x}_\lambda^3)_{[0]} \otimes S^{-1}(F^2 \tilde{y}_\lambda^2) \cdot_{\text{op}} S^{-1}((\tilde{x}_\lambda^3)_{[-1]}) \\ & \quad \otimes S^{-1}(F^1 \tilde{y}_\lambda^1)_1 \cdot_{\text{op}} S^{-1}(f^2 \tilde{x}_\lambda^2) \otimes S^{-1}(F^1 \tilde{y}_\lambda^1)_2 \cdot_{\text{op}} S^{-1}(f^1 \tilde{x}_\lambda^1) \\ (3.9, 3.10) \quad &= \tilde{X}_\rho^1(\tilde{Y}_\rho^1)_{(0)} \otimes \tilde{X}_\rho^2 \cdot_{\text{op}} (\tilde{Y}_\rho^1)_{(1)} \otimes (\tilde{X}_\rho^3)_1 \cdot_{\text{op}} \tilde{Y}_\rho^2 \otimes (\tilde{X}_\rho^3)_2 \cdot_{\text{op}} \tilde{Y}_\rho^3, \end{aligned}$$

where we denote by \cdot_{op} the multiplication of H^{op} and by $F^1 \otimes F^2$ another copy of the Drinfeld twist f .

Finally, it is not hard to see that if the invertible element $\mathbb{U} = \mathbb{U}^1 \otimes \mathbb{U}^2 \in H \otimes \mathfrak{B}$ provides a twist equivalence between the left H -comodule algebras $(\mathfrak{B}, \lambda, \Phi_\lambda)$ and $(\mathfrak{B}, \lambda', \Phi_{\lambda'})$ then the invertible element $\mathbb{V} = \mathbb{U}^2 \otimes S^{-1}(\mathbb{U}^1) \in \mathfrak{B} \otimes H$ provides a twist equivalence between the associated right H^{op} -comodule algebras $(\mathfrak{B}, \rho, \Phi_\rho)$ and $(\mathfrak{B}, \rho', \Phi_{\rho'})$, respectively. \square

Proposition 3.3. *Let H be a quasi-Hopf algebra and $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda, \rho})$ an H -bicomodule algebra. We define two right $H^{\text{op}} \otimes H$ -coactions*

$$\rho_1, \rho_2 : \mathbb{A} \rightarrow \mathbb{A} \otimes (H^{\text{op}} \otimes H)$$

on \mathbb{A} , and corresponding elements $\Phi_{\rho_1}, \Phi_{\rho_2} \in \mathbb{A} \otimes (H^{\text{op}} \otimes H) \otimes (H^{\text{op}} \otimes H)$ as follows:

$$(3.11) \quad \rho_1(u) = u_{\langle 0 \rangle_{[0]}} \otimes \left(S^{-1}(u_{\langle 0 \rangle_{[-1]}}) \otimes u_{\langle 1 \rangle} \right),$$

$$(3.12) \quad \begin{aligned} \Phi_{\rho_1} &= (\tilde{X}_\rho^1)_{[0]} \tilde{x}_\lambda^3 \theta_{[0]}^2 \otimes \left(S^{-1}(f^2(\tilde{X}_\rho^1)_{[-1]_2} \tilde{x}_\lambda^2 \theta_{[-1]}^2) \otimes \tilde{X}_\rho^2 \theta^3 \right) \\ &\otimes \left(S^{-1}(f^1(\tilde{X}_\rho^1)_{[-1]_1} \tilde{x}_\lambda^1 \theta^1) \otimes \tilde{X}_\rho^3 \right), \end{aligned}$$

and

$$(3.13) \quad \rho_2(u) = u_{[0]_{\langle 0 \rangle}} \otimes \left(S^{-1}(u_{[-1]}) \otimes u_{[0]_{\langle 1 \rangle}} \right),$$

$$(3.14) \quad \begin{aligned} \Phi_{\rho_2} &= (\tilde{x}_\lambda^3)_{\langle 0 \rangle} \tilde{X}_\rho^1 \Theta_{\langle 0 \rangle}^2 \otimes \left(S^{-1}(f^2 \tilde{x}_\lambda^2 \Theta^1) \otimes (\tilde{x}_\lambda^3)_{\langle 1 \rangle_1} \tilde{X}_\rho^2 \Theta_{\langle 1 \rangle}^2 \right) \\ &\otimes \left(S^{-1}(f^1 \tilde{x}_\lambda^1) \otimes (\tilde{x}_\lambda^3)_{\langle 1 \rangle_2} \tilde{X}_\rho^3 \Theta^3 \right). \end{aligned}$$

Then $(\mathbb{A}, \rho_1, \Phi_{\rho_1})$ and $(\mathbb{A}, \rho_2, \Phi_{\rho_2})$ are twist equivalent right $H^{\text{op}} \otimes H$ -comodule algebras.

Proof. The statement follows from Lemma 3.2. Indeed, we have seen at the end of Section 1 that \mathbb{A} has two twist equivalent left $H \otimes H^{\text{op}}$ -comodule algebra structures. Identifying $(H \otimes H^{\text{op}})^{\text{op}}$ and $H^{\text{op}} \otimes H$, and computing the induced right coactions we obtain the structures defined in (3.11-3.14). We point out that the reassociator, the antipode and the Drinfeld twist corresponding to $H \otimes H^{\text{op}}$ are given by

$$\begin{aligned} \Phi_{H \otimes H^{\text{op}}} &= (X^1 \otimes x^1) \otimes (X^2 \otimes x^2) \otimes (X^3 \otimes x^3), \\ S_{H \otimes H^{\text{op}}} &= S \otimes S^{-1}, \quad f_{H \otimes H^{\text{op}}} = (f^1 \otimes S^{-1}(g^2)) \otimes (f^2 \otimes S^{-1}(g^1)), \end{aligned}$$

where, as usual, $g^1 \otimes g^2$ is the inverse of $f = f^1 \otimes f^2$. \square

Let H be a quasi-Hopf algebra, C an H -bimodule coalgebra and \mathbb{A} an H -bicomodule algebra. In the sequel, \mathbb{A}^1 and \mathbb{A}^2 will be our notation for the right $H^{\text{op}} \otimes H$ -comodule algebras $(\mathbb{A}, \rho_1, \Phi_{\rho_1})$ and $(\mathbb{A}, \rho_2, \Phi_{\rho_2})$. By the above arguments, it make sense to consider the left-right Doi-Hopf module categories ${}_{\mathbb{A}^1} \mathcal{M}(H^{\text{op}} \otimes H)^C$ and ${}_{\mathbb{A}^2} \mathcal{M}(H^{\text{op}} \otimes H)^C$. It will follow from Proposition 3.4 that these two categories are isomorphic.

Proposition 3.4. *Let H be a quasi-bialgebra, C a left H -module coalgebra and $\mathfrak{A}^1 = (\mathfrak{A}, \rho, \Phi_\rho)$ and $\mathfrak{A}^2 = (\mathfrak{A}, \rho', \Phi_{\rho'})$ two twist equivalent right H -comodule algebras. Then the categories ${}_{\mathfrak{A}^1} \mathcal{M}(H)^C$ and ${}_{\mathfrak{A}^2} \mathcal{M}(H)^C$ are isomorphic.*

Proof. If \mathfrak{A}^1 and \mathfrak{A}^2 are twist equivalent, then there exists $\mathbb{V} \in \mathfrak{A} \otimes H$ satisfying (1.22-1.24). Take $M \in {}_{\mathfrak{A}^1} \mathcal{M}(H)^C$; M becomes an object in ${}_{\mathfrak{A}^2} \mathcal{M}(H)^C$ by keeping the same left \mathfrak{A} -module structure and defining

$$\rho'_M : M \rightarrow M \otimes C, \quad \rho'_M(m) = \mathbb{V} \cdot \rho_M(m).$$

Conversely, take $M \in {}_{\mathfrak{A}^2} \mathcal{M}(H)^C$ via the structures \cdot and ρ'_M . Then M can be viewed as a left-right (H, \mathfrak{A}^1, C) -Hopf module via the same left \mathfrak{A} -action \cdot and the right C -coaction ρ_M defined by

$$\rho_M : M \rightarrow M \otimes C, \quad \rho_M(m) = \mathbb{V}^{-1} \cdot \rho'_M(m).$$

These correspondences define two functors which act as the identity on morphisms and produce inverse isomorphisms. \square

Remark 3.5. Let H be a quasi-bialgebra, and $F \in H \otimes H$ a gauge transformation. We can consider the twisted quasi-bialgebra H_F (see (1.7-1.8)), and it is well-known that the categories of left H -modules and left H_F -modules are isomorphic. We have a similar property for Doi-Hopf modules.

Let $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right H -comodule algebra, and let $\Phi_{\rho_F} = (1_{\mathfrak{A}} \otimes F)\Phi_\rho$. Then $\mathfrak{A}_F := (\mathfrak{A}, \rho, \Phi_{\rho_F})$ is a right H_F -comodule algebra, see [14].

Now let C be a left H -module coalgebra, and define a new comultiplication $\underline{\Delta}_F$ as follows: $\underline{\Delta}_F(c) = F\underline{\Delta}(c)$, for all $c \in C$. Then straightforward computations show that $(C, \underline{\Delta}_F, \varepsilon)$ is a left H_F -module coalgebra and that the categories ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ and ${}_{\mathfrak{A}_F}\mathcal{M}(H_F)^{C_F}$ are isomorphic. Of course, a similar result holds for left comodule algebras.

3.2. Yetter-Drinfeld modules and Doi-Hopf modules. Our next aim is to show that the category of left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to the category of Doi-Hopf modules ${}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C$ and, a fortiori, to ${}_{\mathbb{A}^1}\mathcal{M}(H^{\text{op}} \otimes H)^C$, by Proposition 3.4. We have divided the proof over a few lemmas.

Lemma 3.6. *Let H be a quasi-Hopf algebra, \mathbb{A} an H -bicomodule algebra and C an H -bimodule coalgebra. We have a functor*

$$F : {}_{\mathbb{A}}\mathcal{YD}(H)^C \rightarrow {}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C$$

which acts as the identity on objects and morphisms. If M is a left-right (H, \mathbb{A}, C) -Yetter-Drinfeld module then $F(M) = M$ as a left \mathbb{A} -module, and with the newly defined right C -coaction

$$(3.15) \quad \rho'_M(m) = m_{(0)'} \otimes m_{(1)'} = (\tilde{p}_\lambda^2 \cdot m)_{(0)} \otimes (\tilde{p}_\lambda^2 \cdot m)_{(1)} \cdot \tilde{p}_\lambda^1,$$

for all $m \in M$. Here $\tilde{p}_\lambda = \tilde{p}_\lambda^1 \otimes \tilde{p}_\lambda^2$ is the element defined by (1.25).

Proof. It is not hard to see that (1.33) and (1.5) imply

$$(3.16) \quad \Theta_{[0]}^2 \tilde{p}_\lambda^2 \otimes \Theta_{[-1]}^2 \tilde{p}_\lambda^1 S^{-1}(\Theta^1) \otimes \Theta^3 = \theta^2 (\tilde{p}_\lambda^2)_{(0)} \otimes \theta^1 \tilde{p}_\lambda^1 \otimes \theta^3 (\tilde{p}_\lambda^2)_{(1)}.$$

We now show that $F(M)$ satisfies the relations (2.17-2.19). Let $\tilde{P}_\lambda^1 \otimes \tilde{P}_\lambda^2$ be another copy of \tilde{p}_λ , and compute

$$\begin{aligned} & \Phi_{\rho_2} \cdot (\rho'_M \otimes id_C)(\rho_M(m)) \\ & \stackrel{(3.14, 3.15, 3.6)}{=} (\tilde{x}_\lambda^3)_{(0)} \tilde{X}_\rho^1 \Theta_{(0)}^2 \cdot (\tilde{P}_\lambda^2 \cdot (\tilde{p}_\lambda^2 \cdot m)_{(0)})_{(0)} \\ & \quad \otimes (\tilde{x}_\lambda^3)_{(1)_1} \tilde{X}_\rho^2 \Theta_{(1)}^2 \cdot (\tilde{P}_\lambda^2 \cdot (\tilde{p}_\lambda^2 \cdot m)_{(0)})_{(1)} \cdot \tilde{P}_\lambda^1 S^{-1}(f^2 \tilde{x}_\lambda^2 \Theta^1) \\ & \quad \otimes (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \Theta^3 \cdot (\tilde{p}_\lambda^2 \cdot m)_{(1)} \cdot \tilde{p}_\lambda^1 S^{-1}(f^1 \tilde{x}_\lambda^1) \\ & \stackrel{(3.8, 3.16)}{=} (\tilde{x}_\lambda^3)_{(0)} \tilde{X}_\rho^1 \cdot (\theta^2 (\tilde{P}_\lambda^2)_{(0)} \cdot (\tilde{p}_\lambda^2 \cdot m)_{(0)})_{(0)} \\ & \quad \otimes (\tilde{x}_\lambda^3)_{(1)_1} \tilde{X}_\rho^2 \cdot (\theta^2 (\tilde{P}_\lambda^2)_{(0)} \cdot (\tilde{p}_\lambda^2 \cdot m)_{(0)})_{(1)} \cdot \theta^1 \tilde{P}_\lambda^1 S^{-1}(f^2 \tilde{x}_\lambda^2) \\ & \quad \otimes (\tilde{x}_\lambda^3)_{(1)_2} \tilde{X}_\rho^3 \theta^3 (\tilde{P}_\lambda^2)_{(1)} \cdot (\tilde{p}_\lambda^2 \cdot m)_{(1)} \cdot \tilde{p}_\lambda^1 S^{-1}(f^1 \tilde{x}_\lambda^1) \end{aligned}$$

$$\begin{aligned}
& \stackrel{(3.8,3.7)}{=} (\tilde{x}_\lambda^3)_{\langle 0 \rangle} \cdot (\tilde{y}_\lambda^3(\tilde{P}_\lambda^2)_{[0]}\tilde{p}_\lambda^2 \cdot m)_{\langle 0 \rangle} \\
& \quad \otimes (\tilde{x}_\lambda^3)_{\langle 1 \rangle_1} \cdot (\tilde{y}_\lambda^3(\tilde{P}_\lambda^2)_{[0]}\tilde{p}_\lambda^2 \cdot m)_{\langle 1 \rangle_1} \cdot \tilde{y}_\lambda^1 \tilde{P}_\lambda^1 S^{-1}(f^2 \tilde{x}_\lambda^2) \\
& \quad \otimes (\tilde{x}_\lambda^3)_{\langle 1 \rangle_2} \cdot (\tilde{y}_\lambda^3(\tilde{P}_\lambda^2)_{[0]}\tilde{p}_\lambda^2 \cdot m)_{\langle 1 \rangle_2} \cdot \tilde{y}_\lambda^2(\tilde{P}_\lambda^2)_{[-1]}\tilde{p}_\lambda^1 S^{-1}(f^1 \tilde{x}_\lambda^1) \\
(3.3,3.8,1.30) \quad & \stackrel{=}{=} (\tilde{p}_\lambda^2 \cdot m)_{\langle 0 \rangle} \otimes (\tilde{p}_\lambda^2 \cdot m)_{\langle 1 \rangle_1} \cdot (\tilde{p}_\lambda^1)_1 \otimes (\tilde{p}_\lambda^2 \cdot m)_{\langle 1 \rangle_2} \cdot (\tilde{p}_\lambda^1)_2 \\
(3.3,3.15) \quad & \stackrel{=}{=} m_{\langle 0 \rangle'} \otimes m_{\langle 1 \rangle'_1} \otimes m_{\langle 1 \rangle'_2} = (id_M \otimes \underline{\Delta})(\rho'_M(m)),
\end{aligned}$$

for all $m \in M$, as needed. The relation (2.18) is trivial and (2.19) follows from (1.26), (3.13) and (3.6). \square

Lemma 3.7. *Let H be a quasi-Hopf algebra, C an H -bimodule coalgebra and \mathbb{A} an H -bicomodule algebra. Then we have a functor*

$$G : {}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C \rightarrow {}_{\mathbb{A}}\mathcal{YD}(H)^C$$

which acts as the identity on objects and morphisms. Let M be a left-right $(H^{\text{op}} \otimes H, \mathbb{A}^2, C)$ -Hopf module, with left \mathbb{A} -action \cdot and right C -coaction ρ'_M , $\rho'_M(m) = m_{\langle 0 \rangle'} \otimes m_{\langle 1 \rangle'}$. Then $G(M) = M$ as a left \mathbb{A} -module, with new right C -coaction $\bar{\rho}_M : M \rightarrow M \otimes C$, given by the formula

$$(3.17) \quad \bar{\rho}_M(m) = m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle} = (\tilde{q}_\lambda^2)_{\langle 0 \rangle} \cdot m_{\langle 0 \rangle'} \otimes (\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot m_{\langle 1 \rangle'} \cdot S^{-1}(\tilde{q}_\lambda^1),$$

for all $m \in M$. Here $\tilde{q}_\lambda = \tilde{q}_\lambda^1 \otimes \tilde{q}_\lambda^2$ is the element defined in (1.25).

Proof. The most difficult part is to show that $G(M)$ satisfies the relations (3.7) and (3.8). M is a left-right (H, \mathbb{A}^2, C) -Hopf module, so we have by (2.17), (3.13), (3.14) and (1.14):

$$\begin{aligned}
& \tilde{X}_\rho^1((\tilde{x}_\lambda^3)_{\langle 0 \rangle} \Theta^2)_{\langle 0 \rangle} \cdot m_{\langle 0,0 \rangle'} \otimes \tilde{X}_\rho^2((\tilde{x}_\lambda^3)_{\langle 0 \rangle} \Theta^2)_{\langle 1 \rangle} \cdot m_{\langle 0,1 \rangle'} \cdot S^{-1}(f^2 \tilde{x}_\lambda^2 \Theta^1) \\
(3.18) \quad & \otimes \tilde{X}_\rho^3(\tilde{x}_\lambda^3)_{\langle 1 \rangle} \Theta^3 \cdot m_{\langle 1 \rangle'} \cdot S^{-1}(f^1 \tilde{x}_\lambda^1) = m_{\langle 0 \rangle'} \otimes m_{\langle 1 \rangle'_1} \otimes m_{\langle 1 \rangle'_2},
\end{aligned}$$

$$(3.19) \quad (u \cdot m)_{\langle 0 \rangle'} \otimes (u \cdot m)_{\langle 1 \rangle'} = u_{[0]_{\langle 0 \rangle}} \cdot m_{\langle 0 \rangle'} \otimes u_{[0]_{\langle 1 \rangle}} \cdot m_{\langle 1 \rangle'} \cdot S^{-1}(u_{[-1]}),$$

for all $m \in M$ and $u \in \mathbb{A}$. Also, (1.33) and (1.5) imply that

$$(3.20) \quad S(\theta^1) \tilde{q}_\lambda^1 \theta_{[-1]}^2 \otimes \tilde{q}_\lambda^2 \theta_{[0]}^2 \otimes \theta^3 = \tilde{q}_\lambda^1 \Theta^1 \otimes (\tilde{q}_\lambda^2)_{\langle 0 \rangle} \Theta^2 \otimes (\tilde{q}_\lambda^2)_{\langle 1 \rangle} \Theta^3.$$

Let $\tilde{Q}_\lambda^1 \otimes \tilde{Q}_\lambda^2$ be another copy of \tilde{q}_λ ; for all $m \in M$, we compute that

$$\begin{aligned}
& (\theta^2 \cdot m_{\langle 0 \rangle})_{\langle 0 \rangle} \otimes (\theta^2 \cdot m_{\langle 0 \rangle})_{\langle 1 \rangle} \otimes \theta^3 \cdot m_{\langle 1 \rangle} \\
(3.17,3.19) \quad & \stackrel{=}{=} (\tilde{Q}_\lambda^2 \theta_{[0]}^2 (\tilde{q}_\lambda^2)_{\langle 0 \rangle_{[0]}})_{\langle 0 \rangle} \cdot m_{\langle 0,0 \rangle'} \otimes (\tilde{Q}_\lambda^2 \theta_{[0]}^2 (\tilde{q}_\lambda^2)_{\langle 0 \rangle_{[0]}})_{\langle 1 \rangle} \cdot m_{\langle 0,1 \rangle'} \\
& \cdot S^{-1}(\tilde{Q}_\lambda^1 \theta_{[-1]}^2 (\tilde{q}_\lambda^2)_{\langle 0 \rangle_{[-1]}}) \theta^1 \otimes \theta^3 (\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot m_{\langle 1 \rangle'} \cdot S^{-1}(\tilde{q}_\lambda^1) \\
(3.20,1.32) \quad & \stackrel{=}{=} ((\tilde{Q}_\lambda^2 (\tilde{q}_\lambda^2)_{[0]})_{\langle 0 \rangle} \Theta^2)_{\langle 0 \rangle} \cdot m_{\langle 0,0 \rangle'} \otimes ((\tilde{Q}_\lambda^2 (\tilde{q}_\lambda^2)_{[0]})_{\langle 0 \rangle} \Theta^2)_{\langle 1 \rangle} \cdot m_{\langle 0,1 \rangle'} \\
& \cdot S^{-1}(\tilde{Q}_\lambda^1 (\tilde{q}_\lambda^2)_{[-1]} \Theta^1) \otimes (\tilde{Q}_\lambda^2 (\tilde{q}_\lambda^2)_{[0]})_{\langle 1 \rangle} \Theta^3 \cdot m_{\langle 1 \rangle'} \cdot S^{-1}(\tilde{q}_\lambda^1)
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(1.31,3.18,1.10)}{=} (\tilde{q}_\lambda^2(\tilde{x}_\lambda^3)_{[0]})_{\langle 0,0 \rangle} \tilde{x}_\rho^1 \cdot m_{(0)'} \\
& \quad \otimes (\tilde{q}_\lambda^2(\tilde{x}_\lambda^3)_{[0]})_{\langle 0,1 \rangle} \tilde{x}_\rho^2 \cdot m_{(1)'} \cdot S^{-1}(\tilde{q}_\lambda^1(\tilde{x}_\lambda^3)_{[-1]})_1 \tilde{x}_\lambda^1 \\
& \quad \otimes (\tilde{q}_\lambda^2(\tilde{x}_\lambda^3)_{[0]})_{\langle 1 \rangle} \tilde{x}_\rho^3 \cdot m_{(1)'} \cdot S^{-1}(\tilde{q}_\lambda^1(\tilde{x}_\lambda^3)_{[-1]})_2 \tilde{x}_\lambda^2 \\
& \stackrel{(1.14,1.33,3.19)}{=} \tilde{x}_\rho^1 \cdot ((\tilde{q}_\lambda^2)_{\langle 0 \rangle} \cdot (\tilde{x}_\lambda^3 \cdot m)_{(0)'}) \\
& \quad \otimes \tilde{x}_\rho^2 \cdot ((\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot (\tilde{x}_\lambda^3 \cdot m)_{(1)'}) \cdot S^{-1}(\tilde{q}_\lambda^1)_{\perp} \cdot \tilde{x}_\lambda^1 \\
& \quad \otimes \tilde{x}_\rho^3 \cdot ((\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot (\tilde{x}_\lambda^3 \cdot m)_{(1)'}) \cdot S^{-1}(\tilde{q}_\lambda^1)_{\underline{2}} \cdot \tilde{x}_\lambda^2 \\
& \stackrel{(3.17)}{=} \tilde{x}_\rho^1 \cdot (\tilde{x}_\lambda^3 \cdot m)_{\overline{(0)}} \otimes \tilde{x}_\rho^2 \cdot (\tilde{x}_\lambda^3 \cdot m)_{\overline{(1)_{\perp}}} \cdot \tilde{x}_\lambda^1 \otimes \tilde{x}_\rho^3 \cdot (\tilde{x}_\lambda^3 \cdot m)_{\overline{(1)_{\underline{2}}}} \cdot \tilde{x}_\lambda^2,
\end{aligned}$$

as needed. (3.8) also holds since

$$\begin{aligned}
& u_{\langle 0 \rangle} \cdot m_{\overline{(0)}} \otimes u_{\langle 1 \rangle} \cdot m_{\overline{(1)}} \\
& \stackrel{(3.17,1.27)}{=} (\tilde{q}_\lambda^2 u_{[0,0]})_{\langle 0 \rangle} \cdot m_{(0)' } \otimes (\tilde{q}_\lambda^2 u_{[0,0]})_{\langle 1 \rangle} \cdot m_{(1)' } \cdot S^{-1}(\tilde{q}_\lambda^1 u_{[0,-1]}) u_{[-1]} \\
& \stackrel{(3.19)}{=} (\tilde{q}_\lambda^2)_{\langle 0 \rangle} \cdot (u_{[0]} \cdot m)_{(0)' } \otimes (\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot (u_{[0]} \cdot m)_{(1)' } \cdot S^{-1}(\tilde{q}_\lambda^1) u_{[-1]} \\
& \stackrel{(3.17)}{=} (u_{[0]} \cdot m)_{\overline{(0)}} \otimes (u_{[0]} \cdot m)_{\overline{(1)}} \cdot u_{[-1]},
\end{aligned}$$

for all $u \in \mathbb{A}$ and $m \in M$. The remaining details are left to the reader. \square

Theorem 3.8 is the main result of this Section, and generalizes [8, Theorem 2.3] to the quasi-Hopf algebra setting.

Theorem 3.8. *Let H be a quasi-Hopf algebra, \mathbb{A} an H -bicomodule algebra and C an H -bimodule coalgebra. Then the categories ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ and ${}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C$ are isomorphic. In particular ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is a Grothendieck category, and therefore it has enough injective objects.*

Proof. We show that the functors F and G from Lemmas 3.6 and 3.7 are inverses. Since F and G act as the identity functor at the level of \mathbb{A} -modules we have only to show that F and G are inverses at the level of C -coactions.

Let $M \in {}_{\mathbb{A}}\mathcal{YD}(H)^C$ and $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ its right C -coaction. We denote by $\bar{\rho}_M(m) = m_{\overline{(0)}} \otimes m_{\overline{(1)}}$ the right C -coaction of $G(F(M))$ obtained using first Lemma 3.6 and then Lemma 3.7. For all $m \in M$ we then have

$$\begin{aligned}
\bar{\rho}_M(m) & \stackrel{(3.17)}{=} (\tilde{q}_\lambda^2)_{\langle 0 \rangle} \cdot m_{(0)' } \otimes (\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot m_{(1)' } \cdot S^{-1}(\tilde{q}_\lambda^1) \\
& \stackrel{(3.15)}{=} (\tilde{q}_\lambda^2)_{\langle 0 \rangle} \cdot (\tilde{p}_\lambda^2 \cdot m)_{(0)' } \otimes (\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot (\tilde{p}_\lambda^2 \cdot m)_{(1)' } \cdot \tilde{p}_\lambda^1 S^{-1}(\tilde{q}_\lambda^1) \\
& \stackrel{(3.8)}{=} ((\tilde{q}_\lambda^2)_{[0]} \tilde{p}_\lambda^2 \cdot m)_{(0)' } \otimes ((\tilde{q}_\lambda^2)_{[0]} \tilde{p}_\lambda^2 \cdot m)_{(1)' } \cdot (\tilde{q}_\lambda^2)_{[-1]} \tilde{p}_\lambda^1 S^{-1}(\tilde{q}_\lambda^1) \\
& \stackrel{(1.28)}{=} m_{(0)' } \otimes m_{(1)' } = \rho_M(m).
\end{aligned}$$

Conversely, let $M \in {}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C$ and denote by $\rho'_M(m) = m_{(0)' } \otimes m_{(1)' }$ its right C -coaction. If $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ is the right C -coaction on $F(G(M))$ obtained using first Lemma 3.7 and then Lemma 3.6 we have

$$\begin{aligned}
\rho_M(m) & \stackrel{(3.15)}{=} (\tilde{p}_\lambda^2 \cdot m)_{\overline{(0)}} \otimes (\tilde{p}_\lambda^2 \cdot m)_{\overline{(1)}} \cdot \tilde{p}_\lambda^1 \\
& \stackrel{(3.17)}{=} (\tilde{q}_\lambda^2)_{\langle 0 \rangle} \cdot (\tilde{p}_\lambda^2 \cdot m)_{(0)' } \otimes (\tilde{q}_\lambda^2)_{\langle 1 \rangle} \cdot (\tilde{p}_\lambda^2 \cdot m)_{(1)' } \cdot S^{-1}(\tilde{q}_\lambda^1) \tilde{p}_\lambda^1 \\
& \stackrel{(3.19)}{=} (\tilde{q}_\lambda^2(\tilde{p}_\lambda^2)_{[0]})_{\langle 0 \rangle} \cdot m_{(0)' } \otimes (\tilde{q}_\lambda^2(\tilde{p}_\lambda^2)_{[0]})_{\langle 1 \rangle} \cdot m_{(1)' } \cdot S^{-1}(\tilde{q}_\lambda^1(\tilde{p}_\lambda^2)_{[-1]}) \tilde{p}_\lambda^1 \\
& \stackrel{(1.29)}{=} m_{(0)' } \otimes m_{(1)' } = \rho'_M(m),
\end{aligned}$$

for all $m \in M$, so the proof is finished. \square

Let H be a quasi-bialgebra, \mathfrak{A} a right H -comodule algebra and C a left H -module coalgebra. Identifying the category of left-right (H, \mathfrak{A}, C) -Hopf modules to the category of right-left $(H^{\text{op}}, C^{\text{cop}}, \mathfrak{A}^{\text{op}}, C^{\text{cop}})$ -Hopf modules using the construction preceding Proposition 2.2, we obtain the functor, after permuting the tensor factors:

$$\mathcal{F}' = \bullet \otimes C : {}_{\mathfrak{A}}\mathcal{M} \rightarrow {}_{\mathfrak{A}}\mathcal{M}(H)^C.$$

If M is a left \mathfrak{A} -module then $\mathcal{F}'(M) = M \otimes C$ with structure maps

$$\begin{aligned} \mathbf{a} \cdot (m \otimes c) &= \mathbf{a}_{(0)} \cdot m \otimes \mathbf{a}_{(1)} \cdot c, \\ \rho_M(m \otimes c) &= \tilde{x}_\rho^1 \cdot m \otimes \tilde{x}_\rho^2 \cdot c_{\underline{1}} \otimes \tilde{x}_\rho^3 \cdot c_{\underline{2}}, \end{aligned}$$

for all $\mathbf{a} \in \mathfrak{A}$, $m \in M$ and $c \in C$. For a morphism v in ${}_{\mathfrak{A}}\mathcal{M}$, we have that $\mathcal{F}'(v) = v \otimes id_C$. In particular, we obtain that $\mathfrak{A} \otimes C$ is a left-right (H, \mathfrak{A}, C) -Hopf module.

Moreover, \mathcal{F}' is a right adjoint of the forgetful functor $\mathcal{U}^C : {}_{\mathfrak{A}}\mathcal{M}(H)^C \rightarrow {}_{\mathfrak{A}}\mathcal{M}$, and it is a left adjoint of the functor $\text{Hom}_{\mathfrak{A}}^C(\mathfrak{A} \otimes C, \bullet) : {}_{\mathfrak{A}}\mathcal{M}(H)^C \rightarrow {}_{\mathfrak{A}}\mathcal{M}$ defined as follows. For $M \in {}_{\mathfrak{A}}\mathcal{M}(H)^C$, $\text{Hom}_{\mathfrak{A}}^C(\mathfrak{A} \otimes C, \bullet)(M) = \text{Hom}_{\mathfrak{A}}^C(\mathfrak{A} \otimes C, M)$, the set of morphisms in ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ between $\mathfrak{A} \otimes C$ and M , viewed as a left \mathfrak{A} -module via

$$(\mathbf{a} \cdot \eta)(\mathbf{a}' \otimes c) = \eta(\mathbf{a}'\mathbf{a} \otimes c),$$

for all $\eta \in \text{Hom}_{\mathfrak{A}}^C(\mathfrak{A} \otimes C, M)$, $\mathbf{a}, \mathbf{a}' \in \mathfrak{A}$ and $c \in C$. The functor $\text{Hom}_{\mathfrak{A}}^C(\mathfrak{A} \otimes C, \bullet)$ sends a morphism κ from ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ to the morphism $\vartheta \mapsto \kappa \circ \vartheta$.

Corollary 3.9. *Let H be a quasi-Hopf algebra, \mathbb{A} an H -bicomodule algebra and C an H -bimodule coalgebra. We have a functor $\mathfrak{F} = \bullet \otimes C : {}_{\mathbb{A}}\mathcal{M} \rightarrow {}_{\mathbb{A}}\mathcal{YD}(H)^C$. The structure maps on $\mathfrak{F}(M) = M \otimes C$ are the following:*

$$\begin{aligned} u \cdot (m \otimes c) &= u_{[0]_{(0)}} \cdot m \otimes u_{[0]_{(1)}} \cdot c \cdot S^{-1}(u_{[-1]}), \\ \rho_{M \otimes C}(m \otimes c) &= \theta_{(0)}^2 \tilde{x}_\rho^1 ((\tilde{q}_\lambda^2)_{[0]} \tilde{X}_\lambda^3)_{(0)} \cdot m \otimes \theta_{(1)}^2 \tilde{x}_\rho^2 ((\tilde{q}_\lambda^2)_{[0]} \tilde{X}_\lambda^3)_{(1)_1} \cdot c_{\underline{1}} \\ &\quad \cdot S^{-1}(\theta^1 (\tilde{q}_\lambda^2)_{[-1]} \tilde{X}_\lambda^2 g^2) \otimes \theta^3 \tilde{x}_\rho^3 ((\tilde{q}_\lambda^2)_{[0]} \tilde{X}_\lambda^3)_{(1)_2} \cdot c_{\underline{2}} \cdot S^{-1}(\tilde{q}_\lambda^1 \tilde{X}_\lambda^1 g^1), \end{aligned}$$

for all $u \in \mathbb{A}$, $m \in M$ and $c \in C$. In particular, $\mathbb{A} \otimes C$ is a left-right (H, \mathbb{A}, C) -Yetter-Drinfeld module. Moreover, the following assertions hold:

- i) \mathfrak{F} is right adjoint to the forgetful functor $\mathfrak{U}^C : {}_{\mathbb{A}}\mathcal{YD}(H)^C \rightarrow {}_{\mathbb{A}}\mathcal{M}$.
- ii) \mathfrak{F} is left adjoint to the functor $\text{Hom}_{\mathbb{A}}^C(\mathbb{A} \otimes C, \bullet) : {}_{\mathbb{A}}\mathcal{YD}(H)^C \rightarrow {}_{\mathbb{A}}\mathcal{M}$. If $M \in {}_{\mathbb{A}}\mathcal{YD}(H)^C$ then $\text{Hom}_{\mathbb{A}}^C(\mathbb{A} \otimes C, \bullet)(M) = \text{Hom}_{\mathbb{A}}^C(\mathbb{A} \otimes C, M)$, the set of Yetter-Drinfeld morphisms from $\mathbb{A} \otimes C$ to M , viewed as a right \mathbb{A} -module via the action

$$(u \cdot \eta)(u' \otimes c) = \eta(u'u \otimes c),$$

for all $\eta \in \text{Hom}_{\mathbb{A}}^C(\mathbb{A} \otimes C, M)$, $u, u' \in \mathbb{A}$ and $c \in C$.

Proof. The functor \mathfrak{F} is well defined because it is the composition

$$\mathfrak{F} : {}_{\mathbb{A}}\mathcal{M} = {}_{\mathbb{A}^2}\mathcal{M} \xrightarrow{\mathcal{F}'} {}_{\mathbb{A}^2}\mathcal{M}(H^{\text{op}} \otimes H)^C \xrightarrow{G} {}_{\mathbb{A}}\mathcal{YD}(H)^C,$$

where \mathcal{F}' is the functor described above but now for the context given by (3.13-3.14), and G is the functor from Lemma 3.7. All the others details are left to the reader. \square

Let H be a quasi-bialgebra, \mathfrak{A} a right H -comodule algebra and C a left H -module coalgebra. Since the category ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ can be identified to a category of right-left Doi-Hopf modules it follows that ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to a category of comodules over the coring $\mathcal{C}' = C \otimes \mathfrak{A}$, with \mathfrak{A} -bimodule structure given by

$$\mathbf{a} \cdot (c \otimes \mathbf{a}') \cdot \mathbf{a}'' = \mathbf{a}_{\langle 1 \rangle} \cdot c \otimes \mathbf{a}_{\langle 0 \rangle} \mathbf{a}' \mathbf{a}''$$

and comultiplication and counit given by

$$\Delta_{\mathcal{C}'}(c \otimes \mathbf{a}) = (\tilde{x}_\rho^3 \cdot c_\perp \otimes 1_{\mathfrak{A}}) \otimes_{\mathfrak{A}} (\tilde{x}_\rho^2 \cdot c_\perp \otimes \tilde{x}_\rho^1 \mathbf{a}), \quad \varepsilon_{\mathcal{C}'}(c \otimes \mathbf{a}) = \underline{\varepsilon}(c) \mathbf{a}$$

for all $\mathbf{a}, \mathbf{a}', \mathbf{a}'' \in \mathfrak{A}$ and $c \in C$. Using arguments similar to the ones given in [3, Theorem 5.4], one can easily check that ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to ${}^C\mathcal{M}$.

Corollary 3.10. *Let H be a quasi-Hopf algebra, \mathbb{A} an H -bicomodule algebra and C an H -bimodule coalgebra. Then the category of left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to the category of left comodules over the coring $\mathcal{C}' = C \otimes \mathbb{A}$, with the following structure maps. The \mathbb{A} -bimodule is given by*

$$u \cdot (c \otimes u') \cdot u'' = u_{[0]_{\langle 1 \rangle}} \cdot c \cdot S^{-1}(u_{[-1]}) \otimes u_{[0]_{\langle 0 \rangle}} u' u'',$$

and the comultiplication and counit are defined by the formulas

$$\begin{aligned} \Delta_{\mathcal{C}'}(c \otimes u) &= \left(\theta^3 \tilde{x}_\rho^3 (\tilde{X}_\lambda^3)_{\langle 1 \rangle_2} \cdot c_\perp \cdot S^{-1}(\tilde{X}_\lambda^1 g^1) \otimes 1_{\mathbb{A}} \right) \\ &\quad \otimes_{\mathbb{A}} \left(\theta_{\langle 1 \rangle}^2 \tilde{x}_\rho^2 (\tilde{X}_\lambda^3)_{\langle 1 \rangle_1} \cdot c_\perp \cdot S^{-1}(\theta^1 \tilde{X}_\lambda^2 g^2) \otimes \theta_{\langle 0 \rangle}^2 \tilde{x}_\rho^1 (\tilde{X}_\lambda^3)_{\langle 0 \rangle} u \right), \\ \varepsilon_{\mathcal{C}'}(c \otimes u) &= \underline{\varepsilon}(c) u, \end{aligned}$$

for all $u, u', u'' \in \mathbb{A}$ and $c \in C$.

Proof. This is a direct consequence of the above comments and Theorem 3.8. \square

3.3. The category of Yetter-Drinfeld modules as a module category. Our next aim is to describe the category of Yetter-Drinfeld modules as a category of modules. We will need the (right) generalized diagonal crossed product construction, as introduced in [14, 7].

Let H be a quasi-bialgebra and \mathcal{A} an H -bicomodule algebra. In the sequel, \mathcal{A} will be an H -bimodule algebra. This means that \mathcal{A} is an H -bimodule which has a multiplication and a usual unit $1_{\mathcal{A}}$ such that for all $\varphi, \psi, \xi \in \mathcal{A}$ and $h \in H$ the following relations hold:

$$\begin{aligned} (\varphi\psi)\xi &= (X^1 \cdot \varphi \cdot x^1)[(X^2 \cdot \psi \cdot x^2)(X^3 \cdot \xi \cdot x^3)], \\ h \cdot (\varphi\psi) &= (h_1 \cdot \varphi)(h_2 \cdot \psi), \quad (\varphi\psi) \cdot h = (\varphi \cdot h_1)(\psi \cdot h_2), \\ h \cdot 1_{\mathcal{A}} &= \varepsilon(h)1_{\mathcal{A}}, \quad 1_{\mathcal{A}} \cdot h = \varepsilon(h)1_{\mathcal{A}}. \end{aligned}$$

If H is a quasi-bialgebra, then H^* , the linear dual of H , is an H -bimodule via the H -actions

$$\langle h \rightharpoonup \varphi, h' \rangle = \varphi(h'h), \quad \langle \varphi \leftarrow h, h' \rangle = \varphi(hh'),$$

for all $\varphi \in H^*$, $h, h' \in H$. The convolution $\langle \varphi\psi, h \rangle = \varphi(h_1)\psi(h_2)$, $\varphi, \psi \in H^*$, $h \in H$, is a multiplication on H^* which is not associative in general, but with this multiplication H^* becomes an H -bimodule algebra.

Let H be a quasi-bialgebra, \mathcal{A} an H -bimodule algebra and $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda, \rho})$ an H -bicomodule algebra. In the sequel (δ, Ψ) will be the pair

$$\begin{aligned} \delta_l &= (\lambda \otimes id_H) \circ \rho, \\ \Psi &= (id_H \otimes \lambda \otimes id_H^{\otimes 2}) ((\Phi_{\lambda, \rho} \otimes 1_H)(\lambda \otimes id_H^{\otimes 2})(\Phi_\rho^{-1})) [\Phi_\lambda \otimes 1_H^{\otimes 2}], \end{aligned}$$

or

$$\begin{aligned}\delta_r &= (id_H \otimes \rho) \circ \lambda, \\ \Psi_r &= (id_H^{\otimes 2} \otimes \rho \otimes id_H) \left((1_H \otimes \Phi_{\lambda, \rho}^{-1})(id_H^{\otimes 2} \otimes \rho)(\Phi_\lambda) \right) [1_H^{\otimes 2} \otimes \Phi_\rho^{-1}].\end{aligned}$$

$\Omega_{L_{l/r}}, \Omega_{R_{l/r}} \in H^{\otimes 2} \otimes \mathbb{A} \otimes H^{\otimes 2}$ are defined by the following formulas

$$\begin{aligned}\Omega_{L_{l/r}} &= (id_H^{\otimes 2} \otimes id_{\mathbb{A}} \otimes S^{-1} \otimes S^{-1})(\Psi_{l/r}^{-1}) \cdot [1_H^{\otimes 2} \otimes 1_{\mathbb{A}} \otimes S^{-1}(f^1) \otimes S^{-1}(f^2)], \\ \Omega_{R_{l/r}} &= [S^{-1}(g^1) \otimes S^{-1}(g^2) \otimes 1_{\mathbb{A}} \otimes 1_H^{\otimes 2}] \cdot (S^{-1} \otimes S^{-1} \otimes id_{\mathbb{A}} \otimes id_H^{\otimes 2})(\Psi_{l/r}).\end{aligned}$$

Here $f = f^1 \otimes f^2$ is the Drinfeld twist and $f^{-1} = g^1 \otimes g^2$ is its inverse. We will use the notation

$$\Omega_{L_{l/r}} = \Omega_{L_{l/r}}^1 \otimes \cdots \otimes \Omega_{L_{l/r}}^5.$$

Let $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ and $\mathcal{A} \blacktriangleright_{\delta_r} \mathbb{A}$ be the vector space $\mathcal{A} \otimes \mathbb{A}$ furnished with the multiplication given respectively by the following formulas:

$$(3.21) \quad \begin{aligned} &(\varphi \bowtie u)(\psi \bowtie u') \\ &= (\Omega_{L_l}^1 \cdot \varphi \cdot \Omega_{L_l}^5)(\Omega_{L_l}^2 u_{\langle 0 \rangle_{[-1]}} \cdot \psi \cdot S^{-1}(u_{\langle 1 \rangle}) \Omega_{L_l}^4) \bowtie \Omega_{L_l}^3 u_{\langle 0 \rangle_{[0]}} u', \end{aligned}$$

$$(3.22) \quad \begin{aligned} &(\varphi \blacktriangleright u)(\psi \blacktriangleright u') \\ &= (\Omega_{L_r}^1 \cdot \varphi \cdot \Omega_{L_r}^5)(\Omega_{L_r}^2 u_{[-1]} \cdot \psi \cdot S^{-1}(u_{[0]_{\langle 1 \rangle}}) \Omega_{L_r}^4) \blacktriangleright \Omega_{L_r}^3 u_{[0]_{\langle 0 \rangle}} u', \end{aligned}$$

for all $\varphi, \psi \in \mathcal{A}$ and $u, u' \in \mathbb{A}$. We write $\varphi \bowtie u$ (respectively $\varphi \blacktriangleright u$) for $\varphi \otimes u$ considered as an element of $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ (respectively $\mathcal{A} \blacktriangleright_{\delta_r} \mathbb{A}$). $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ and $\mathcal{A} \blacktriangleright_{\delta_r} \mathbb{A}$ are isomorphic associative algebras with unit $1_{\mathcal{A}} \bowtie 1_{\mathbb{A}}$ (respectively $1_{\mathcal{A}} \blacktriangleright 1_{\mathbb{A}}$), containing $\mathbb{A} \cong 1_{\mathcal{A}} \bowtie \mathbb{A}$ (respectively $\mathbb{A} \cong 1_{\mathcal{A}} \blacktriangleright \mathbb{A}$) as unital subalgebra. These algebras are called the left generalized diagonal crossed products.

The right generalized diagonal crossed products are introduced in a similar way: denote

$$\Omega_{R_{l/r}} = \Omega_{R_{l/r}}^1 \otimes \cdots \otimes \Omega_{R_{l/r}}^5,$$

and let $\mathbb{A} \bowtie_{\delta_l} \mathcal{A}$ and $\mathbb{A} \blacktriangleright_{\delta_r} \mathcal{A}$ be the vector space $\mathbb{A} \otimes \mathcal{A}$ with the following product:

$$(3.23) \quad \begin{aligned} &(u \bowtie \varphi)(u' \bowtie \psi) \\ &= uu'_{\langle 0 \rangle_{[0]}} \Omega_{R_l}^3 \bowtie (\Omega_{R_l}^2 S^{-1}(u'_{\langle 0 \rangle_{[-1]}}) \cdot \varphi \cdot u'_{\langle 1 \rangle}) \Omega_{R_l}^4 (\Omega_{R_l}^1 \cdot \psi \cdot \Omega_{R_l}^5), \end{aligned}$$

$$(3.24) \quad \begin{aligned} &(u \blacktriangleright \varphi)(u' \blacktriangleright \psi) \\ &= uu'_{[0]_{\langle 0 \rangle}} \Omega_{R_r}^3 \blacktriangleright (\Omega_{R_r}^2 S^{-1}(u'_{[-1]}) \cdot \varphi \cdot u'_{[0]_{\langle 1 \rangle}}) \Omega_{R_r}^4 (\Omega_{R_r}^1 \cdot \psi \cdot \Omega_{R_r}^5), \end{aligned}$$

for all $u, u' \in \mathbb{A}$ and $\varphi, \psi \in \mathcal{A}$. $\mathbb{A} \bowtie_{\delta_l} \mathcal{A}$ and $\mathbb{A} \blacktriangleright_{\delta_r} \mathcal{A}$ are isomorphic associative algebras with unit $1_{\mathbb{A}} \bowtie 1_{\mathcal{A}}$ and $1_{\mathbb{A}} \blacktriangleright 1_{\mathcal{A}}$, containing \mathbb{A} as a unital subalgebra.

As algebras, the left and right generalized crossed product algebras are isomorphic, see [14, 7]. If H is a quasi-Hopf algebra then $\mathcal{A} = H^*$ is an H -bimodule algebra. In this particular case the left and right generalized diagonal crossed products are exactly the left and the right diagonal crossed products constructed in [14]. In this way Hausser and Nill gave four explicit realizations of $D(H)$, the quantum double of a finite dimensional quasi-Hopf algebra H . Two of them are built on $H^* \otimes H$ and the other two on $H \otimes H^*$. All these are, as algebras, diagonal crossed products. The first two realizations built on $H^* \otimes H$ coincide to $H^* \bowtie_{\delta_l} H$ and $H^* \blacktriangleright_{\delta_r} H$, and the last two realizations built on $H \otimes H^*$ coincide to $H \bowtie_{\delta_l} H^*$ and $H \blacktriangleright_{\delta_r} H^*$.

Proposition 3.11. *Let H be a quasi-Hopf algebra, \mathbb{A} an H -bicomodule algebra and C an H -bimodule coalgebra. Then $\mathbb{A}^1 \rtimes C^* \equiv \mathbb{A} \bowtie_{\delta_l} C^*$ and $\mathbb{A}^2 \rtimes C^* \equiv \mathbb{A} \blacktriangleright_{\delta_r} C^*$, as algebras. In particular, the algebras $\mathbb{A}^1 \rtimes C^*$ and $\mathbb{A}^2 \rtimes C^*$ are isomorphic to each other, and also to the four generalized diagonal crossed products.*

Proof. Keeping the above concepts and notations it is not hard to see that for an H -bicomodule algebra \mathbb{A} the reassociators Φ_{ρ_1} and Φ_{ρ_2} defined in (3.12) and (3.14), respectively, can be rewritten as

$$\begin{aligned}\Phi_{\rho_1} &= \tilde{\Omega}_{R_l}^3 \otimes (\tilde{\Omega}_{R_l}^2 \otimes \tilde{\Omega}_{R_l}^4) \otimes (\tilde{\Omega}_{R_l}^1 \otimes \tilde{\Omega}_{R_l}^5), \\ \Phi_{\rho_2} &= \tilde{\Omega}_{R_r}^3 \otimes (\tilde{\Omega}_{R_r}^2 \otimes \tilde{\Omega}_{R_r}^4) \otimes (\tilde{\Omega}_{R_r}^1 \otimes \tilde{\Omega}_{R_r}^5),\end{aligned}$$

where we used the notation

$$\Omega_{R_l/r}^{-1} = \tilde{\Omega}_{R_l/r}^1 \otimes \cdots \otimes \tilde{\Omega}_{R_l/r}^5.$$

Now, if C is an H -bimodule coalgebra viewed as a left $H^{\text{op}} \otimes H$ -module coalgebra via the structure defined in (3.6) then C^* , the linear dual space of C , is a right $H^{\text{op}} \otimes H$ -module algebra. The multiplication of C^* is the convolution, that is $(c^* d^*)(c) = c^*(c_1) d^*(c_2)$, the unit is $\underline{\varepsilon}$ and the right $H^{\text{op}} \otimes H$ -module action is given by the formula $(c^* \cdot (h \otimes h'))(c) = c^*(h' \cdot c \cdot h) = (h \rightarrow c^* \leftarrow h')(c)$, for all $h, h' \in H$, $c^*, d^* \in C^*$, $c \in C$.

Since $\mathbb{A}^{1/2}$ are right $H^{\text{op}} \otimes H$ -comodule algebras and C^* is a right $H^{\text{op}} \otimes H$ -module coalgebra, it makes sense to consider the right generalized smash product algebras $\mathbb{A}^{1/2} \rtimes C^*$ (cf. Remark 2.13 i)). It also follows easily from Remark 2.13 i) that the multiplication on $\mathbb{A}^1 \rtimes C^*$ is given by

$$\begin{aligned}(u \rtimes c^*)(u' \rtimes d^*) &= uu'_{(0)[0]} \Omega_{R_l}^3 \otimes \left((c^* \cdot (\Omega_{R_l}^2 S^{-1}(u'_{(0)[-1]}) \otimes u'_{(1)} \Omega_{R_l}^4))(d^* \cdot (\Omega_{R_l}^1 \otimes \Omega_{R_l}^5)) \right) \\ &= uu'_{(0)[0]} \Omega_{R_l}^3 \otimes (\Omega_{R_l}^2 S^{-1}(u'_{(0)[-1]}) \rightarrow c^* \leftarrow u'_{(1)} \Omega_{R_l}^4)(\Omega_{R_l}^1 \rightarrow d^* \leftarrow \Omega_{R_l}^5).\end{aligned}$$

On the other hand, it is easy to see that the linear dual space C^* of an H -bimodule coalgebra C is an H -bimodule algebra. The multiplication of C^* is the convolution, the unit is $\underline{\varepsilon}$ and the H -bimodule structure is given by the formula $(h \rightarrow c^* \leftarrow h')(c) = c^*(h' \cdot c \cdot h)$, for all $h, h' \in H$, $c^* \in C^*$, $c \in C$. So we can consider the generalized right diagonal crossed product $\mathbb{A} \bowtie_{\delta_l} C^*$. From (3.23) it follows that the multiplication rule on $\mathbb{A} \bowtie_{\delta_l} C^*$ coincides with the multiplication of $\mathbb{A}^1 \rtimes C^*$, hence the algebras $\mathbb{A} \bowtie_{\delta_l} C^*$ and $\mathbb{A}^1 \rtimes C^*$ are equal. In a similar way we can show the equality of the k -algebras $\mathbb{A}^2 \rtimes C^*$ and $\mathbb{A} \blacktriangleright_{\delta_r} C^*$. \square

Remark 3.12. It was shown in [7] that the left generalized crossed product algebras $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ and $\mathcal{A} \blacktriangleright_{\delta_r} \mathbb{A}$ coincide with the left generalized smash product algebras $\mathcal{A} \blacktriangleright \mathbb{A}_1$ and $\mathcal{A} \blacktriangleright \mathbb{A}_2$, respectively. The generalized smash products are made over $H \otimes H^{\text{op}}$ and by \mathbb{A}_1 and \mathbb{A}_2 we denote the left $H \otimes H^{\text{op}}$ -comodule algebra structures on \mathbb{A} defined at the end of Section 1.

Let H be a quasi-bialgebra, \mathfrak{A} a right H -comodule algebra and C a left H -module algebra. Viewing the category ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ as a category of right-left Doi-Hopf modules, we deduce from Theorem 2.6 that ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to the category of rational $\mathfrak{A} \rtimes C^*$ -modules, $\text{Rat}({}_{\mathfrak{A}} \rtimes C^* \mathcal{M})$, and that $\text{Rat}({}_{\mathfrak{A}} \rtimes C^* \mathcal{M}) = \sigma_{\mathfrak{A}} \rtimes C^* [{}_{\mathfrak{A}} \otimes C]$.

A rational $\mathfrak{A} \bowtie C^*$ -module is a left $\mathfrak{A} \bowtie C^*$ -module M such that for any $m \in M$ there exist two finite families $\{c_i\}_i \subseteq C$ and $\{m_i\}_i \subseteq M$ such that

$$(\mathfrak{a} \bowtie c^*) \cdot m = c^*(c_i)(\mathfrak{a} \bowtie \varepsilon) \cdot m_i,$$

for all $\mathfrak{a} \in \mathfrak{A}$ and $c^* \in C^*$.

Corollary 3.13. *Let H be a quasi-Hopf algebra, \mathbb{A} an H -bicomodule algebra and C an H -bimodule coalgebra. Then the following assertions hold:*

- i) *The category of left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to the category of rational $\mathbb{A} \bowtie_{\delta_r} C^*$ -modules $\text{Rat}(\mathbb{A} \bowtie_{\delta_r} C^* \mathcal{M})$, which is also equal to the category $\sigma_{\mathbb{A} \bowtie_{\delta_r} C^*}[\mathbb{A} \otimes C]$.*
- ii) *If C is finite dimensional then ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to the category of left $C^* \bowtie_{\delta_l} \mathbb{A}$ -modules.*

Proof. The assertion i) follows easily from the above comments and Theorem 3.8. ii) If C is finite dimensional then $\text{Rat}(\mathbb{A} \bowtie_{\delta_r} C^* \mathcal{M}) = \mathbb{A} \bowtie_{\delta_r} C^* \mathcal{M}$. Moreover, $\mathbb{A} \bowtie_{\delta_r} C^*$ is always isomorphic to $C^* \bowtie_{\delta_l} \mathbb{A}$ as an algebra. We note that another proof of this result can be found in [7]. \square

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