

# INVOLUTORY QUASI-HOPF ALGEBRAS

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*To Fred Van Oystaeyen on the occasion of his 60th birthday.*

ABSTRACT. We introduce and investigate the basic properties of an involutory (dual) quasi-Hopf algebra. We also study the representations of an involutory quasi-Hopf algebra and prove that an involutory dual quasi-Hopf algebra with non-zero integral is cosemisimple.

## INTRODUCTION

One of the aims in [8] was to find a plausible definition of an involutory quasi-Hopf algebra. Since the definition of a quasi-Hopf algebra  $H$  is given in such a way that the category of its finite dimensional representations  ${}_H\mathcal{M}^{\text{fd}}$  has a rigid monoidal structure it seems to be natural to relate the involutory notion to a certain property of this category.

On one hand, in the classical case of a Hopf algebra, a categorical interpretation for the involutory notion was given by Majid in [26]. Namely, he has shown that for a finite dimensional Hopf algebra  $H$ , the trace of the square of the antipode  $S$  of  $H$ ,  $\text{Tr}(S^2)$ , arises in a very natural way as the representation-theoretic rank of the Schrödinger representation of  $H$ ,  $\underline{\dim}(H)$ , or as the representation-theoretic rank of the canonical representation of the quantum double,  $\underline{\dim}(D(H))$ . Relating this to results of Larson and Radford [20, 21] and Etingof and Gelaki [15], we obtain that the above rank coincide to the classical dimension of  $H$  if  $H$  is involutory; otherwise it is zero.

This is why, for a finite dimensional quasi-Hopf algebra  $H$ , we computed in [8] the representation-theoretic rank of a finite dimensional quasi-Hopf algebra  $H$ , and of its associated quantum double  $D(H)$ , within the category of finite dimensional left modules over  $D(H)$ ,  ${}_{D(H)}\mathcal{M}^{\text{fd}}$ . More precisely, for a quasi-Hopf algebra  $H$ , we have

$$\underline{\dim}(H) = \underline{\dim}(D(H)) = \text{Tr}(h \mapsto S^{-2}(S(\beta)\alpha h\beta S(\alpha))).$$

Therefore, we call a quasi-Hopf algebra  $H$  involutory if  $S^2(h) = S(\beta)\alpha h\beta S(\alpha)$ , for all  $h \in H$ .

On the other hand, due to some recent results of Etingof, Nikshych and Ostrik [13], it seems that the notion of involutory quasi-Hopf algebra should be given in such a

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way that, at least over an algebraic closed field of characteristic zero, its category of finite dimensional representations is a fusion category and, moreover, has that unique pivotal structure with respect to which the categorical dimensions of simple objects coincide with their usual dimensions. In other words, an involutory quasi-Hopf algebra, say  $H$ , should be semisimple as an algebra and such that the identity functor and the second duality functor  $(-)^{**}$  of the category  ${}_H\mathcal{M}^{\text{fd}}$  are tensor isomorphic via a tensor functor, say  $j$ . In addition, we should have

$$\dim_k(V) = \text{ev}_{V^*} \circ (j_V \otimes \text{id}_{V^*}) \circ \text{coev}_V,$$

for any simple finite dimensional left  $H$ -module  $V$ , where  $\text{ev}_{V^*}$  and  $\text{coev}_V$  are the evaluation map of  $V^*$  and the coevaluation map of  $V$ , respectively. Note that the composition on the right hand side of the above equality was defined in [13] as being the categorical dimension of  $V$ .

We have to stress the fact that our definition for an involutory quasi-Hopf algebra agrees with this point of view. More precisely, over a field of characteristic zero (or, more generally, if  $\dim(H) \neq 0$  in  $k$ ) any involutory quasi-Hopf algebra is semisimple because of the trace formula for quasi-Hopf algebras proved in [8]. Also, by Lemma 2.2 below for an involutory quasi-Hopf algebra the square of the antipode in an inner automorphism defined by  $S(\beta)\alpha$ . Now, using some results from [25, 29] we will prove that the invertible element  $\mathfrak{g}$  which defines  $j$  is exactly  $\beta S(\alpha)$ , the inverse of  $S(\beta)\alpha$ . From here we conclude that the family of left  $H$ -module isomorphisms

$$j_V : V \in v \mapsto (v^* \mapsto v^*(S(\beta)\alpha \cdot v)) \in V^{**}$$

endows  ${}_H\mathcal{M}^{\text{fd}}$  with the unique pivotal structure with respect to which the categorical dimension of any simple object  $V$  of  ${}_H\mathcal{M}^{\text{fd}}$  coincides with its usual dimension. All the details are presented in Section 3.

Then the aim of this paper is to see that with our definition of an involutory (dual) quasi-Hopf algebra some results of Lorenz [24] can be generalized to quasi-Hopf algebras, and that the main result of Dăscălescu, Năstăsescu and Torrecillas in [10] has a counterpart for involutory dual quasi-Hopf algebras.

The paper is organized as follows. In Section 2 we introduce the involutory notion and then we study their basic properties. Starting with  $H(2)$ , the (involutory) quasi-Hopf algebra presented in [14], using different constructions as transmutation, bosonization or quantum double, we will be able to construct three involutory quasi-Hopf algebras of dimension 4. It comes out that all of them are  $k[C_2 \times C_2]$ , the group Hopf algebra associated to the Klein group, viewed as quasi-Hopf algebras via some non-equivalent 3-cocycles. Moreover, all of them are not coming from twisting  $k[C_2 \times C_2]$  by a gauge transformation. Also, we will prove in Section 3 that, under a “natural” condition, the involutory property survives when we pass from  $H$  to its quantum double. Note that this “natural” condition is satisfied if we work over an algebraic closed field of characteristic zero.

In Section 4 we study the representations of an involutory quasi-Hopf algebra. Namely, we show that if  $H$  is an involutory semisimple quasi-Hopf algebra over a field  $k$  then the characteristic of  $k$  does not divide the dimension of any finite dimensional absolutely simple  $H$ -module. We also prove that if  $H$  is a non-semisimple involutory quasi-Hopf algebra then the characteristic of  $k$  divides the dimension of any finite dimensional projective  $H$ -module. Both results generalize well-known results in Hopf algebra theory due to Larson and Lorenz, see [22, 24].

The first goal of Section 5 is to introduce and study the dual concept: involutory dual quasi-Hopf algebras. Then due to some recent results proved in [3] we will be able to show that any involutory co-Frobenius dual quasi-Hopf algebra over a field of characteristic zero is cosemisimple. This result was proved for Hopf algebras in [10] and it can be viewed as an extension of the results of Larson and Radford to the infinite dimensional case. For Hopf algebras it is well-known that the converse property is true only in the finite dimensional case.

### 1. PRELIMINARIES

We work over a commutative field  $k$ . All algebras, linear spaces etc. will be over  $k$ ; unadorned  $\otimes$  means  $\otimes_k$ . Following Drinfeld [12], a quasi-bialgebra is a four-tuple  $(H, \Delta, \varepsilon, \Phi)$  where  $H$  is an associative algebra with unit,  $\Phi$  is an invertible element in  $H \otimes H \otimes H$ , and  $\Delta : H \rightarrow H \otimes H$  and  $\varepsilon : H \rightarrow k$  are algebra homomorphisms satisfying the identities

$$(1.1) \quad (id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1},$$

$$(1.2) \quad (id \otimes \varepsilon)(\Delta(h)) = h, \quad (\varepsilon \otimes id)(\Delta(h)) = h,$$

for all  $h \in H$ , and  $\Phi$  has to be a 3-cocycle, in the sense that

$$(1.3) \quad (1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi),$$

$$(1.4) \quad (id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1.$$

The map  $\Delta$  is called the coproduct or the comultiplication,  $\varepsilon$  the counit and  $\Phi$  the reassociator. As for Hopf algebras we denote  $\Delta(h) = h_1 \otimes h_2$ , but since  $\Delta$  is only quasi-coassociative we adopt the further convention (summation understood):

$$(\Delta \otimes id)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all  $h \in H$ . We will denote the tensor components of  $\Phi$  by capital letters, and the ones of  $\Phi^{-1}$  by small letters, namely

$$\begin{aligned} \Phi &= X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \dots \\ \Phi^{-1} &= x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \dots \end{aligned}$$

$H$  is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism  $S$  of the algebra  $H$  and elements  $\alpha, \beta \in H$  such that, for all  $h \in H$ , we have:

$$(1.5) \quad S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1\beta S(h_2) = \varepsilon(h)\beta,$$

$$(1.6) \quad X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2\beta S(x^3) = 1.$$

Our definition of a quasi-Hopf algebra is different from the one given by Drinfeld [12] in the sense that we do not require the antipode to be bijective. Nevertheless, in the finite dimensional or quasi-triangular case this condition can be deleted because it follows from the other axioms, see [3] and [5].

The definition of a quasi-Hopf algebra is “twist coinvariant” in the following sense. An invertible element  $F \in H \otimes H$  is called a gauge transformation or twist if  $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$ . If  $H$  is a quasi-Hopf algebra and  $F = F^1 \otimes F^2 \in H \otimes H$  is a gauge transformation with inverse  $F^{-1} = G^1 \otimes G^2$ , then we can define a new quasi-Hopf algebra  $H_F$  by keeping the multiplication, unit, counit and antipode of

$H$  and replacing the comultiplication, reassociator and the elements  $\alpha$  and  $\beta$  by

$$(1.7) \quad \Delta_F(h) = F\Delta(h)F^{-1},$$

$$(1.8) \quad \Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1),$$

$$(1.9) \quad \alpha_F = S(G^1)\alpha G^2, \quad \beta_F = F^1\beta S(F^2).$$

For a quasi-Hopf algebra the antipode is determined uniquely up to a transformation  $\alpha \mapsto \alpha_{\mathbb{U}} := \mathbb{U}\alpha$ ,  $\beta \mapsto \beta_{\mathbb{U}} := \beta\mathbb{U}^{-1}$ ,  $S(h) \mapsto S_{\mathbb{U}}(h) := \mathbb{U}S(h)\mathbb{U}^{-1}$ , where  $\mathbb{U} \in H$  is invertible. In this case we will denote by  $H^{\mathbb{U}}$  the new quasi-Hopf algebra  $(H, \Delta, \varepsilon, \Phi, S_{\mathbb{U}}, \alpha_{\mathbb{U}}, \beta_{\mathbb{U}})$ .

If  $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$  is a quasi-bialgebra or a quasi-Hopf algebra then  $H^{\text{op}}$ ,  $H^{\text{cop}}$  and  $H^{\text{op,cop}}$  are also quasi-bialgebras (respectively quasi-Hopf algebras), where "op" means opposite multiplication and "cop" means opposite comultiplication. The structures are obtained by putting  $\Phi_{\text{op}} = \Phi^{-1}$ ,  $\Phi_{\text{cop}} = (\Phi^{-1})^{321}$ ,  $\Phi_{\text{op,cop}} = \Phi^{321}$ ,  $S_{\text{op}} = S_{\text{cop}} = (S_{\text{op,cop}})^{-1} = S^{-1}$ ,  $\alpha_{\text{op}} = S^{-1}(\beta)$ ,  $\beta_{\text{op}} = S^{-1}(\alpha)$ ,  $\alpha_{\text{cop}} = S^{-1}(\alpha)$ ,  $\beta_{\text{cop}} = S^{-1}(\beta)$ ,  $\alpha_{\text{op,cop}} = \beta$  and  $\beta_{\text{op,cop}} = \alpha$ .

The axioms for a quasi-Hopf algebra imply that  $\varepsilon \circ S = \varepsilon$  and  $\varepsilon(\alpha)\varepsilon(\beta) = 1$ , so, by rescaling  $\alpha$  and  $\beta$ , we may assume without loss of generality that  $\varepsilon(\alpha) = \varepsilon(\beta) = 1$ . The identities (1.2), (1.3) and (1.4) also imply that

$$(1.10) \quad (\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1.$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation  $f \in H \otimes H$  such that

$$(1.11) \quad f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\text{cop}}(h)), \quad \text{for all } h \in H,$$

where  $\Delta^{\text{cop}}(h) = h_2 \otimes h_1$ . The element  $f$  can be computed explicitly. First set

$$(1.12) \quad A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}),$$

$$(1.13) \quad B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1)$$

and then define  $\gamma, \delta \in H \otimes H$  by

$$(1.14) \quad \gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = B^1\beta S(B^4) \otimes B^2\beta S(B^3).$$

With this notation  $f$  and  $f^{-1}$  are given by the formulas

$$(1.15) \quad f = (S \otimes S)(\Delta^{\text{op}}(x^1))\gamma\Delta(x^2\beta S(x^3)),$$

$$(1.16) \quad f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{cop}}(x^3)).$$

Moreover,  $f$  satisfies the following relations:

$$(1.17) \quad f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta,$$

$$(1.18) \quad (1 \otimes f)(id \otimes \Delta)(f)\Phi(\Delta \otimes id)(f^{-1})(f^{-1} \otimes 1) = (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1).$$

In a Hopf algebra  $H$ , we obviously have the identity

$$h_1 \otimes h_2 S(h_3) = h \otimes 1, \quad \text{for all } h \in H.$$

We will need the generalization of this formula to quasi-Hopf algebras. Following [16, 17], we define

$$(1.19) \quad p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3), \quad q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2,$$

$$(1.20) \quad p_L = \tilde{p}^1 \otimes \tilde{p}^2 = X^2 S^{-1}(X^1 \beta) \otimes X^3, \quad q_L = \tilde{q}^1 \otimes \tilde{q}^2 = S(x^1) \alpha x^2 \otimes x^3.$$

For all  $h \in H$ , we then have

$$(1.21) \quad \Delta(h_1)p_R(1 \otimes S(h_2)) = p_R(h \otimes 1),$$

$$(1.22) \quad (1 \otimes S^{-1}(h_2))q_R\Delta(h_1) = (h \otimes 1)q_R,$$

$$(1.23) \quad \Delta(h_2)p_L(S^{-1}(h_1) \otimes 1) = p_L(1 \otimes h),$$

$$(1.24) \quad (S(h_1) \otimes 1)q_L\Delta(h_2) = (1 \otimes h)q_L.$$

Furthermore, the following relations hold

$$(1.25) \quad (1 \otimes S^{-1}(p^2))q_R\Delta(p^1) = 1 \otimes 1,$$

$$(1.26) \quad \Delta(q^1)p_R(1 \otimes S(q^2)) = 1 \otimes 1,$$

$$(1.27) \quad (S(\tilde{p}^1) \otimes 1)q_L\Delta(\tilde{p}^2) = 1 \otimes 1,$$

$$(1.28) \quad \Delta(\tilde{q}^2)p_L(S^{-1}(\tilde{q}^1) \otimes 1) = 1 \otimes 1.$$

Finally, for further use we need the notion of quasi-triangular quasi-Hopf algebra. Recall that a quasi-Hopf algebra  $H$  is quasi-triangular ( $QT$  for short) if there exists an element  $R = R^1 \otimes R^2 = r^1 \otimes r^2 \in H \otimes H$  such that

$$(1.29) \quad (\Delta \otimes id)(R) = X^2 R^1 x^1 Y^1 \otimes X^3 x^3 r^1 Y^2 \otimes X^1 R^2 x^2 r^2 Y^3$$

$$(1.30) \quad (id \otimes \Delta)(R) = x^3 R^1 X^2 r^1 y^1 \otimes x^1 X^1 r^2 y^2 \otimes x^2 R^2 X^3 y^3$$

$$(1.31) \quad \Delta^{\text{cop}}(h)R = R\Delta(h), \text{ for all } h \in H$$

$$(1.32) \quad (\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1.$$

To any finite dimensional quasi-Hopf algebra  $H$  we can associate a  $QT$  quasi-Hopf algebra  $D(H)$ , the quantum double of  $H$ . From [16, 17, 4], we recall the definition of the quantum double  $D(H)$ . Let  $\{e_i\}_{i=1, \dots, n}$  be a basis of  $H$ , and  $\{e^i\}_{i=1, \dots, n}$  the corresponding dual basis of  $H^*$ . We can easily see that  $H^*$ , the linear dual of  $H$ , is not a quasi-Hopf algebra. But  $H^*$  has a dual structure coming from the initial structure of  $H$ . So  $H^*$  is a coassociative coalgebra, with comultiplication

$$\widehat{\Delta}(\varphi) = \varphi_1 \otimes \varphi_2 = \sum_{i,j=1}^n \varphi(e_i e_j) e^i \otimes e^j,$$

or, equivalently,  $\widehat{\Delta}(\varphi) = \varphi_1 \otimes \varphi_2 \Leftrightarrow \varphi(hh') = \varphi_1(h)\varphi_2(h')$ , for all  $h, h' \in H$ .  $H^*$  is also an  $H$ -bimodule, by

$$\langle h \rightharpoonup \varphi, h' \rangle = \langle \varphi, h'h \rangle, \quad \langle \varphi \leftarrow h, h' \rangle = \langle \varphi, hh' \rangle.$$

The convolution is a multiplication on  $H^*$ ; it is not associative, but only quasi-associative:

$$[\varphi\psi]\xi = (X^1 \rightharpoonup \varphi \leftarrow x^1)[(X^2 \rightharpoonup \psi \leftarrow x^2)(X^3 \rightharpoonup \xi \leftarrow x^3)], \quad \forall \varphi, \psi, \xi \in H^*.$$

We also introduce  $\overline{S} : H^* \rightarrow H^*$  as the coalgebra antimorphism dual to  $S$ , this means  $\langle \overline{S}(\varphi), h \rangle = \langle \varphi, S(h) \rangle$ , for all  $\varphi \in H^*$  and  $h \in H$ .

Now consider  $\Omega \in H^{\otimes 5}$  given by

$$(1.33) \quad \begin{aligned} \Omega &= \Omega^1 \otimes \Omega^2 \otimes \Omega^3 \otimes \Omega^4 \otimes \Omega^5 \\ &= X_{(1,1)}^1 y^1 x^1 \otimes X_{(1,2)}^1 y^2 x_1^2 \otimes X_2^1 y^3 x_2^2 \otimes S^{-1}(f^1 X^2 x^3) \otimes S^{-1}(f^2 X^3), \end{aligned}$$

where  $f \in H \otimes H$  is the element defined in (1.15). We define the quantum double  $D(H) = H^* \bowtie H$  as follows: as a  $k$ -linear space,  $D(H)$  equals  $H^* \otimes H$ , and the

multiplication is given by

$$(1.34) \quad (\varphi \bowtie h)(\psi \bowtie h') = [(\Omega^1 \rightharpoonup \varphi \leftarrow \Omega^5)(\Omega^2 h_{(1,1)} \rightharpoonup \psi \leftarrow S^{-1}(h_2)\Omega^4)] \bowtie \Omega^3 h_{(1,2)} h'.$$

From [16, 17] we know that  $D(H)$  is an associative algebra with unit  $\varepsilon \bowtie 1$ , and  $H$  is a unital subalgebra via the morphism  $i_D : H \rightarrow D(H)$ ,  $i_D(h) = \varepsilon \bowtie h$ . Moreover,  $D(H)$  is a quasi-triangular quasi-Hopf algebra with the following structure:

$$(1.35) \quad \Delta_D(\varphi \bowtie h) = (\varepsilon \bowtie X^1 Y^1)(p_1^1 x^1 \rightharpoonup \varphi_2 \leftarrow Y^2 S^{-1}(p^2) \bowtie p_2^1 x^2 h_1) \otimes (X_1^2 \rightharpoonup \varphi_1 \leftarrow S^{-1}(X^3) \bowtie X_2^2 Y^3 x^3 h_2)$$

$$(1.36) \quad \varepsilon_D(\varphi \bowtie h) = \varepsilon(h)\varphi(S^{-1}(\alpha))$$

$$(1.37) \quad \Phi_D = (i_D \otimes i_D \otimes i_D)(\Phi)$$

$$(1.38) \quad S_D(\varphi \bowtie h) = (\varepsilon \bowtie S(h)f^1)(p_1^1 U^1 \rightharpoonup \overline{S}^{-1}(\varphi) \leftarrow f^2 S^{-1}(p^2) \bowtie p_2^1 U^2)$$

$$(1.39) \quad \alpha_D = \varepsilon \bowtie \alpha, \quad \beta_D = \varepsilon \bowtie \beta$$

$$(1.40) \quad R_D = \sum_{i=1}^n (\varepsilon \bowtie S^{-1}(p^2) e_i p_1^1) \otimes (e^i \bowtie p_2^1).$$

Here  $p_R = p^1 \otimes p^2$  and  $f = f^1 \otimes f^2$  are the elements defined by (1.19) and (1.15), respectively, and  $U = U^1 \otimes U^2 \in H \otimes H$  is the following element

$$(1.41) \quad U = U^1 \otimes U^2 = g^1 S(q^2) \otimes g^2 S(q^1),$$

where  $f^{-1} = g^1 \otimes g^2$  and  $q_R = q^1 \otimes q^2$  are the elements defined by (1.16) and (1.19), respectively.

## 2. INVOLUTORY QUASI-HOPF ALGEBRAS

The purpose of this Section is to introduce and study involutory quasi-Hopf algebras. As we have already explained in the Introduction there is a categorical interpretation for this definition, see [8].

**Definition 2.1.** A quasi-Hopf algebra is called involutory if the following formula holds for all  $h \in H$ :

$$(2.1) \quad S^2(h) = S(\beta)\alpha h \beta S(\alpha).$$

Examples of involutory quasi-Hopf algebras will be presented at the end of this Section.

Next we prove that for an involutory quasi-Hopf algebra the square of the antipode is inner via an element depending of  $\alpha$  and  $\beta$ .

**Lemma 2.2.** *Let  $H$  be an involutory quasi-Hopf algebra. Then  $S(\beta)\alpha$  is an invertible element and  $(S(\beta)\alpha)^{-1} = \beta S(\alpha)$ . In particular,  $S^2$  is inner and therefore  $S$  is bijective. Moreover,  $\alpha$  and  $\beta$  are invertible elements and*

$$(2.2) \quad \alpha^{-1} = S^{-1}(\alpha\beta)\beta = \beta S(\beta\alpha),$$

$$(2.3) \quad \beta^{-1} = S(\beta\alpha)\alpha = \alpha S^{-1}(\alpha\beta).$$

*Proof.* For simplicity denote  $u = S(\beta)\alpha$  and  $v = \beta S(\alpha)$ . Then  $S^2(h) = uhv$ , for all  $h \in H$ . Since  $S^2$  is an algebra map, we have that  $1 = S^2(1) = uv$ . This implies that  $S^2(u) = uvv = u$  and  $S^2(v) = uvv = v$ . Then we find that  $vu = S^2(v)S^2(u) = S^2(vu) = uvv = 1$ . This shows that  $u^{-1} = v$ . The relation  $vu = 1$  comes out as  $\beta S(\beta\alpha)\alpha = 1$ . Thus  $\beta$  has a right inverse, namely  $S(\beta\alpha)\alpha$ , and  $\alpha$  has a left inverse,

namely  $\beta S(\beta\alpha)$ . Similarly, from  $uv = 1$  we obtain that  $S(\beta)\alpha\beta S(\alpha) = 1$ . Since  $S$  is bijective this is equivalent to  $\alpha S^{-1}(\alpha\beta)\beta = 1$ , so  $\beta$  has also a left inverse, namely  $\alpha S^{-1}(\alpha\beta)$ , and  $\alpha$  has a right inverse, namely  $S^{-1}(\alpha\beta)\beta$ . It follows now that  $\alpha$  and  $\beta$  are invertible and that their inverses are given by (2.2,2.3).  $\square$

*Remark 2.3.* Following the ideas in [8], there are two seemingly different ways to introduce the notion of involutory quasi-Hopf algebra. The first definition is obtained from the formula of the categorical representation rank of  $H$  and  $D(H)$ , the quantum double of  $H$ , and requires that the map sending  $h$  to  $S^{-2}(S(\beta)\alpha h\beta S(\alpha))$  is the identity of  $H$ ; this is clearly equivalent to (2.1) in Definition 2.1. The second definition involves  $S^2$  and is obtained from the trace formula for quasi-Hopf algebras proved in [8]. It requires that the map sending  $h \in H$  to  $\beta S(\alpha)S^2(h)S(\beta)\alpha$  is the identity of  $H$ . It follows immediately from Lemma 2.2 that these two definitions are equivalent. Moreover, we can easily verify that  $H$  is involutory if and only if  $H^{\text{op}}$  is involutory, if and only if  $H^{\text{cop}}$  is involutory, if and only if  $H^{\text{op,cop}}$  is involutory.

For a Hopf algebra  $H$  it is well-known that  $S^2 = id_H$  if and only if  $S(h_2)h_1 = \varepsilon(h)1$  for any  $h \in H$ , if and only if  $h_2S(h_1) = \varepsilon(h)1$  for any  $h \in H$  (see for instance [11, Proposition 4.2.7]). For quasi-Hopf algebras we have the following result.

**Proposition 2.4.** *Let  $H$  be an involutory quasi-Hopf algebra. Then for all  $h \in H$  the following relations hold:*

$$(2.4) \quad S(h_2)\beta^{-1}h_1 = \varepsilon(h)\beta^{-1} \text{ and } h_2\alpha^{-1}S(h_1) = \varepsilon(h)\alpha^{-1}.$$

*Proof.* As we have seen, if  $\mathbb{U} \in H$  is invertible then we can define a new quasi-Hopf algebra  $H^{\mathbb{U}} = (H, \Delta, \varepsilon, \Phi, S_{\mathbb{U}}, \alpha_{\mathbb{U}}, \beta_{\mathbb{U}})$ , where  $\alpha_{\mathbb{U}} = \mathbb{U}\alpha$ ,  $\beta_{\mathbb{U}} = \beta\mathbb{U}^{-1}$  and  $S_{\mathbb{U}}(h) = \mathbb{U}S(h)\mathbb{U}^{-1}$ .

Now, consider  $\mathbb{U} = \alpha^{-1}$ . We know from Lemma 2.2 that  $\mathbb{U}$  is invertible, so it makes sense to consider the quasi-Hopf algebra  $H^{\mathbb{U}}$ . In this particular case we have that  $\alpha_{\mathbb{U}} = 1$ ,  $\beta_{\mathbb{U}} = \beta\alpha$  and

$$S_{\mathbb{U}}(h) = \alpha^{-1}S(h)\alpha = \beta S(\beta\alpha)S(h)\alpha = \beta S(\alpha)S(\beta^{-1}h\beta)S(\beta)\alpha = S^{-1}(\beta^{-1}h\beta).$$

Since  $S_{\mathbb{U}}(h_1)\alpha_{\mathbb{U}}h_2 = \varepsilon(h)\alpha_{\mathbb{U}}$  for all  $h \in H$ , we get that  $S^{-1}(\beta^{-1}h_1\beta)h_2 = \varepsilon(h)1$ , and this is equivalent to  $S(h_2)\beta^{-1}h_1\beta = \varepsilon(h)1$ , for all  $h \in H$ . It follows now that  $S(h_2)\beta^{-1}h_1 = \varepsilon(h)\beta^{-1}$  for all  $h \in H$ , as needed.

Similarly, using the fact that  $h_1\beta_{\mathbb{U}}S(h_2) = \varepsilon(h)\beta_{\mathbb{U}}$  for all  $h \in H$ , one can prove that  $h_2\alpha^{-1}S(h_1) = \varepsilon(h)\alpha^{-1}$  for all  $h \in H$ , the details are left to the reader.  $\square$

Let us now present some examples of involutory quasi-Hopf algebras.

**Examples 2.5.** 1) To any finite dimensional cocommutative Hopf algebra  $H$  and any normalized 3-cocycle  $\omega$  on  $H$  we can associate a quasi-Hopf algebra,  $D^{\omega}(H)$ , see [7]. For  $D^{\omega}(H)$  we have  $\alpha = 1$ ,  $\beta^{-1} = S(\beta)$  and  $S^2(h) = \beta^{-1}h\beta$ , for all  $h \in D^{\omega}(H)$ . Thus  $D^{\omega}(H)$  is an involutory quasi-Hopf algebra.

2) Let  $k$  be a field of characteristic different from 2,  $C_2$  the cyclic group of order two, and  $g$  the generator of  $C_2$ . Following [14], the two dimensional quasi-Hopf algebra  $H(2)$  is the bialgebra  $k[C_2]$ , the group algebra associated to  $C_2$ , viewed as a quasi-Hopf algebra via the non-trivial reassociator  $\Phi = 1 - 2p_- \otimes p_- \otimes p_-$ , where  $p_- = \frac{1}{2}(1 - g)$ . The antipode  $S$  is the identity map and the distinguished elements  $\alpha$  and  $\beta$  are  $\alpha = g$  and  $\beta = 1$ , respectively. It is well-known that  $H(2)$  is not twist

equivalent to a Hopf algebra. It is easy to see that  $H(2)$  is an involutory quasi-Hopf algebra.

We will now study further properties of  $H(2)$ . First we will show that if  $k$  contains a primitive fourth root of unity then there are exactly two quasi-triangular structures on  $H(2)$ . In what follows,  $p_{\pm} = \frac{1}{2}(1 \pm g)$ . One can easily check that  $\{p_{\pm}\}$  is a basis for  $H(2)$  consisting of orthogonal idempotents, and that  $p_+ + p_- = 1$ ,  $p_+ - p_- = g$ .

**Proposition 2.6.** *Suppose that  $k$  is a field of characteristic different from 2 containing a primitive fourth root of unity  $i$ . Then there are exactly two different  $R$ -matrices for  $H(2)$ , namely,  $R_{\pm} = 1 - (1 \pm i)p_- \otimes p_-$ .*

*Proof.* Suppose that  $R = a1 \otimes 1 + b1 \otimes g + cg \otimes 1 + dg \otimes g$  is an  $R$ -matrix for  $H(2)$ , where  $a, b, c, d \in k$ . By (1.32) we have that  $a + c = a + b = 1$  and  $b + d = c + d = 0$ , and therefore  $b = c = -d$  and  $a = 1 - b$ . Hence,  $R$  should be on the form

$$R = (1 - b)1 \otimes 1 + b1 \otimes g + bg \otimes 1 - bg \otimes g = 1 - b(1 - g) \otimes (1 - g) = 1 - \omega p_- \otimes p_-,$$

where we denoted  $4b = \omega$ . Now, we can easily see that

$$(2.5) \quad \Phi^{-1} = \Phi = 1 - 2p_- \otimes p_- \otimes p_-,$$

and since  $X^2 \otimes X^3 \otimes X^1 = \Phi$ , the above relation implies

$$X^2 R^1 x^1 \otimes X^3 x^3 \otimes X^1 R^2 x^2 = 1 - \omega p_- \otimes 1 \otimes p_-,$$

and therefore, after some computations, we get

$$\begin{aligned} & X^2 R^1 x^1 Y^1 \otimes X^3 x^3 r^1 Y^2 \otimes X^1 R^2 x^2 r^2 Y^3 \\ &= 1 - \omega p_- \otimes p_+ \otimes p_- - \omega p_+ \otimes p_- \otimes p_- - (2 - 2\omega + \omega^2)p_- \otimes p_- \otimes p_-. \end{aligned}$$

On the other hand, we have  $\Delta(p_-) = p_- \otimes p_+ + p_+ \otimes p_-$ , so

$$(\Delta \otimes id)(R) = 1 - \omega p_- \otimes p_+ \otimes p_- - \omega p_+ \otimes p_- \otimes p_-.$$

We conclude that (1.29) holds if and only if  $2 - 2\omega + \omega^2 = 0$ , and this is equivalent to  $\omega = 1 \pm i$ .

Using (2.5) for  $\Phi^{-1}$ , we obtain in a similar way that

$$x^3 R^1 X^2 \otimes x^1 X^1 \otimes x^2 R^2 X^3 = 1 - \omega p_- \otimes 1 \otimes p_-.$$

Using this formula, it can be proved that

$$\begin{aligned} & x^3 R^1 X^2 r^1 y^1 \otimes x^1 X^1 r^2 y^2 \otimes x^2 R^2 X^3 y^3 \\ &= 1 - \omega p_- \otimes p_- \otimes p_+ - \omega p_- \otimes p_+ \otimes p_- - (2 - 2\omega + \omega^2)p_- \otimes p_- \otimes p_-. \end{aligned}$$

It is easy to see that

$$(id \otimes \Delta)(R) = 1 - \omega p_- \otimes p_- \otimes p_+ - \omega p_- \otimes p_+ \otimes p_-,$$

so the relation in (1.30) holds if and only if  $2 - 2\omega + \omega^2 = 0$ . The relation in (1.31) is automatically satisfied because of the commutativity and cocommutativity of  $H(2)$ . Thus the  $R$ -matrices for  $H(2)$  are in bijective correspondence with the solutions of the equation  $2 - 2\omega + \omega^2 = 0$ , from where we deduce that  $R_{\pm} = 1 - (1 \pm i)p_- \otimes p_-$  are the only quasi-triangular structures on  $H(2)$ .  $\square$

*Remark 2.7.* It is not difficult to show that  $H(2)_+ = (H(2), R_+)$  and  $H(2)_- = (H(2), R_-)$  are non-isomorphic  $QT$  quasi-Hopf algebras, this means that there is no quasi-Hopf algebra isomorphism  $\nu : H(2) \rightarrow H(2)$  satisfying  $(\nu \otimes \nu)(R_+) = R_-$ . Indeed, if such a  $\nu$  exists then  $(1 + i)\nu(p_-) \otimes \nu(p_-) = (1 - i)p_- \otimes p_-$ . If we write

$\nu(p_-) = ap_- + bp_+$ , for some scalars  $a, b \in k$ , then from the above relation we obtain that  $a^2 = -i$  and  $b = 0$ . Since  $p_{\pm}^2 = p_{\pm}$  and  $\nu$  is an algebra map we get that  $ap_- = \nu(p_-) = \nu(p_-^2) = (ap_-)^2 = -ip_-$ , and we conclude that  $a = -i$ . But  $a^2 = -i$ , so  $i \in \{-1, 0\}$ , a contradiction.

To any quasi-triangular quasi-Hopf algebra  $(H, R)$ , we can associate a new quasi-Hopf algebra  $\text{bos}(H_0)$ , called the bosonisation of  $H_0$  (see [6, Corollary 5.3]).  $H_0$  equals  $H$  as a vector space, with a newly defined multiplication  $\circ$  given by the formula

$$(2.6) \quad h \circ h' = X^1 h S(x^1 X^2) \alpha x^2 X_1^3 h' S(x^3 X_2^3),$$

and left  $H$ -module structure given by  $h \triangleright h' = h_1 h' S(h_2)$ , for all  $h, h' \in H$ . Then  $\text{bos}(H_0)$  is the  $k$ -vector space  $H_0 \otimes H$  with the following quasi-Hopf algebra structure:

$$(2.7) \quad (b \times h)(b' \times h') = (x^1 \triangleright b) \circ (x^2 h_1 \triangleright b') \times x^3 h_2 h',$$

$$\Delta(b \times h)$$

$$(2.8) \quad = y^1 X^1 \triangleright b_1 \times y^2 Y^1 R^2 x^2 X_1^3 h_1 \otimes y_1^3 Y^2 R^1 x^1 X^2 \triangleright b_2 \times y_2^3 Y^3 x^3 X_2^3 h_2,$$

$$(2.9) \quad \Phi_{\text{bos}(H_0)} = \beta \times X^1 \otimes \beta \times X^2 \otimes \beta \times X^3,$$

$$(2.10) \quad s(b \times h) = (\beta \times S(X^1 x_1^1 R^2 h) \alpha)(X^2 x_2^1 R^1 \triangleright S_{H_0}(b) \times X^3 x^2 \beta S(x^3)),$$

for all  $b, b', h, h' \in H$ , where we write  $b \times h$  and  $b' \times h'$  in place of  $b \otimes h$  and respectively  $b' \otimes h'$  to distinguish the new structure on  $H_0 \otimes H$ , and where

$$(2.11) \quad \underline{\Delta}(b) = b_1 \otimes b_2 := x^1 X^1 b_1 g^1 S(x^2 R^2 y^3 X_2^3) \otimes x^3 R^1 \triangleright y^1 X^2 b_2 g^2 S(y^2 X_1^3),$$

$$(2.12) \quad S_{H_0}(b) = X^1 R^2 p^2 S(q^1 (X^2 R^1 p^1 \triangleright b) S(q^2) X^3),$$

for all  $b \in H$ . Here  $R = R^1 \otimes R^2$  and  $f^{-1} = g^1 \otimes g^2$ ,  $p_R = p^1 \otimes p^2$  and  $q_R = q^1 \otimes q^2$  are the elements defined by (1.16) and (1.19), respectively.

The unit for  $\text{bos}(H_0)$  is  $\beta \times 1$ , the counit is  $\varepsilon(b \times h) = \varepsilon(b)\varepsilon(h)$ , for all  $b, h \in H$ , and the distinguished elements  $\alpha$  and  $\beta$  are given by  $\beta \times \alpha$  and  $\beta \times \beta$ , respectively. Our next goal is to compute the quasi-Hopf algebra structure on  $\text{bos}(H_0)$ , in the case where  $H = H(2)_+$  or  $H = H(2)_-$ .

**Proposition 2.8.**  *$\text{bos}(H(2)_+) = \text{bos}(H(2)_-) = k[C_2 \times C_2]$  as bialgebras, viewed as a quasi-Hopf algebra via the non-trivial reassociator  $\Phi_x := 1 - 2p^x \otimes p^x \otimes p^x$ , where  $x$  is one of the generators of  $C_2 \times C_2$ , and where  $p^x := \frac{1}{2}(1 - x)$ . The antipode is the identity map and the distinguished elements  $\alpha$  and  $\beta$  are given by  $\alpha = x$  and  $\beta = 1$ , respectively. In particular,  $\text{bos}(H(2)_+) = \text{bos}(H(2)_-)$  is an involutory quasi-Hopf algebra.*

*Proof.* Since  $H(2)_{\pm}$  are commutative algebras and  $\beta = 1$ , it follows from (1.5) and (1.6) that the multiplication  $\circ$  defined in (2.6) coincides with the original multiplication of  $H(2)$ . Also, from the definition of  $H(2)$  it follows that the action  $\triangleright$  is trivial, i.e.  $h \triangleright h' = \varepsilon(h)h'$ , for all  $h, h' \in H(2)$ . Using (2.7) we obtain that the multiplication on  $\text{bos}(H_0)$  is the componentwise multiplication, and by (2.11) we get that the comultiplication  $\underline{\Delta}$  on  $H$  reduces to

$$\underline{\Delta}(b) = X^1 b_1 g^1 S(y^3 X_2^3) \otimes y^1 X^2 b_2 g^2 S(y^2 X_1^3) = \Delta(b)(X^1 X_2^3 y^3 \otimes X^2 X_1^3 y^1 y^2) f^{-1}.$$

We have that

$$(id \otimes id \otimes \Delta)(\Phi) = 1 - 2p_- \otimes p_- \otimes p_+ \otimes p_- - 2p_- \otimes p_- \otimes p_- \otimes p_+,$$

and therefore  $X^1 X_2^3 \otimes X^2 X_1^3 = 1$ . Also, by (2.5) we have

$$y^3 \otimes y^1 y^2 = 1 - 2p_- \otimes p_- = 1 \otimes p_+ + g \otimes p_-,$$

and a straightforward computation ensures us that the Drinfeld twist  $f$  and its inverse  $f^{-1}$  for  $H(2)$  are given by

$$f = f^{-1} = g \otimes p_- + 1 \otimes p_+.$$

Combining all these facts we get  $\underline{\Delta} = \Delta$ , and keeping in mind that the action  $\triangleright$  is trivial we conclude that the comultiplication in (2.8) is the componentwise comultiplication on  $H(2) \otimes H(2)$ . Thus  $\text{bos}(H(2)_+) = \text{bos}(H(2)_-) = H(2) \otimes H(2)$  as bialgebras. Hence  $\text{bos}(H(2)_+) = \text{bos}(H(2)_-)$  is generated as an algebra by  $x = 1 \times g$  and  $y = g \times 1$ , with relations  $x^2 = y^2 = 1$  and  $xy = yx$ . The elements  $x$  and  $y$  are grouplike elements, so  $\text{bos}(H(2)_+) = \text{bos}(H(2)_-) = k[C_2 \times C_2]$  as bialgebras. According to (2.9) the reassociator of  $\text{bos}(H(2)_+) = \text{bos}(H(2)_-)$  is given by

$$\begin{aligned} \Phi_x &= 1 \times X^1 \otimes 1 \times X^2 \otimes 1 \times X^3 \\ &= 1 \times 1 \otimes 1 \times 1 \otimes 1 \times 1 - 2 \times p_- \otimes 1 \times p_- \otimes 1 \times p_- \\ &= 1 - 2p_-^x \otimes p_-^x \otimes p_-^x \end{aligned}$$

since  $1 \times p_- = \frac{1}{2}(1 \times 1 - 1 \times g) = \frac{1}{2}(1 - x) = p_-^x$ . Finally, using that  $\triangleright$  is trivial,  $\beta = 1$  and the axiom (1.6), we obtain  $S_{H_0} = S$ , the antipode of  $H(2)$ . From (2.10) and (1.6) we deduce that the antipode of  $\text{bos}(H(2)_+) = \text{bos}(H(2)_-)$  is the identity map. Clearly, the distinguished elements  $\alpha$  and  $\beta$  are respectively  $1 \times g = x$  and  $1 \times 1 = 1$ .  $\square$

**Example 2.9.** If  $H$  and  $K$  are two quasi-Hopf algebras then  $H \otimes K$  is also a quasi-Hopf algebra with the componentwise structure. In particular, the reassociator of  $H \otimes K$  is

$$\Phi_{H \otimes K} = (X_H^1 \otimes X_K^1) \otimes (X_H^2 \otimes X_K^2) \otimes (X_H^3 \otimes X_K^3),$$

where  $\Phi_H = X_H^1 \otimes X_H^2 \otimes X_H^3$  and  $\Phi_K = X_K^1 \otimes X_K^2 \otimes X_K^3$  are the reassociators of respectively  $H$  and  $K$ . Therefore  $H(2) \otimes H(2)$  has a second quasi-Hopf algebra structure.  $H(2) \otimes H(2) = k[C_2 \times C_2]$  as a bialgebra but now it is viewed as a quasi-Hopf algebra via the reassociator

$$\begin{aligned} \Phi_{x,y} &= 1 - 2(1 \otimes p_-) \otimes (1 \otimes p_-) \otimes (1 \otimes p_-) \\ &\quad - 2(p_- \otimes 1) \otimes (p_- \otimes 1) \otimes (p_- \otimes 1) + 4(p_- \otimes p_-) \otimes (p_- \otimes p_-) \otimes (p_- \otimes p_-). \end{aligned}$$

If  $x = 1 \otimes g$  and  $y = g \otimes 1$  are the algebra generators of  $H(2) \otimes H(2)$  then

$$1 \otimes p_- = \frac{1}{2}(1 - 1 \otimes g) = \frac{1}{2}(1 - x) := p_-^x, \quad p_- \otimes 1 = \frac{1}{2}(1 - g \otimes 1) = \frac{1}{2}(1 - y) := p_-^y,$$

and  $p_- \otimes p_- = \frac{1}{4}(1 - g) \otimes (1 - g) = \frac{1}{4}(1 - x - y + xy) = p_-^x p_-^y$ . Therefore,  $\Phi_{x,y}$  comes out as

$$\Phi_{x,y} = (1 - 2p_-^x \otimes p_-^x \otimes p_-^x)(1 - 2p_-^y \otimes p_-^y \otimes p_-^y).$$

Note that the distinguished elements are  $\alpha = xy$  and  $\beta = 1$ , and that the antipode is the identity map. Consequently,  $H(2) \otimes H(2)$  is an involutory quasi-Hopf algebra.

Another example of involutory quasi-Hopf algebra is the quantum double of  $H(2)$ .

**Proposition 2.10.** *The quantum double of  $H(2)$  is the unital algebra generated by  $X$  and  $Y$  with relations*

$$X^2 = 1, \quad Y^2 = X, \quad XY = YX.$$

The coalgebra structure on  $D(H(2))$  is given by the formulas:

$$\begin{aligned} \Delta(X) &= X \otimes X, \quad \varepsilon(X) = 1, \\ \Delta(Y) &= -\frac{1}{2}(Y \otimes Y + XY \otimes Y + Y \otimes XY - XY \otimes XY), \quad \varepsilon(Y) = -1. \end{aligned}$$

The reassociator, the distinguished elements  $\alpha$  and  $\beta$ , and the antipode are respectively given by

$$\Phi_X = 1 - 2p_-^X \otimes p_-^X \otimes p_-^X, \quad \alpha = X, \quad \beta = 1, \quad S(X) = X, \quad S(Y) = Y,$$

where we denoted  $p_-^X = \frac{1}{2}(1 - X)$ . Moreover,  $D(H(2))$  is an involutory quasi-Hopf algebra.

*Proof.* Using the commutativity and cocommutativity of  $H(2)$ , (1.5), and the fact that  $\beta = 1$ , we find that the multiplication rule (1.34) takes the following form on  $D(H(2))$ :

$$(\varphi \bowtie h)(\varphi' \bowtie h') = (\Omega^1 \Omega^5 \rightharpoonup \varphi)(\Omega^2 \Omega^4 \rightharpoonup \varphi') \bowtie \Omega^3 h h',$$

for all  $\varphi, \varphi' \in H(2)^*$  and  $h, h' \in H(2)$ . Now, from the definition (1.33) of  $\Omega$  we find out that

$$\Omega^1 \Omega^5 \otimes \Omega^2 \Omega^4 \otimes \Omega^3 = X_{(1,1)}^1 X^3 y^1 x^1 f^2 \otimes X_{(1,2)}^1 X^2 y^2 x_1^2 x^3 f^1 \otimes X_2^1 y^3 x_2^2.$$

Using the expressions of  $\Phi$  and  $\Phi^{-1}$  in (2.5) we easily compute that

$$\begin{aligned} X_{(1,1)}^1 X^3 \otimes X_{(1,2)}^1 X^2 \otimes X_2^1 &= 1 - 2p_- \otimes p_- \otimes p_-, \\ x^1 \otimes x_1^2 x^3 \otimes x_2^2 &= 1 - 2p_- \otimes p_- \otimes p_+, \\ \Phi^{-1}(f^2 \otimes f^1 \otimes 1) &= p_- \otimes 1 \otimes p_- + p_- \otimes g \otimes p_+ + p_+ \otimes 1 \otimes 1, \end{aligned}$$

where  $f = g \otimes p_- + 1 \otimes p_+$  is the Drinfeld twist of  $H(2)$ . By the above relations the multiplication of  $D(H(2))$  comes out explicitly as

$$(\varphi \bowtie h)(\varphi' \bowtie h') = \varphi \varphi' \bowtie h h' - 2(p_- \rightharpoonup \varphi)(p_- \rightharpoonup \varphi') \bowtie p_- h h'.$$

Now, let  $\{P_1, P_g\}$  be the dual basis of  $H(2)^*$  corresponding to the basis  $\{1, g\}$  of  $H(2)$ . Then  $\varepsilon = P_1 + P_g$  and  $\{\varepsilon, \mu = P_1 - P_g\}$  is clearly a basis for  $H(2)^*$ . Now let  $X = \varepsilon \bowtie g$  and  $Y = \mu \bowtie 1$ . Since  $p_- \rightharpoonup P_1 = \frac{1}{2}\mu$  and  $p_- \rightharpoonup P_g = -\frac{1}{2}\mu$ , we obtain that

$$\begin{aligned} X^2 &= (\varepsilon \bowtie g)(\varepsilon \bowtie g) = \varepsilon \bowtie 1 - 2(p_- \rightharpoonup \varepsilon)(p_- \rightharpoonup \varepsilon) = 1, \\ XY &= YX = \mu \bowtie g, \\ Y^2 &= \mu^2 \bowtie 1 - 2(p_- \rightharpoonup \mu)^2 \bowtie p_- = \varepsilon \bowtie 1 - \varepsilon \bowtie 2p_- = \varepsilon \bowtie g = X, \end{aligned}$$

which are the multiplication rules that we stated. A computation as in the proof of Proposition 2.8 shows that the reassociator of  $D(H(2))$  has the desired form. Since  $H(2)$  is commutative and cocommutative,  $\beta = 1$  and  $\Phi^{-1} = \Phi = Y^2 \otimes Y^1 \otimes Y^3$ , by (1.35) we have

$$\Delta(\varphi \bowtie h) = p_1^1 p^2 \rightharpoonup \varphi_2 \bowtie p_2^1 X^1 h_1 \otimes X_1^2 X^3 \rightharpoonup \varphi_1 \bowtie X_2^2 h_2,$$

for all  $\varphi \in H(2)^*$  and  $h \in H(2)$ . On the other hand,

$$\begin{aligned} p_1^1 p^2 \otimes p_2^1 X^1 \otimes X_1^2 X^3 \otimes X_2^2 \\ &= (p_+ \otimes 1 \otimes 1 \otimes 1 - p_- \otimes g \otimes 1 \otimes 1)(1 - 2 \otimes p_- \otimes p_- \otimes p_+) \\ &= p_+ \otimes 1 \otimes 1 \otimes 1 - p_- \otimes g \otimes 1 \otimes 1 - 2 \otimes p_- \otimes p_- \otimes p_+. \end{aligned}$$

We then have

$$\Delta(X) = \Delta(\varepsilon \bowtie g) = \varepsilon \bowtie g \otimes \varepsilon \bowtie g = X \otimes X,$$

and since  $\Delta(P_1) = P_1 \otimes P_1 + P_g \otimes P_g$  and  $\Delta(P_g) = P_1 \otimes P_g + P_g \otimes P_1$  we get that  $\Delta(\mu) = \Delta(P_1 - P_g) = (P_1 - P_g) \otimes (P_1 - P_g) = \mu \otimes \mu$ , and therefore

$$\begin{aligned} \Delta(Y) &= p_+ \rightarrow \mu \bowtie 1 \otimes \mu \bowtie 1 - p_- \rightarrow \mu \bowtie g \otimes \mu \bowtie 1 - 2\mu \bowtie p_- \otimes p_- \rightarrow \mu \bowtie p_+ \\ &= -XY \otimes Y - \frac{1}{2}(Y - XY) \otimes (Y + XY) \\ &= -\frac{1}{2}(Y \otimes Y + XY \otimes Y + Y \otimes XY - XY \otimes XY), \end{aligned}$$

as needed. It follows from (1.36) that  $\varepsilon(X) = 1$  and  $\varepsilon(Y) = -1$ .

Finally, in our particular situation (1.38) takes the form

$$S(\varphi \bowtie h) = p_1^1 p^2 U^1 f^2 \rightarrow \varphi \bowtie p_2^1 U^2 f^1 h$$

for all  $\varphi \in H(2)^*$  and  $h \in H$ . But  $p_1^1 p^2 f^2 \otimes p_2^1 f^1 = U^1 \otimes U^2 = g \otimes 1$ , so the antipode for  $D(H(2))$  is the identity map. Obviously,  $\alpha = \varepsilon \bowtie g = X$ ,  $\beta = \varepsilon \bowtie 1 = 1$ , and our proof is complete.  $\square$

*Remark 2.11.* Since  $p_1^1 p^2 \otimes p_2^1 = p_+ \otimes 1 - p_- \otimes g$  we have from (1.40) that the canonical  $R$ -matrix for  $D(H(2))$  is  $R = p_+^X \otimes 1 - p_-^X \otimes XY$ , where, as usual,  $p_{\pm}^X = \frac{1}{2}(1 \pm X)$ .

At first sight there is no relationship between  $D(H(2))$  and  $H(2) \otimes H(2)$ , so it comes as a surprise that these two Hopf algebras are twisted equivalent. To show this, we will use the structure of the quantum double associated to a factorizable quasi-Hopf algebra, see [2].

Recall from [2] that a  $QT$  quasi-Hopf algebra  $(H, R)$  is called factorizable if the  $k$ -linear map  $\mathcal{Q} : H^* \rightarrow H$  given for all  $\chi \in H^*$  by

$$(2.13) \quad \mathcal{Q}(\chi) = \langle \chi, S(X_2^2 \tilde{p}^2) f^1 R^2 r^1 U^1 X^3 \rangle X^1 S(X_1^2 \tilde{p}^1) f^2 R^1 r^2 U^2,$$

is bijective. Here  $r^1 \otimes r^2$  is another copy of  $R$ , and  $U = U^1 \otimes U^2$ ,  $f^{-1} = g^1 \otimes g^2$  and  $p_L = \tilde{p}^1 \otimes \tilde{p}^2$  are the elements defined by (1.41), (1.16) and (1.20), respectively. The first step is to prove that  $H(2)$  is a factorizable quasi-Hopf algebra.

**Proposition 2.12.** *For  $(H(2), R)$  with  $R$  as in Proposition 2.6 the map  $\mathcal{Q}$  from (2.13) has the following form for all  $\chi \in H(2)^*$*

$$\mathcal{Q}(\chi) = \chi(1)p_- + \chi(g)p_+.$$

Since  $\{p_-, p_+\}$  and  $\{1, g\}$  are bases for  $H(2)$  it follows that  $\mathcal{Q}$  is bijective, so  $H(2)$  is factorizable.

*Proof.* For  $H(2)$  the element  $p_L$  has the form

$$p_L = X^1 X^2 \otimes X^3 = 1 - 2p_- \otimes p_- = 1 - (1 - g) \otimes p_- = 1 \otimes p_+ + g \otimes p_- = f.$$

Also, we can easily see that  $X^1 X_1^2 \otimes X_2^2 X^3 = 1$  and since  $f$  is an involution we conclude that

$$X_2^2 X^3 \tilde{p}^2 f^1 \otimes X^1 X_1^2 \tilde{p}^1 f^2 = 1.$$

On the other hand, since  $\omega^2 - 2\omega = -2$  it follows that  $R^2r^1 \otimes R^1r^2 = (1 - \omega p_- \otimes p_-)^2 = 1 - 2p_- \otimes p_-$ . We have already seen that  $U = g \otimes 1$ , and therefore

$$\begin{aligned} S(X_2^2 \tilde{p}^2) f^1 R^2 r^1 U^1 X^3 \otimes X^1 S(X_1^2 \tilde{p}^1) f^2 R^1 r^2 U^2 \\ = (1 - 2p_- \otimes p_-)(g \otimes 1) = 1 \otimes p_- + g \otimes p_+. \end{aligned}$$

It is now clear that  $\mathcal{Q}(\chi) = \chi(1)p_- + \chi(g)p_+$ , for all  $\chi \in H(2)^*$ , and this finishes the proof.  $\square$

The structure of the quantum double of a finite dimensional factorizable quasi-Hopf algebra  $(H, R)$  can be found in [2, Theorem 5.4]. More precisely, since  $(H, R)$  is quasi-triangular there exist two quasi-Hopf algebra morphisms  $\pi, \tilde{\pi} : D(H) \rightarrow H$  covering the natural inclusion  $i_D : H \rightarrow D(H)$ . They are given by the formulas

$$\pi(\varphi \bowtie h) = \varphi(q^2 R^1) q^1 R^2 h \text{ and } \tilde{\pi}(\varphi \bowtie h) = \varphi(q^2 \bar{R}^2) q^1 \bar{R}^1 h,$$

for all  $\varphi \in H^*$  and  $h \in H$ , where  $R^{-1} = \bar{R}^1 \otimes \bar{R}^2$ . Now, if we define

$$(2.14) \quad \mathbf{F} = Y_1^1 x^1 X^1 y_1^1 \otimes Y_2^1 x^2 R^2 X^3 y^2 \otimes Y^2 x^3 R^1 X^2 y_2^1 \otimes Y^3 y^3,$$

and  $\mathbf{U} = \bar{R}^1 g^2 \otimes \bar{R}^2 g^1$ , then  $F$  is a twist on  $H \otimes H$ ,  $\mathbf{U}$  is an invertible element of  $H \otimes H$  and  $\zeta : D(H) \rightarrow (H \otimes H)_{\mathbf{F}}^{\mathbf{U}}$ , given by  $\zeta(\mathbf{D}) = \tilde{\pi}(\mathbf{D}_1) \otimes \pi(\mathbf{D}_2)$ , for all  $\mathbf{D} \in D(H)$ , is a quasi-Hopf algebra isomorphism. We make this result explicit for the quasitriangular quasi-Hopf algebra  $(H(2), R)$ .

**Proposition 2.13.** *Let  $X, Y$  be the algebra generators of  $D(H(2))$  defined in Proposition 2.10, and  $x, y$  the generators of  $H(2) \otimes H(2) \cong k[C_2 \times C_2]$ , the quasi-Hopf algebra described in Example 2.9. Let  $\omega_{\pm} = 1 \pm i$  and consider the elements*

$$\begin{aligned} \mathbf{U}_{\pm} &= p_+ \otimes 1 + p_- \otimes g + \omega_{\pm} p_- \otimes p_-, \\ \mathbf{F}_{\pm} &= 1 - 2p_-^x p_-^y \otimes p_-^x p_+^y - 2p_+^x p_-^y \otimes p_-^x p_-^y - \omega_{\pm} p_-^x \otimes p_-^y, \end{aligned}$$

where  $p_{\pm}^x = \frac{1}{2}(1 \pm x)$  and  $p_{\pm}^y = \frac{1}{2}(1 \pm y)$ . Then the maps  $\zeta_{\pm} : D(H(2)) \rightarrow (H(2) \otimes H(2))_{\mathbf{F}_{\pm}}^{\mathbf{U}_{\pm}}$ , given by

$$\zeta_{\pm}(X) = xy, \quad \zeta_{\pm}(Y) = -1 + \omega_{\pm} p_-^x + \omega_{\mp} p_-^y = -\frac{1}{2}(\omega_{\pm} x + \omega_{\mp} y),$$

are quasi-Hopf algebra isomorphisms.

*Proof.* Since  $H(2)$  is a factorizable quasi-Hopf algebra everything will follow from the general isomorphism presented above. For  $H(2)$  we have  $q_R = 1 \otimes p_+ - g \otimes p_-$ . Also, it is easy to see that the inverse of  $R_{\pm} = 1 - \omega_{\pm} p_- \otimes p_-$  is  $R_{\mp} = 1 - \omega_{\mp} p_- \otimes p_-$ , and therefore

$$\begin{aligned} q^2 R_{\pm}^1 \otimes q^1 R_{\pm}^2 &= p_+ \otimes 1 - p_- \otimes g - \omega_{\pm} p_- \otimes p_-, \\ q^2 \bar{R}_{\pm}^2 \otimes q^1 \bar{R}_{\pm}^1 &= q^2 R_{\mp}^1 \otimes q^1 R_{\mp}^2 = p_+ \otimes 1 - p_- \otimes g - \omega_{\mp} p_- \otimes p_-. \end{aligned}$$

From the structure of  $D(H(2))$  in Proposition 2.10 we compute that  $\pi(X) = \tilde{\pi}(X) = g$ ,

$$\pi(Y) = \pi(\mu \bowtie 1) = \mu(p_+)1 - \mu(p_-)g - \omega_{\pm} \mu(p_-)p_- = -g - \omega_{\pm} p_-,$$

and, in a similar way,  $\tilde{\pi}(Y) = -g - \omega_{\mp} p_-$ . We get that  $\pi(XY) = -1 + \omega_{\pm} p_-$  and  $\tilde{\pi}(XY) = -1 + \omega_{\mp} p_-$ , so  $\zeta_{\pm}(X) = g \otimes g = xy$  and

$$\zeta_{\pm}(Y) = -\frac{1}{2}(\pi \otimes \tilde{\pi})(Y \otimes Y + XY \otimes Y + Y \otimes XY - XY \otimes XY).$$

After some straightforward computations we obtain

$$\begin{aligned}\pi(Y) \otimes \tilde{\pi}(Y) &= xy + 2p_-^x p_-^y + \omega_{\pm} x p_-^y + \omega_{\mp} y p_-^x, \\ \pi(XY) \otimes \tilde{\pi}(Y) &= x - 2p_-^x p_-^y - \omega_{\pm} x p_-^y + \omega_{\mp} p_-^x, \\ \pi(Y) \otimes \tilde{\pi}(XY) &= y - 2p_-^x p_-^y + \omega_{\pm} p_-^y - \omega_{\mp} y p_-^x, \\ \pi(XY) \otimes \tilde{\pi}(XY) &= 1 + 2p_-^x p_-^y - \omega_{\pm} p_-^y - \omega_{\mp} p_-^x.\end{aligned}$$

Thus, we can compute:

$$\begin{aligned}\zeta_{\pm}(Y) &= -\frac{1}{2}(xy + x + y - 1 - 4p_-^x p_-^y + 2\omega_{\pm} p_-^y + 2\omega_{\mp} p_-^x) \\ &= 1 - x - y - \omega_{\pm} p_-^y - \omega_{\mp} p_-^x \\ &= -1 + (2 - \omega_{\mp})p_-^x + (2 - \omega_{\pm})p_-^y = -1 + \omega_{\pm} p_-^x + \omega_{\mp} p_-^y,\end{aligned}$$

and this is exactly what we need. Finally, one can easily see that the corresponding elements  $\mathbf{U}_{\pm}$  and  $\mathbf{F}_{\pm}$  for  $(H(2), R_{\pm})$  are exactly the ones defined in the statement, we leave the details to the reader.  $\square$

*Remarks 2.14.* 1) Keeping the notation used in Proposition 2.13, we have that

$$(\zeta_{\pm} \otimes \zeta_{\pm} \otimes \zeta_{\pm})(\Phi_X) = \Phi_{xy} := 1 - 2p_-^{xy} \otimes p_-^{xy} \otimes p_-^{xy},$$

where  $\Phi_X$  is the reassociator of  $D(H(2))$  and  $p_-^{xy} := \frac{1}{2}(1 - xy)$ . We then have that  $\Phi_{xy}$  is a 3-cocycle for  $k[C_2 \times C_2]$  and  $\Phi_{xy} = (\Phi_{x,y})_{\mathbf{F}} = (\Phi_x \Phi_y)_{\mathbf{F}}$ , because of (1.8). Here  $\Phi_y = 1 - 2p_-^y \otimes p_-^y \otimes p_-^y$  is the 3-cocycle on  $k[C_2 \times C_2]$  corresponding to  $y$ . In other words we have proved that the 3-cocycles  $\Phi_{xy}$  and  $\Phi_x \Phi_y$  are equivalent.

2) It follows from Proposition 2.13 that  $k[C_4]$  and  $k[C_2 \times C_2]$  are isomorphic as algebras if  $\text{char}(k) \neq 2$  and  $k$  contains a primitive fourth root of 1. This is well-known and can be easily seen directly:  $k[C_4]$  and  $k[C_2 \times C_2]$  are isomorphic (as Hopf algebras even) to their duals and the two duals are both isomorphic to  $k^4$  as algebras. More explicitly, we have the following:  $\zeta_{\pm} = \gamma \circ \beta \circ \alpha$ , where  $\alpha, \beta, \gamma$  are the following three algebra isomorphisms.  $\{e_1, e_2, e_3, e_4\}$  is the standard basis of  $k^4$ .

$$\begin{aligned}\alpha : k[C_4] &= k[Y]/(Y^4 - 1) \rightarrow k^4, \quad \alpha(Y) = e_1 + ie_2 - e_3 - ie_4; \\ \beta : k^4 &\rightarrow k[C_2 \times C_2] = k[x, y], \\ &\quad \beta(e_1) = p_+^x p_+^y, \quad \beta(e_2) = p_-^x p_+^y, \quad \beta(e_3) = p_+^x p_-^y; \quad \beta(e_4) = p_-^x p_-^y; \\ \gamma : k[C_2 \times C_2] &\rightarrow k[C_2 \times C_2], \quad \gamma(x) = xy, \quad \gamma(y) = -x, \quad \gamma(xy) = -y.\end{aligned}$$

### 3. THE PIVOTAL STRUCTURE OF ${}_H\mathcal{M}^{\text{fd}}$ WHEN $H$ IS INVOLUTORY

If  $\mathcal{C}$  is a monoidal category with left duality, then the functor  $(-)^{**} : \mathcal{C} \rightarrow \mathcal{C}$  is strongly monoidal: we have an isomorphism  $\phi_0 : I \rightarrow I^{**}$ , and for  $V, W \in \mathcal{C}$ , we have the following family of isomorphisms in  $\mathcal{C}$ :

$$\phi_{V,W} : V^{**} \otimes W^{**} \xrightarrow{\lambda_{W^*,V^*}} (W^* \otimes V^*)^* \xrightarrow{(\lambda_{V,W}^{-1})^*} (V \otimes W)^{**}.$$

$\lambda_{V,W} : W^* \otimes V^* \rightarrow (V \otimes W)^*$  is the isomorphism described in [19, Proposition XIV.2.2]), and  $(\lambda_{V,W}^{-1})^*$  is the transpose in  $\mathcal{C}$  of the morphism  $\lambda_{V,W}^{-1}$ , see [19, XIV.2]. By definition, a pivotal structure on  $\mathcal{C}$  is an isomorphism  $i$  between the strongly monoidal functors  $\text{Id}$  and  $(-)^{**}$ . This means that  $i$  is a natural transformation satisfying the coherence conditions

$$(3.1) \quad \phi_{V,W} \circ (i_V \otimes i_W) = i_{V \otimes W}, \quad \phi_0 = i_I.$$

The importance of pivotal structures lies in the following fundamental result of Etingof, Nikshych and Ostrik [13, Prop. 8.24 and 8.23]. Recall first that the category  $\mathcal{C} = {}_H\mathcal{M}^{\text{fd}}$  of finite dimensional left modules over a quasi-Hopf algebra  $H$  is monoidal with left duality. The structure is the following. If  $U, V, W$  are left  $H$ -modules, define  $a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$  by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)).$$

Then  ${}_H\mathcal{M}$  becomes a monoidal category with tensor product  $\otimes$  given via  $\Delta$ , associativity constraints  $a_{U,V,W}$ , unit  $k$  as a trivial  $H$ -module and the usual left and right unit constraints. In addition, every object  $V$  of  $\mathcal{C}$  has a (left) dual object  $V^*$ , the linear dual of  $V$ , with left  $H$ -action  $\langle h \cdot \varphi, v \rangle = \langle \varphi, S(h) \cdot v \rangle$ . The evaluation and coevaluation maps are given by the formulas

$$\text{ev}_V(\varphi \otimes v) = \varphi(\alpha \cdot v), \quad \text{coev}_V(1) = \sum_i \beta \cdot v_i \otimes v^i,$$

for all  $\varphi \in V^*$  and  $v \in V$ . Here  $\{v_i\}_i$  is a basis of  $V$  with dual basis  $\{v^i\}_i$  in  $V^*$ . The next result was proved in [13, Prop. 8.24 and 2.3].

**Theorem 3.1.** *If  $H$  is a semisimple quasi-Hopf algebra over an algebraically closed field of characteristic zero then  ${}_H\mathcal{M}^{\text{fd}}$  has a unique pivotal structure  $j : \text{Id} \rightarrow (-)^{**}$  such that for any simple object  $V$  of  $\mathcal{C}$ ,  $\underline{\dim}_k(V) = \dim_k(V)$ , where  $\underline{\dim}_k(V) := \text{ev}_{V^*} \circ (j_V \otimes \text{id}_{V^*}) \circ \text{coev}_V$  is the (categorical) dimension of  $V$  in  $\mathcal{C}$ .*

The aim of this Section is to compute explicitly this unique pivotal structure for an involutory quasi-Hopf algebra  $H$ . Note that, due to the trace formula proved in [8],  $H$  is semisimple, so that Theorem 3.1 can be applied.

We first give a description of all pivotal structures on  ${}_H\mathcal{M}^{\text{fd}}$ , when  $H$  is finite dimensional. For this, we do not need any assumption on the groundfield  $k$ .

**Proposition 3.2.** *Let  $H$  be a finite dimensional quasi-Hopf algebra over a field  $k$ . Then we have a bijective correspondence between pivotal structures  $i$  on  $\mathcal{C} = {}_H\mathcal{M}^{\text{fd}}$  and invertible elements  $\mathfrak{g}_i \in H$  satisfying*

$$(3.2) \quad S^2(h) = \mathfrak{g}_i^{-1} h \mathfrak{g}_i,$$

for all  $h \in H$  and

$$(3.3) \quad \Delta(\mathfrak{g}_i) = (\mathfrak{g}_i \otimes \mathfrak{g}_i)(S \otimes S)(f_{21}^{-1})f,$$

where  $f = f^1 \otimes f^2$  is the Drinfeld twist defined in (1.15) and  $f_{21} = f^2 \otimes f^1$ .

*Proof.* Since  $H$  is finite dimensional its antipode  $S$  is bijective, cf. [3].

Let  $V \in {}_H\mathcal{M}^{\text{fd}}$ . We have a  $k$ -linear isomorphism  $V \rightarrow V^{**}$ , and  $V^{**}$  can be regarded as  $V$  with newly defined left  $H$ -action  $h \cdot v = S^2(h)v$ .

It has been pointed out in the literature (see [25, 29]), that there is a bijective correspondence between natural isomorphisms between the functors  $\text{Id}$  and  $(-)^{**}$  and invertible elements  $\mathfrak{g}_i \in H$  satisfying (3.2). Let us sketch this correspondence. Let  $i : \text{Id} \rightarrow (-)^{**}$  be a natural isomorphism, and let  $\mathfrak{g}_i = S^2(i_H^{-1}(1))$ ,  $\mathfrak{k}_i = i_H(1)$ . For all  $h \in H$ , we have  $i_H(h) = S^2(h)\mathfrak{k}_i$  and  $i_H^{-1}(h) = S^{-2}(h\mathfrak{g}_i)$ . In particular,  $1 = i_H^{-1}(\mathfrak{k}_i) = S^{-2}(\mathfrak{k}_i\mathfrak{g}_i)$  and  $1 = i_H(S^{-2}(\mathfrak{g}_i)) = \mathfrak{g}_i\mathfrak{k}_i$ , hence  $\mathfrak{k}_i = \mathfrak{g}_i^{-1}$ . Now take  $V \in {}_H\mathcal{M}^{\text{fd}}$ , and fix  $v \in V$ . From the naturality of  $i$ , we deduce that  $i_V(v) = (i_V \circ f)(1) = (f \circ i_H)(1) = \mathfrak{g}_i^{-1}v$ . This means that  $i$  is completely determined by  $\mathfrak{g}_i$ :

$$(3.4) \quad i_V(v) = \mathfrak{g}_i^{-1}v.$$

Now take  $V = H$  and  $v = h$ . (3.4) tells us that  $S^2(h)\mathfrak{g}_i^{-1} = i_H(h) = \mathfrak{g}_i^{-1}h$ , and it follows that (3.2) is satisfied.

For  $H$  a quasi-Hopf algebra and  $\mathcal{C} = {}_H\mathcal{M}^{\text{fd}}$  the isomorphisms  $\lambda_{V,W}$  were computed in [1, Proposition 4.2], namely

$$\lambda_{V,W}(w^* \otimes v^*)(v \otimes w) = \langle v^*, f^1 \cdot v \rangle \langle w^*, f^2 \cdot w \rangle,$$

for all  $v \in V$ ,  $w \in W$ ,  $v^* \in V^*$  and  $w^* \in W^*$ , where  $f = f^1 \otimes f^2$  is the Drinfeld twist defined in (1.15). Observe that the isomorphism  $\lambda_{V,W}$  is denoted  $\phi_{W,V}^*$  in [1]. It is not difficult to see at this point that  $\lambda_{V,W}^{-1}$  is given by

$$\lambda_{V,W}^{-1}(\psi) = \sum_{i,j} \psi(g^1 \cdot v_i \otimes g^2 \cdot w_j) w^j \otimes v^i,$$

for all  $\psi \in (V \otimes W)^*$ , where  $\{v_i\}_i$  and  $\{v^i\}_i$  are dual bases of  $V$  and  $V^*$ ,  $\{w_j\}_j$  and  $\{w^j\}_j$  are dual bases of  $W$  and  $W^*$ , and  $f^{-1} = g^1 \otimes g^2$  is the inverse of the Drinfeld twist.

It is then easy to establish that the first condition from (3.1) is equivalent to the following equivalent conditions:

$$\begin{aligned} \phi_{V,W}(i_V(v) \otimes i_W(w)) &= i_{V \otimes W}(v \otimes w) \\ \Leftrightarrow \lambda_{W^*,V^*}(i_V(v) \otimes i_W(w)) \circ \lambda_{V,W}^{-1} &= i_{V \otimes W}(v \otimes w) \\ \Leftrightarrow \lambda_{W^*,V^*}(i_V(v) \otimes i_W(w))(w^* \otimes v^*) &= i_{V \otimes W}(v \otimes w)(\lambda_{V,W}(w^* \otimes v^*)) \\ \Leftrightarrow i_W(w)(f^1 \cdot w^*) i_V(v)(f^2 \cdot v^*) &= \lambda_{V,W}(w^* \otimes v^*)(\mathfrak{g}_i^{-1} \cdot (v \otimes w)) \\ \Leftrightarrow v^*(S(f^2)\mathfrak{g}_i^{-1} \cdot v) w^*(S(f^1)\mathfrak{g}_i^{-1} \cdot w) &= v^*(f^1(\mathfrak{g}_i^{-1})_1 \cdot v) w^*(f^2(\mathfrak{g}_i^{-1})_2 \cdot w), \end{aligned}$$

for all  $V, W \in \mathcal{C}$ ,  $v \in V$ ,  $w \in W$ ,  $v^* \in V^*$  and  $w^* \in W^*$ . Since  $H$  is an object of  $\mathcal{C}$ , this last condition is equivalent to

$$\Delta(\mathfrak{g}_i^{-1}) = f^{-1}(S \otimes S)(f_{21})(\mathfrak{g}_i^{-1} \otimes \mathfrak{g}_i^{-1}),$$

which is equivalent to (3.3). Now take  $x \in k$ . It follows from (3.4) that  $j_k(x) = \varepsilon(g_i^{-1})x$ . Since  $\phi_0 : k \rightarrow k^{**}$  is the identity, we see that the second condition from (3.1) is equivalent to  $\varepsilon(\mathfrak{g}_i) = 1$ . If (3.3) is satisfied, then  $\varepsilon(\mathfrak{g}_i)^2 = \varepsilon(\mathfrak{g}_i)$  (apply  $\varepsilon \otimes \varepsilon$  to (3.3)), and it follows that  $\varepsilon(\mathfrak{g}_i) = 1$ . This completes our proof.  $\square$

The categorical dimension corresponding to  $i$  is now given by the formula

$$(3.5) \quad \underline{\dim}_k(V) = \sum_i v^i(\mathfrak{g}_i^{-1}\beta S(\alpha) \cdot v_i) = \sum_i v^i(\mathfrak{g}_i S(\beta)\alpha \cdot v_i),$$

for all  $V \in \mathcal{C}$ .

The element  $\mathfrak{g}$  corresponding to the unique pivotal structure  $j$  in Theorem 3.1 was computed in [25]. Since  $H$  is semisimple there exists a (left and right) integral  $\Lambda$  in  $H$  such that  $\varepsilon(\Lambda) = 1$ , cf. [28]. By [25, Corollary 8.5] we then have

$$\mathfrak{g} = q^2 \Lambda_2 p^2 S(q^1 \Lambda_1 p^1),$$

where  $p_R = p^1 \otimes p^2$  and  $q_R = q^1 \otimes q^2$  are the elements defined in (1.19). We should note that the above formula for  $\mathfrak{g}$  can be immediately obtained from the trace formula proved in [8]. An alternative way to compute  $\mathfrak{g}$  can be found in [29, Lemma 3.1], in the case where  $\beta$  is invertible.

If  $H$  is an involutory quasi-Hopf algebra, then  $S^2$  is inner, and induced by  $\beta S(\alpha)$ . We will now show that the invertible element  $\mathfrak{g}$  corresponding to unique pivotal

structure on  $\mathcal{C}$  is precisely  $\beta S(\alpha)$ . In particular,  $\mathfrak{g}^{-1} = S(\beta)\alpha$ , and the equalities in (3.5) become trivial.

**Proposition 3.3.** *Let  $H$  be a finite dimensional involutory quasi-Hopf algebra over an algebraic closed field of characteristic zero. Then the element  $\mathfrak{g}$  corresponding to the pivotal structure  $j$  in Theorem 3.1 is equal to  $\beta S(\alpha)$ .*

*Proof.* By [3, Lemma 2.1], we have for any left integral  $t$  in  $H$  that

$$t_1 \otimes t_2 = \beta q^1 t_1 \otimes q^2 t_2 = q^1 t_1 \otimes S^{-1}(\beta) q^2 t_2.$$

Replacing  $H$  by  $H^{\text{op}}$  we find for any right integral  $r$  in  $H$  that,

$$r_1 \otimes r_2 = r_1 p^1 S^{-1}(\alpha) \otimes r_2 p^2 = r_1 p^1 \otimes r_2 p^2 \alpha.$$

Now, since  $\Lambda$  is both a left and right integral in  $H$  we compute:

$$\begin{aligned} \mathfrak{g} &= q^2 \Lambda_2 p^2 S(q^1 \Lambda^1 p^1) = S^{-1}(\beta^{-1}) \Lambda_2 p^2 S(\Lambda_1 p^1) = S^{-1}(\beta^{-1}) \Lambda_2 \alpha^{-1} S(\Lambda_1) \\ &\stackrel{(2.4)}{=} \varepsilon(\Lambda) S^{-1}(\beta^{-1}) \alpha^{-1} = \beta S(\alpha) S(\beta^{-1}) S(\beta) \alpha \alpha^{-1} = \beta S(\alpha). \end{aligned}$$

In the penultimate equality, we used (2.1) in its equivalent form,  $S^{-1}(h) = \beta S(\alpha) S(h) S(\beta) \alpha$ , for all  $h \in H$ .  $\square$

Assume that  $H$  is an involutory quasi-Hopf algebra. It is a natural question to ask whether the Drinfeld double  $D(H)$  is also involutory. We will present a sufficient condition in Proposition 3.4. In order to simplify the computations we need the following formulas

$$(3.6) \quad f_1^1 p^1 \otimes f_2^1 p^2 S(f^2) = g^1 S(\tilde{q}^2) \otimes g^2 S(\tilde{q}^1),$$

$$(3.7) \quad S(U^1) \tilde{q}^1 U_1^2 \otimes \tilde{q}^2 U_2^2 = f,$$

where  $p_R = p^1 \otimes p^2$  and  $q_L = \tilde{q}^1 \otimes \tilde{q}^2$  are the elements defined in (1.19) and (1.20), and  $f = f^1 \otimes f^2$  is the Drinfeld's twist defined in (1.15) with its inverse  $f^{-1} = g^1 \otimes g^2$  as in (1.16). (3.6,3.7) follow easily from the axioms and the basic properties of a quasi-Hopf algebra.

**Proposition 3.4.** *Let  $H$  be an involutory quasi-Hopf algebra such that*

$$(3.8) \quad \Delta(S(\beta)\alpha) = f^{-1}(S \otimes S)(f_{21})(S(\beta)\alpha \otimes S(\beta)\alpha),$$

where  $f_{21} = f^2 \otimes f^1$ . Then  $D(H)$  is an involutory quasi-Hopf algebra.

*Proof.* For all  $\varphi \in H^*$  and  $h \in H$  we compute:

$$\begin{aligned}
& S_D^2(\varphi \bowtie h) \\
&= S_D(p_1^1 U^1 \rightharpoonup \bar{S}^{-1}(\varphi) \leftarrow f^2 S^{-1}(p^2) \bowtie p_2^1 U^2)(\varepsilon \bowtie S(S(h)f^1)) \\
&\stackrel{(1.38)}{=} (\varepsilon \bowtie S(p_2^1 U^2)F^1)(P_1^1 \mathcal{U}^1 \rightharpoonup \bar{S}^{-1}(p_1^1 U^1 \rightharpoonup \bar{S}^{-1}(\varphi) \leftarrow f^2 S^{-1}(p^2)) \\
&\quad \leftarrow F^2 S^{-1}(P^2) \bowtie P_2^1 \mathcal{U}^2)(\varepsilon \bowtie S(f^1))(\varepsilon \bowtie S^2(h)) \\
&\stackrel{(1.34,3.6)}{=} (S(p_2^1 U^2)_1 g^1 S(\tilde{q}^2))_1 \mathcal{U}^1 p^2 S(f^2) \rightharpoonup \bar{S}^{-2}(\varphi) \\
&\quad \leftarrow S(p_1^1 U^1) \tilde{q}^1 S^{-1}(S(p_2^1 U^2)_2 g^2) \bowtie (S(p_2^1 U^2)_1 g^1 S(\tilde{q}^2))_2 \mathcal{U}^2 S(f^1))(\varepsilon \bowtie S^2(h)) \\
&\stackrel{(1.11,1.24)}{=} (g_1^1 S(p^1 \tilde{q}^2 U_2^2)_1 \mathcal{U}^1 p^2 S(f^2) \rightharpoonup \bar{S}^{-2}(\varphi) \leftarrow S(U^1) \tilde{q}^1 U_1^1 S^{-1}(g^2) \\
&\quad \bowtie g_2^1 S(p^1 \tilde{q}^2 U_2^2)_2 \mathcal{U}^2 S(f^1))(\varepsilon \bowtie S^2(h)) \\
&\stackrel{(1.41,1.11,1.21)}{=} (g_1^1 G^1 S(f^2 \tilde{q}_2^2 U_{(2,2)}^2) \rightharpoonup \bar{S}^{-2}(\varphi) \leftarrow S(U^1) \tilde{q}^1 U_1^1 S^{-1}(g^2) \\
&\quad \bowtie g_2^1 G^2 S(f^1 \tilde{q}_1^2 U_{(2,1)}^2))(\varepsilon \bowtie S^2(h)) \\
&\stackrel{(3.7)}{=} g_1^1 G^1 S(f^2 F_2^2) \rightharpoonup \bar{S}^{-2}(\varphi) \leftarrow F^1 S^{-1}(g^2) \bowtie g_2^1 G^2 S(f^1 F_1^2) S^2(h),
\end{aligned}$$

where  $\mathcal{U}^1 \otimes \mathcal{U}^2$ ,  $F^1 \otimes F^2$  and  $G^1 \otimes G^2$  are second copies of  $U$ ,  $f$  and  $f^{-1}$ , respectively. Using (3.8) twice and (1.11) we obtain that

$$\begin{aligned}
& (\Delta \otimes id)(\Delta(S(\beta)\alpha)) \\
&= g_1^1 G^1 S(f^2 F_2^2) S(\beta)\alpha \otimes g_2^1 G^2 S(f^1 F_1^2) S(\beta)\alpha \otimes g^2 S(F^1) S(\beta)\alpha.
\end{aligned}$$

By (2.1) and (2.2) we have

$$S(\beta S(\alpha)) = S^2(\alpha) S(\beta) = S(\beta) \alpha^2 \beta S(\beta \alpha) = S(\beta) \alpha^2 \alpha^{-1} = S(\beta) \alpha,$$

or, equivalently,  $S^{-1}(S(\beta)\alpha) = \beta S(\alpha)$ . We then have

$$\begin{aligned}
& (\varepsilon \bowtie S(\beta)\alpha)(\varphi \bowtie h)(\varepsilon \bowtie \alpha S(\beta)) \\
&\stackrel{(1.34)}{=} g_1^1 G^1 S(f^2 F_2^2) S(\beta)\alpha \rightharpoonup \varphi \leftarrow \beta S(\alpha) S^{-1}(g^2 S(F^1)) \bowtie g_2^1 G^2 S(F^1 f_1^2) S^2(h) \\
&= g_1^1 G^1 S(f^2 F_2^2) \rightharpoonup \bar{S}^{-2}(\varphi) \leftarrow F^1 S^{-1}(g^2) \bowtie g_2^1 G^2 S(f^1 F_1^2) S^2(h) \\
&= S_D^2(\varphi \bowtie h),
\end{aligned}$$

for all  $\varphi \in H^*$  and  $h \in H$ , and this means that  $D(H)$  is an involutory quasi-Hopf algebra.  $\square$

*Remarks 3.5.* 1) The formula (3.8) holds for any finite dimensional involutory quasi-Hopf algebra over an algebraic closed field of characteristic zero. Indeed, by Proposition 3.3 we have  $\mathfrak{g}^{-1} = S(\beta)\alpha$ , so (3.8) follows from (3.3). Moreover, we believe that (3.8) is satisfied for an arbitrary involutory quasi-Hopf algebra; this would imply that  $H$  is involutory if and only if  $D(H)$  is involutory. Somehow this should follow naturally from the equality  $\Delta(S^2(h)) = \Delta(S(\beta)\alpha h \beta S(\alpha))$  which is equivalent to

$$\begin{aligned}
& \Delta(S(\beta)\alpha) \Delta(h) \Delta(\beta S(\alpha)) \\
&= f^{-1}(S \otimes S)(f_{21})(S(\beta)\alpha \otimes S(\beta)\alpha) \Delta(h) (\beta S(\alpha) \otimes \beta S(\alpha)) (S \otimes S)(f_{21}^{-1}) f.
\end{aligned}$$

2) For  $H(2)$ , the condition (3.8) reduces to  $\Delta(g) = g \otimes g$  which is just part of the definition of  $H(2)$ . So Proposition 3.4 gives a new proof for the fact that  $D(H(2))$  is involutory.

3) One of the aims in [25] was to compute the Frobenius-Schur indicator for the irreducible representations of  $D^\omega(G)$ . Note that  $D^\omega(G)$  is a particular case of the quasi-Hopf algebra  $D^\omega(H)$  roughly described in Example 2.5 1). Thus  $D^\omega(G)$  is an involutory quasi-Hopf algebra, so the element  $\mathfrak{g}$  corresponding to  $D^\omega(G)$  is  $\beta$ , and its inverse is  $\beta^{-1} = S(\beta)$ .

#### 4. REPRESENTATIONS OF INVOLUTORY QUASI-HOPF ALGEBRAS

The goal of this Section is to study the representations of an involutory quasi-Hopf algebra  $H$  over a field  $k$ . In the case where  $H$  is semisimple, we will show that the characteristic of  $k$  does not divide the dimension of any finite dimensional absolutely simple  $H$ -module. Following [9, Definition 3.42] a left  $H$ -module  $V$  is called absolutely simple if for every field extension  $k \subseteq K$ ,  $K \otimes V$  is a simple  $K \otimes H$ -module or, equivalently, if every  $H$ -endomorphism of  $V$  is on the form  $c \text{id}_V$  for some scalar  $c \in k$  (see [9, Theorem 3.43]).

The case when  $H$  is not semisimple is treated as well. In this case the characteristic of  $k$  divides the dimension of any finite dimensional projective  $H$ -module.

In the case of Hopf algebras, the first result was initially proved for Hopf algebras by Larson in [22] and afterwards by Lorenz [24] in a slightly more general shape. The Hopf algebra version of the second result is also due to Lorenz [24].

Before we are able to prove these results over involutory quasi-Hopf algebras, we need some preliminary results.

Let  $V$  and  $W$  be left  $H$ -modules and denote by  $\text{Hom}_H(V, W)$  the set of  $H$ -linear morphisms between  $V$  and  $W$ . If  $V$  is finite dimensional then  $\text{Hom}_H(V, W) \cong \text{Hom}_H(k, W \otimes V^*)$  as  $k$ -vector spaces. Actually, this isomorphism works in any monoidal category  $\mathcal{C}$ , in the sense that for any object  $V$  of  $\mathcal{C}$  admitting a left dual object  $V^*$  and for any  $W \in \mathcal{C}$  we have  $\text{Hom}_{\mathcal{C}}(V, W) \cong \text{Hom}_{\mathcal{C}}(\mathbb{1}, W \otimes V^*)$ , where  $\mathbb{1}$  is the unit object of  $\mathcal{C}$ . Indeed, it is well-known that

$$\text{Hom}_{\mathcal{C}}(V, W) \ni \vartheta \mapsto \left( \mathbb{1} \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{\vartheta \otimes \text{Id}_{V^*}} W \otimes V^* \right) \in \text{Hom}_{\mathcal{C}}(\mathbb{1}, W \otimes V^*)$$

is an isomorphism in  $\mathcal{C}$ ; its inverse is given by

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\mathbb{1}, W \otimes V^*) \ni \varsigma \mapsto & \left( V \xrightarrow{l_V^{-1}} \mathbb{1} \otimes V \xrightarrow{\varsigma \otimes \text{Id}_V} (W \otimes V^*) \otimes V \right. \\ & \left. \xrightarrow{a_{W, V^*, V}} W \otimes (V^* \otimes V) \xrightarrow{\text{Id}_W \otimes \text{ev}_V} W \otimes \mathbb{1} \xrightarrow{r_W} W \right) \in \text{Hom}_{\mathcal{C}}(V, W), \end{aligned}$$

where  $a, l, r$  are the associativity, and the left and right unit constraints of  $\mathcal{C}$ .

In particular, if  $V \in {}_H\mathcal{M}^{\text{fd}}$  then  $\text{End}_H(V) := \text{Hom}_H(V, V)$  is isomorphic to  $\text{Hom}_H(k, V \otimes V^*)$ , as  $k$ -vector space.

Secondly, for any morphism  $a : V \rightarrow V^{**}$  in  ${}_H\mathcal{M}^{\text{fd}}$  the categorical trace  $\text{Tr}_V(a)$  of  $a$  is defined as the scalar corresponding to

$$k \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{a \otimes \text{Id}_{V^*}} V^{**} \otimes V^* \xrightarrow{\text{ev}_{V^*}} k.$$

We are now able to prove one of the main results of this Section.

**Theorem 4.1.** *Let  $H$  be a semisimple involutory quasi-Hopf algebra over a field  $k$  of characteristic  $p \geq 0$ . Then  $p$  does not divide the dimension of any finite dimensional absolutely simple  $H$ -module.*

*Proof.* Since  $H$  is involutory we have  $S^2(h) = \mathbf{g}^{-1}h\mathbf{g}$ , for all  $h \in H$ , where  $\mathbf{g} = \beta S(\alpha)$  and  $\mathbf{g}^{-1} = S(\beta)\alpha$  is its inverse in  $H$ . Then for any  $V \in {}_H\mathcal{M}^{\text{fd}}$  we get that  $a : V \rightarrow V^{**}$  defined by  $a(v)(v^*) = v^*(\mathbf{g}^{-1} \cdot v)$ , for all  $v \in V$  and  $v^* \in V^*$ , is an isomorphism in  ${}_H\mathcal{M}^{\text{fd}}$ . Moreover, its categorical trace  $\underline{\text{Tr}}_V(a)$  coincides with the usual  $k$ -dimension of  $V$ . Indeed, if  $\{v_i\}_{i=1, \overline{n}}$  is a basis in  $V$  with dual basis  $\{v^i\}_{i=1, \overline{n}}$  in  $V^*$  we then have

$$\underline{\text{Tr}}_V(a) = \sum_{i=1}^n a(\beta \cdot v_i)(\alpha \cdot v^i) = \sum_{i=1}^n (\alpha \cdot v^i)(\mathbf{g}^{-1} \beta \cdot v_i) = \sum_{i=1}^n v^i(S(\beta)\alpha)\alpha\beta \cdot v_i = \dim_k(V),$$

because of Lemma 2.2.

Assume now that  $V$  is a finite dimensional absolutely simple  $H$ -module. Thus  $V \in {}_H\mathcal{M}^{\text{fd}}$  and  $\dim_k(\text{End}_H(V)) = 1$ . We claim that  $\underline{\text{Tr}}_V(a) \neq 0$ ; this would imply that  $\dim_k(V) \neq 0$ , as required.

By the way of contradiction, if  $\underline{\text{Tr}}_V(a) = 0$  we then obtain a sequence

$$0 \rightarrow k \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{\text{ev}_{V^*}(a \otimes \text{Id}_{V^*})} k \rightarrow 0$$

in  ${}_H\mathcal{M}^{\text{fd}}$  with  $\text{Im}(\text{coev}_V) \subseteq \text{Ker}(\text{ev}_{V^*}(a \otimes \text{Id}_{V^*}))$ . Since  $H$  is semisimple  $k$  is a projective left  $H$ -module, and so we get a sequence in  ${}_k\mathcal{M}$ ,

$$0 \rightarrow k \xrightarrow{\zeta} \text{Hom}_H(k, V \otimes V^*) \xrightarrow{v} k \rightarrow 0,$$

with  $\zeta$  and  $v$   $k$ -linear morphisms satisfying  $\text{Im}(\zeta) \subseteq \text{Ker}(v)$ . Now, if  $\psi : k \rightarrow \text{Hom}_H(k, V \otimes V^*)$  is a  $k$ -linear map such that  $v\psi = \text{Id}_k$  and  $\theta := \psi v$  then  $\theta^2 = \theta$ , hence  $\text{Hom}_H(k, V \otimes V^*) = \text{Im}(\theta) \oplus \text{Ker}(\theta)$ . Taking into account that  $\text{Im}(\theta) = \text{Im}(\psi) \cong k$  and that  $\text{Ker}(\theta) = \text{Ker}(v) \supseteq \text{Im}(\zeta) \cong k$  it follows that  $\dim_k(\text{Hom}_H(k, V \otimes V^*)) \geq 2$ . Thus  $1 = \dim_k(\text{End}_H(V)) = \dim_k(\text{Hom}_H(k, V \otimes V^*)) \geq 2$ , a contradiction.  $\square$

*Remark 4.2.* Some arguments in the preceding proof have also been used in [13] in the context of fusion categories.

Using Schur's Lemma and Theorem 4.1, we immediately obtain the following result.

**Corollary 4.3.** *Let  $H$  be a semisimple involutory quasi-Hopf algebra over an algebraically closed field of characteristic  $p \geq 0$ . Then  $p$  does not divide the dimension of any finite dimensional simple  $H$ -module.*

We will now focus attention to the case where  $H$  is not semisimple. Let  $V$  and  $W$  be two left  $H$ -modules. Then  $\text{Hom}_k(V, W)$  is a left  $H$ -module, with structure given by the formula  $(h \cdot \psi)(v) = h_1 \cdot \psi(S(h_2) \cdot v)$ , for all  $h \in H$ ,  $\psi \in \text{Hom}_k(V, W)$  and  $v \in V$ . Furthermore, if  $V$  is finite dimensional then  $\text{Hom}_k(V, W) \cong W \otimes V^*$  as left  $H$ -modules. To see this take  $\{v_i\}_{i=1, \overline{n}}$  a basis in  $V$  with dual basis  $\{v^i\}_{i=1, \overline{n}}$  and define

$$\xi : \text{Hom}_k(V, W) \ni \chi \mapsto \sum_{i=1}^n \chi(v_i) \otimes v^i \in W \otimes V^*.$$

Then  $\xi$  is a left  $H$ -linear isomorphism with

$$\xi^{-1} : W \otimes V^* \ni w \otimes v^* \mapsto (v \mapsto v^*(v)w) \in \text{Hom}_k(V, W).$$

(The verification of all these details is left to the reader.) In particular, if  $V$  is a finite dimensional left  $H$ -module then  $\text{End}_k(V) := \text{Hom}_k(V, V) \cong V \otimes V^*$ , as left  $H$ -modules.

In order to simplify the proof of Theorem 4.5, we first need the following result.

**Proposition 4.4.** *Let  $H$  be a finite dimensional quasi-Hopf algebra and  $P, Q$  finite dimensional projective left  $H$ -modules. Then:*

- (i)  $P^* = \text{Hom}_k(P, k)$  is a projective left  $H$ -module;
- (ii)  $P \otimes Q$  is a projective left  $H$ -module, where the  $H$ -module structure of  $P \otimes Q$  is defined by the comultiplication  $\Delta$  of  $H$ ;
- (iii)  $\text{End}_k(P)$  is a projective left  $H$ -module.

*Proof.* (i) Note that, if  $V$  is a left  $H$ -module then  $V^*$ , the linear dual space of  $V$ , is a left  $H$ -module via the  $H$ -action  $(h \cdot v^*)(v) = v^*(S(h) \cdot v)$ , for all  $v^* \in V^*$ ,  $h \in H$  and  $v \in V$ .

Since  $P$  is finite dimensional it follows that  $P$  is a finitely generated projective left  $H$ -module. Therefore, there exists  $n \in \mathbb{N}$  and a left  $H$ -module  $P'$  such that  $P \oplus P' \cong H^n$  as left  $H$ -modules. Thus

$$\begin{aligned} (H^n)^* &= \text{Hom}_k(H^n, k) \cong \text{Hom}_k(P \oplus P', k) \\ &\cong \text{Hom}_k(P, k) \oplus \text{Hom}_k(P', k) = P^* \oplus P'^*, \end{aligned}$$

as left  $H$ -modules. Now,  $H$  is finite dimensional, so from the proof of [18, Theorem 4.3] we know that the map

$$H \ni h \mapsto (h' \mapsto \lambda(h'S(h))) \in H^*$$

is bijective. Here  $\lambda$  is a non-zero left cointegral on  $H$ , we refer to [18] for the definition of a left cointegral. Replacing  $H$  by  $H^{\text{op}, \text{cop}}$  we get that  $H \cong H^*$  as left  $H$ -modules.

Then we obtain that  $H^n \cong (H^*)^n \cong (H^n)^* \cong P^* \oplus P'^*$ , as left  $H$ -modules, so  $P^*$  is a projective left  $H$ -module.

(ii) We follow the same line as above. There exist  $n, m \in \mathbb{N}$  and two left  $H$ -modules  $P'$  and  $Q'$  such that  $P \oplus P' \cong H^n$  and  $Q \otimes Q' \cong H^m$ , as left  $H$ -modules. We then have

$$(H \otimes H)^{nm} \cong H^n \otimes H^m \cong (P \otimes Q) \oplus (P \otimes Q') \oplus (P' \otimes Q) \oplus (P' \otimes Q'),$$

as left  $H$ -modules. It now suffices to show that  $H \otimes H$ , with the diagonal  $H$ -action, is free as a left  $H$ -module. To this end, we will show that the map

$$\mu : .H \otimes .H \rightarrow .H \otimes H, \quad \mu(h \otimes h') = \tilde{q}^2 h'_2 \otimes S^{-1}(\tilde{q}^1 h'_1)h,$$

is a left  $H$ -linear isomorphism. Here we denoted by  $.H \otimes .H$  and  $.H \otimes H$  the  $k$ -vector space  $H \otimes H$ , respectively with the diagonal left  $H$ -action, and left  $H$ -action given by left multiplication.  $q_L = \tilde{q}^1 \otimes \tilde{q}^2$  is the element defined in (1.20). For all  $h, h', h'' \in H$  we have that

$$\begin{aligned} \mu(h'' \cdot (h \otimes h')) &= \mu(h''_1 h \otimes h''_2 h') = \tilde{q}^2 h''_{(2,2)} h'_2 \otimes S^{-1}(\tilde{q}^1 h''_{(2,1)} h'_1) h''_1 h \\ &\stackrel{(1.24)}{=} h'' \tilde{q}^2 h'_2 \otimes S^{-1}(\tilde{q}^1 h'_1) h = h'' \mu(h \otimes h'), \end{aligned}$$

proving that  $\mu$  is left  $H$ -linear. It is easy check that the map

$$\mu^{-1} : .H \otimes H \rightarrow .H \otimes .H, \quad \mu^{-1}(h \otimes h') = h_1 \tilde{p}^1 h' \otimes h_2 \tilde{p}^2$$

is the inverse of  $\mu$ . More precisely, (1.23) and (1.28) imply that  $\mu^{-1} \circ \mu = id$ , while (1.24) and (1.27) imply that  $\mu \circ \mu^{-1} = id$ , we leave the verification of the details to the reader.

(iii) If  $P$  is finite dimensional, then  $\text{End}_k(P) \cong P \otimes P^*$  as left  $H$ -modules. (iii) is now an immediate application of (i) and (ii).  $\square$

We are now able to prove the second important result of this Section.

**Theorem 4.5.** *Let  $H$  be a finite dimensional involutory quasi-Hopf algebra over a field of characteristic  $p \geq 0$ . If  $H$  is not semisimple, then  $p$  divides the dimension of any finite dimensional projective  $H$ -module.*

*Proof.* Let  $P$  be a finite dimensional projective left  $H$ -module and suppose that  $p$  does not divide  $\dim_k(P)$ . If  $a : P \rightarrow P^{**}$  is the isomorphism in  ${}_H\mathcal{M}^{fd}$  defined in the proof of Theorem 4.1, specialized for  $V = P$ , then  $\text{Tr}_P(a) = \dim_k(P) \neq 0$  in  $k$ , and so  $\text{coev}_P : k \rightarrow P \otimes P^*$  is a split monomorphism. Hence,  $k$  is isomorphic to a direct summand of  $P \otimes P^*$  which is a projective left  $H$ -module by Proposition 4.4. It follows that  $k$  is a projective left  $H$ -module.

Now, since  $\varepsilon$  is left  $H$ -linear and surjective, we obtain that there exists an  $H$ -linear map  $\vartheta : k \rightarrow H$  such that  $\varepsilon \circ \vartheta = \text{id}_k$ . Then  $t = \vartheta(1_k)$  is a left integral in  $H$  since

$$ht = h\vartheta(1_k) = \vartheta(h \cdot 1_k) = \varepsilon(h)\vartheta(1_k) = \varepsilon(h)t,$$

for all  $h \in H$ . Moreover,  $\varepsilon(t) = \varepsilon(\vartheta(1_k)) = 1_k$ , so by the Maschke-type Theorem proved in [28] we obtain that  $H$  is semisimple, a contradiction.  $\square$

## 5. INVOLUTORY DUAL QUASI-HOPF ALGEBRAS WITH NON-ZERO INTEGRALS

Let  $H$  be a finite dimensional quasi-Hopf algebra over a field of characteristic  $p \geq 0$ . It was proved in [8] that

$$\text{Tr}(h \mapsto \beta S(\alpha)S^2(h)S(\beta)\alpha) = \varepsilon(r)\lambda(S^{-1}(\alpha)\beta),$$

where  $r$  is a non-zero right integral in  $H$  (this means a left integral in  $H^{\text{op}}$ ) and  $\lambda$  is a non-zero left cointegral on  $H$  such that  $\lambda(S(r)) = 1$ . In particular, if  $H$  is involutory it follows from Lemma 2.2 that  $S(\alpha)S^2(h)S(\beta)\alpha = h$ , and

$$\dim_k(H) = \text{Tr}(\text{id}_H) = \varepsilon(r)\lambda(S^{-1}(\alpha)\beta).$$

Thus a finite dimensional involutory quasi-Hopf algebra is both semisimple and cosemisimple if and only if  $\dim_k(H) \neq 0$  in  $k$ . We recall from [18, 8] that  $H$  is called cosemisimple if there is a left cointegral on  $H$  such that  $\lambda(S^{-1}(\alpha)\beta) \neq 0$ . Consequently, we obtain that a finite dimensional involutory quasi-Hopf algebra over a field of characteristic zero is always semisimple and cosemisimple.

By duality, we obtain that a finite dimensional involutory dual quasi-Hopf algebra over a field of characteristic zero is both semisimple and cosemisimple. The aim of this Section is to study the infinite dimensional case. In fact, we will prove that an involutory co-Frobenius dual quasi-Hopf algebra over a field of characteristic zero is cosemisimple. Our approach is based on the methods developed in [10].

Throughout this Section,  $A$  will be a dual quasi-Hopf algebra. Following [27], a dual quasi-bialgebra  $A$  is a coassociative coalgebra  $A$  with comultiplication  $\Delta$  and counit  $\varepsilon$  together with coalgebra morphisms  $m_A : A \otimes A \rightarrow A$  (the multiplication; we write  $m_A(a \otimes b) = ab$ ) and  $\eta_A : k \rightarrow A$  (the unit; we write  $\eta_A(1) = 1$ ), and an invertible element  $\varphi \in (A \otimes A \otimes A)^*$  (the reassociator), such that for all  $a, b, c, d \in A$  the following relations hold (summation understood):

$$(5.1) \quad a_1(b_1c_1)\varphi(a_2, b_2, c_2) = \varphi(a_1, b_1, c_1)(a_2b_2)c_2,$$

$$(5.2) \quad 1a = a1 = a,$$

$$(5.3) \quad \varphi(a_1, b_1, c_1d_1)\varphi(a_2b_2, c_2, d_2) = \varphi(b_1, c_1, d_1)\varphi(a_1, b_2c_2, d_2)\varphi(a_2, b_3, c_3),$$

$$(5.4) \quad \varphi(a, 1, b) = \varepsilon(a)\varepsilon(b).$$

$A$  is called a dual quasi-Hopf algebra if, moreover, there exist an anti-morphism  $S$  of the coalgebra  $A$  and elements  $\alpha, \beta \in H^*$  such that, for all  $a \in A$ :

$$(5.5) \quad S(a_1)\alpha(a_2)a_3 = \alpha(a)1, \quad a_1\beta(a_2)S(a_3) = \beta(a)1,$$

$$(5.6) \quad \varphi(a_1\beta(a_2), S(a_3), \alpha(a_4)a_5) = \varphi^{-1}(S(a_1), \alpha(a_2)a_3, \beta(a_4)S(a_5)) = \varepsilon(a).$$

It follows from the axioms that  $S(1) = 1$  and  $\alpha(1)\beta(1) = 1$ , so we can assume that  $\alpha(1) = \beta(1) = 1$ . Moreover (5.3) and (5.4) imply

$$(5.7) \quad \varphi(1, a, b) = \varphi(a, b, 1) = \varepsilon(a)\varepsilon(b), \quad \forall a, b \in A.$$

Note that, if  $A$  is a dual quasi-bialgebra then  $A^*$ , the linear dual space of  $A$ , is an algebra with multiplication given by the convolution and unit  $\varepsilon$ .

We call a dual quasi-Hopf algebra  $A$  involutory if

$$(5.8) \quad S^2(a) = \beta(S(a_1))\alpha(a_2)a_3\beta(a_4)\alpha(S(a_5)), \quad \forall a \in A.$$

The proof of the following result is formally dual to the proof of Lemma 2.2 and Proposition 2.4, and is left to the reader.

**Proposition 5.1.** *Let  $A$  be an involutory dual quasi-Hopf algebra. Then  $(\beta \circ S)\alpha$  is convolution invertible with  $((\beta \circ S)\alpha)^{-1} = \beta(\alpha \circ S)$ . In particular, the square of the antipode is coinver, so the antipode  $S$  is bijective.*

*Moreover, the elements  $\alpha$  and  $\beta$  are convolution invertible and for all  $a \in A$  the following relations hold:*

$$S(a_3)\alpha^{-1}(a_2)a_1 = \alpha^{-1}(a)1 \text{ and } a_3\beta^{-1}(a_2)S(a_1) = \beta^{-1}(a)1.$$

Let  $A$  be a dual quasi-Hopf algebra. The connection between integrals and the ideal  $A^{\text{rat}}$  was given in [3].  $A^{\text{rat}}$  is our notation for the left rational part of  $A^*$ , and coincides with the right rational part of  $A^*$ , see [3]. Also recall that a left integral on  $A$  is an element  $T \in A^*$  such that  $a^*T = a^*(1)T$ , for all  $a^* \in A^*$ , and that  $\int_l$  is the standard notation for the space of left integrals on  $A$ . Finally, note that  $A^{\text{rat}} \neq 0$  if and only if  $\int_l \neq 0$ , if and only if  $A$  is a left or right co-Frobenius coalgebra. In this case  $\dim_k(\int_l) = 1$  (see [3]).

Now let  $\sigma : A \otimes A \rightarrow A^*$  be defined by  $\sigma(a \otimes b)(c) = \varphi(c, a, b)$ , for all  $a, b, c \in A$ .  $\sigma$  is convolution invertible, with inverse given by  $\sigma^{-1}(a \otimes b)(c) = \varphi^{-1}(c, a, b)$ , for all  $a, b, c \in A$ . Now define  $\theta^* : \int_l \otimes A \rightarrow A^{\text{rat}}$  by

$$\theta^*(T \otimes a) = \sigma(S(a_5) \otimes \alpha(a_6)a_7)(T \leftarrow a_4)\sigma^{-1}(S(a_3) \otimes \beta(S(a_2))S^2(a_1)),$$

for all  $T \in \int_l$  and  $a \in A$ . The right  $A$ -action  $\leftarrow$  on  $A^*$  is given by  $(a^* \leftarrow a)(b) = \langle a^*, bS(a) \rangle$ , for all  $a^* \in A^*$  and  $a, b \in A$ . It is proved in [3, Proposition 4.2] that  $\theta^*$  is a well-defined isomorphism of right  $A$ -comodules.

Assume that  $S$  is bijective, and define the elements  $p_R, q_R \in (A \otimes A)^*$  by

$$(5.9) \quad p_R(a, b) = \varphi^{-1}(a, b_1, S(b_3))\beta(b_2), \quad q_R(a, b) = \varphi(a, b_3, S^{-1}(b_1))\alpha(S^{-1}(b_2)).$$

Then the map  $\theta^*$  can be written in the following form:

$$\theta^*(T \otimes a)(b) = q_R(b_1, S(a_3))T(b_2S(a_2))p_R(b_3, S(a_1)),$$

for all  $a, b \in A$  and  $T \in \int_l$ .

Let  $A$  be a co-Frobenius dual quasi-Hopf algebra with non-zero left integral  $T$ . For every  $a^* \in A^{\text{rat}}$ , there exists  $a \in A$  such that

$$(5.10) \quad a^*(b) = \omega_T(b, S(a)),$$

for all  $b \in A$ . The map  $\omega_T : A \otimes A \rightarrow k$  is defined by the fomrula

$$(5.11) \quad \omega_T(b, a) = q_R(b_1, a_1)T(b_2a_2)p_R(b_3, a_3),$$

for all  $a, b \in A$ . The next two Lemmas will be crucial in the sequel.

**Lemma 5.2.** *Let  $A$  be an involutory dual quasi-Hopf algebra, and  $T$  a left integral on  $A$ . The map  $\omega_T$  satisfies the following formula, for all  $a \in A$ :*

$$(5.12) \quad \omega_T(a_2, S(a_1)) = T(1)\beta(S(a_1))\alpha(a_2).$$

*Proof.* For all  $a, b \in A$ , we have

$$(5.13) \quad q_R(a_1, S(b_2))\beta(b_3)T(a_2S(b_1)) = T(aS(b)).$$

This formula is the formal dual of the first equality in [3, Lemma 2.1, (2.3)].

Also, from (5.8) and Proposition 5.1 we have for all  $a \in A$  that

$$(5.14) \quad a = \beta(a_1)\alpha(S(a_2))S^2(a_3)\beta(S(a_4))\alpha(a_5).$$

We then compute, for all  $a \in A$  that

$$\begin{aligned} \omega_T(a_2, S(a_1)) &\stackrel{(5.14)}{=} \beta(a_2)\alpha(S(a_3))\omega_T(S^2(a_4), S(a_1))\beta(S(a_5))\alpha(a_6) \\ &\stackrel{(5.11)}{=} q_R(S^2(a_6), S(a_3))\beta(a_4)\alpha(S(a_5))T(S^2(a_7)S(a_2)) \\ &\quad \times p_R(S^2(a_8), S(a_1))\beta(S(a_9))\alpha(a_{10}) \\ &\stackrel{(5.13)}{=} \alpha(S(a_3))T(S^2(a_4)S(a_2))p_R(S^2(a_5), S(a_1))\beta(S(a_6))\alpha(a_7) \\ &\stackrel{(5.5)}{=} T(1)\alpha(S(a_2))p_R(S^2(a_3), S(a_1))\beta(S(a_4))\alpha(a_5) \\ &\stackrel{(5.9, 5.6)}{=} T(1)\varepsilon(S(a_1))\beta(S(a_2))\alpha(a_3) = T(1)\beta(S(a_1))\alpha(a_2), \end{aligned}$$

as claimed, and this completes the proof.  $\square$

Our next result is a generalization of [10, Lemma 1].

**Lemma 5.3.** *Let  $H$  be an involutory dual quasi-Hopf algebra with a non-zero left integral  $T$  and  $J$  a finite dimensional non-zero left  $A$ -subcomodule of  $A$  which is a direct summand of  $A$  as a left  $A$ -comodule. Then  $\dim_k(J) = cT(1)$ , for some scalar  $c \in k$ .*

*Proof.* Let  $J'$  be a left  $A$ -subcomodule of  $A$  such that  $J \oplus J' = A$ , and define  $a^* \in A^*$  by  $a^*(a + a') = \varepsilon(a)$ , for  $a \in J$  and  $a' \in J'$ . Since  $J$  is finite dimensional it follows that  $\text{Ker}(a^*)$  contains a left  $A$ -subcomodule of  $A$  of finite codimension. It then follows from [11, Corollary 2.2.16] that  $a^* \in A^{*\text{rat}}$ , and we have an element  $a \in A$  such that  $a^*(b) = \omega_T(b, S(a))$ , for all  $b \in A$ .

For all  $b, c \in A$ , we have the following formula (cf. [3, (4.16)])

$$q_R(b_2, S(c_2))T(b_3S(c_1))b_1 = q_R(b_1, S(c_2))T(b_2S(c_1))c_3,$$

Using this formula and the definition of  $\omega_T$  we compute that

$$\begin{aligned} a^* \dashv b &= a^*(b_2)b_1 = \omega_T(b_2, S(a))b_1 = q_R(b_2, S(a_3))T(b_3S(a_2))p_R(b_4, S(a_1))b_1 \\ &= q_R(b_1, S(a_3))T(b_2S(a_2))p_R(b_3, S(a_1))a_4 = \omega_T(b, S(a_1))a_2, \end{aligned}$$

for all  $b \in B$ . Now consider a basis  $\{a_i\}_{i=1, \overline{n}}$  for  $J$  and  $\{a'_\lambda\}_{\lambda \in \Lambda}$  a basis for  $J'$  and then write

$$\Delta(a) = \sum_{i=1}^n b_i \otimes a_i + \sum_{\lambda \in \Lambda} b'_\lambda \otimes a'_\lambda$$

for some elements  $b_i, b'_\lambda \in A$ . We then have

$$\begin{aligned} a_j &= \varepsilon((a_j)_2)(a_j)_1 = a^*((a_j)_2)(a_j)_1 = a^* \rightharpoonup a_j = \omega_T(a_j, S(a_1))a_2 \\ &= \sum_{i=1}^n \omega_T(a_j, S(b_i))a_i + \sum_{\lambda \in \Lambda} \omega_T(a_j, S(b'_\lambda))a'_\lambda, \end{aligned}$$

for any  $j \in \{1, \dots, n\}$ , so

$$\omega_T(a_j, S(b_i)) = \delta_{i,j} \text{ and } \omega_T(a_j, S(b'_\lambda)) = 0,$$

for all  $i, j \in \{1, \dots, n\}$  and  $\lambda \in \Lambda$ , where  $\delta_{i,j}$  is the Kronecker's symbol.

In a similar way, we compute for all  $\lambda' \in \Lambda$  that

$$\begin{aligned} 0 &= a^*((a'_{\lambda'})_2)(a'_{\lambda'})_1 = a^* \rightharpoonup a'_{\lambda'} = \omega_T(a'_{\lambda'}, S(a_1))a_2 \\ &= \sum_{i=1}^n \omega_T(a'_{\lambda'}, S(b_i))a_i + \sum_{\lambda \in \Lambda} \omega_T(a'_{\lambda'}, S(b'_\lambda))a'_\lambda \end{aligned}$$

We find that  $\omega_T(a'_{\lambda'}, S(b_i)) = \omega_T(a'_{\lambda'}, S(b'_\lambda)) = 0$ , for all  $i \in \{1, \dots, n\}$  and  $\lambda, \lambda' \in \Lambda$ . It follows now that

$$\dim_k(J) = \sum_{i=1}^n \omega_T(a_i, S(b_i)) + \sum_{\lambda \in \Lambda} \omega_T(a'_\lambda, S(b'_\lambda)) = \omega_T(a_2, S(a_1)).$$

From Lemma 5.2 we conclude that  $\dim_k(J) = T(1)\beta(S(a_1))\alpha(a_2)$ , so  $\dim_k(J) = cT(1)$  for  $c = \beta(S(a_1))\alpha(a_2)$ , as claimed.  $\square$

We can now prove the main result of this Section, generalizing [10, Theorem 2]. The remaining arguments are a purely coalgebraic flavour, and are identical to the arguments in [10].

**Theorem 5.4.** *Let  $A$  be an involutory dual quasi-Hopf algebra with non-zero integral over a field of characteristic zero. Then  $A$  is cosemisimple.*

*Proof.* We have that  $A$  is an injective left  $A$ -comodule, so there exists an injective envelope  $J$  of  $k1_A$  such that  $J \subseteq A$ . Being injective,  $J$  is a direct summand of  $A$ . Since  $A$  is co-Frobenius by [23, Theorem 3] we obtain that  $J$  is finite dimensional. Applying Lemma 5.3 we deduce that  $T(1) \neq 0$  and from [3, Theorem 4.10] we conclude that  $A$  is cosemisimple.  $\square$

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