

# PROJECTIVITY AND FLATNESS OVER THE ENDOMORPHISM RING OF A FINITELY GENERATED MODULE

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ABSTRACT. Let  $A$  be a ring, and  $\Lambda$  a finitely generated  $A$ -module. We give necessary and sufficient conditions for projectivity and flatness of a module over the endomorphism ring of  $\Lambda$ .

## INTRODUCTION

Suppose that a group  $G$  acts as a group of automorphisms on a ring  $R$ . Jøndrup [11] has studied when  $R$  is projective as an  $R^G$ -module. Del Río and García [9] have continued Jøndrup's work, and gave necessary and sufficient conditions for the flatness and projectivity of  $R$  as an  $R^G$ -module. In [12], the second author generalized the results from [9] in the following way: let  $k$  be a commutative ring,  $H$  a Hopf algebra over  $k$ , and  $\Lambda$  a left  $H$ -module algebra. Then we can consider the smash product  $\Lambda \# H$  and the subring of invariants  $\Lambda^H$ . Then we can give necessary and sufficient conditions for the projectivity and flatness over  $\Lambda^H$  of a left  $\Lambda \# H$ -module  $P$ . The results from [9] are recovered if we take  $k = \mathbb{Z}$ ,  $H = kG$ , and  $P = R$ .

The authors discovered recently that the same methods can be applied to the following situation: let  $j : S \rightarrow \Lambda$  be a ring morphism, and consider the centralizer  $\Lambda^S = \{b \in \Lambda \mid bj(s) = j(s)b, \text{ for all } s \in S\}$ . Then we can give necessary and sufficient conditions for projectivity and flatness over  $\Lambda^S$  of an  $(S, \Lambda)$ -bimodule.

In fact, both results are special cases of a more general result which happens to be very elementary: let  $\Lambda$  be a finitely generated module over a ring  $A$ , and  $B$  the endomorphism ring of  $\Lambda$ . Adapting the techniques developed in [9] and [12], we give in Section 1 some necessary and sufficient conditions for projectivity and flatness of a  $B$ -module.

It turns out that in our two motivating examples,  $\Lambda$  is a cyclic module. Such modules appeared recently in [7], and are obtained by considering a ring morphism with a grouplike functional. This will be explained in Section 2.

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## 1. MAIN RESULT

Let  $A$  be an associative ring with unit, and  $\Lambda \in \mathcal{M}_A$ , where  $\mathcal{M}_A$  is the category of right  $A$ -modules and right  $A$ -linear maps. All our results hold for left modules as well, but the formulas are slightly more elegant in the right handed case, a consequence of the fact that we write from left to right. Let  $B = \text{End}_A(\Lambda)$ . Then  $\Lambda$  is a  $(B, A)$ -bimodule, the left action is just evaluation:  $b \cdot \lambda = b(\lambda)$ , for all  $b \in B$  and  $\lambda \in \Lambda$ . Then we have an adjoint pair of functors  $(F, G)$  connecting the categories of right  $B$ -modules and right  $A$ -modules:

$$F = \bullet \otimes_B \Lambda : \mathcal{M}_B \rightarrow \mathcal{M}_A ; G = \text{Hom}_A(\Lambda, \bullet) : \mathcal{M}_A \rightarrow \mathcal{M}_B$$

For  $M \in \mathcal{M}_A$ ,  $G(M) = \text{Hom}_A(\Lambda, M)$  is a right  $B$ -module, by  $(f \cdot b)(\lambda) = f(b\lambda)$ . The unit and counit of the adjunction are defined as follows, for  $M \in \mathcal{M}_A$ ,  $N \in \mathcal{M}_B$ :

$$\eta_N : N \rightarrow \text{Hom}_A(\Lambda, N \otimes_B \Lambda), \eta_N(n)(\lambda) = n \otimes \lambda$$

$$\varepsilon_M : \text{Hom}_A(\Lambda, M) \otimes_B \Lambda \rightarrow M, \varepsilon_M(f \otimes \lambda) = f(\lambda)$$

The adjointness property then means that we have

$$(1) \quad G(\varepsilon_M) \circ \eta_{G(M)} = I_{G(M)} ; \varepsilon_{F(N)} \circ F(\eta_N) = I_{F(N)}$$

In particular,  $\eta_{G(M)}$  is monic, for every  $M \in \mathcal{M}_A$ .

Recall from [14] that  $\Lambda \in \mathcal{M}_A$  is called semi- $\Sigma$ -projective if the functor  $\text{Hom}_A(\Lambda, \bullet) : \mathcal{M}_A \rightarrow \underline{\text{Ab}}$  sends an exact sequence in  $\mathcal{M}_A$  of the form

$$\Lambda^{(J)} \rightarrow \Lambda^{(I)} \rightarrow M \rightarrow 0$$

to an exact sequence in  $\underline{\text{Ab}}$ . Obviously a projective module is semi- $\Sigma$ -projective.

**Lemma 1.1.** *Let  $\Lambda$  be a finitely generated right  $A$ -module, and  $B = \text{End}_A(\Lambda)$ . For every set  $I$  we have*

- (1) *the natural map  $\kappa : B^{(I)} = \text{Hom}_A(\Lambda, \Lambda)^{(I)} \rightarrow \text{Hom}_A(\Lambda, \Lambda^{(I)})$  is an isomorphism*
- (2)  *$\varepsilon_{\Lambda^{(I)}}$  is an isomorphism;*
- (3)  *$\eta_{B^{(I)}}$  is an isomorphism;*
- (4) *if  $\Lambda$  is semi- $\Sigma$ -projective as a right  $A$ -module, then  $\eta$  is a natural isomorphism, in other words, the induction functor  $F = \bullet \otimes_A \Lambda$  is fully faithful.*

*Proof.* 1. is an easy exercise.

2. It is straightforward to check that the canonical isomorphism  $B^{(I)} \otimes_B \Lambda \cong \Lambda^{(I)}$  is the composition of the maps

$$B^{(I)} \otimes_B \Lambda \xrightarrow{\kappa \otimes I_\Lambda} \text{Hom}_A(\Lambda, \Lambda^{(I)}) \otimes_B \Lambda \xrightarrow{\varepsilon_{\Lambda^{(I)}}} \Lambda^{(I)}$$

and it follows from 1. that  $\varepsilon_{\Lambda^{(I)}}$  is an isomorphism.

3. Putting  $M = \Lambda^{(I)}$  in (1), we find

$$\mathrm{Hom}_A(\Lambda, \varepsilon_{\Lambda^{(I)}}) \circ \eta_{\mathrm{Hom}_A(\Lambda, \Lambda^{(I)})} = I_{\mathrm{Hom}_A(\Lambda, \Lambda^{(I)})}$$

From 2., it follows that  $\mathrm{Hom}_A(\Lambda, \varepsilon_{\Lambda^{(I)}})$  is an isomorphism, and from 1. that  $\mathrm{Hom}_A(\Lambda, \Lambda^{(I)}) \cong B^{(I)}$ , hence  $\eta_{B^{(I)}}$  is an isomorphism.

4. follows from [14, Theorem 2.1]. We include a short proof for the sake of completeness. Take a free resolution  $B^{(J)} \rightarrow B^{(I)} \rightarrow N \rightarrow 0$  of a right  $B$ -module  $N$ .  $\eta$  is natural, so we have a commutative diagram

$$\begin{array}{ccccccc} B^{(J)} & \longrightarrow & B^{(I)} & \longrightarrow & N & \longrightarrow & 0 \\ \downarrow \eta_{B^{(J)}} & & \downarrow \eta_{B^{(I)}} & & \downarrow \eta_N & & \\ GF(B^{(J)}) & \longrightarrow & GF(B^{(I)}) & \longrightarrow & GF(N) & \longrightarrow & 0 \end{array}$$

The top row is exact; the bottom row is exact, since  $GF(B^{(I)}) = \mathrm{Hom}_A(\Lambda, \Lambda^{(I)})$  and  $\Lambda$  is semi- $\Sigma$ -projective. By 3.,  $\eta_{B^{(I)}}$  and  $\eta_{B^{(J)}}$  are isomorphisms, and it follows from the Lemma of five that  $\eta_N$  is an isomorphism.  $\square$

We can now give equivalent conditions for the projectivity and flatness of  $P \in \mathcal{M}_B$ .

**Theorem 1.2.** *Assume that  $\Lambda \in \mathcal{M}_A$  is finitely generated, and let  $B = \mathrm{End}_A(\Lambda)$ . For  $P \in \mathcal{M}_B$ , we consider the following statements.*

- (1)  $P \otimes_B \Lambda$  is projective in  $\mathcal{M}_A$  and  $\eta_P$  is injective;
- (2)  $P$  is projective as a right  $B$ -module;
- (3)  $P \otimes_B \Lambda$  is a direct summand in  $\mathcal{M}_A$  of some  $\Lambda^{(I)}$ , and  $\eta_P$  is bijective;
- (4) there exists  $Q \in \mathcal{M}_A$  such that  $Q$  is a direct summand of some  $\Lambda^{(I)}$ , and  $P \cong \mathrm{Hom}_A(\Lambda, Q)$  in  $\mathcal{M}_B$ ;
- (5)  $P \otimes_B \Lambda$  is a direct summand in  $\mathcal{M}_A$  of some  $\Lambda^{(I)}$ .

Then  $1. \Rightarrow 2. \Leftrightarrow 3. \Leftrightarrow 4. \Rightarrow 5.$  If  $\Lambda$  is semi- $\Sigma$ -projective (in particular,  $\Lambda$  is quasi-projective, see [1, Proposition 1.8]) as a right  $A$ -module, then  $5. \Rightarrow 3.$ ; if  $\Lambda$  is projective as a right  $A$ -module, then  $2. \Rightarrow 1.$

*Proof.*  $2. \Rightarrow 3.$  If  $P$  is projective as a right  $B$ -module, then we can find a set  $I$  and  $P' \in \mathcal{M}_B$  such that  $B^{(I)} \cong P \oplus P'$ . Then obviously  $\Lambda^{(I)} \cong B^{(I)} \otimes_B \Lambda \cong (P \otimes_B \Lambda) \oplus (P' \otimes_B \Lambda)$ .  $\eta$  is a natural transformation, so we have a commutative diagram

$$\begin{array}{ccc} B^{(I)} & \xrightarrow{\cong} & P \oplus P' \\ \downarrow \eta_{B^{(I)}} & & \downarrow \eta_P \oplus \eta_{P'} \\ \mathrm{Hom}_A(\Lambda, \Lambda^{(I)}) & \xrightarrow{\cong} & \mathrm{Hom}_A(\Lambda, P \otimes_B \Lambda) \oplus \mathrm{Hom}_A(\Lambda, P' \otimes_B \Lambda) \end{array}$$

From the fact that  $\eta_{B^{(I)}}$  is an isomorphism, it follows that  $\eta_P$  (and  $\eta_{P'}$ ) are isomorphisms.

3.  $\Rightarrow$  5. Take  $Q = P \otimes_B \Lambda$ .

4.  $\Rightarrow$  2. If  $f : \Lambda^{(I)} \rightarrow Q$  is a split epimorphism in  $\mathcal{M}_A$ , then

$$\mathrm{Hom}_A(\Lambda, f) : \mathrm{Hom}_A(\Lambda, \Lambda^{(I)}) \cong B^{(I)} \rightarrow \mathrm{Hom}_A(\Lambda, Q) \cong P$$

is also split surjective, hence  $P$  is projective as a right  $B$ -module.

4.  $\Rightarrow$  5. is trivial.

5.  $\Rightarrow$  3. under the assumption that  $\Lambda$  is semi- $\Sigma$ -projective: follows immediately from part 4. of Lemma 1.1.

1.  $\Rightarrow$  2. Take an epimorphism  $f : B^{(I)} \rightarrow P$  in  $\mathcal{M}_B$ . Then  $f \otimes_B I_\Lambda : \Lambda^{(I)} \rightarrow P \otimes_B \Lambda$  is also surjective, and split in  $\mathcal{M}_A$ , since  $P \otimes_B \Lambda$  is projective. Consider the commutative diagram

$$\begin{array}{ccccc} B^{(I)} & \xrightarrow{f} & P & \longrightarrow & 0 \\ \eta_{B^{(I)}} \downarrow & & \downarrow \eta_P & & \\ \mathrm{Hom}_A(\Lambda, \Lambda^{(I)}) & \xrightarrow{GF(f)} & \mathrm{Hom}_A(\Lambda, P \otimes_B \Lambda) & \longrightarrow & 0 \end{array}$$

The bottom row is split exact, since any functor, in particular  $\mathrm{Hom}_A(\Lambda, \bullet)$  preserves split exact sequences.  $\eta_{B^{(I)}}$  is an isomorphism, by Lemma 1.1, and a diagram chasing tells us that  $\eta_P$  is surjective. By assumption,  $\eta_P$  is injective, so  $\eta_P$  is bijective, and the top row is isomorphic to the bottom row, and therefore split. Thus  $P \in \mathcal{M}_B$  is projective.

2.  $\Rightarrow$  1. If  $\Lambda$  is projective in  $\mathcal{M}_A$ , then  $\Lambda$  is a direct summand of  $A^{(J)}$  in  $\mathcal{M}_A$  for some set  $J$ . Now  $P \otimes_B \Lambda$  is a direct summand of  $\Lambda^{(I)}$ , and also of  $A^{(J \times I)}$ , so  $P \otimes_B \Lambda \in \mathcal{M}_A$  is projective. Since 2. implies 3., it follows that  $\eta_P$  is injective.  $\square$

The Govorov-Lazard Theorem ([10, 13]) states that  $M \in \mathcal{M}_A$  is flat if and only if it is the inductive limit of a directed system of free modules of finite type. Theorem 1.3 can be viewed as a generalization of this classical result.

**Theorem 1.3.** *Assume that  $\Lambda \in \mathcal{M}_A$  is finitely presented, and let  $B = \mathrm{End}_A(\Lambda)$ . For  $P \in \mathcal{M}_B$ , the following assertions are equivalent.*

- (1)  $P$  is flat as a right  $B$ -module;
- (2)  $P \otimes_B \Lambda = \varinjlim Q_i$ , where  $Q_i \cong \Lambda^{n_i}$  in  $\mathcal{M}_A$  for some positive integer  $n_i$ , and  $\eta_P$  is bijective;

- (3)  $P \otimes_B \Lambda = \varinjlim Q_i$ , where  $Q_i \in \mathcal{M}_A$  is a direct summand of some  $\Lambda^{(I)}$ , and  $\eta_P$  is bijective;
- (4) there exists  $Q = \varinjlim Q_i \in \mathcal{M}_A$ , such that  $Q_i \cong \Lambda^{n_i}$  for some positive integer  $n_i$  and  $\text{Hom}_A(\Lambda, Q) \cong P$  in  $\mathcal{M}_B$ ;
- (5) there exists  $Q = \varinjlim Q_i \in \mathcal{M}_A$ , such that  $Q_i$  is a direct summand of  $\Lambda^{(I)}$  as a right  $A$ -module and  $\text{Hom}_A(\Lambda, Q) \cong P$  in  $\mathcal{M}_B$ .

If  $\Lambda$  is semi- $\Sigma$ -projective as a right  $A$ -module, then these conditions are also equivalent to conditions 2. and 3., without the assumption that  $\eta_P$  is bijective.

*Proof.* 1.  $\Rightarrow$  2.  $P = \varinjlim N_i$ , with  $N_i = B^{n_i}$ . Take  $Q_i = \Lambda^{n_i}$ , then

$$\varinjlim Q_i \cong \varinjlim (N_i \otimes_B \Lambda) \cong \left( \varinjlim N_i \right) \otimes_B \Lambda \cong P \otimes_B \Lambda$$

Consider the following commutative diagram:

$$\begin{array}{ccc} P = \varinjlim N_i & \xrightarrow{\varinjlim \eta_{N_i}} & \varinjlim \text{Hom}_A(\Lambda, N_i \otimes_B \Lambda) \\ \eta_P \downarrow & & \downarrow f \\ \text{Hom}_A(\Lambda, \left( \varinjlim N_i \right) \otimes_B \Lambda) & \xrightarrow{\cong} & \text{Hom}_A(\Lambda, \varinjlim (N_i \otimes_B \Lambda)) \end{array}$$

By Lemma 1.1, the  $\eta_{N_i}$  are isomorphisms; the natural homomorphism  $f$  is an isomorphism, because  $\Lambda$  is finitely presented. Hence  $\eta_P$  is an isomorphism.

2.  $\Rightarrow$  3. and 4.  $\Rightarrow$  5. are obvious.

2.  $\Rightarrow$  4. and 3.  $\Rightarrow$  5. Put  $Q = P \otimes_B \Lambda$ . Then  $\eta_P : P \rightarrow \text{Hom}_A(\Lambda, P \otimes_B \Lambda)$  is the required isomorphism.

5.  $\Rightarrow$  1. We have a split exact sequence

$$0 \rightarrow N_i \rightarrow P_i = \Lambda^{(I)} \rightarrow Q_i \rightarrow 0$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & FG(N_i) & \longrightarrow & FG(P_i) & \longrightarrow & FG(Q_i) \longrightarrow 0 \\ & & \downarrow \varepsilon_{N_i} & & \downarrow \varepsilon_{P_i} & & \downarrow \varepsilon_{Q_i} \\ 0 & \longrightarrow & N_i & \longrightarrow & P_i & \longrightarrow & Q_i \longrightarrow 0 \end{array}$$

We know from Lemma 1.1 that  $\varepsilon_{P_i}$  is an isomorphism. Both rows in the sequence are split exact, so it follows that  $\varepsilon_{N_i}$  and  $\varepsilon_{Q_i}$  are also isomorphisms.

Next consider the commutative diagram

$$\begin{array}{ccc}
\left(\varinjlim \mathrm{Hom}_A(\Lambda, Q_i)\right) \otimes_B \Lambda & \xrightarrow{f \otimes_B I_\Lambda} & \mathrm{Hom}_A(\Lambda, Q) \otimes_B \Lambda \\
\uparrow h & & \downarrow \varepsilon_Q \\
\varinjlim \left(\mathrm{Hom}_A(\Lambda, Q_i) \otimes_B \Lambda\right) & \xrightarrow{\varinjlim \varepsilon_{Q_i}} & Q
\end{array}$$

where  $h$  and  $f$  are the natural homomorphisms.  $h$  is an isomorphism, because  $\bullet \otimes_B \Lambda$  preserves inductive limits, and  $f$  is an isomorphism because  $\Lambda \in \mathcal{M}_A$  is finitely presented.  $\varinjlim \varepsilon_{Q_i}$  is an isomorphism, because every  $\varepsilon_{Q_i}$  is an isomorphism, so it follows that  $\varepsilon_Q$  is an isomorphism. From (1), we deduce

$$\mathrm{Hom}_A(\Lambda, \varepsilon_Q) \circ \eta_{\mathrm{Hom}_A(\Lambda, Q)} = I_{\mathrm{Hom}_A(\Lambda, Q)}$$

$\mathrm{Hom}_A(\Lambda, \varepsilon_Q)$  is an isomorphism, so  $\eta_{\mathrm{Hom}_A(\Lambda, Q)}$  is also an isomorphism, and, since  $\mathrm{Hom}_A(\Lambda, Q) \cong P$ ,  $\eta_P$  is an isomorphism. Consider the isomorphisms

$$\begin{array}{ccc}
P & \xrightarrow{\eta_P} & \mathrm{Hom}_A(\Lambda, P \otimes_B \Lambda) \xrightarrow{\cong} \mathrm{Hom}_A(\Lambda, \mathrm{Hom}_A(\Lambda, Q) \otimes_B \Lambda) \\
\mathrm{Hom}(\Lambda, \varepsilon_Q) & \longrightarrow & \mathrm{Hom}_A(\Lambda, Q) \xrightarrow{f} \varinjlim \mathrm{Hom}_A(\Lambda, Q_i)
\end{array}$$

It follows from Lemma 1.1 that  $\mathrm{Hom}_A(\Lambda, P_i) \cong B^{(I_i)}$  is projective as a right  $B$ -module, hence  $\mathrm{Hom}_A(\Lambda, Q_i)$  is also projective as a right  $B$ -module, and we conclude that  $P \in \mathcal{M}_B$  is flat.

The final statement is an immediate consequence of part 4 of Lemma 1.1.  $\square$

*Remark 1.4.* To  $\Lambda \in \mathcal{M}_A$ , we can associate a Morita context (cf. [2]), namely  $(A, B, \Lambda, \Lambda^* = \mathrm{Hom}_A(\Lambda, A), \tau, \mu)$  with connecting maps

$$\tau : \Lambda \otimes_A \Lambda^* \rightarrow B, \quad \tau(\lambda \otimes \lambda^*)(\mu) = \lambda \lambda^*(\mu)$$

$$\mu : \Lambda^* \otimes_B \Lambda \rightarrow A, \quad \mu(\lambda^* \otimes \lambda) = \lambda^*(\lambda)$$

$\tau$  is surjective (and a fortiori bijective) if and only if  $\Lambda$  is finitely generated and projective as a right  $A$ -module. In this situation, the natural transformation

$$\gamma : G_1 = \bullet \otimes_A \mathrm{Hom}_A(\Lambda, A) \rightarrow G = \mathrm{Hom}_A(\Lambda, \bullet)$$

with  $\gamma_M(m \otimes_A \lambda^*)(\lambda) = m \lambda^*(\lambda)$  is a natural isomorphism.

For any  $N \in \mathcal{M}_B$ , we easily compute that

$$\gamma_{N \otimes_B \Lambda} = \eta_N \circ (I_N \otimes \tau)$$

We can conclude that  $\eta$  is a natural isomorphism if  $\Lambda$  is finitely generated and projective as a right  $A$ -module. This is an alternative proof of a special case of Lemma 1.1.

## 2. MAIN EXAMPLE

Let  $\Lambda$  and  $A$  be rings,  $i : \Lambda \rightarrow A$  a ring morphism, and  $\chi : A \rightarrow \Lambda$  a right grouplike functional. This means that  $\chi$  satisfies the following properties, for all  $a, b \in A$  (see [7]):

- (1)  $\chi$  is right  $\Lambda$ -linear;
- (2)  $\chi(\chi(a)b) = \chi(ab)$ ;
- (3)  $\chi(1_A) = 1_\Lambda$ .

Then  $\Lambda \in \mathcal{M}_A$ , with action

$$\lambda \leftarrow a = \chi(\lambda a)$$

$\Lambda$  is finitely generated, in fact it is a cyclic module, since

$$\lambda = \chi(1_A)\lambda = \chi(1_A\lambda) = \chi(1_A\lambda 1_\Lambda) = 1_\Lambda \leftarrow (1_A\lambda)$$

for all  $\lambda \in \Lambda$ . It is easy to verify that

$$\mathrm{Hom}_A(\Lambda, M) \cong M^A = \{m \in M \mid ma = m\chi(a), \text{ for all } a \in A\}$$

In particular

$$B = \mathrm{Hom}_A(\Lambda, \Lambda) \cong \Lambda^A = \{b \in \Lambda \mid b\chi(a) = \chi(ba), \text{ for all } a \in A\}$$

We can apply Theorems 1.2 and 1.3, and we find necessary and sufficient conditions for  $P \in \mathcal{M}_B$  to be projective or flat. Observe that  $\Lambda$  is semi- $\Sigma$ -projective if the functor  $(\bullet)^A$  sends an exact sequence  $\Lambda^{(J)} \rightarrow \Lambda^{(I)} \rightarrow M \rightarrow 0$  in  $\mathcal{M}_A$  to a free resolution  $B^{(J)} \rightarrow B^{(I)} \rightarrow M^A \rightarrow 0$ .  $\Lambda \in \mathcal{M}_A$  is projective if  $(\bullet)^A$  is exact.  $\Lambda$  is finitely presented as a right  $A$ -module if  $\mathrm{Ker} \chi$  is a finitely generated right ideal of  $A$ . We now give some particular examples of ring morphisms with a grouplike functional. More examples can be found in [7].

**Example 2.1.** Let  $H$  be a Hopf algebra over a commutative ring  $k$ , and  $\Lambda$  a left  $H$ -module algebra. The smash product  $A = \Lambda \# H$  is equal to  $\Lambda \otimes H$  as a  $k$ -module, with multiplication defined by

$$(\lambda \# h)(\mu \# k) = \lambda(h_{(1)} \cdot \mu) \# h_{(2)}k$$

Let  $\eta_H$  and  $\varepsilon_H$  be the unit and counit of  $H$ . Then  $i = I_\Lambda \otimes \eta_H : \Lambda \rightarrow \Lambda \# H$  is a ring morphism, and  $\chi = I_\Lambda \otimes \varepsilon_H : \Lambda \# H \rightarrow \Lambda$  is a left grouplike functional. It is easy to compute that  $B = \Lambda^A = \Lambda^H$  is the subring of invariants of  $\Lambda$ . Thus we find necessary and sufficient conditions for projectivity and flatness of a module over the subring of invariants, and we recover the results obtained by the second author in [12].

In the particular situation where  $H = kG$  is a group ring and  $\Lambda$  is a  $G$ -module algebra,  $A$  is a skew group ring, and we recover [9, Theorems 3 and 5], in which necessary and sufficient conditions for projectivity and flatness of  $\Lambda$  as a  $\Lambda^G$ -module are given.

This example can be extended to factorization structures, we refer to [7] for detail.

**Example 2.2.** Let  $j : S \rightarrow \Lambda$  be a morphism of algebras over a commutative ring  $k$ . Then  $i : \Lambda \rightarrow A = S^{\text{op}} \otimes \Lambda$ ,  $i(\lambda) = 1_S \otimes \lambda$  is a ring morphism, and  $\chi : A \rightarrow \Lambda$ ,  $\chi(s \otimes \lambda) = j(s)\lambda$  is a right grouplike functional. We easily compute that

$$B = \Lambda^A = \{b \in \Lambda \mid bj(s) = j(s)b, \text{ for all } s \in S\} = \Lambda^S$$

is the centralizer of  $S$  in  $\Lambda$ . Thus we obtain necessary and sufficient conditions for projectivity and flatness over the centralizer.

**Example 2.3.** Let  $\Lambda$  be a ring, and  $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$  a  $\Lambda$ -coring. (this is a coalgebra in the monoidal category  ${}_{\Lambda}\mathcal{M}_{\Lambda}$ ). Corings were introduced by Sweedler [15]. The language of corings allows us to generalize, unify and formulate more elegantly many results about generalized Hopf modules. This is due to the fact that Hopf modules (and their generalizations) are in fact comodules over a coring. This observation can be found implicitly in the proof of [8, Theorem 2.11]. It is tributed to Takeuchi in a Mathematical Review (MR 2000c 16047) written by Masuoka, but the referee informs us that it was already known by Sweedler. Only recently the importance of the coring viewpoint became clear. [3] is the first of a series of papers on this subject, we refer to [5] for more references and the actual state of the art.  $x \in \mathcal{C}$  is called grouplike if  $\Delta_{\mathcal{C}}(x) = x \otimes_{\Lambda} x$  and  $\varepsilon_{\mathcal{C}}(x) = 1_{\Lambda}$  (see [4]). The left dual of  $\mathcal{C}$  is then a ring:  $A = {}_{\Lambda}\text{Hom}(\mathcal{C}, \Lambda)$  has an associative multiplication

$$(f\#g)(c) = g(c_{(1)})f(c_{(2)})$$

The unit is  $\varepsilon_{\mathcal{C}}$ . We have a ring morphism

$$i : \Lambda \rightarrow A, \quad i(\lambda)(c) = \varepsilon_{\mathcal{C}}(c)\lambda$$

and a right grouplike functional

$$\chi : A \rightarrow \Lambda, \quad \chi(f) = f(x).$$

We easily compute that

$$B = \Lambda^A = \{b \in \Lambda \mid bf(x) = f(xb), \text{ for all } f \in A\},$$

and we find necessary and sufficient conditions for projectivity and flatness of a right  $B$ -module.

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