FROBENIUS AND SEPARABLE FUNCTORS FOR THE CATEGORY OF ENTWINED MODULES OVER MONOIDAL COWREATHS. APPLICATIONS

D. BULACU, S. CAENEPEEL, AND B. TORRECILLAS

Abstract. Entwined modules over cowreaths in a monoidal category are introduced. Monoidal cowreaths can be identified to coalgebras in an appropriate monoidal category. It is investigated when such coalgebras are Frobenius (resp. separable), and when the forgetful functor from entwined modules to representations of the underlying algebra is Frobenius (resp. separable). These properties are equivalent when the unit object of the category is an ⊗-generator, a property stronger than the generator property. Our results are applied to Doi-Hopf modules, two-sided Hopf modules and Yetter-Drinfeld modules over a quasi-Hopf algebra.

Introduction

Central elements in the enveloping algebra $A\otimes A^{\rm op}$ of an algebra A are often called Casimir elements, and they play a crucial role in the theory of separable and of Frobenius algebras. The fact that they appear in both theories is well understood, and has a categorical explanation related to the properties that an algebra is Frobenius if the restriction of scalars functor G is Frobenius, that is, its right adjoint is also a left adjoint, and that it is separable if and only if G is separable in the sense of [31]. This can be exploited in order to study Frobenius and separable functors simultaneously. This idea originated in the study of separability and Frobenius properties for Doi-Hopf modules in [18, 19, 20], and was later refined and applied to entwined modules, see [4].

Entwined modules over entwining structures were introduced by Brzeziński in [3]. One of the attractive aspects is that many structures that appear in Hopf algebra theory, such as relative Hopf modules, Doi-Hopf and Yetter-Drinfeld modules, turn out to be special cases. An entwining structure is a kind of local braiding between an algebra and a coalgebra. In fact an entwining structure with underlying algebra A can be viewed as a coalgebra in the monoidal category $\mathcal{T}_A$ of transfer morphisms through A as introduced by Tambara in [37]. Tambara's construction can be obtained from Street's formal theory of monads, see [35]. Monads in a 2-category $\mathcal{C}$ can be organized into a new 2-category ${\rm Mnd}(\mathcal{C})$. For an algebra (or monad) in a strict monoidal category $\mathcal{C}$ (a 2-category with single 0-cell), Tambara's category $\mathcal{T}_A$ is the category ${\rm Mnd}(\mathcal{C})(A,A)$ of endomorphisms of A in ${\rm Mnd}(\mathcal{C})$.

There is a second way to organize monads into a 2-category, see [27]; the second 2-category is the Eilenberg-Moore 2-category ${\rm EM}(\mathcal{C})$. It coincides with ${\rm Mnd}(\mathcal{C})$ at the level of 0-cells and 1-cells, but has different 2-cells. A cowreath in $\mathcal{C}$ is a comonad in ${\rm EM}(\mathcal{C})$, and consists of an algebra in $\mathcal{C}$ together with a coalgebra in $\mathcal{T}_A^\# = {\rm EM}(\mathcal{C})(A,A)$, the category of endomorphisms of A in the

---

2010 *Mathematics Subject Classification.* Primary 16T05; Secondary 18D10; 16T15; 16S40.

Key words and phrases. Module category, cowreath, generalized entwined module, Frobenius functor, separable functor, Frobenius coalgebra, coseparable coalgebra, quasi-Hopf algebra, Doi-Hopf module, Yetter-Drinfeld module.

The first author was supported by the UEFISCDI Grant PN-II-ID-PCE-2011- 3-0635, contract no. 253/5.10.2011 of CNCSIS. The second author was supported by research project G.0117.10 "Equivariant Brauer groups and Galois deformations" from FWO-Vlaanderen. The third author was partially supported by FQM 211 from Junta Andalucía and by research project MTM2014-54439 from MEC. The first author thanks the Vrije Universiteit Brussel and the Universidad de Almería for their support and warm hospitality. The authors also thank Bodo Pareigis for sharing his "diagrams" program.

Eilenberg-Moore 2-category. We can introduce entwined modules over a monoidal cowreath. The initial aim of this paper is to study when the forgetful functor from entwined modules to A-modules is Frobenius or separable. This is related to the question when a coalgebra in $\mathcal{T}_A^\#$ is a Frobenius or a coseparable coalgebra.
Compared to the classical situation, we have a two-fold generalization: first of all, the category of vector spaces is replaced by an arbitrary (strict) monoidal category $\mathcal{C}$. The best results are obtained in the situation where the unit object $\underline{1}$ is an $\otimes$-generator of the monoidal category $\mathcal{C}$, as introduced in [15]. The following monoidal categories satisfy this condition: the category of vector spaces, the category of bimodules ${}_R\mathcal{M}_R$ over an Azumaya k-algebra R, the category of finite dimensional Hilbert complex vector spaces FdHilb, and the category $\mathcal{Z}_k$ as introduced in [16]. We refer to [15, Examples 3.2].
Secondly, we work over monoidal cowreaths which can be viewed as generalized entwining structures. Our motivation to investigate monoidal cowreaths (which we also term generalized entwining structures) comes from the applications that we have in mind, namely the study of categories of Doi-Hopf modules, two-sided Hopf modules and Yetter-Drinfeld modules over a quasi-Hopf algebra, which can be viewed as entwined modules over a generalized entwining structure.

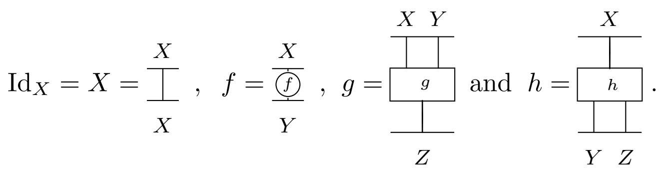
This paper consists of two parts. In the first part, consisting of Sections 2-6, we present our general theory. The second part consists of Sections 7-9, and contains applications to categories of modules defined by quasi-Hopf algebras. In Section 1, we present preliminary results on monoidal categories, bimodules and quasi-Hopf algebras. In Section 2, we introduce cowreaths in monoidal categories, and entwined modules over them. In Section 3, we introduce generalized factorization structures; these are algebras in $\mathcal{T}_A^\#$, or, equivalently, wreaths in $\mathcal{C}$. Given a generalized factorization structure, we can define an algebra in $\mathcal{C}$, called the wreath product algebra or the generalized smash product. Duality arguments turn monoidal cowreaths into generalized factorization structures, and the category of entwined modules is isomorphic to the category of modules over the generalized smash product, see Theorem 3.4. In Section 4, we discuss when the forgetful functor F is Frobenius. F always has a right adjoint G; in order to investigate when G is also a left adjoint, we need to investigate natural transformations from the identity functor to FG, and from GF to the identity functor. Propositions 4.6 and 4.7 tell us that the necessary and sufficient information that is needed to produce such natural transformations is encoded in the so-called Frobenius elements and Casimir morphisms, at least in the case where $\underline{1}$ is an $\otimes$-generator. Using these results, it is straightforward to prove the main Theorem 4.8, stating that F is a Frobenius functor if and only if the coalgebra corresponding to the given cowreath is Frobenius. In Section 5 it is shown that there is a strong monoidal functor from the category of generalized transfer morphisms $\mathcal{T}_A^\#$ to the category of A-bimodules, as introduced in the preliminary Section 1.2. Consequently, a cowreath produces an A-coring, that is a coalgebra in the category of A-bimodules. The main result is that this A-coring is Frobenius if and only if the corresponding coalgebra (A, X) in $\mathcal{T}_A^\#$ is Frobenius, see Theorem 5.2. Under the assumption that X has a right adjoint Y, we have additional results, see Theorem 5.6. Separability is investigated in Section 6. The main result is Theorem 6.5 stating that a coalgebra (X, ψ) in $\mathcal{T}_A^\#$ is coseparable if and only if the forgetful functor is coseparable. Again, additional results can be stated if X has a right adjoint.
In Section 7, we apply our results to Doi-Hopf modules over a quasi-Hopf algebra H, as introduced in [7]. Let A be a right H-comodule algebra. To a coalgebra C in the monoidal category $\mathcal{M}_H$, we can associate a coalgebra C in $\mathcal{T}_A^\#$. If C is a Frobenius coalgebra in $\mathcal{M}_H$, then the forgetful functor from the category of Doi-Hopf modules to the category of A-modules is Frobenius, see Proposition 7.4. The converse property holds if $A = H$, see Theorem 7.5. We have a similar result on the separability of the forgetful functor, see Proposition 7.7. Moreover, the relationship between coseparability of C as a coalgebra in $\mathcal{M}_H$ and in $\mathcal{T}_A^\#$ is well-understood: there is a bijection between

normalized Casimir morphisms for C in $\mathcal{M}_H$ and normalized Casimir morphisms for C in $\mathcal{T}_A^{\#}$ that satisfy the additional condition (7.23), see Theorem 7.8.
In Section 8 we focus on the category of two-sided Hopf modules, see [7]. Since this category is isomorphic to a suitable category of Doi-Hopf modules, we can apply the results of Section 7. Some interesting results can be obtained in the situation where $A = C = H$. The forgetful functor $F : {}_H\mathcal{M}_H^H \to \mathcal{M}_H$ is separable if and only if H is unimodular, see Theorem 8.7. H is a coseparable coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and cosemisimple, see Proposition 8.6. Thus the coseparability of the coalgebra H in ${}_H\mathcal{M}_H$ is not equivalent to the separability of the functor F, as it happens in the Frobenius case: F is Frobenius if and only if F is separable, if and only if H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$, i.e. H is unimodular.
Yetter-Drinfeld modules (see [12]) are studied in Section 9. The category of Yetter-Drinfeld modules is also isomorphic to a suitable category of Doi-Hopf modules, so that the results of Section 7 can be applied, see Corollary 9.3. We obtain complicated conditions, but better results can be obtained in the situation where $A = C = H$, that is, when we deal with classical Yetter-Drinfeld modules. The first main result is Theorem 9.6, telling that $F : \mathcal{YD}_H^H \to \mathcal{M}_H$ is Frobenius if and only if H is finite dimensional and unimodular, if and only if H is finite dimensional and Frobenius as a coalgebra in ${}_H\mathcal{M}_H$. The second main result is Theorem 9.10, saying that $F : \mathcal{YD}_H^H \to \mathcal{M}_H$ is separable if and only if H is a coseparable coalgebra in ${}_H\mathcal{M}_H$, i.e. H is unimodular and cosemisimple. If H is finite dimensional, then we can consider the Drinfeld double $D(H)$, and the algebra extension $H \hookrightarrow D(H)$ is Frobenius (resp. separable) if and only if H is unimodular (resp. unimodular and cosemisimple). Note that the proofs of these results are based on the structure theorem for two-sided Hopf H-bimodules, see [23].

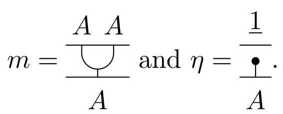
1. Preliminaries

1.1. **Monoidal categories.**

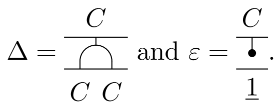
Monoidal categories. A monoidal category is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product, an object $\underline{1} \in \mathcal{C}$, called the unit object, and natural isomorphisms $a : \otimes \circ (\otimes \times \mathrm{Id}) \to \otimes \circ (\mathrm{Id} \times \otimes)$ (the associativity constraint), $l : \otimes \circ (\underline{1} \times \mathrm{Id}) \to \mathrm{Id}$ (the left unit constraint) and $r : \otimes \circ (\mathrm{Id} \times \underline{1}) \to \mathrm{Id}$ (the right unit constraint) satisfying appropriate coherence conditions, see for example [25, XI.2] for a detailed discussion. $\mathcal{C}$ is called strict if a, l and r are the identity natural transformations. It is well-known that every monoidal category is monoidally equivalent to a strict monoidal category, and this enables us to assume without loss of generality that $\mathcal{C}$ is strict. We will often delete the tensor symbol $\otimes$, and write $X \otimes Y = XY$. We write X^n for the tensor product of n copies of X. The identity morphism of an object $X \in \mathcal{C}$ will be denoted by Id_X or simply X. For morphisms $\mathrm{Id}_X = X : X \to X$, $f : X \to Y$, $g : XY \to Z$ and $h : X \to YZ$ in $\mathcal{C}$, we adopt the following graphical notation



Algebras and coalgebras. An algebra in $\mathcal{C}$ is a triple (A, m, η), where A is an object in $\mathcal{C}$ and $m : AA \to A$ (the multiplication) and $\eta : \underline{1} \to A$ (the unit) are morphisms in $\mathcal{C}$ satisfying the associativity and unit conditions $m \circ mA = m \circ Am$ and $m \circ \eta A = m \circ A\eta = A$. The graphical notation for m and η is the following:



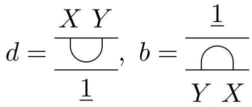
We use A as a shorter notation for the algebra (A, m, η); the multiplication on an algebra A is typically denoted by m, and the unit by η; we put subscripts whenever convenient, so that we can write $A = (A, m_A, \eta_A)$. Similar conventions are used for other structures, such as coalgebras, modules over an algebra, adjunctions, entwining structures etc.
A coalgebra in $\mathcal{C}$ is a triple $C = (C,\ \Delta :\ C \to CC,\ \varepsilon :\ C \to \underline{1})$, satisfying the appropriate coassociativity and counit conditions. The graphical notation takes the form



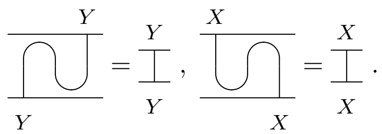
Adjunctions. An adjunction $X \dashv Y$ in $\mathcal{C}$ is a quadruple (X, Y, b, d), with X, Y objects in $\mathcal{C}$ and morphisms $b :\ \underline{1} \to YX$ and $d :\ XY \to \underline{1}$ satisfying

$$Yd \circ bY = Y \text{ and } dX \circ Xb = X. \tag{1.1}$$

With the graphical notation



(1.1) can be rewritten as



Y is called a right adjoint of X, and X a left adjoint of Y. Right adjoints are unique in the following sense. If (X, Y', b', d') is another adjunction, then $\lambda = Y'd \circ b'Y :\ Y \to Y'$ is an isomorphism with inverse $\lambda^{-1} = Yd' \circ bY'$. It is easy to show that λ is designed in such a way that

$$b' = \lambda X \circ b \text{ and } d = d' \circ X\lambda. \tag{1.2}$$

For two adjunctions (X, Y, b, d) and (X', Y', b', d'), we have a new adjunction

$$(XX', Y'Y, b \cdot b' = Y'bX' \circ b', d \cdot d' = d \circ Xd'Y).$$

In particular, we have an adjunction $(X^2, Y^2, b^2 = b \cdot b, d^2 = d \cdot d)$. Given a morphism $f :\ X \to X'$, we have

$$g = Yd' \circ YfY' \circ bY' :\ Y' \to X',$$

which reproduces $f = dX' \circ XgX' \circ Xb'$.
If every object in $\mathcal{C}$ has a right (resp. left) adjoint, then we say that $\mathcal{C}$ has right (resp. left) duality; $\mathcal{C}$ is called rigid if it has left and right duality. Assume that $\mathcal{C}$ has right duality, and choose a right dual ${}^*X$ for every object X. For every morphism $f :\ X \to X'$ in $\mathcal{C}$, we have ${}^*f :\ {}^*X' \to {}^*X$, and this defines a functor ${}^*(-) :\ \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$. If $\mathcal{C}$ is rigid, then $\big({}^*(-), (-)^*\big)$ is a pair of inverse equivalences between $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$.
For $X, X' \in \mathcal{C}$, ${}^*(XX')$ and ${}^*X'{}^*X$ are right duals of XX', so we have an isomorphism $\varphi_2(X, X') :\ {}^*(XX') \to {}^*X'{}^*X$. $(\underline{1}, \underline{1}, \underline{1}, \underline{1})$ is an adjunction, so we can put ${}^*\underline{1} = \underline{1}$, and define $\varphi_0 = \underline{1} :\ \underline{1} \to \underline{1}$. $({}^*(-), \varphi_0, \varphi_2) :\ \mathcal{C} \to \mathcal{C}^{\mathrm{oprev}}$ is a strong monoidal functor.
Let C be a coalgebra in $\mathcal{C}$, and assume that we have an adjunction $C \dashv A$. Then A is an algebra, with structure maps

$$m = Ad^2 \circ A\Delta AA \circ bAA :\ AA \to A \text{ and } \eta = A\varepsilon \circ b :\ \underline{1} \to A. \tag{1.3}$$

In this situation C is a right A-module (the definition of an A-module is given below), with structure map

$$\mu = Cd \circ \Delta A. \tag{1.4}$$

In a similar way, if a coalgebra C has a left adjoint A, then A is an algebra, and C is a left A-module.

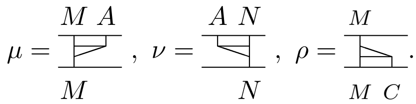
Module categories. Let $\mathcal{C}$ be a monoidal category. A right $\mathcal{C}$-category is a quadruple $(\mathcal{D}, \diamond, \Psi, r)$, where $\mathcal{D}$ is a category, $\diamond :\ \mathcal{D}\times\mathcal{C}\to\mathcal{D}$ is a functor, and $\Psi :\ \diamond\circ(\diamond\times \mathrm{Id})\to\diamond\circ(\mathrm{Id}\times\otimes)$ and $\mathbf{r} :\ \diamond\circ(\mathrm{Id}\times\underline{1})\to \mathrm{Id}$ are natural isomorphisms such that the diagrams

$$\begin{array}{ccccc} ((MX)Y)Z & \xrightarrow{\Psi_{MX,Y,Z}} & (MX)(YZ) & \xrightarrow{\Psi_{M,X,YZ}} & M(X(YZ)) \\ \downarrow{\scriptstyle \Psi_{M,X,Y}Z} & & & \nearrow{\scriptstyle Ma_{X,Y,Z}} & \\ (M(XY))Z & \xrightarrow{\Psi_{M,XY,Z}} & M((XY)Z) & & \end{array} \quad\text{and}\quad \begin{array}{ccc} (M\underline{1})X & \xrightarrow{\Psi_{M,\underline{1},X}} & M(\underline{1}X) \\ & \searrow{\scriptstyle \mathbf{r}_M X} & \downarrow{\scriptstyle Ml_X} \\ & & X \end{array}$$

commute, for all $M\in\mathcal{D}$ and $X,Y,Z\in\mathcal{C}$. Obviously $\mathcal{C}$ is itself a right $\mathcal{C}$-category. As before, we deleted the diamond product symbols, and wrote $MX = M\diamond X$, for $M\in\mathcal{D}$ and $X\in\mathcal{C}$. The above mentioned coherence theorem for monoidal categories can be extended to $\mathcal{C}$-categories, enabling us to assume throughout that Ψ and $\mathbf{r}$ are natural identities. In the literature, $\mathcal{C}$-categories are also named module categories.

Let $\mathcal{D}$ be a right $\mathcal{C}$-category, and consider an algebra A in $\mathcal{C}$. A right A-module in $\mathcal{D}$ is a pair $M = (M,\mu)$, with $M\in\mathcal{D}$ and $\mu :\ MA\to M$ satisfying $\mu\circ M\eta = M$ and $\mu\circ\mu A = \mu\circ Mm$. A morphism $f :\ M\to N$ between two right A-modules M and N in $\mathcal{D}$ is called right A-linear if $f\circ\mu = \mu\circ fA$. $\mathcal{D}_A$ will be the category of right A-modules and right A-linear morphisms in $\mathcal{D}$. In a similar way, we can define left A-modules $N = (N,\nu)$ in a left $\mathcal{C}$-category $\mathcal{E}$ and the category ${}_A\mathcal{E}$. We will typically use the notation μ for a right action and ν for a left action. The next step is to introduce two-sided $\mathcal{C}$-categories, and two-sided A-bimodules in a two-sided $\mathcal{C}$-category. We leave it to the reader to formulate the precise definitions.

We can also define the notions of a right C-comodule (M,ρ) in a right $\mathcal{C}$-category $\mathcal{D}$, and right C-colinearity of a morphism between two right C-comodules in $\mathcal{D}$. The category of right comodules and right C-colinear morphisms in $\mathcal{D}$ will be denoted as $\mathcal{D}^C$. We will use the following diagrammatic notation for actions and coactions:



1.2. **The category of bimodules.** The results in this Subsection will be needed in Sections 5 and 6. The results are well-known, see for example [32], [34] or [8]. What follows is an original reformulation, which is why we decided to keep the details. Let $\mathcal{C}$ be a (strict) monoidal category with coequalizers. Recall that $X\in\mathcal{C}$ is called left coflat if the functor $-X :\ \mathcal{C}\to\mathcal{C}$ preserves coequalizers. Let A be an algebra in $\mathcal{C}$. For $X\in\mathcal{C}_A$ and $Y\in{}_A\mathcal{C}$, $(X\otimes_A Y, q)$ is the coequalizer of the parallel morphisms $\mu Y,\ X\nu :\ XAY\to XY$:

$$XAY \underset{\mu Y}{\overset{X\nu}{\rightrightarrows}} XY \xrightarrow{q} X\otimes_A Y.$$

We compactify our notation by writing $X\otimes_A Y = X\bullet Y$. Now let $f :\ X\to X'$ in $\mathcal{C}_A$ and $g :\ Y\to Y'$ in ${}_A\mathcal{C}$. The universal property of coequalizers tells us that there is a unique $f\otimes_A g = f\bullet g$ in $\mathcal{C}$ such that (1.5) commutes.

$$\begin{array}{ccccc} XAY & \underset{\mu Y}{\overset{X\nu}{\rightrightarrows}} & XY & \xrightarrow{q} & X\bullet Y \\ \downarrow{\scriptstyle fAg} & & \downarrow{\scriptstyle fg} & & \vdots{\scriptstyle \exists ! f\bullet g} \\ X'AY' & \underset{\mu Y'}{\overset{X'\nu}{\rightrightarrows}} & X'Y' & \xrightarrow{q} & X'\bullet Y' \end{array} \tag{1.5}$$

Proposition 1.1. *Let $X \in \mathcal{C}_A$ and $M \in \mathcal{C}$. Then $(AM, mM) \in {}_A\mathcal{C}$, and*

$$XAAM \underset{\mu AM}{\overset{XmM}{\rightrightarrows}} XAM \xrightarrow{\mu M} XM$$

is a coequalizer in $\mathcal{C}$. If $M \in \mathcal{C}_A$ (resp. $X \in {}_A\mathcal{C}_A$), then this is also a coequalizer in $\mathcal{C}_A$ (resp. ${}_A\mathcal{C}$).

Proof. Let $f : XAM \to P$ be such that $f \circ \mu AM = f \circ XmM$. We have to prove the existence and uniqueness of $g : XM \to P$ such that $f = g \circ \mu M$.
If g exists, then it is unique since

$$f \circ X\eta M = g \circ \mu M \circ X\eta M = g. \tag{1.6}$$

$g = f \circ X\eta M$ is such that $g \circ \mu M = f \circ X\eta M \circ \mu M = f \circ \mu AM \circ XA\eta M = f \circ XmM \circ XA\eta M = f$. Finally, let $M \in \mathcal{C}_A$, and assume that f is right A-linear. The morphism g defined by (1.6) is also right A-linear, and this shows that $(XM, \mu M)$ is also a coequalizer in $\mathcal{C}_A$. Similar arguments hold in the case where $X \in {}_A\mathcal{C}_A$. □

It follows from Proposition 1.1 that we have a unique isomorphism $\Upsilon : X \bullet (AM) \to XM$ such that $\Upsilon \circ q = \mu M$, and $\Upsilon^{-1} = q \circ X\eta M$. Otherwise stated, there is a unique isomorphism of coequalizers $(X \bullet (AM), q) \cong (XM, \mu M)$. Now coequalizers are defined only up to isomorphisms, so we can go one step further, and declare $(X \bullet (AM), q) = (XM, \mu M)$. This identification will also be useful at the level of morphisms. Before we explain this, we state the following Lemma, which is folklore.

Lemma 1.2. *For $M \in \mathcal{C}$ and $Y \in {}_A\mathcal{C}$, we have an isomorphism $\alpha : {}_A\mathcal{C}(AM, Y) \to \mathcal{C}(M, Y)$, given by the formulas*

$$\alpha(\underline{f}) = \underline{f} \circ \eta N \ ; \ \alpha^{-1}(f) = \nu_Y \circ Af.$$

Now take $f : X \to X'$ in $\mathcal{C}_A$, $M, M' \in \mathcal{C}$ and $\underline{g} : AM \to AM'$ in ${}_A\mathcal{C}$. Let $g = \alpha(\underline{g}) : M \to AM'$. Making the identification $X \bullet (AM) = XM$ and $X' \bullet (AM') = X'M'$, we have $f \bullet \underline{g} : XM \to X'M'$. According to (1.5), $f \bullet \underline{g}$ is determined by the commutativity of the diagram

$$\begin{array}{ccc} XAM & \xrightarrow{\mu M} & XM \\ {\scriptstyle f\underline{g}}\downarrow & & \downarrow{\scriptstyle f\bullet\underline{g}} \\ X'AM' & \xrightarrow{\mu M'} & X'M' \end{array}$$

It follows from (1.6) that

$$f \bullet \underline{g} = \mu M' \circ f\underline{g} \circ X\eta M = \mu M' \circ fg. \tag{1.7}$$

Definition 1.3. Let A be an algebra in $\mathcal{C}$. $Y \in {}_A\mathcal{C}$ is called robust as a left A-module if

$$MXAY \underset{M\mu Y}{\overset{MX\nu}{\rightrightarrows}} MXY \xrightarrow{Mq} M(X \bullet Y)$$

is a coequalizer in $\mathcal{C}$, for all $M \in \mathcal{C}$ and $X \in \mathcal{C}_A$.

This definition can be restated as follows: the universal property of coequalizers implies the existence of a unique $\theta : (MX) \bullet Y \to M(X \bullet Y)$ such that $\theta \circ q = Mq$. Y is robust if and only if θ is an isomorphism for all X and M.

Proposition 1.4. *For all $N \in \mathcal{C}$, $AN \in {}_A\mathcal{C}$ is robust.*

Proof. Take the coequalizer from Proposition 1.1, with X replaced by NX. This is precisely the coequalizer in Definition 1.3. □

Proposition 1.5. (1) *Let $X \in \mathcal{C}_A$ and $Y \in {}_A\mathcal{C}_A$. If A is left A-coflat, then $X \bullet Y \in \mathcal{C}_A$, and $(X \bullet Y, q)$ is also a coequalizer in $\mathcal{C}_A$.*

(2) *Let $X \in {}_A\mathcal{C}_A$ and $Y \in {}_A\mathcal{C}_A$. If Y is robust as a left A-module, then $X \bullet Y \in {}_A\mathcal{C}$, and $(X \bullet Y, q)$ is also a coequalizer in ${}_A\mathcal{C}$.*

(3) *If both X and Y are A-bimodules, A is left coflat and Y is left A-robust, then $X \bullet Y \in {}_A\mathcal{C}_A$, and $(X \bullet Y, q)$ is also a coequalizer in ${}_A\mathcal{C}_A$.*

Proof. (1) Consider the diagram

$$\begin{array}{ccccc} XAYA & \overset{X\nu A}{\underset{\mu YA}{\rightrightarrows}} & XYA & \xrightarrow{qA} & (X\bullet Y)A \\ \downarrow{\scriptstyle XA\mu} & & \downarrow{\scriptstyle X\mu} & & \downarrow{\scriptstyle \exists!\mu} \\ XAY & \overset{X\nu}{\underset{\mu Y}{\rightrightarrows}} & XY & \xrightarrow{q} & X\bullet Y \end{array} \tag{1.8}$$

The top row is a coequalizer since A is left coflat, and this implies the existence of μ. μ satisfies the unit property. The diagram

$$\begin{array}{ccc} XY & \xrightarrow{q} & X\bullet Y \\ \downarrow{\scriptstyle XY\eta} & & \downarrow{\scriptstyle (X\bullet Y)\eta} \\ XYA & \xrightarrow{qA} & (X\bullet Y)A \\ \downarrow{\scriptstyle X\mu} & & \downarrow{\scriptstyle \mu} \\ XY & \xrightarrow{q} & X\bullet Y \end{array}$$

commutes. $X\mu \circ XY\eta = XY$, the identity, and it follows from the uniqueness in the universal property of the coequalizer that $\mu \circ (X \bullet Y)\eta = X \bullet Y$. We omit the proof of the associativity of μ, since it is similar to the proof of the compatibility of μ and ν in the third part of the proof.
Let $f : XY \to P$ be a morphism in $\mathcal{C}_A$ such that $f \circ \mu Y = f \circ X\nu$. We know that there exists a unique $g : X \bullet Y \to P$ such that $g \circ f = q$. In follows that $(X \bullet A, q)$ is a coequalizer in $\mathcal{C}_A$ if we can show that g is right A-linear. We first compute that

$$g \circ \mu \circ qA \overset{(1.8)}{=} g \circ q \circ X\mu = f \circ X\mu \overset{(*)}{=} \mu \circ fA = \mu \circ gA \circ qA.$$

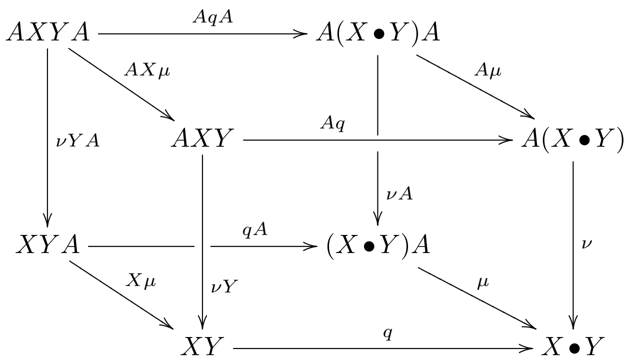
At $(*)$, we used the fact that f is right A-linear. $((X\bullet Y)A, qA)$ is a coequalizer since A is left coflat, ant it follows that $g \circ \mu = \mu \circ gA$, which is precisely what we need.

(2) If Y is left A-robust, then the top row in the diagram

$$\begin{array}{ccccc} AXAY & \overset{AX\nu}{\underset{A\mu Y}{\rightrightarrows}} & AXY & \xrightarrow{Aq} & A(X\bullet Y) \\ \downarrow{\scriptstyle \nu AY} & & \downarrow{\scriptstyle \nu Y} & & \downarrow{\scriptstyle \exists!\nu} \\ XAY & \overset{X\nu}{\underset{\mu Y}{\rightrightarrows}} & XY & \xrightarrow{q} & X\bullet Y \end{array} \tag{1.9}$$

is a coequalizer, and the universal property brings the left action ν on $X \bullet Y$. The rest of the proof of part (2) is left to the reader.

(3) Now we assume that both X and Y are bimodules. We show that the actions μ and ν on $X \bullet Y$ are compatible. To this end, consider the cubic diagram



Commutativity of the top and bottom faces follows from the definition of μ, and commutativity of the front and back faces follows from the definition of ν. It is obvious that the left face commutes. From this we deduce that

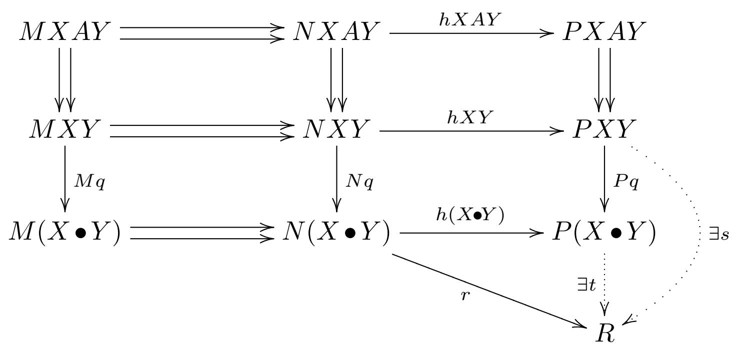
$$\mu \circ \nu A \circ AqA = \mu \circ qA \circ \nu YA = q \circ X\mu \circ \nu YA = q \circ \nu Y \circ AX\mu = \nu \circ Aq \circ AX\mu = \nu \circ A\mu \circ AqA.$$

From the robustness of Y, we know that $(A(X \bullet Y), Aq)$ is a coequalizer, and from the left coflatness of A that $(A(X \bullet Y)A, AqA)$ is a coequalizer. It then follows that $\mu \circ \nu A = \nu \circ A\mu$, which is the compatibility that we need. The proof of the associativity of μ and ν follows by similar arguments. □

Let A be a left coflat algebra in $\mathcal{C}$, and let ${}^!_A\mathcal{C}_A$ be the full subcategory of ${}_A\mathcal{C}_A$ consisting of bimodules that are left coflat as objects in $\mathcal{C}$, and robust as left A-modules. Our aim is to show that ${}^!_A\mathcal{C}_A$ is a monoidal category, with tensor product $\otimes_A$ and unit object A.

Lemma 1.6. *Let A be left coflat. If $X, Y \in {}^!_A\mathcal{C}_A$, then $X \bullet Y$ is left coflat.*

Proof. It is easy to show that the tensor product (in $\mathcal{C}$) of two left coflat objects is left coflat. Let (P, h) be the coequalizer of two parallel morphisms $f, g : M \to N$ in $\mathcal{C}$. We have to show that $(P(X \bullet Y), h(X \bullet Y))$ is the coequalizer of $f(X \bullet Y), g(X \bullet Y)$. Take $r : N(X \bullet Y) \to R$ such that $r \circ f(X \bullet Y) = r \circ g(X \bullet Y)$ and consider the diagram



The first two rows are coequalizers since XY and XAY are left coflat, and the three columns are coequalizers since Y is left A-robust. The rectangles in the diagram commute, and we easily compute that

$$r \circ Nq \circ fXY = r \circ f(X \bullet Y) \circ Mq = r \circ g(X \bullet Y) \circ Mq = r \circ Nq \circ gXY,$$

and it follows that there exists $s:\ PXY\to R$ such that $s\circ hXY=r\circ Nq$. Then we compute that

$$s\circ P\nu Y\circ hXAY=s\circ hXY\circ N\nu Y=r\circ Nq\circ N\nu Y=r\circ Nq\circ NX\mu=s\circ PX\mu\circ hXAY,$$

hence $s\circ P\nu Y=s\circ PX\mu$, since $hXAY$ is an epimorphism. This implies the existence of $t:$ $P(X\bullet Y)\to R$ such that $t\circ Pq=s$. It is now easy to see that

$$t\circ h(X\bullet Y)\circ Nq=t\circ Pq\circ hXY=s\circ hXY=r\circ Nq,$$

hence $t\circ h(X\bullet Y)=r$, as needed. The uniqueness can be easily obtained as follows: if $t':\ P(X\bullet Y)\to R$ is such that $t'\circ h(X\bullet Y)=r$, then $t'\circ Pq\circ hXY=r\circ Nq=t\circ Pq\circ hXY$, and $t'=t$. □

Recall that coequalizers are colimits, see for example [29, III.3]; in particular, the tensor product $X\bullet Y$ of $X\in\mathcal{C}_A$ and $Y\in{}_A\mathcal{C}$ is a colimit: consider the category J with two objects labelled xay and xy, and two non-identity arrows $my,\ xn:\ xay\to xy$, and let $F:\ J\to\mathcal{C}$ be the following functor:

$$F(xay)=XAY,\ F(xy)=XY,\ F(my)=\mu Y,\ F(xn)=X\nu.$$

Cones from F to the vertex P in $\mathcal{C}$ correspond to morphisms $f:\ XY\to P$ such that $f\circ\mu Y=f\circ X\nu$, and the colimit $\mathrm{Colim}F=(X\bullet Y,q)$ consists of an object $X\bullet Y\in\mathcal{C}$ and a universal cone q from F to $X\bullet Y$.
We now generalize this construction. Let J_2 be the category with four objects $xayaz$, $xyaz$, $xayz$ and xyz, and morphisms

$$\begin{array}{ccc}
xayaz & \overset{myaz}{\underset{xnaz}{\rightrightarrows}} & xyaz\\
{\scriptstyle xamz}\downdownarrows{\scriptstyle xayn} & & {\scriptstyle xmz}\downdownarrows{\scriptstyle xyn}\\
xayz & \overset{myz}{\underset{xnz}{\rightrightarrows}} & xyz
\end{array}$$

and their compositions, subject to the relations

$$\begin{array}{ccc}
xmz\circ myaz=myz\circ xamz=mmz & ; & xyn\circ xmaz=myz\circ xayn=myn;\\
xmz\circ xnaz=xnz\circ xamz=xmnz & ; & xyn\circ xnaz=xnz\circ xayn=xnn.
\end{array}$$

Consider $X\in\mathcal{C}_A$, $Y\in{}_A\mathcal{C}_A$ and $Z\in{}_A\mathcal{C}$. $F_2:\ J_2\to\mathcal{C}$ is defined in the following way: $F_2(xayaz)=XAYAZ$, $F_2(xyaz)=XYAZ$, $\cdots$, $F_2(xmaz)=X\mu AZ$ etc. We can also consider the full subcategory J_2' of J_2, with objects $xyaz$, $xayz$ and xyz, and the restriction F_2' of F_2 to J_2'. It is easy to establish that cones from F_2 to $P\in\mathcal{C}$ correspond bijectively to cones from F_2' to P, so that F_2 and F_2' have the same colimit. We now define

$$\mathrm{Colim}F_2'=(X\bullet Y\bullet Z,q_2).$$

Proposition 1.7. *Let A be a coflat algebra in $\mathcal{C}$, and consider $X\in\mathcal{C}_A$ and $Y,Z\in{}_A^{!}\mathcal{C}_A$. Then we have isomorphisms of cones*

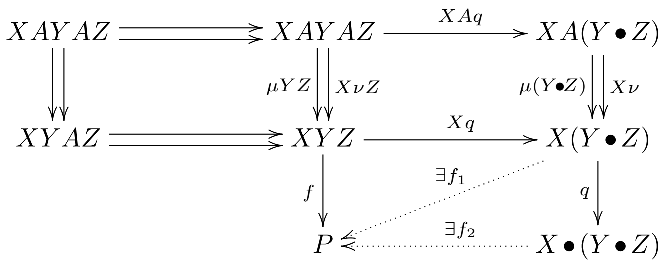
$$(X\bullet Y\bullet Z,q_2)\cong(X\bullet(Y\bullet Z),q\circ Xq)\cong((X\bullet Y)\bullet Z,q\circ qZ).$$

If X is an A-bimodule, then F_2 and F_2' corestrict to functors with values in ${}_A\mathcal{C}_A$, and the above cones are also the colimits of these corestrictions.

Proof. We will show that $(X\bullet(Y\bullet Z),q\circ Xq)$ satisfies the universal property of cones. Assume that $f:\ XYZ\to P$ in $\mathcal{C}$ is such that

$$f\circ\mu YZ=f\circ X\nu Z\ \text{and}\ f\circ x\mu Z=f\circ XY\nu. \tag{1.10}$$

Consider the diagram



The two top rows are coequalizers since Z is robust as a left A-module. The second equation in (1.10) implies the existence of $f_1 :\ X(Y \bullet Z) \to P$ such that $f_1 \circ Xq = f$.
The two squares in the top right corner of the diagram commute. The commutativity of the one on the left is obvious, and the commutativity of the one on the right is a consequence of the definition of the left action ν on $Y \bullet Z$, see (1.9). We now easily find that

$$f_1 \circ X\nu \circ XAq = f_1 \circ Xq \circ X\nu Z = f \circ X\nu Z \overset{(1.10)}{=} f \circ \mu YZ = f_1 \circ Xq \circ \mu YZ = f_1 \circ \mu(Y \bullet Z) \circ XAq,$$

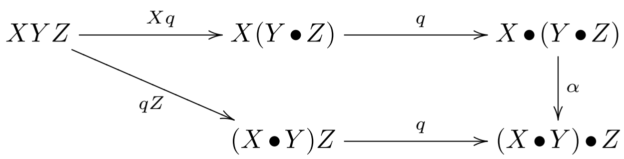
hence $f_1 \circ X\nu = f_1 \circ \mu(Y \bullet Z)$, so there exists $f_2 :\ X \bullet (Y \bullet Z) \to P$ such that $f_2 \circ q = f_1$, and $f_2 \circ q \circ Xq = f$, as needed. The uniqueness of f_2 follows from the fact that q and Xq are epimorphisms. If f is a morphism in ${}_A\mathcal{C}_A$, then it follows from Proposition 1.5 that f_1 and f_2 are also in ${}_A\mathcal{C}_A$, and this shows that $(X \bullet (Y \bullet Z), q \circ Xq)$ is the colimit of the corestriction of F_2' to ${}_A\mathcal{C}_A$.
Similar arguments show that $((X \bullet Y) \bullet Z, q \circ qZ)$ is a colimit of F_2'. □

Take $X, Y, Z \in {}^!_A\mathcal{C}_A$. It follows from the universal property of colimits that there exists a unique isomorphism

$$\alpha = \alpha_{X,Y,Z} :\ X \bullet (Y \bullet Z) \to (X \bullet Y) \bullet Z$$

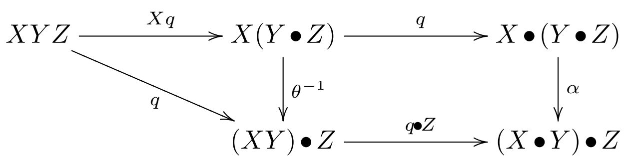
in ${}_A\mathcal{C}_A$ such that the diagram

(1.11)



commutes. The diagram

(1.12)



commutes. Indeed, the commutativity of the pentangle follows from (1.11) combined with (1.5); the triangle commutes: this is the definition of θ. Then we compute that

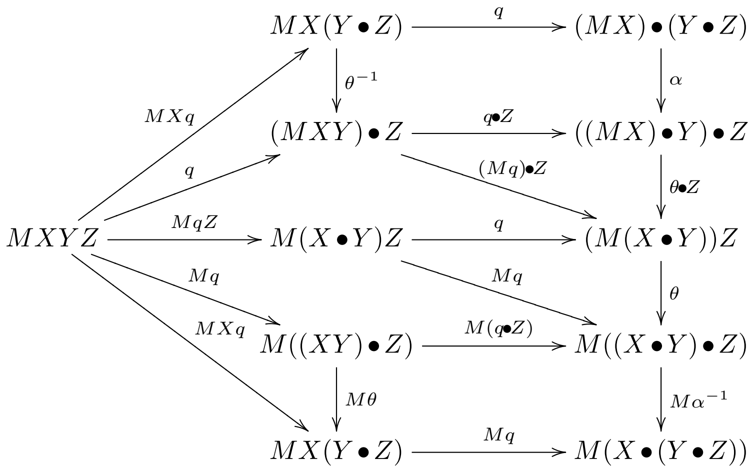
$$\alpha \circ q \circ \theta \circ q = \alpha \circ q \circ Xq = q \bullet Z \circ q,$$

hence $\alpha \circ q \circ \theta = Z \circ q$, so the rectangle commutes.

Proposition 1.8. *If A is left coflat , then ${}^!_A\mathcal{C}_A$ is closed under the tensor product over A.*

Proof. Take $Y, Z \in {}^!_A\mathcal{C}_A$. We know from Lemma 1.6 that $Y \bullet Z$ is left coflat. We are left to show that $Y \bullet Z$ is robust as a left A-module. Take $M \in \mathcal{C}$ and $X \in \mathcal{C}_A$ and consider the diagram

(1.13)



Commutativity of the top and bottom triangles and rectangles follows from (1.12). The commutativity of the two remaining triangles follows from the definition of θ, and the commutativity of the two remaining quadrangles follows from (1.5). We conclude that the whole diagram commutes. Now let

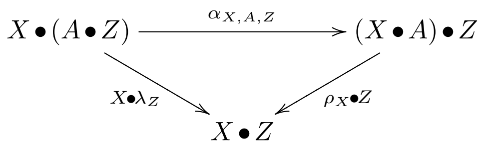
$$\Theta = M\alpha^{-1} \circ \theta \circ \theta \bullet Z \circ \alpha : \ (MX) \bullet (Y \bullet Z) \to M(X \bullet (Y \bullet Z)).$$

Θ is an isomorphism, and $\Theta \circ q \circ MXq = Mq \circ MXq$, so that $\Theta \circ q = Mq$. It follows from the (reformulation of) Definition 1.3 that $Y \bullet Z$ is robust as a left A-module. □

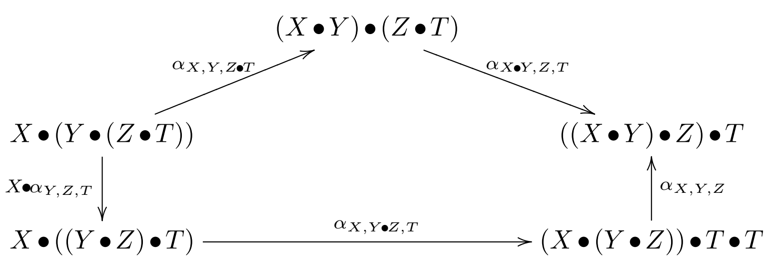
Theorem 1.9. *Let A be a left coflat algebra in $\mathcal{C}$. Then we have a monoidal category $({}^!_A\mathcal{C}_A, \otimes_A = \bullet, A, \alpha, \lambda, \rho)$. The category $\mathcal{C}_A$ is a right ${}^!_A\mathcal{C}_A$-category.*

Proof. We have shown in Proposition 1.8 that the tensor product over A of two objects in ${}^!_A\mathcal{C}_A$ is again in ${}^!_A\mathcal{C}_A$. The associativity constraint α was defined as an application of Proposition 1.7.
The unit constraint follows as an application of Proposition 1.1. (X, μ) and $(X \bullet A, q)$ are both coequalizers in $\mathcal{C}$ (and in ${}_A\mathcal{C}_A$) of $Xm, \mu A : \ XAA \to XA$, so there exists a unique isomorphism $\rho_X : \ X \bullet A \to X$ in ${}_A\mathcal{C}_A$ such that $\rho_X \circ q = \mu$, with inverse $\rho_X^{-1} = q \circ X\eta$. In a similar way, we have a unique isomorphism $\lambda_X : \ A \bullet X \to X$ in ${}_A\mathcal{C}_A$ such that $\lambda_X \circ q = \nu$, with inverse $\lambda_X^{-1} = q \circ \eta X$.
We are left to show that the coherence conditions are satisfied.
Take $X, Y, Z, T \in {}^!_A\mathcal{C}_A$. We have to show that the following diagrams commute.

(1.14)



(1.15)



$\alpha_{X,A,Z}$ is the unique morphism that makes the diagram (1.11) commutative. If we can show that $(\rho_X \bullet Z)^{-1} \circ X \bullet \lambda_Z$ has the same property, then it follows that (1.14) commutes. This means that we have to show that the diagram

$$\begin{array}{ccccc} XAZ & \xrightarrow{Xq} & X(A\bullet Z) & \xrightarrow{q} & X\bullet(A\bullet Y) \\ {\scriptstyle qZ}\downarrow & & & & \downarrow{\scriptstyle X\lambda_Y} \\ (X\bullet A)Z & \xrightarrow{q} & (X\bullet A)\bullet Z & \xrightarrow{\rho_X\bullet Y} & X\bullet Y \end{array}$$

commutes. This is an easy computation:

$$X\bullet\lambda_Y \circ q \circ Xq \overset{(1.5)}{=} q \circ X\lambda_Y \circ Xq = q \circ X\nu = q \circ \mu Y = q \circ \rho_X Y \circ qY \overset{(1.5)}{=} \rho_X \bullet Y \circ q \circ qZ.$$

Now consider the category J_3', consisting of four objects and six morphisms that are not identities:

$$\begin{array}{ccccc} & & xyazt & & \\ & & {\scriptstyle xmzt}\downarrow\downarrow{\scriptstyle xynt} & & \\ xayzt & \underset{xnzt}{\overset{myzt}{\rightrightarrows}} & xyzt & \underset{xyzn}{\overset{xymt}{\leftleftarrows}} & xyzat \end{array}$$

We define $F_3' : J_3' \to \mathcal{C}$ in the obvious way: $F_3'(xyazt) = XYAZT$, $F_3'(xmzt) = X\mu ZT$, etc. The fourfold tensor product is defined as the colimit of F_3': $\mathrm{Colim}F_3' = (X\bullet Y\bullet Z\bullet T, q_3)$. Proceeding as in Proposition 1.7, we can show that $\big(X\bullet(Y\bullet(Z\bullet T)), q\circ Xq\circ XYq\big)$, $\big((X\bullet Y)\bullet(Z\bullet T), q\circ qq\big)$, $\big(((X\bullet Y)\bullet Z)\bullet T, q\circ qT\circ qZT\big)$, $\big(X\bullet((Y\bullet Z)\bullet T), q\circ Xq\circ XqT\big)$ and $\big((X\bullet(Y\bullet Z))\bullet T, q\circ qT\circ XqT\big)$ are all colimits of F_3' (and of the corestriction of F_3' to ${}_A\mathcal{C}_A$). This means that these five cones are isomorphic. For example, the isomorphism between the first two cones is the unique morphism that makes the diagram

$$\begin{array}{ccccccc} XYZT & \xrightarrow{XYq} & XY(Z\bullet T) & \xrightarrow{Xq} & X(Y\bullet(Z\bullet T)) & \xrightarrow{q} & X\bullet(Y\bullet(Z\bullet T)) \\ & \searrow^{XYq} & \downarrow{=} & & & & \vdots\ \exists! \\ & & XY(Z\bullet T) & \xrightarrow{q(Z\bullet T)} & (X\bullet Y)(Z\bullet T) & \xrightarrow{q} & (X\bullet Y)\bullet(Z\bullet T) \end{array}$$

commutative. Here we used the equality $qq = q(Z\bullet T)\circ XYq$. In view of (1.11), this morphism is $\alpha_{X,Y,Z\bullet T}$. In a similar way, we can prove that the maps in the diagram (1.15) establish isomorphisms between the 5 coequalizers above. Therefore the two compositions in the diagram also establish isomorphisms between coequalizers, hence they are equal, since these isomorphisms are unique. This tells us that (1.15) commutes. □

A coalgebra C in ${}_A^!\mathcal{C}_A$ is called an A-coring. The category of right C-comodules and right C-colinear morphisms in $\mathcal{C}_A$ is denoted by $\mathcal{C}^C$.

1.3. **Quasi-bialgebras and quasi-Hopf algebras.** The results in this Subsection will be needed in Sections 7, 8 and 9. Recall that a quasi-bialgebra over a field k is an associative unital algebra H, with a comultiplication $\Delta : H \to H\otimes H$ that is coassociative up to conjugation by an invertible element $\Phi\in H\otimes H\otimes H$, called the reassociator:

$$(H\otimes\Delta)(\Delta(h)) = \Phi(\Delta\otimes H)(\Delta(h))\Phi^{-1}, \tag{1.16}$$

for all $h\in H$. In addition, Δ is an algebra morphism and is counital via an algebra map $\varepsilon : H\to k$. Φ is a normalized 3-cocycle in the following sense:

$$(H\otimes H\otimes\Delta)(\Phi)(\Delta\otimes H\otimes H)(\Phi) = (1_H\otimes\Phi)(H\otimes\Delta\otimes H)(\Phi)(\Phi\otimes 1_H), \tag{1.17}$$

$$(H\otimes\varepsilon\otimes H)(\Phi) = 1_H\otimes 1_H. \tag{1.18}$$

The following notation is used for the reassociator Φ and its inverse Φ^{-1}:

$$\Phi = X^1 \otimes X^2 \otimes X^3,\ \Phi^{-1} = x^1 \otimes x^2 \otimes x^3\ \in H \otimes H \otimes H.$$

Summation is implicitly understood. We use Sweedler's notation $\Delta(h) = h_1 \otimes h_2$, for the comultiplication. Since Δ is not coassociative, we have to write

$$(\Delta \otimes H)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2\ ,\ (H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},\ \text{etc.}$$

A quasi-Hopf algebra is a quasi-bialgebra H together with an algebra morphism $S:\ H \to H^{\rm op}$ and elements $\alpha, \beta \in H$ such that

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \text{ and } S(h_1)\beta h_2 = \varepsilon(h)\beta \tag{1.19}$$

$$X^1\beta S(X^2)\alpha X^3 = 1 \text{ and } S(x^1)\alpha x^2\beta S(x^3) = 1. \tag{1.20}$$

The antipode of a quasi-Hopf algebra is an anti-coalgebra morphism in the following sense: there exists an invertible element $f = f^1 \otimes f^2 \in H \otimes H$, called the Drinfeld twist or gauge transformation, such that $\varepsilon(f^1)f^2 = \varepsilon(f^2)f^1 = 1$ and

$$f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\rm cop}(h)), \tag{1.21}$$

for all $h \in H$. Note that f and f^{-1} can be described explicitly:

$$\begin{aligned} f &= (S \otimes S)(\Delta^{\rm cop}(x^1))\gamma\Delta(x^2\beta S(x^3)),\\ f^{-1} &= \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\rm cop}(x^3)), \end{aligned}$$

where $\gamma, \delta \in H \otimes H$ are given by the formulas

$$\gamma = S(x^1X^2)\alpha x^2X^3_1 \otimes S(X^1)\alpha x^3X^3_2 = S(X^2x^1_2)\alpha X^3x^2 \otimes S(X^1x^1_1)\alpha x^3, \tag{1.22}$$

$$\delta = X^1_1x^1\beta S(X^3) \otimes X^1_2x^2\beta S(X^2x^3) = x^1\beta S(x^3_2X^3) \otimes x^2X^1\beta S(x^3_1X^2). \tag{1.23}$$

Furthermore, $f = f^1 \otimes f^2 = F^1 \otimes F^2$ and f^{-1} have the following properties:

$$f\Delta(\alpha) = \gamma,\ \Delta(\beta)f^{-1} = \delta \text{ and} \tag{1.24}$$

$$f^1X^1 \otimes F^1f^2_1X^2 \otimes F^2f^2_2X^3 = S(X^3)f^1F^1_1 \otimes S(X^2)f^2F^1_2 \otimes S(X^1)F^2. \tag{1.25}$$

The category $\mathcal{M}_H$ of right H-modules over H is monoidal. The tensor product of $M, N \in \mathcal{M}_H$ is $M \otimes N$ with right H-action $(m \otimes n) \cdot h = m \cdot h_1 \otimes n \cdot h_2$, for $m \in M$, $n \in N$ and $h \in H$. The unit object is the groundfield k with H-action induced by ε. The associativity constraint is given by the formula

$$a_{M,N,P}((m \otimes n) \otimes p) = m \cdot x^1 \otimes (n \cdot x^2 \otimes p \cdot x^3),$$

for all $m \in M$, $n \in N$ and $p \in P$.

2. Entwined modules over monoidal cowreaths

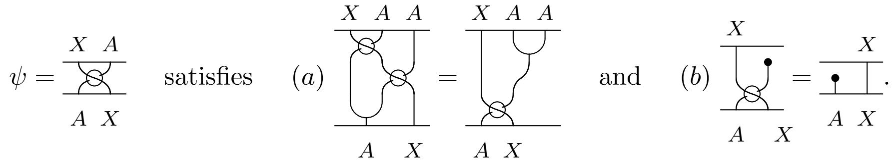
A (strict) monoidal category $\mathcal{C}$ can be viewed as a 2-category with a single 0-cell, hence we can consider the 2-categories Mnd($\mathcal{C}$) [35] and EM($\mathcal{C}$) [27]. These have the same 0-cells and 1-cells, but are different at the level of 2-cells. The 0-cells are algebras (or monads) in $\mathcal{C}$. Fix an algebra A in $\mathcal{C}$ and consider the endomorphism categories

$$\mathcal{T}(\mathcal{C})_A = \mathcal{T}_A = \mathrm{Mnd}(\mathcal{C})(A, A) \text{ and } \mathcal{T}(\mathcal{C})^\#_A = \mathcal{T}^\#_A = \mathrm{EM}(\mathcal{C})(A, A).$$

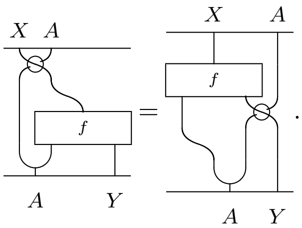
The notation $\mathcal{T}(\mathcal{C})_A$ is taken from [37], where $\mathcal{T}(\mathcal{C})_A$ appears in a different context, and where it is called the category of right transfer morphisms through A. $\mathcal{T}_A$ and $\mathcal{T}^\#_A$ are (strict) monoidal categories. A monad in Mnd($\mathcal{C}$) is called a distributive law [35, Sec. 6] or a factorization structure. A comonad in Mnd($\mathcal{C}$) is called a mixed distributive law or an entwining structure [2]. A monad in EM($\mathcal{C}$) is called a wreath in $\mathcal{C}$ [27], an alternative name suggested in [27] is generalized distributive law. A comonad in EM($\mathcal{C}$) is called a cowreath in $\mathcal{C}$ or a generalized entwining structure.

A cowreath in $\mathcal{C}$ consists of an algebra A in $\mathcal{C}$ and a coalgebra in $\mathcal{T}^\#_A$. In a similar way, a wreath in $\mathcal{C}$ consists of an algebra A in $\mathcal{C}$ and an algebra in $\mathcal{T}_A^{\#}$. For later use, we spell out the explicit definition of a cowreath.

2.1. **The monoidal categories $\mathcal{T}_A$ and $\mathcal{T}_A^{\#}$.** Let A be an algebra in $\mathcal{C}$. A (right) transfer morphism through A is a pair $X = (X, \psi)$, with $X \in \mathcal{C}$ and $\psi:\ XA \to AX$ in $\mathcal{C}$ such that $\psi \circ Xm = Xm \circ A\psi \circ \psi A$ and $\psi \circ X\eta = \eta X$; in diagrammatic notation

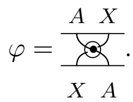
(2.1)


The categories $\mathcal{T}_A$ and $\mathcal{T}_A^{\#}$ coincide at the level of objects; their objects are right transfer morphisms through A. A morphism $X \to Y$ in $\mathcal{T}_A$ is a morphism $f:\ X \to Y$ in $\mathcal{C}$ such that $\psi \circ fA = Af \circ \psi$. A morphism $X \to Y$ in $\mathcal{T}_A^{\#}$ is a morphism $f:\ X \to AY$ in $\mathcal{C}$ such that $mY \circ Af \circ \psi = mY \circ A\psi \circ fA$, or

(2.2)


The composition of two morphisms $f:\ X \to Y$ and $g:\ Y \to Z$ in $\mathcal{T}_A^{\#}$ is $g \bullet f = mZ \circ Ag \circ f$. The identity on (X, ψ) is ηX. The tensor product of X and Y is $XY = (XY, \psi_X \cdot \psi_Y = \psi_X Y \circ X\psi_Y)$. The tensor product of $f:\ X \to X'$ and $g:\ Y \to Y'$ in $\mathcal{T}_A^{\#}$ is given by the composition $mXY \circ A\psi Y \circ fg$. The unit object is $(\underline{1}, A)$. $\mathcal{T}_A$ and $\mathcal{T}_A^{\#}$ are strict monoidal categories, and we have a strong monoidal functor $F:\ \mathcal{T}_A \to \mathcal{T}_A^{\#}$, which is the identity on objects, and $F(f) = \eta f$, for $f:\ X \to Y$ in $\mathcal{T}_A$. If a morphism in $\mathcal{T}_A^{\#}$ is of the form ηf, with $f:\ X \to Y$ in $\mathcal{C}$, then f is a morphism in $\mathcal{T}_A$.

In a similar way we introduce left transfer morphisms through A, consisting of pairs $X = (X, \varphi)$, with $\varphi:\ AX \to XA$, with diagrammatic notation

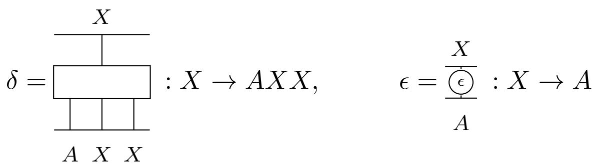


We leave it to the reader to write down the precise definition of the categories ${}_A\mathcal{T}$ and ${}_A^{\#}\mathcal{T}$. In fact

$${}_A\mathcal{T} = \mathrm{Mnd}(\mathcal{C}^{\mathrm{op}})(A, A) \text{ and } {}_A^{\#}\mathcal{T} = \mathrm{EM}(\mathcal{C}^{\mathrm{op}})(A, A).$$

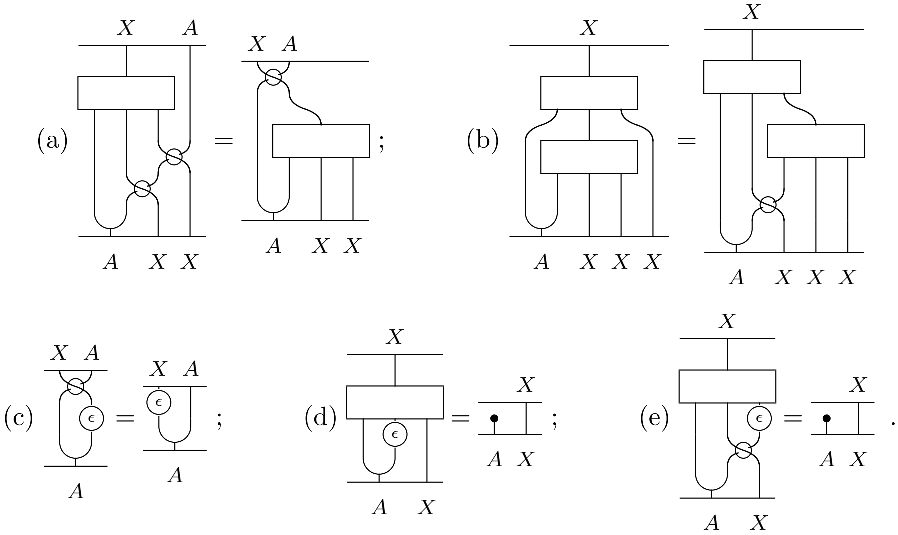
The tensor product in ${}_A\mathcal{T}$ and ${}_A^{\#}\mathcal{T}$ is given by the formula $XY = (XY, \varphi_X \cdot \varphi_Y = X\varphi_Y \circ \varphi_X Y)$.

2.2. **Monoidal cowreaths.** A cowreath (or generalized entwining structure) in $\mathcal{C}$ is a triple (A, X, ψ), where A is an algebra in $\mathcal{C}$, and (X, ψ) is a coalgebra in $\mathcal{T}_A^{\#}$, which is an object $(X, \psi) \in \mathcal{T}_A^{\#}$ together with morphisms



in $\mathcal{C}$ such that the following relations hold:

(2.3)



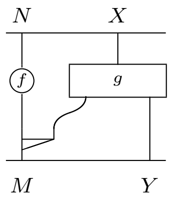
Conditions (a) and (c) mean that δ and ϵ define morphisms $X \to XX$ and $X \to \underline{1}$ in $\mathcal{T}_A^{\#}$. (b) is the coassociativity of the comultiplication δ and (d) and (e) are the left and right counit property.

2.3. **Entwined modules over monoidal cowreaths.** Let $\mathcal{D}$ be a right $\mathcal{C}$-category, and let A be an algebra in $\mathcal{C}$. Then $\mathcal{D}_A$ is a right $\mathcal{T}_A$-category, see [14, Prop. 4.3]. We will now show that it is also a right $\mathcal{T}_A^{\#}$-category.

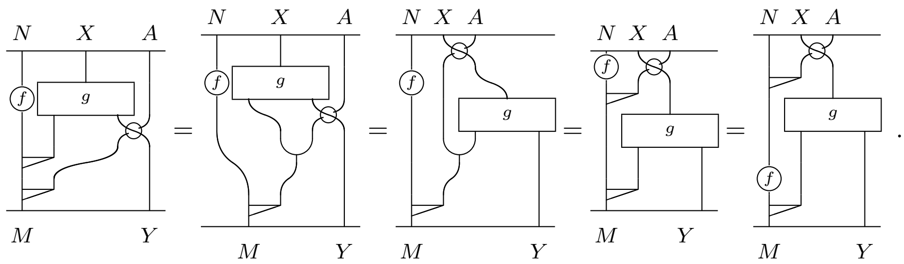
Proposition 2.1. *Let A be an algebra in a (strict) monoidal category $\mathcal{C}$, and let $\mathcal{D}$ be a (strict) right $\mathcal{C}$-category. Then $\mathcal{D}_A$ is a right $\mathcal{T}_A^{\#}$-category. The tensor product of $N \in \mathcal{D}_A$ and $X \in \mathcal{T}_A^{\#}$ is given by the formula $N \diamond X = (NX, \mu_{NX} = \mu X \circ N\psi)$. The tensor product of $f : N \to M$ in $\mathcal{D}_A$ and $g : X \to Y$ in $\mathcal{T}_A^{\#}$ is given by the formula $f \diamond g = \mu Y \circ fg$.*

Proof. It follows from [14, Prop. 4.3] that $N \diamond X$ is a right A-module Let us now show that

$$f \diamond g = \mu Y \circ fg =$$



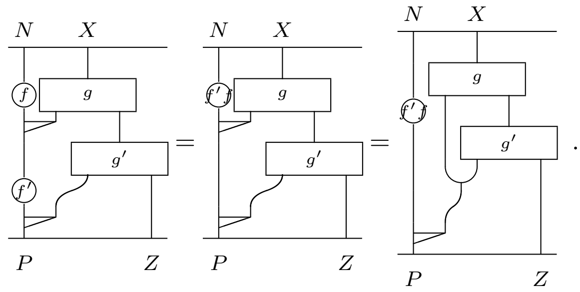
is right A-linear:



Now take $f' : M \to P$ in $\mathcal{D}_A$ and $g' : Y \to Z$ in $\mathcal{T}_A^{\#}$. We have to show that

$$(f' \diamond g') \circ (f \diamond g) = (f' \circ f) \diamond (g' \bullet g).$$

This is a result of the following computation:



Finally observe that $\mathrm{Id}_M \diamond \mathrm{Id}_X = \mu X \circ M\eta X = MX = \mathrm{Id}_{M\diamond X}$. □

Proposition 2.1 justifies the following definition.

Definition 2.2. Let (A, X, ψ) be a cowreath in $\mathcal{C}$. An entwined module in $\mathcal{D}$ over (A, X, ψ) is a right (X, ψ)-comodule in $\mathcal{D}_A$.

An entwined module over (A, X, ψ) consists of an object $M \in \mathcal{D}_A$ and a morphism $\rho : M \to MX$ in $\mathcal{D}_A$ satisfying

$$\rho X \circ \rho = \mu XX \circ M\delta \circ \rho; \tag{2.4}$$

$$\mu \circ M\epsilon \circ \rho = M. \tag{2.5}$$

(2.4) is the coassociativity of the coaction, and (2.5) is the counit property. The fact that ρ is right A-linear is expressed by the formula

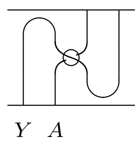
$$\rho \circ \mu = \mu X \circ M\psi \circ \rho A. \tag{2.6}$$

A mixed distributive law (or entwining structure) (A, X, ψ) can be considered as a cowreath (or generalized entwining structure): take $\delta = \eta\Delta$ and $\epsilon = \eta\varepsilon$. It is easy to see that entwined modules over (A, X, ψ) considered as a mixed distributive law coincide with entwined modules over (A, X, ψ) considered as a monoidal cowreath.
A morphism between two entwined modules M and N is a right A-linear morphism $f : M \to N$ such that $fX \circ \rho = \rho \circ f$. The category of entwined modules in $\mathcal{D}_A$ over $(X, \psi, \delta, \epsilon)$ will be denoted as $\mathcal{D}(\psi)_A^X$.

3. Monoidal wreaths, wreath product algebras and duality

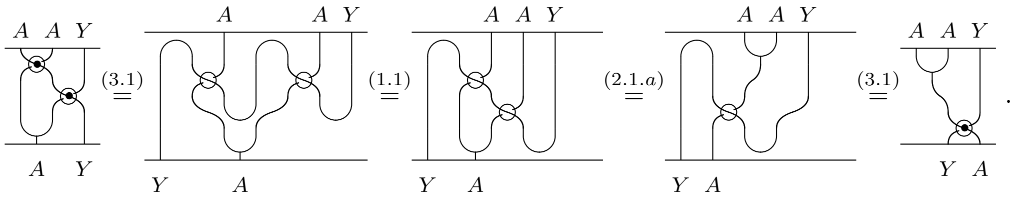
3.1. Duality between left and right transfer morphisms.

Theorem 3.1. *Let A be an algebra in a (strict) monoidal category $\mathcal{C}$. Take $X \in \mathcal{T}_A$, and assume that $X \dashv Y$ in $\mathcal{C}$. Consider*

(3.1) $\overline{\psi} = \varphi = YAd \circ Y\psi Y \circ bAY =$  $: AY \to YA.$

Then $(Y, \varphi) \in {}_A\mathcal{T}$. If $\mathcal{C}$ has right duality, then we have strong monoidal functors ${}^*(-) : \mathcal{T}_A^\# \to {}_A^\#\mathcal{T}^{\rm oprev}$ and ${}^*(-) : \mathcal{T}_A \to {}_A\mathcal{T}^{\rm oprev}$.

Proof. We first compute that



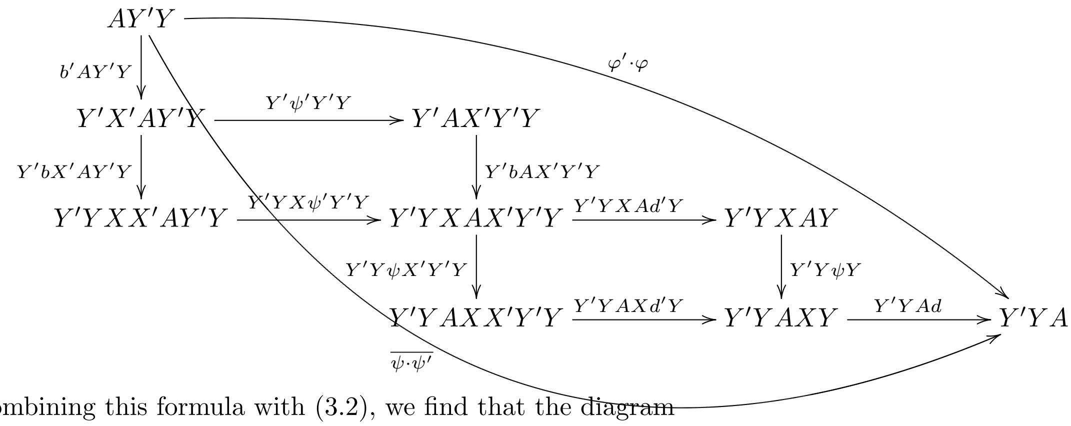
It follows immediately from (1.1, 2.1) that $\varphi \circ \eta Y = Y\eta$, hence $(Y,\varphi) \in {}_A\mathcal{T}$.
φ is independent of the choice of the right adjoint Y of X in the following sense. If (X, Y, b', d') is another adjunction, leading to $\varphi' : AY' \to Y'A$, then it follows from (1.2). that

$$\lambda A \circ \varphi = \varphi' \circ A\lambda, \tag{3.2}$$

where $\lambda = Y'd \circ b'Y : Y \to Y'$.
Let (X,ψ), $(X',\psi') \in \mathcal{T}_A$, $X \dashv Y$ and $X' \dashv Y'$. For $f : X \to X'$ in $\mathcal{T}_A^{\#}$,

$$g = YAd' \circ YfY' \circ bY' : Y' \to YA \tag{3.3}$$

is a morphism $g : Y' \to Y$ in ${}_A^{\#}\mathcal{T}$.
Assuming that $\mathcal{C}$ has right duality, and fixing a right dual ${}^*X$ for every $X \in \mathcal{C}$, we obtain a functor ${}^*(-) : \mathcal{T}_A^{\#} \to {}_A^{\#}\mathcal{T}^{\rm op}$, putting ${}^*(X,\psi) = ({}^*X, \varphi)$ and ${}^*f = g$. We leave it to the reader to verify that ${}^*(f' \bullet f) = {}^*f \bullet {}^*f'$ and ${}^*\mathrm{Id}_{(X,\psi)} = \mathrm{Id}_{({}^*X,\varphi)}$.
Let us finally show that ${}^*(-)$ is strong monoidal. It suffices to show that, for $X, X' \in \mathcal{T}_A$, $\varphi_2(X,X') : {}^*(XX') \to {}^*X'{}^*X$ defines an isomorphism $({}^*(XX'), \overline{\psi \cdot \psi'}) \to ({}^*X'{}^*X, \varphi' \cdot \varphi)$ in ${}_A\mathcal{T}$, and, a fortiori, in ${}_A^{\#}\mathcal{T}$. We have $X \dashv Y = {}^*X$, $X' \dashv Y' = {}^*X'$ and $XX' \dashv Y'Y$, and we claim that $\varphi' \cdot \varphi = \overline{\psi \cdot \psi'}$. To this end it suffices to observe that the following diagram commutes.



Combining this formula with (3.2), we find that the diagram

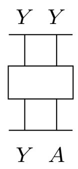
$$\begin{array}{ccc}
A^*(XX') & \xrightarrow{\overline{\psi\cdot\psi'}} & {}^*(XX')A \\
\downarrow{\scriptstyle A\varphi_2(X,X')} & & \downarrow{\scriptstyle \varphi_2(X,X')A} \\
A^*X'{}^*X & \xrightarrow{\varphi'\cdot\varphi} & {}^*X'{}^*XA
\end{array}$$

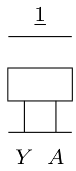
commutes, which is precisely what we need. □

3.2. **Factorization structures.**

Definition 3.2. Let $\mathcal{C}$ be a (strict) monoidal category. A left wreath (or left generalized factorization structure) in $\mathcal{C}$ is a triple (A, X, ψ), where A is an algebra in $\mathcal{C}$, and (X,ψ) is an algebra in $\mathcal{T}_A^{\#}$. A right wreath is a triple (A, Y, φ), where A is an algebra in $\mathcal{C}$ and (Y,φ) is an algebra in ${}_A^{\#}\mathcal{T}$.

Explicitly, a right wreath is a triple (A, Y, φ), where A is an algebra in $\mathcal{C}$, and $(Y, \varphi) \in {}^{\#}_{A}\mathcal{T}$, together with morphisms

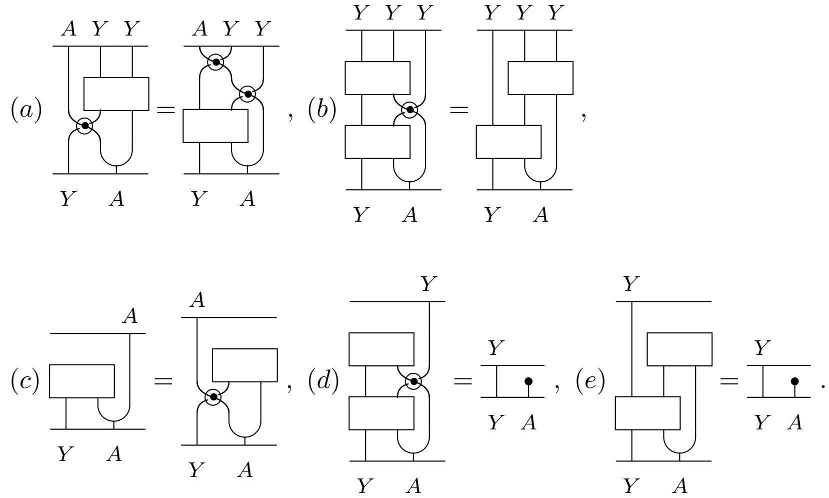
$m_Y =$ 

$: YY \to YA$ and $\eta_Y =$ 

$: \underline{1} \to YA$

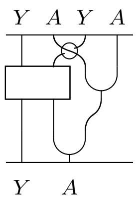
in $\mathcal{C}$ such that

(3.4)



m_A and η_A are the multiplication and unit of the algebra (Y, φ). (a) and (c) express the fact that $m_A : YY \to Y$ and $\eta_A : \underline{1} \to Y$ are morphisms in ${}^{\#}_{A}\mathcal{T}$; (b) is the associativity and (d) and (e) are the unit conditions.

If (A, Y, φ) is a right wreath, then YA is an algebra in $\mathcal{C}$ with multiplication

$m_{\#} =$ 

and unit $\eta_{\#} = \eta_Y : \underline{1} \to YA$, see for example [10]. In the literature, this algebra is called the wreath product or generalized smash product, and is denoted as $Y\#_{\varphi}A$.

If $F : \mathcal{C} \to \mathcal{D}$ is strong monoidal, and C is a coalgebra in $\mathcal{C}$, then $F(C)$ is a coalgebra in $\mathcal{D}$ with comultiplication and counit given by the formulas

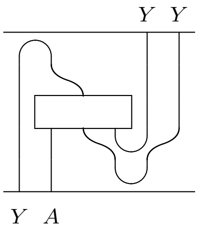
$$\Delta_{F(C)} = \varphi_2^{-1}(C, C) \circ F(\Delta) \; ; \; \varepsilon_{F(C)} = \varphi_0^{-1} \circ F(\varepsilon). \tag{3.5}$$

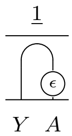
Let (A, X, ψ) be a cowreath, and assume that $X \dashv Y = {}^*X$ in $\mathcal{C}$. Then (X, ψ) is a coalgebra in $\mathcal{T}^{\#}_{A}$, and (Y, φ) is coalgebra in ${}^{\#}_{A}\mathcal{T}^{\mathrm{oprev}}$, by Theorem 3.1, and therefore an algebra in ${}^{\#}_{A}\mathcal{T}$, so that (A, Y, φ) is a right wreath. We compute the multiplication and unit using (3.3), with $f : X \to X'$ in $\mathcal{T}^{\#}_{A}$ replaced by $\delta : X \to XX$ and $\epsilon : X \to \underline{1}$. We find that

$$m_Y = YAd^2 \circ Y\delta YY \circ bYY : YY \to YA \text{ and } \eta_Y = Y\epsilon \circ b.$$

This proves the first part of Proposition 3.3. The proof of the second part is similar and is left to the reader.

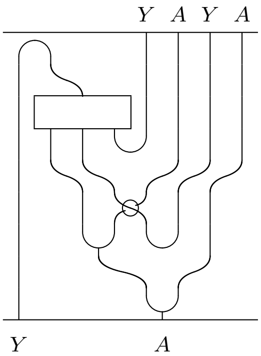
Proposition 3.3. *Let $\mathcal{C}$ be a (strict) monoidal category.*
(i) If (A, X, ψ) is a cowreath and $X \dashv Y$ in $\mathcal{C}$, then (A, Y, φ), with φ given by (3.1), is a right wreath, with multiplication m_Y and unit η_Y given by the formulas

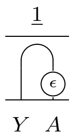
$m_Y =$ 

and $\eta_Y =$ 

.

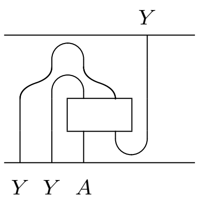
The wreath product YA is an algebra in $\mathcal{C}$, with structure maps

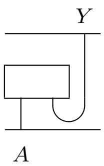
(3.6) $m_{\#} =$ 

and $\eta_{\#} =$ 

.

(ii) If (A, X, ψ) is a left wreath then (A, Y, φ) is a left cowreath (a coalgebra in ${}^{\#}_{A}\mathcal{T}$), with comultiplication and counit given by the formulas

$\overline{\delta} =$ 

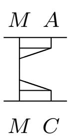
and $\overline{\epsilon} =$ 

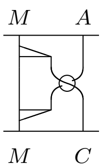
.

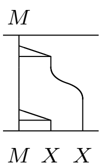
3.3. **Modules versus entwined modules.** Theorem 3.4 is the main result of this Section. It is a generalization of [24, Cor. 6.3].

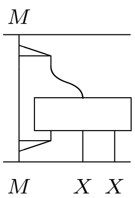
Theorem 3.4. *Let A be an algebra in a (strict) monoidal category $\mathcal{C}$, let $\mathcal{D}$ be a right $\mathcal{C}$-category and let (A, X, ψ) be a cowreath in $\mathcal{C}$. If $X \dashv Y$ in $\mathcal{C}$ then the categories $\mathcal{D}(\psi)_A^X$ and $\mathcal{D}_{YA}$ are isomorphic.*

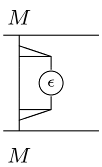
Take $M \in \mathcal{D}(\psi)_A^X$. The coaction $\rho : M \to MX$ is right A-linear, and satisfies (2.4) and (2.5). In diagrammatic notation, these conditions take the form

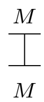
(3.7) 

$=$ 

, 

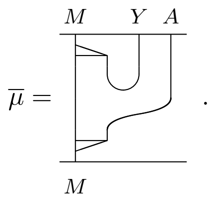
$=$ 

and 

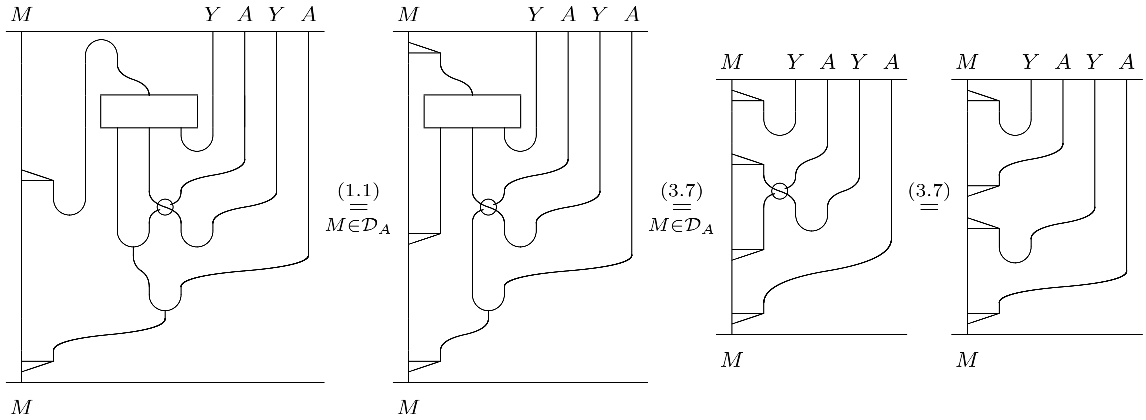
$=$ 

.

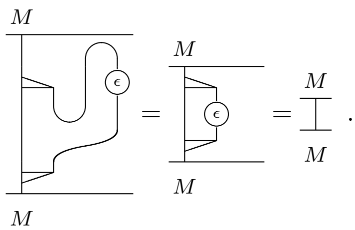
Lemma 3.5. *We have a functor $F : \mathcal{D}(\psi)_A^X \to \mathcal{D}_{YA}$. For $M \in \mathcal{D}(\psi)_A^X$, $F(M) = M \in \mathcal{C}_{YA}$ via*

(3.8)


Proof. $\overline{\mu} \circ Mm$ is equal to

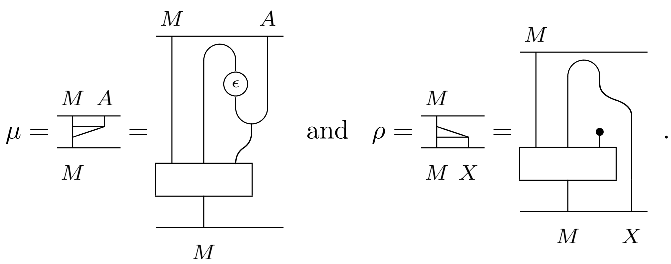


which is equal to $\overline{\mu} \circ (\overline{\mu} \diamond YA)$, as needed. Using (1.1) and (3.7) we obtain that

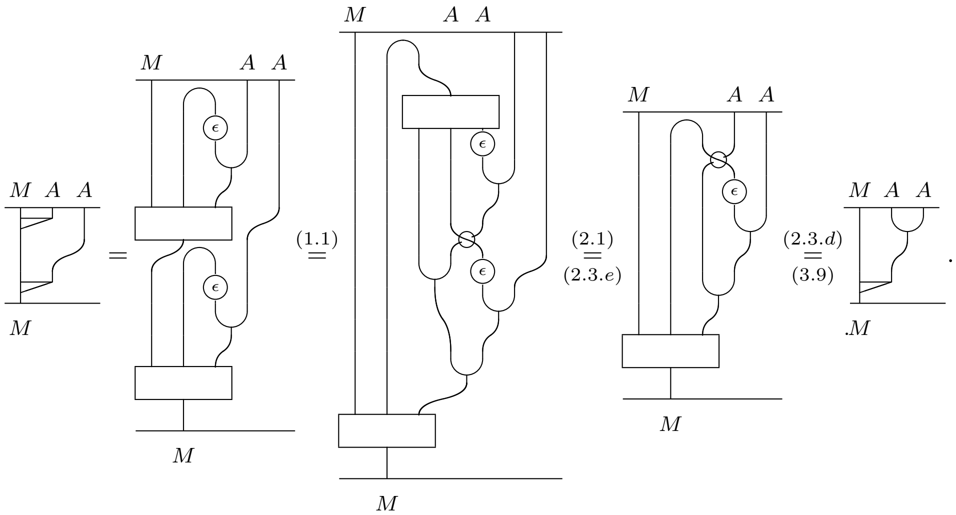


This shows that M is a right YA-module. It is left to the reader to show that a morphism between two entwined modules is also a morphism of right YA-modules. □

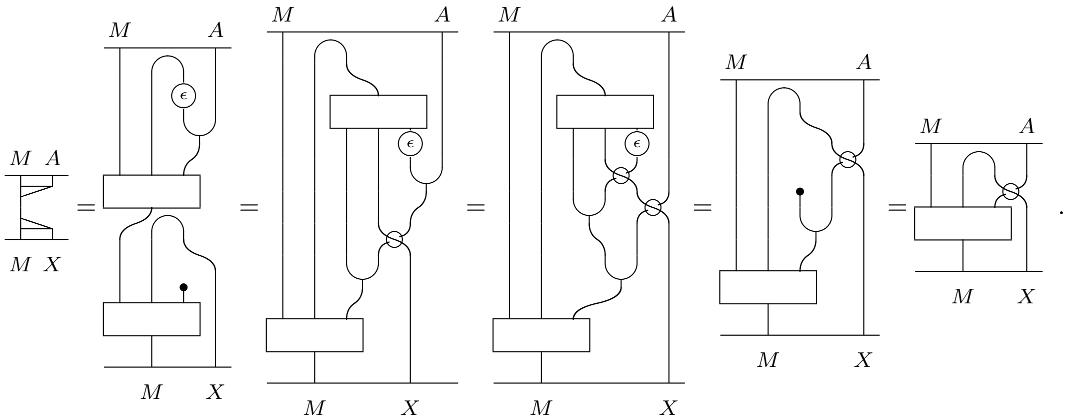
Lemma 3.6. *We have a functor $G : \mathcal{D}_{YA} \to \mathcal{D}(\psi)_A^X$. $G(M) = M \in \mathcal{D}(\psi)_A^X$ via*

(3.9)


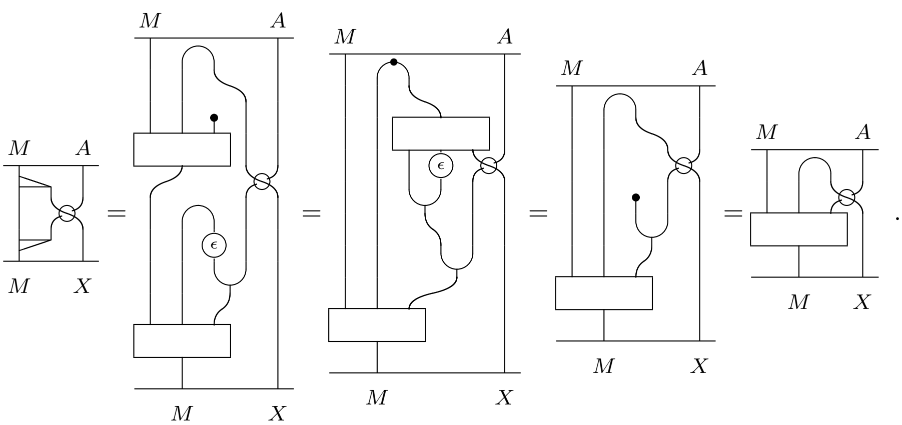
Proof. We first show that $M \in \mathcal{D}_A$. The unit property of μ is easily verified, and the associativity follows from the following computation, which uses $M \in \mathcal{D}_{YA}$ (in the second equality) and the associativity of m (in the third and fourth equality).



Now we prove that (3.7) holds. Applying (1.1, 2.1, 2.3), the fact that M is a right YA-module and the associativity of m, we obtain that

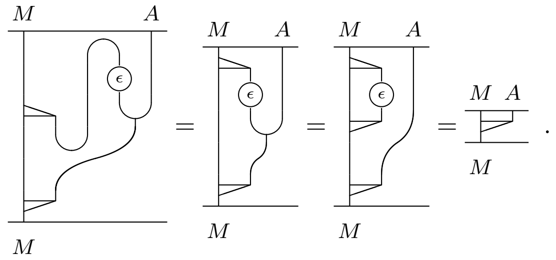


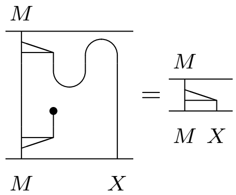
Applying the fact that M is a right YA-module, (1.1) and (2.3.d), we obtain that



It follows that the first equation in (3.7) is satisfied. Similar computations show that the other two also hold. We leave it to the reader to show that a right YA-linear morphism is also a morphism of entwined modules. □

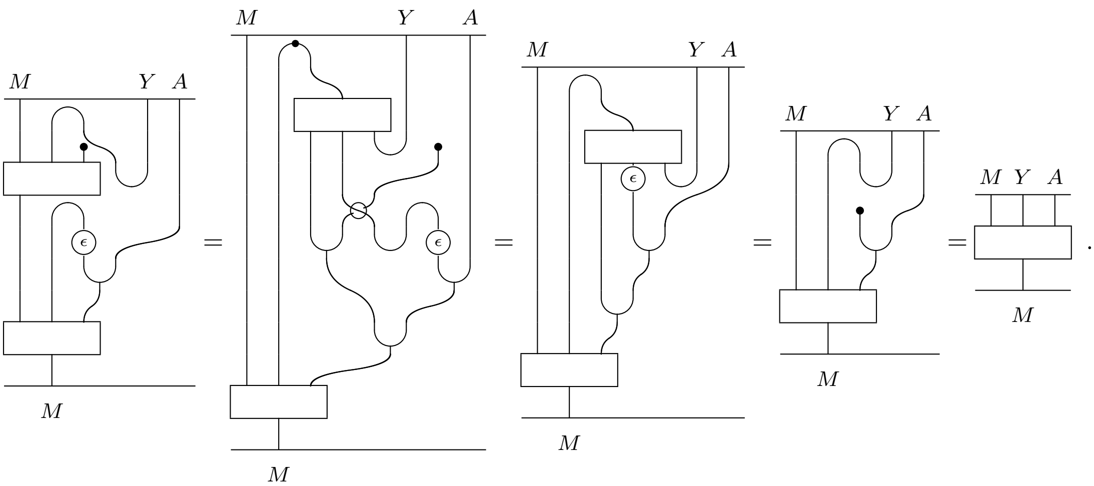
Proof. (of Theorem 3.4). It suffices to show that the functors F and G from Lemmas 3.5 and 3.6 are inverses. Let $M \in \mathcal{D}(\psi)_A^X$. Then $F(M) \in \mathcal{D}_{YA}$ via (3.8). Applying (3.9), $GF(M)$ is a right A-module in $\mathcal{D}$ via



Furthermore $\rho =$ 

. This shows that $G(F(M)) = M$ in $\mathcal{D}(\psi)_A^X$.

Now take $M \in \mathcal{D}_{YA}$. Then $G(M) \in \mathcal{D}(\psi)_A^X$ with structure maps (3.9). According to (3.8), $FG(M)$ is a right YA-module in $\mathcal{D}$ via

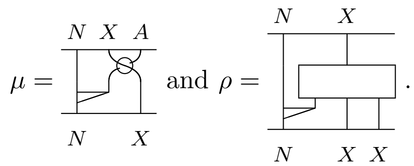


We conclude that the right YA-actions on M and $FG(M)$ coincide and that $FG(M) = M$ in $\mathcal{D}_{YA}$. This completes the proof. □

4. Frobenius functors versus Frobenius coalgebras

4.1. **Frobenius functors.** Throughout this Section (A, X, ψ) is a cowreath in a (strict) monoidal category $\mathcal{C}$. Recall that a Frobenius functor is a functor that has a right adjoint which is also a left adjoint. The aim of this Section is to investigate when the forgetful functor $F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ is Frobenius. Lemma 4.1 tells us that F always has a right adjoint G, so that our problem reduces to examining whether G is a left adjoint of F.

Lemma 4.1. *Let $\mathcal{D}$ be a right $\mathcal{C}$-category. The forgetful functor F has a right adjoint $G : \mathcal{D}_A \to \mathcal{D}(\psi)_A^X$, defined as follows: $G(N) = NX$ is an object of $\mathcal{D}(\psi)_A^X$ via*

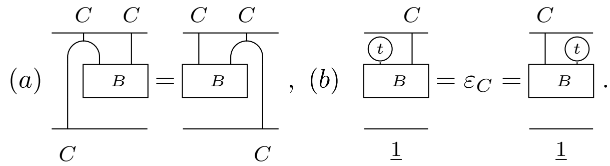


Proof. The unit and the counit of the adjunction are given by the formulas, $\eta_M = \rho : M \to GF(M) = MX$ and $\varepsilon_N = \mu \circ N\epsilon : \ FG(N) = NX \to N$, for all $M \in \mathcal{D}(\psi)_A^X$ and $N \in \mathcal{D}_A$. □

4.2. **Frobenius coalgebras.** The notion of Frobenius algebra in a monoidal category (as introduced in [36], see also [15, Def. 4.1]) can be dualized: a coalgebra in a monoidal category is Frobenius if and only if the corresponding algebra in the opposite category is Frobenius. This leads to the following definition.

Definition 4.2. A coalgebra C in $\mathcal{C}$ is called Frobenius if there exists a Frobenius system (t, B) consisting of morphisms $t : \ \underline{1} \to C$ (the Frobenius element) and $B : \ CC \to \underline{1}$ (the Casimir morphism) in $\mathcal{C}$ such that

(4.1)



Remark 4.3. Several equivalent characterizations of a Frobenius algebra are known, see for example [15, Theorem 5.1]. Now a Frobenius coalgebra is a Frobenius algebra in the opposite category, and this leads to the following equivalent characterizations of a Frobenius coalgebra.

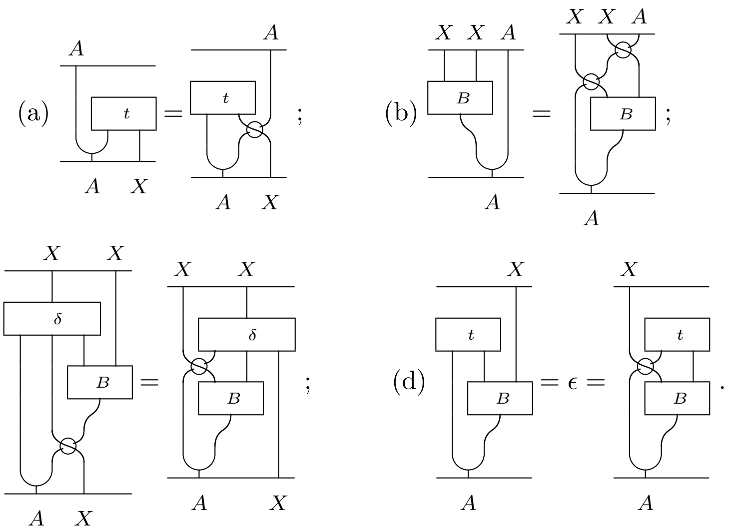
(i) There is an adjunction $C \dashv A$, and C is isomorphic to A as a right A-module. Recall from (1.3-1.4) that A is an algebra, and that C is a right A-module.
(ii) There is an adjunction $A' \dashv C$ and C is isomorphic to A' as a left A'-module.
(iii) C is an algebra in $\mathcal{C}$ with C-bicolinear multiplication.
(iv) There is an adjunction $A' \dashv C$ and there exists a balanced right non-degenerate morphism $B_r : \ CC \to \underline{1}$ in $\mathcal{C}$. This means that $CB_r \circ \Delta C = B_r C \circ C\Delta_C$ (balanced) and that $\Phi_{B_r} = B_r A' \circ Cb : \ C \to A'$ is an isomorphism (non-degenerate).
(v) There is an adjunction $C \dashv A$ and there exists a balanced left non-degenerate morphism $B_l : \ CC \to \underline{1}$ in $\mathcal{C}$. The fact that B_l is left non-degenerate means that $\Psi_{B_l} = AB_l \circ b'C : C \to A$ is an isomorphism.
(vi) There is an adjunction $(b, d) : \ C \dashv C$ such that $C\Delta \circ b = \Delta C \circ b$.
(vii) There is an adjunction $(b, d) : \ C \dashv C$ such that $b = \Delta \circ t$, for some $t : \underline{1} \to C$ in $\mathcal{C}$.

A Frobenius coalgebra C is also a Frobenius algebra (and vice versa), with multiplication $m = CB \circ \Delta C = BC \circ C\Delta$, unit $\eta = t$, and Frobenius system $(\varepsilon, \Delta \circ \eta)$.

Specializing Definition 4.2 to coalgebras in $\mathcal{T}_A^\#$, we obtain the following result.

Lemma 4.4. *A coalgebra (X, ψ) in $\mathcal{T}_A^{\#}$ is Frobenius if and only if there exist morphisms $t : \underline{1} \to AX$ and $B : XX \to A$ in $\mathcal{C}$ such that*

(4.2)



Proof. This is basically a reformulation of Definition 4.2 in the special case where $\mathcal{C} = \mathcal{T}_A^{\#}$. (a) and (b) express the fact that t and B are morphisms in $\mathcal{T}_A^{\#}$, and (c) and (d) are a reformulation of (a) and (b) in (4.1). We leave the verification of the details to the reader. □

4.3. **Natural transformations, Frobenius elements and Casimir morphisms.**

Definition 4.5. [15, Def. 3.1] An object P in a monoidal category $\mathcal{C}$ is called a left $\otimes$-generator if the following condition is satisfied: if $f,\ g :\ YZ \to W$ are morphisms in $\mathcal{C}$ such that $f \circ hZ = g \circ hZ$ for all $h :\ P \to Y$ in $\mathcal{C}$, then $f = g$.

It is easy to see that a left $\otimes$-generator is a generator in the classical sense. If $\underline{1}$ is a left $\otimes$-generator for $\mathcal{C}$, then C is a Frobenius coalgebra if and only if the forgetful functor $\mathcal{C}^C \to \mathcal{C}$ is Frobenius. In Theorem 4.8, we will prove the following result: under the hypothesis that $\underline{1}$ is a left $\otimes$-generator in $\mathcal{C}$, the forgetful functor $F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ is Frobenius if and only if (X, ψ) is a Frobenius coalgebra in $\mathcal{T}_A^{\#}$. We have to determine when G is a left adjoint of F, an to this end we have to investigate natural transformations $\theta :\ \mathrm{Id}_{\mathcal{C}_A} \to FG$ and $\vartheta :\ GF \to \mathrm{Id}_{\mathcal{C}(\psi)_A^C}$. We show that these natural transformations correspond to Frobenius elements (Proposition 4.6) and Casimir morphisms (Proposition 4.7). We recall from Lemma 1.2 that, for $N \in \mathcal{C}_A$, we have an isomorphism $\alpha :\ \mathcal{C}_A(A, N) \to \mathcal{C}(\underline{1}, N)$, $\alpha(\underline{h}) = h$, with $h = \underline{h} \circ \eta$ and $\underline{h} = \mu \circ hA$.

Proposition 4.6. *Let (A, X, ψ) be a cowreath in $\mathcal{C}$. If $\underline{1}$ is a left $\otimes$-generator for $\mathcal{C}$, then we have an isomorphism* $\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}_A}, FG) \cong \mathcal{T}_A^{\#}(\underline{1}, X)$.

Proof. Consider a natural transformation $\theta :\ \mathrm{Id}_{\mathcal{C}_A} \to FG$. We claim that $t = \alpha(\theta_A) = \theta_A \circ \eta :$ $\underline{1} \to AX$ is a morphism $\underline{1} \to X$ in $\mathcal{T}_A^{\#}$. Take $h \in \mathcal{C}(\underline{1}, A)$. From the naturality of θ, it follows that $\theta_A \circ \underline{h} = \underline{h}X \circ \theta_A$, hence

$$mX \circ At \circ h = mX \circ hAX \circ t = \underline{h}X \circ \theta_A \circ \eta = \theta_A \circ \underline{h} \circ \eta = \theta_A \circ m \circ A\eta \circ h = \theta_A \circ h,$$

so that

$$mX \circ At = \theta_A, \tag{4.3}$$

since $\underline{1}$ is a left $\otimes$-generator. Using the right A-linearity of θ_A, we find that

$$\theta_A = \theta_A \circ m \circ \eta A = mX \circ A\psi \circ \theta_A A \circ \eta A = mX \circ A\psi \circ tA.$$

It follows that $\theta_A = mX\circ At = mX\circ A\psi\circ tA$, which is precisely (4.2.a), expressing that $t\in \mathcal{T}_A^{\#}(\underline{1},X)$. Our next aim is to show that θ is completely determined by t. θ_A is given by (4.3). Take $N\in\mathcal{C}_A$ and $h:\ \underline{1}\to N$ in $\mathcal{C}$. Then

$$\theta_N\circ\mu\circ hA = \theta_N\circ \underline{h}\overset{(*)}{=} g_hX\circ\theta_A = \mu X\circ hAX\circ\theta_A = \mu X\circ N\theta_A\circ hA.$$

At $(*)$, we used the naturality of θ. From the fact that $\underline{1}$ is a left $\otimes$-generator, it follows that $\theta_N\circ\mu = \mu X\circ N\theta_A$ (4.2.a) and $\theta_N = \theta_N\circ\mu\circ N\eta = \mu X\circ N\theta_A\circ N\eta = \mu X\circ Nt$. We conclude that

$$\theta_N = \mu X\circ Nt \tag{4.4}$$

is completely determined by t.
Finally, for $t\in\mathcal{T}_A^{\#}(\underline{1},X)$, we define θ using (4.4). We show that θ_N is right A-linear, for all $N\in\mathcal{C}_A$.

$$\begin{aligned}\mu_{NX}\circ\theta_N A &\overset{(4.4)}{=} \mu X\circ N\psi\circ\mu XA\circ NtA = \mu X\circ\mu AX\circ NA\psi\circ NtA\\ &\overset{(x)}{=} \mu X\circ NmX\circ NA\psi\circ NtA\overset{(y)}{=}\mu X\circ NmX\circ NAt\\ &\overset{(x)}{=} \mu X\circ\mu AX\circ NAt = \mu X\circ Nt\circ\mu\overset{(4.4)}{=}\theta_N\circ\mu.\end{aligned}$$

At (x) we used the associativity of μ, and at (y), we used the fact that $t\in\mathcal{T}_A^{\#}(\underline{1},X)$. □

Proposition 4.7. *Let (A,X,ψ) be a cowreath in $\mathcal{C}$. If $\underline{1}$ is a left $\otimes$-generator for $\mathcal{C}$, then we have a bijective correspondence between* $\mathrm{Nat}(GF,\mathrm{Id}_{\mathcal{C}(\psi)_A^X})$ *and the set of Casimir morphisms for (X,ψ), that is the subset of $\mathcal{T}_A^{\#}(XX,\underline{1})$ consisting of morphisms $B:\ XX\to A$ satisfying (4.2.c).*

Proof. Consider a natural transformation $\vartheta:\ GF\to\mathrm{Id}_{\mathcal{C}(\psi)_A^X}$, and take $N\in\mathcal{C}_A$. Then $G(X)=NX$ is an entwined module, and for all $h\in\mathcal{C}(\underline{1},N)$, we have that

$$\begin{aligned}\vartheta_{NX}\circ hXX &= \vartheta_{AX}\circ\underline{h}XX\circ\eta XX\overset{(a)}{=}\underline{h}X\circ\vartheta_{AX}\circ\eta XX\\ &= \mu X\circ hAX\circ\vartheta_{AX}\circ\eta XX = \mu X\circ N\vartheta_{AX}\circ hAXX\circ\eta XX\\ &= \mu X\circ N\vartheta_{AX}\circ N\eta XX\circ hXX.\end{aligned}$$

At (a), we used the naturality of ϑ. Let $\zeta = \vartheta_{AX}\circ\eta XX$. From the fact that $\underline{1}$ is a left $\otimes$-generator for $\mathcal{C}$, it follows that

$$\vartheta_{NX} = \mu X\circ N\zeta. \tag{4.5}$$

For an entwined module M, the coaction $\rho:\ M\to MX$ is a morphism of entwined modules, and it follows from the naturality of ϑ that

$$\rho\circ\vartheta_M = \vartheta_{MX}\circ\rho X. \tag{4.6}$$

This enables us to compute that

$$\begin{aligned}\vartheta_M &\overset{(2.5)}{=} \mu\circ M\epsilon\circ\rho\circ\vartheta_M\overset{(4.5,4.6)}{=}\mu\circ M\epsilon\circ\mu X\circ M\zeta\circ\rho X\\ &= \mu\circ Mm\circ MA\epsilon\circ M\zeta\circ\rho X = \mu\circ MB\circ\rho X,\end{aligned} \tag{4.7}$$

with

$$B = m\circ A\epsilon\circ\zeta = m\circ A\epsilon\circ\vartheta_{AX}\circ\eta XX:\ XX\to A. \tag{4.8}$$

This shows that ϑ is completely determined by B.

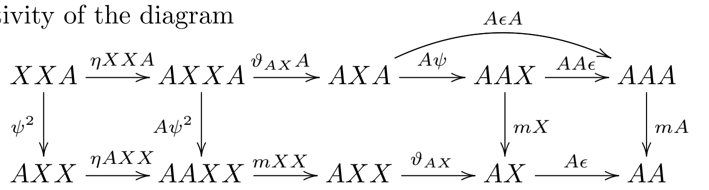
We claim that $B\in\mathcal{T}_A^{\#}(XX,\underline{1})$. To this end, we need to show that (4.2.b) holds, that is,

$$m^2\circ A\epsilon A\circ\vartheta_{AX}A\circ\eta XXA = m\circ A\epsilon\circ mX\circ A\vartheta_{AX}\circ A\eta XX\circ\psi^2. \tag{4.9}$$

Observe that

$$mX\circ A\vartheta_{AX}\circ A\eta XX = \vartheta_{AX}\circ mXX\circ A\eta XX = \vartheta_{AX} = \vartheta_{AX}\circ mXX\circ\eta AXX,$$

so that the right hand side of (4.9) equals $m \circ A\epsilon \circ mX \circ A\vartheta_{AX} \circ \eta AXX \circ \psi^2$. (4.9) then follows from the commutativity of the diagram



The commutativity of the two squares is obvious, and the commutativity of the rectangle in the middle follows from the right A-linearity of ϑ_{AX}. It follows from (2.3) that $AA\epsilon \circ A\psi = A\epsilon A$.

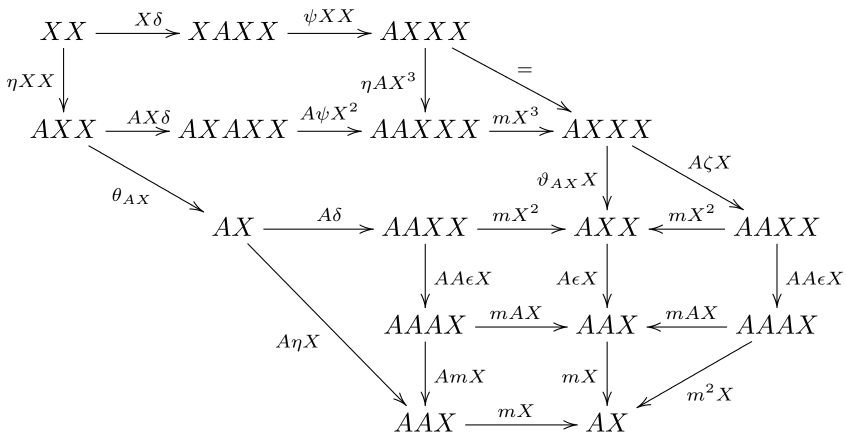
Our next step is to show that B as defined in (4.8) satisfies (4.2.c), or

$$mX \circ A\psi \circ AXm \circ AXA\epsilon \circ AX\zeta \circ \delta X = m^2X \circ AA\epsilon X \circ A\zeta X \circ \psi XX \circ X\delta. \tag{4.10}$$

We will show that the two sides of (4.10) are equal to ζ. First

$$\begin{aligned}
\zeta &= \vartheta_{AX} \circ \eta XX \\
&\overset{(4.7)}{=} mX \circ A\psi \circ AXm \circ AXA\epsilon \circ AX\zeta \circ mXXX \circ A\delta X \circ \eta XX \\
&= mX \circ A\psi \circ AXm \circ AXA\epsilon \circ AX\zeta \circ mXXX \circ \eta AXXX \circ \delta X \\
&= mX \circ A\psi \circ AXm \circ AXA\epsilon \circ AX\zeta \circ \delta X,
\end{aligned}$$

the left hand side of (4.10). The diagram below is commutative.



The septangle in the middle commutes because θ_{AX} preserves the right X-coaction. The commutativity of all the other parts of the diagram is obvious. It follows from the commutativity of the diagram that $\zeta = mX \circ A\eta X \circ \vartheta_{AX} \circ \eta XX$ is equal to the right hand side of (4.10).

Finally, for $B \in \mathcal{T}_A^{\#}(XX, \underline{1})$ satisfying (4.2.c), we define ϑ using the formula

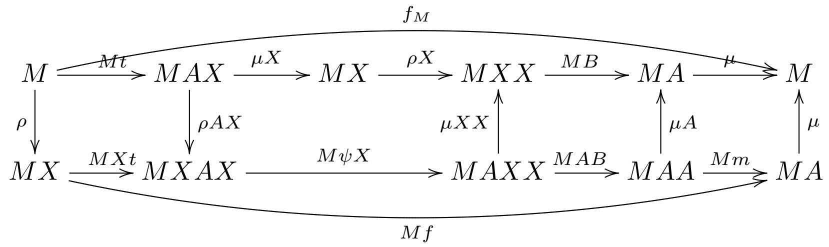
$$\vartheta_M = \mu \circ MB \circ \rho X : \; MX \to M, \tag{4.11}$$

for any entwined module M. It is left to the reader to show that ϑ_M is a morphism of entwined modules, and that ϑ is natural in M. □

4.4. **Frobenius functors and Frobenius systems.**

Theorem 4.8. *Let (A, X, ψ) be a cowreath in $\mathcal{C}$. If $\underline{1}$ is a left $\otimes$-generator for $\mathcal{C}$, then the forgetful functor $F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ is Frobenius if and only if (X, ψ) is a Frobenius coalgebra in $\mathcal{T}_A^{\#}$.*

Proof. Let $F \dashv G$ be the adjunction described in Lemma 4.1. The functor F is Frobenius if and only if $G \dashv F$, and this is equivalent to the existence of $\theta \in \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}_A}, FG)$ and $\vartheta \in \mathrm{Nat}(GF, \mathrm{Id}_{\mathcal{C}(\psi)^X_A})$ such that $F(\vartheta_M) \circ \theta_{F(M)} = F(M)$ and $\vartheta_{G(N)} \circ G_{\theta_N} = G(N)$, for all $M \in \mathcal{C}(\psi)^C_A$ and $N \in \mathcal{C}_A$.
Fix $\theta \in \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}_A}, FG)$ and $\vartheta \in \mathrm{Nat}(GF, \mathrm{Id}_{\mathcal{C}(\psi)^X_A})$, and let $t \in \mathcal{T}^{\#}_A(\underline{1}, X)$ and $B \in \mathcal{T}^{\#}_A(XX, \underline{1})$ be the Frobenius element and the Casimir morphism corresponding to θ and ϑ, see Propositions 4.6 and 4.7. (t, B) is a Frobenius system if and only if (4.2.d) holds, which comes down to the following: $f = g = \epsilon$, where $f = m \circ AB \circ \psi X \circ Xt$ and $g = m \circ AB \circ tX$. It is easy to verify that $f_M = F(\vartheta_M) \circ \theta_{F(M)} = \mu \circ MB \circ \rho X \circ \mu X \circ Mt$, the composition in the top row of the diagram



The pentangle in the diagram commutes by (2.6). The right square commutes by the associativity of μ, and the commutativity of the two other squares is obvious. We conclude that the diagram commutes.
If $f = \epsilon$, then it follows that $F(\vartheta_M) \circ \theta_{F(M)} = f_M = \mu \circ M\epsilon \circ \rho \overset{(2.5)}{=} M$.
Conversely, if $f_M = M$, for every entwined module M, in particular, $f_{AX} = AX$. From the commutativity of the above diagram, it follows that $\mu_{AX} \circ AXf \circ \rho_{AX} = AX$. Using the unit property of A, we find that

$$m \circ A\epsilon \circ \eta X = m \circ \eta A \circ \epsilon = \epsilon. \tag{4.12}$$

Then we compute that

$$\begin{aligned} \epsilon &= m \circ A\epsilon \circ \mu_{AX} \circ AXf \circ \rho_{AX} \circ \eta X \\ &= m \circ A\epsilon \circ mX \circ A\psi \circ AXf \circ mXX \circ A\delta \circ \eta X \\ &= m \circ mA \circ AA\epsilon \circ A\psi \circ AXf \circ \delta \\ &\overset{(2.3.c)}{=} m \circ mA \circ A\epsilon A \circ AXf \circ \delta \\ &= m \circ Af \circ mX \circ A\epsilon X \circ \delta \\ &\overset{(2.3.d)}{=} M \circ Af \circ \eta X = m \circ \eta A \circ f = f. \end{aligned}$$

We conclude that $f = \epsilon$ if and only if $F(\vartheta_M) \circ \theta_{F(M)} = F(M)$, for all $M \in \mathcal{C}(\psi)^X_A$. In a similar way, we show that $g = \epsilon$ if and only if $g_N = \vartheta_{G(N)} \circ G_{\theta_N}) = G(N)$, for all $N \in \mathcal{C}_A$.

$$g_N = \mu X \circ N\psi \circ NXB \circ \mu XXX \circ N\delta X \circ \mu XX \circ NtN,$$

and some (complicated) diagram chasing arguments using (4.2.a-c) show that

$$g_N = \mu X \circ N\mu X \circ NAgX \circ N\delta. \tag{4.13}$$

If $g = \epsilon$, then it follows that $g_N = \mu X \circ N\mu X \circ NA\epsilon X \circ N\delta \overset{(2.3.d)}{=} \mu X \circ N\eta X = NX$.
The converse implication is more subtle. The tensor product in $\mathcal{T}^{\#}_A$ of $t \in \mathcal{T}^{\#}_A(\underline{1}, X)$ and the identity $\eta X \in \mathcal{T}^{\#}_A(X, X)$ is $tX \in \mathcal{T}^{\#}_A(X, XX)$. The composition in $\mathcal{T}^{\#}_A$ of $tX \in \mathcal{T}^{\#}_A(X, XX)$ and $B \in \mathcal{T}^{\#}_A(XX, \underline{1})$ is then precisely $g \in \mathcal{T}^{\#}_A(X, \underline{1})$, see Section 2. We conclude that g is a morphism in $\mathcal{T}^{\#}_A(X, \underline{1})$, so that it follows from (2.2) that

$$m \circ Ag \circ \psi = m \circ gA. \tag{4.14}$$

If $g_A = AX$, then it follows that

$$\begin{aligned}
\epsilon &\overset{(4.12)}{=} m\circ A\epsilon\circ\eta X = m\circ A\epsilon\circ AX\circ\eta X \overset{(4.13)}{=} m\circ A\epsilon\circ m^2X\circ AAgX\circ A\delta\circ\eta X\\
&= m\circ Am\circ AgA\circ AX\epsilon\circ\delta \overset{(4.14)}{=} m\circ Am\circ AAg\circ A\psi\circ AX\epsilon\circ\delta\\
&= m\circ mA\circ AAg\circ A\psi\circ AX\epsilon\circ\delta = m\circ Ag\circ mX\circ A\psi\circ AX\epsilon\circ\delta\\
&\overset{(2.3.e)}{=} m\circ Ag\circ\eta X = m\circ\eta A\circ g = g.
\end{aligned}$$

□

5. Frobenius coalgebras versus Frobenius corings

Throughout this Section, $\mathcal{C}$ is a (strict) monoidal category with coequalizers, and A is a left coflat algebra in $\mathcal{C}$. ${}^!\mathcal{T}_A^\#$ is the full subcategory of $\mathcal{T}_A^\#$ consisting of right transfer morphisms (X,ψ) with X left coflat in in $\mathcal{C}$.

Lemma 5.1. *We have a fully faithful strong monoidal functor $H:\ {}^!\mathcal{T}_A^\# \to {}_A^!\mathcal{C}_A$.*

Proof. Take $(X,\psi)\in {}^!\mathcal{T}_A^\#$. It follows from Proposition 1.4 that AX is robust as a left A-module; AX is left coflat since A and X are left coflat. AX is an A-bimodule, with left A-action $\nu = mX$ and right A-action $\mu = mX\circ A\psi$. We conclude that $AX\in {}_A^!\mathcal{C}_A$, and we define $H(X,\psi) = AX$.
Let $f:\ X\to Y$ in ${}^!\mathcal{T}_A^\#$, and define $H(f) = mY\circ Af$. It is clear that Hf is left A-linear, and the right A-linearity follows from (2.2). $H:\ {}^!\mathcal{T}_A^\#(X,Y)\to {}_A^!\mathcal{C}_A(AX,AY)$ is bijective. The inverse of $\underline{f}:\ AX\to AY$ is $H^{-1}(\underline{f}) = \underline{f}\circ\eta f$. Thus H is a fully faithful functor; the monoidal structure is the following: $\varphi_0 = A:\ A\to H(\underline{1}) = A$, and $\varphi_2(X,Y):\ AX\bullet AY\to AXY$ is the unique isomorphism of coequalizers, see Proposition 1.1. □

In fact, if we make the identification of coequalizers $(AX\bullet AY, q) = (AXY, \mu_{AX}Y)$, then H becomes strictly monoidal. It follows immediately from Lemma 5.1 that there exists a bijective correspondence between coalgebra structures on $(X,\psi)\in {}^!\mathcal{T}_A^\#$ and A-coring structures on AX. Moreover, Frobenius systems on (A,ψ) correspond bijectively to Frobenius systems on AX, and the following result follows.

Theorem 5.2. *With notation and assumptions as above, (X,ψ) is a Frobenius coalgebra in $\mathcal{T}_A^\#$ if and only if $\mathfrak{C} = AX$ is a Frobenius A-coring, this is a Frobenius coalgebra in ${}_A^!\mathcal{C}_A$.*

Combining Theorems 4.8 and 5.2, we obtain the following result.

Corollary 5.3. *Assume that $\underline{1}$ is a left $\otimes$-generator for $\mathcal{C}$. Let A be a left coflat algebra in $\mathcal{C}$, and take $(X,\psi)\in {}^!\mathcal{T}_A^\#$. Then the following assertions are equivalent:*
- (i) *The forgetful functor $U:\mathcal{C}^{AX}\to\mathcal{C}_A$ is Frobenius;*
- (ii) *AX is a Frobenius A-coring.*

Proof. From [10, Theorem 4.8] we know that the categories $\mathcal{C}^{\mathfrak{C}}$ and $\mathcal{C}(\psi)_A^X$ are isomorphic. Thus U is a Frobenius functor if and only if the forgetful functor $F:\mathcal{C}(\psi)_A^X\to\mathcal{C}_A$ is Frobenius. Since $\underline{1}$ is a left $\otimes$-generator for $\mathcal{C}$, it follows from Theorem 4.8 that F is Frobenius if and only if (X,ψ) is a Frobenius coalgebra in $\mathcal{T}_A^\#$, and this is equivalent to AX being a Frobenius A-coring, by Theorem 5.2. □

Another immediate corollary of Lemma 5.1 is the following. If (X,ψ) has a right adjoint in ${}^!\mathcal{T}_A^\#$, then AX has a right adjoint in ${}_A^!\mathcal{C}_A$. In particular, if (X,ψ) is a Frobenius coalgebra in ${}^!\mathcal{T}_A^\#$, then it is selfadjoint, and AX is selfadjoint in ${}_A^!\mathcal{C}_A$. Proposition 5.4 is a result of the same type, but with a more complicated proof. Take $(X,\psi)\in {}^!\mathcal{T}_A^\#$. If X has a right adjoint Y in $\mathcal{C}$, then $(Y,\varphi)\in {}_A^\#\mathcal{T}$, and YA is an A-bimodule.

Proposition 5.4. *With notation as above, we assume that $YA\in {}_A^!\mathcal{C}_A$. Then $AX\dashv YA$ in ${}_A^!\mathcal{C}_A$.*

Proof. We have an adjunction (X, Y, b, d) in $\mathcal{C}$. Recall that $b:\ \underline{1} \to YX$, $d:\ XY \to \underline{1}$.

$$B = Y\psi \circ bA:\ A \to YAX = YA \bullet AX \text{ and } m \circ AdA : AXYA \to A$$

are morphisms of A-bimodules. Let D be the unique morphism of A-bimodules that makes the following diagram commutative:

$$(5.1)\qquad \begin{array}{ccccc} AXAYA & \overset{AX\nu}{\underset{\mu YA}{\rightrightarrows}} & AXYA & \xrightarrow{q} & AX \bullet YA \\ & & \downarrow{\scriptstyle AdA} & & \downarrow{\scriptstyle \exists! D} \\ & & AA & \xrightarrow{m} & A \end{array}$$

Applying Propositions 1.1 and 1.7, we have isomorphisms of colimits

$$(5.2)\qquad (AX \bullet YA \bullet AX, q_2) \cong (AX \bullet YAX, q \circ AXYmX) \cong ((AX \bullet YA)X, \mu_{AX\bullet YA} \circ qAX).$$

Consider the diagram

$$(5.3)\qquad \begin{array}{ccccc} & & AXA & \xrightarrow{q} & AX \bullet A \\ & & \downarrow{\scriptstyle AXB} & & \downarrow{\scriptstyle AX\bullet B} \\ AXYAAX & \xrightarrow{AXYmX} & AXYAX & \xrightarrow{q} & AX \bullet YAX \\ \downarrow{\scriptstyle =} & & & & \downarrow{\scriptstyle \cong} \\ AXYAAX & \xrightarrow{qAX} & (AX \bullet YA)AX & \xrightarrow{\mu X} & (AX \bullet YA)X \\ \downarrow{\scriptstyle AdAAY} & & \downarrow{\scriptstyle DAX} & & \downarrow{\scriptstyle D\bullet AX} \\ AAAX & \xrightarrow{mAX} & AAX & \xrightarrow{mX} & AX \end{array}\qquad (\delta : AX \bullet YAX \to AX)$$

The commutativity of top and bottom right squares follows from (1.5); the commutativity of the rectangle in the middle follows from (5.2); the commutativity of the bottom left square follows from the definition of D. We conclude that the diagram commutes. Let δ be the composition of $D \bullet AX$ and the isomorphism $AX \bullet YAX \cong (AX \bullet YA)X$, as indicated in the diagram, and consider the diagram

$$(5.4)\qquad \begin{array}{ccccc} AXYAAX & \xrightarrow{AXYmX} & AXYAX & \xrightarrow{q} & AX \bullet YAX \\ \downarrow{\scriptstyle AdAAX} & & \downarrow{\scriptstyle Ad'AX} & & \downarrow{\scriptstyle \delta} \\ AAAX & \xrightarrow{AmX} & AAX & \xrightarrow{mX} & AX \end{array}$$

$$\delta \circ q \circ AXYmX \overset{(5.3)}{=} mX \circ mAX \circ AdAAY = mX \circ AmX \circ AdAAY = mX \circ AdAX \circ AXYmY.$$

We used the associativity of m and the fact that the left square in (5.4) commutes. It follows that $\delta \circ q = mX \circ AdAX$, since $(AXYAX, AXYmX)$ is a coequalizer, see Proposition 1.1. We conclude that the diagram (5.4) commutes. We therefore have commutative diagrams

$$\begin{array}{ccc} AXA & \xrightarrow{q} & AX \bullet A \\ \downarrow{\scriptstyle AXB} & & \downarrow{\scriptstyle AX\bullet B} \\ AXYAX & \xrightarrow{q} & AX \bullet YAX \\ \downarrow{\scriptstyle AdAX} & & \downarrow{\scriptstyle \delta} \\ AAX & \xrightarrow{mX} & AX \end{array} \qquad \text{and} \qquad \begin{array}{ccc} AXA & \xrightarrow{q} & AX \bullet A \\ & \searrow{\scriptstyle \mu_{AX}} & \downarrow{\scriptstyle \cong} \\ & & AX \end{array}$$

Now

$$mX \circ AdAX \circ AXB = mX \circ AdAX \circ AXY\psi \circ AXbA = mX \circ A\psi \circ AdXA \circ AXbA = \mu_{AX}.$$

Since $(AX \bullet A, q)$ is a coequalizer, we conclude that $\delta \circ (AX \bullet B)$ is the canonical isomorphism $AX \bullet A \cong AX$, as needed.
In a similar way, the diagram commutes

$$\begin{array}{ccccc}
 & & AYA & \xrightarrow{\ q\ } & A \bullet YA \\
 & & \Big\downarrow{\scriptstyle BYA} & & \Big\downarrow{\scriptstyle B\bullet YA} \\
YAAXYA & \xrightarrow{\ YmXYA\ } & YAXYA & \xrightarrow{\ q\ } & YAX \bullet YA \\
\Big\downarrow{\scriptstyle =} & & & & \Big\downarrow{\scriptstyle \cong} \\
YAAXYA & \xrightarrow{\ YAq\ } & YA(AX \bullet YA) & \xrightarrow{\ Y\nu\ } & Y(AX \bullet YA) \\
\Big\downarrow{\scriptstyle YAAdA} & & \Big\downarrow{\scriptstyle YAD} & & \Big\downarrow{\scriptstyle YA\bullet D} \\
YAAA & \xrightarrow{\ YAm\ } & YAA & \xrightarrow{\ Ym\ } & YA
\end{array}
\qquad (\delta' : YAX \bullet YA \to YA)$$

Then we consider the diagram

$$\begin{array}{ccccc}
YAAXYA & \xrightarrow{\ YmXYA\ } & YAXYA & \xrightarrow{\ q\ } & YAX \bullet YA \\
\Big\downarrow{\scriptstyle YAAdA} & & \Big\downarrow{\scriptstyle YAdA} & & \Big\downarrow{\scriptstyle \delta} \\
YAAA & \xrightarrow{\ YmA\ } & YAA & \xrightarrow{\ Ym\ } & YA
\end{array}$$

We compute that

$$\delta \circ q \circ YmXYA = Ym \circ YAm \circ YAAdA = Ym \circ YmA \circ YAAdA = Ym \circ YAdA \circ YmXYA,$$

and conclude that $\delta \circ q = Ym \circ YAdA$, so that we have the commutative diagrams

$$\begin{array}{ccc}
AYA & \xrightarrow{\ q\ } & A \bullet YA \\
\Big\downarrow{\scriptstyle BYA} & & \Big\downarrow{\scriptstyle B\bullet YA} \\
YAXYA & \xrightarrow{\ q\ } & YAX \bullet YA \\
\Big\downarrow{\scriptstyle YAdA} & & \Big\downarrow{\scriptstyle \delta'} \\
YAA & \xrightarrow{\ Ym\ } & YA
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
AYA & \xrightarrow{\ q\ } & A \bullet YA \\
 & {\scriptstyle \nu_{YA}} \searrow & \Big\downarrow{\scriptstyle \cong} \\
 & & YA
\end{array}$$

Now

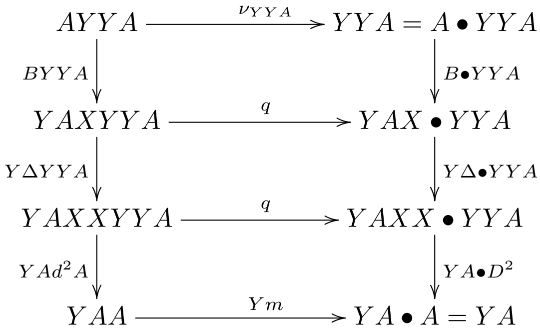
$$Ym \circ YAdA \circ BYA = Ym \circ YAdA \circ Y\psi YA \circ bAYA = \nu_{YA},$$

and we conclude that $\delta' \circ B \bullet YA$ is the canonical isomorphism $Y \bullet YA \cong YA$, finishing the proof. □

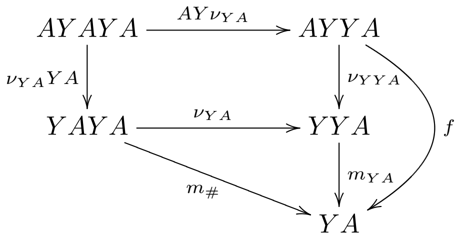
In the setting of Proposition 5.4, assume moreover that (X, ψ) is a coalgebra in ${}^!\mathcal{T}_A^{\#}$. It follows from Lemma 5.1 that AX is an A-coring; we know from Proposition 5.4 that $AX \dashv YA$ in ${}_A^!\mathcal{C}_A$, hence YA is an A-ring, with structure given by (1.3). The structure maps are denoted as m_{YA} : $YA \bullet YA = YYA \to YA$ and $\eta_A : \ A \to YA$.
We have seen in Proposition 3.3 that (A, Y, φ) is a right wreath, and we can consider the wreath product $YA = Y\#_\varphi A$, which is an algebra in $\mathcal{C}$. As in Proposition 3.3, the multiplication and the unit are denoted as $m_\# : YAYA \to YA$ and $\eta_\# : \ 1 \to YA$.
AX is an entwined module, and therefore a right YA-module in $\mathcal{C}$, with right YA-action $\overline{\mu}$: $AXYA \to AX$ given by (3.8). Since $AX \dashv YA$ in ${}_A^!\mathcal{C}_A$, AX is also a right YA-module in ${}_A^!\mathcal{C}_A$, with right YA-action $\tilde{\mu} : \ AX \bullet YA \to AX$ given by (1.4).
In Proposition 5.5 we investigate the relation between these two sets of structures.

Proposition 5.5. *With notation as above,* $m_\# = m_{YA} \circ Y\nu_{YA}$, $\eta_\# = \eta_{YA} \circ \eta$ *and* $\overline{\mu} = \tilde{\mu} \circ q$.

Proof. According to (1.3), $m_{YA} = YA \bullet D^2 \circ Y\Delta \bullet YYA \circ B \bullet YYA$, where $\Delta = mX \circ A\delta$: $AX \to AX \bullet AX = AXX$ is the comultiplication on AX. Consider the commutative diagram



m_{YA} is the unique morphism such that $m_{YA} \circ \nu_{YYA} = f$, where f is southwestern composition $f = Ym \circ YAd^2A \circ Y\Delta YYA \circ BYYA$ in the diagram. We have to show that the triangle in the diagram



commutes. Since $(YAYA, \nu_{YA}YA)$ is a coequalizer, and since the rectangle in the diagram commutes, it suffices to show that

$$m_{\#} \circ \nu_{YA}YA = f \circ AY\nu_{YA}. \tag{5.5}$$

The fact that $\delta :\ X \to AXX$ defines a morphism in $\mathcal{T}_A^{\#}$ is expressed by the formula

$$mXX \circ A\delta \circ \psi = mXX \circ A\psi X \circ AX\psi \circ \delta A = \mu_{AAX} \circ \delta A.$$

Using this formula and the definition of B, we can show that $f \circ AY\nu_{YA} = g_1 \circ g$, with

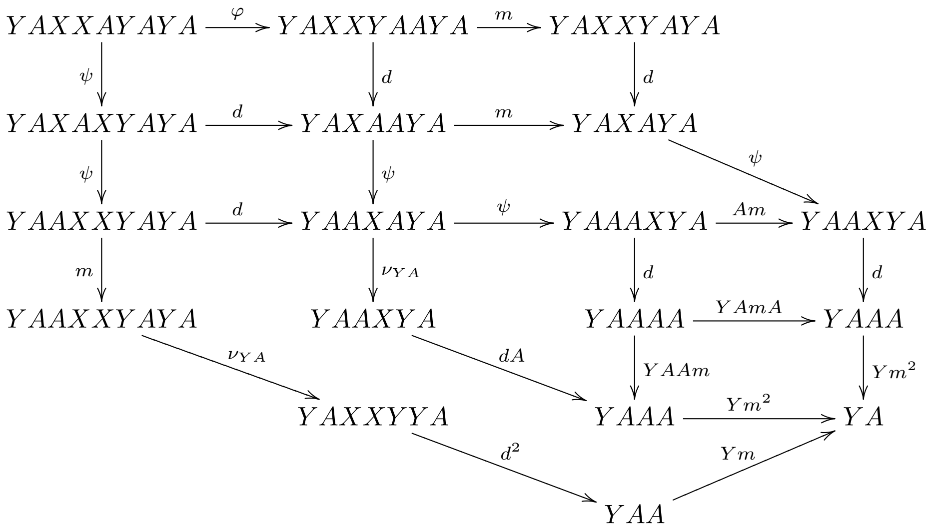
$$\begin{aligned} g_1 &= Ym \circ YAd^2A \circ Y\mu_{AXX}Y\nu_{YA} :\ YAXXAYAYA \to YA; \\ g &= Y\delta AYAYA \circ bAYAYA :\ AYAY \to YAXXAYAYA. \end{aligned}$$

Using (3.6), we compute that $m_{\#} \circ \nu_{YA}YA = g_2 \circ g$, with

$$g_2 = Ym^2 \circ YAAdA \circ Y\psi YA \circ YAXdAYA \circ YAXX\nu_{YA}YA :\ YAXXAYAYA \to YA.$$

Therefore it suffices to show that $g_1 = g_2$. To this end, we consider the following diagram. We slightly simplified the notation for the morphisms in the diagram, deleting identity morphisms on tensor factors; for example, ψ in the top left corner is a shorter notation for $YAX\psi YAYA$. It follows from (5.6) that the top left square in the diagram commutes. We easily deduce from (5.6) that $dA \circ X\nu_{YA} = m \circ AdA = \psi YA$, telling us that the lower pentagon in the diagram commutes. The associativity of m entails that the lower right square commutes. The commutativity of the three remaining squares, the remaining pentangle, and the octangle at the lower left is an obvious consequence of the property that $Cg \circ fB = fD \circ Ag$ for $f :\ A \to C$ and $g :\ B \to D$. Our conclusion is that the diagram commutes. The composition taking $YAXXAYAYA$ in the top left corner to YA via the southwestern route is g_1, and the composition via the northeastern route is

g_2. It follows that $g_1 = g_2$ and (5.5) is satisfied.



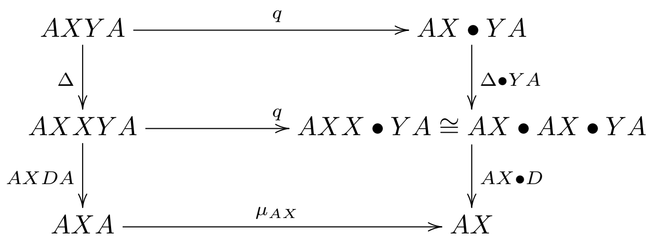
From the definition of φ, see (3.1), it easily follows that

$$Ad \circ \psi Y = dA \circ X\varphi. \tag{5.6}$$

Let us prove the second formula. As an application of (1.7), we find that $YA \bullet \varepsilon_{AX} = Y\varepsilon_{AX} : YA \bullet AX = YAX \to YA \bullet A = YA$.

$$\eta_{YA} \circ \eta \overset{(1.3)}{=} YA \bullet AX \circ B \circ \eta = Y\varepsilon_{AX} \circ Y\psi \circ bA \circ \eta = Ym \circ YA\epsilon \circ Y\psi \circ bA \circ \eta$$
$$\overset{(2.3(c))}{=} Ym \circ Y\epsilon A \circ bA \circ \eta = Ym \circ YA\eta \circ Y\epsilon \circ b \overset{(3.6)}{=} \eta_{\#}.$$

It follows from (1.4) that $\tilde{\mu} = AX \bullet D \ \circ \ \Delta \bullet YA$. This fits into the diagram



It follows that $\tilde{\mu} \circ q = \mu_{AX} \circ AXdA \circ D = \overline{\mu}$. □

From now on we will make the following assumptions: $\mathcal{C}$ is a (strict) monoidal category with coequalizers; every object of $\mathcal{C}$ is left coflat; A is an algebra in $\mathcal{C}$ such that every left A-module in $\mathcal{C}$ is robust. In this situation the categories ${}^{!}_{A}\mathcal{C}_A$ and ${}_A\mathcal{C}_A$ coincide.
An algebra morphism $A \to S$ is also called an algebra extension; algebra extensions correspond to A-rings, these are algebras in the category ${}_A\mathcal{C}_A$. $A \to S$ is called a Frobenius algebra extension if S is a Frobenius algebra in ${}_A\mathcal{C}_A$. Now we can state the main result of this Section.

Theorem 5.6. *Let (A, X, ψ) be a cowreath, and assume that $X \dashv Y$ in $\mathcal{C}$. Then the following assertions are equivalent:*

(i) *(X, ψ) is a Frobenius coalgebra in $\mathcal{T}_A^{\#}$;*
(ii) *YA is a Frobenius A-ring;*
(iii) *(Y, φ) is a Frobenius algebra in ${}^{\#}_{A}\mathcal{T}$;*
(iv) *the algebra extension $Ym \circ \eta_{YA}A : A \to YA$ is Frobenius;*

(v) *AX and YA are isomorphic as left A, right YA-modules in $\mathcal{C}$.*
(vi) *AX and YA are isomorphic as left A-modules and as entwined modules;*
(vii) *there exists $t : \underline{1} \to X$ in $\mathcal{T}_A^\#$ (that is a Frobenius element for (X, ψ) in $\mathcal{T}_A^\#$) such that*

$$\Phi = m^2X \circ AA\psi A \circ AAXdA \circ A\delta YA \circ tYA : \; YA \to AX \tag{5.7}$$

is an isomorphism in $\mathcal{C}$;
(viii) *there exists $B : X \otimes X \to \underline{1}$ in $\mathcal{T}_A^\#$ satisfying (4.2.c) (that is a Casimir morphism for (X, ψ) in $\mathcal{T}_A^\#$) such that*

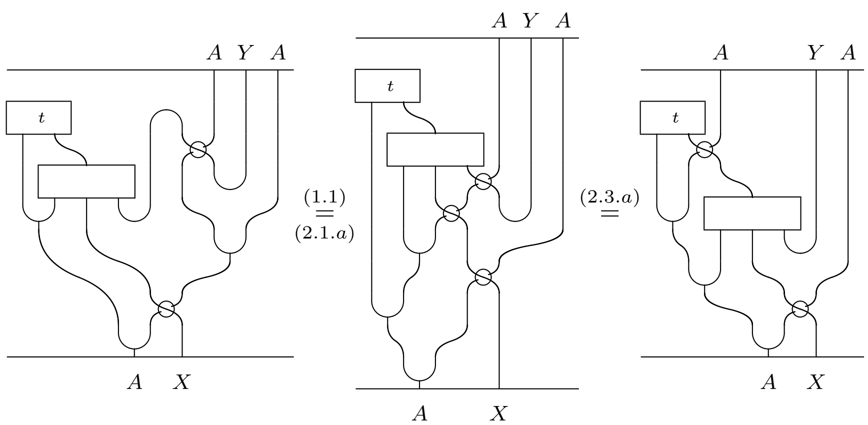
$$\Psi = Ym \circ YAB \circ Y\psi X \circ bAX : \; AX \to YA \tag{5.8}$$

is an isomorphism in $\mathcal{C}$.

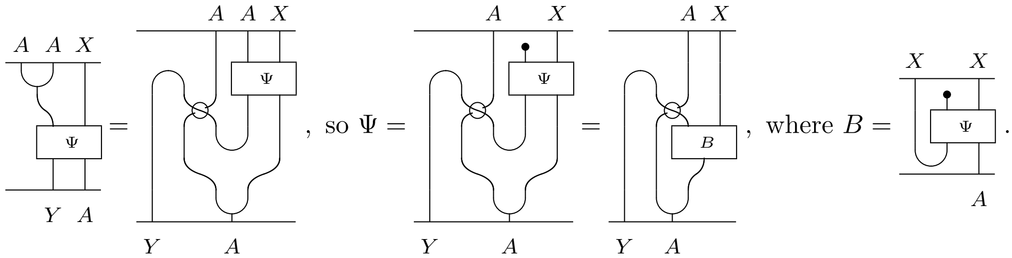
If $\underline{1}$ is a left $\otimes$-generator of $\mathcal{C}$, then these statements are equivalent to

(ix) *The functor $F : \; \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ is a Frobenius functor.*

Proof. *(i) ⇔ (ii)*. By Theorem 5.2, (X, ψ) is a Frobenius coalgebra in $\mathcal{T}_A^\#$ if and only if AX is a Frobenius coalgebra in ${}_A\mathcal{C}_A$. Since $AX \dashv YA$ in ${}_A\mathcal{C}_A$, this is equivalent to YA is a Frobenius algebra in ${}_A\mathcal{C}_A$, which is a Frobenius A-ring.
(ii) ⇔ (iii) ⇔ (iv) follows from [15, Corollary 8.8] applied to the wreath (A, Y, φ) in $\overline{\mathcal{C}}$. Observe that it is possible to give a direct proof of the equivalence of *(ii)* and *(iii)* using arguments similar to the ones in the proof of Theorem 5.2.
(i) ⇔ (v). We use the characterization (i) in Remark 4.3, applied to the colgebra AX in ${}_A\mathcal{C}_A$. We have that (X, ψ) in $\mathcal{T}_A^\#$ is a Frobenius coalgebra if and only if AX is a Frobenius A-coring. Since $AX \dashv YA$ in ${}_A\mathcal{C}_A$, this is equivalent to AX and YA being isomorphic as right YA-modules in ${}_A\mathcal{C}_A$. By Proposition 5.5 it then follows that (X, ψ) is a Frobenius coalgebra if and only if AX and YA are isomorphic as A-bimodules and right YA-modules in $\mathcal{C}$. But the right A-module structure on AX and YA is inherited from the right action of YA on them via the restriction of scalars functor defined by the algebra extension $Ym \circ \eta_{YA} A : \; A \to YA$. So we conclude that (X, ψ) is a Frobenius coalgebra in $\mathcal{T}_A^\#$ if and only if AX and YA are isomorphic as left A, right YA-modules.
(v) ⇔ (vi) is an immediate consequence of Theorem 3.4.
(v) ⇔ (vii) is based on the elementary observation that a right YA-linear morphism $\Phi : YA \to AX$ is completely determined by the morphism $t = \Phi \circ \eta_\# : \; \underline{1} \to AX$ in $\mathcal{C}$. Looking at the right A-action $\overline{\mu} : \; AXYA \to AX$ (see Lemma 3.5 and Proposition 5.5), we easily find that Φ is given by (5.7). Since



we obtain that Φ is left A-linear if and only if t is a morphism in $\mathcal{T}_A^{\#}$.
<u>$(v) \Leftrightarrow (viii)$.</u> If $\Psi:\ AX \to YA$ is left A-linear, then



This shows that any left A-linear morphism $\Psi:\ AX \to YA$ is of the form (5.8), for some $B:\ X\otimes X \to A$ in $\mathcal{C}$. A long but straightforward computation shows that Ψ is right YA-linear if and only if B is a morphism in $\mathcal{T}_A^{\#}$ satisfying (4.2.c).
<u>$(i) \Leftrightarrow (ix)$</u> follows from Theorem 4.8. □

6. Separability properties for entwined modules

The aim of this Section is to study the separability of the forgetful functor $F:\ \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ and its right adjoint G. Separable functors were introduced in [31]. Consider a pair of adjoint functors $F \dashv G$ between two categories $\mathcal{D}$ and $\mathcal{E}$, with unit $\eta: \mathrm{Id}_{\mathcal{D}} \to GF$ and counit $\varepsilon: FG \to \mathrm{Id}_{\mathcal{E}}$. The following result is due to Rafael [33].

- F is separable if and only if the unit η of the adjunction splits: there is a natural transformation $\vartheta:\ GF \to \mathrm{Id}_{\mathcal{D}}$ such that $\vartheta\circ\eta = \mathrm{Id}_{\mathcal{D}}$;
- G is separable if and only if the counit ε of the adjunction cosplits: there is a natural transformation $\theta:\ \mathrm{Id}_{\mathcal{E}} \to FG$ such that $\varepsilon\circ\theta = \mathrm{Id}_{\mathcal{E}}$.

We will apply Rafael's Theorem to $F:\ \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ and its right adjoint G. The natural transformations ϑ and θ will be obtained as an application of Propositions 4.6 and 4.7.

Proposition 6.1. *Assume that $\underline{1}$ is a left $\otimes$-generator of the (strict) monoidal category $\mathcal{C}$, and let (A, X, ψ) be a cowreath.*

1. *The forgetful functor $F:\ \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ is separable if and only if there exists a Casimir morphism $B:\ XX \to A$ for the coalgebra (X,ψ) in $\mathcal{T}_A^{\#}$ such that $m\circ AB\circ\delta = \epsilon$.*
2. *$G:\ \mathcal{C}_A \to \mathcal{C}(\psi)_A^X$ is separable if and only if there exists a morphism $t:\ \underline{1} \to X$ in $\mathcal{T}_A^{\#}$ such that $m\circ A\epsilon\circ t = \eta$.*

Proof. By Rafael's Theorem, F is separable if and only if there exists $\vartheta:\ GF \to \mathrm{Id}_{\mathcal{C}(\psi)_A^X}$ such that $\vartheta\circ\eta$ is the identity natural transformation. Let B be the Casimir morphism corresponding to ϑ, see Proposition 4.7. Then we can easily show that $\vartheta_M\circ\eta_M = \mu\circ MB\circ\rho X\circ\rho = \mu\circ Mh\circ\rho$, where $h = m\circ AB\circ\delta$.
If $h=\epsilon$, then it follows that $\vartheta_M\circ\eta_M = \mu\circ M\epsilon\circ\rho \overset{(2.5)}{=} M$.
Conversely, if $\theta\circ\eta$ is the identity natural transformation, then

$$AX = \theta_{AX}\circ\eta_{AX} = \mu_{AX}\circ AXh\circ\rho_{AX} = mX\circ A\psi\circ AXh\circ mXX\circ A\delta,$$

and

$$\begin{aligned}\epsilon &\overset{(4.12)}{=} m\circ A\epsilon\circ\eta X = m\circ A\epsilon\circ AX\circ\eta X = m^2\circ AA\epsilon\circ A\psi\circ AXh\circ mXX\circ A\delta\circ\eta X\\ &\overset{(2.3.c)}{=} m^2\circ A\epsilon A\circ AXh\circ mXX\circ A\delta\circ\eta X = m\circ Ah\circ mX\circ AmX\circ AA\epsilon X\circ A\delta\circ\eta X\\ &\overset{(2.3.d)}{=} m\circ Ah\circ mX\circ A\eta X\circ\eta X = m\circ Ah\circ\eta X = m\circ\eta A\circ h = h.\end{aligned}$$

The proof of the second statement is similar. G is separable if and only if there exists $\theta:\ \mathrm{Id}_{\mathcal{C}_A} \to FG$ such that $\varepsilon\circ\theta$ is the identity natural transformation, that is, $\varepsilon_N\circ\theta_N = N$, for all $N\in\mathcal{C}_A$. Fix

θ, and let $t \in \mathcal{T}_A^{\#}(\underline{1}, X)$ be the Frobenius element corresponding to θ, see Proposition 4.6. Then $\varepsilon_N \circ \theta_N = \mu \circ N\epsilon \circ \mu X \circ Nt$, fitting into the commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{Nt} & NAX & \xrightarrow{\mu X} & NX \\ & & \downarrow{\scriptstyle NA\epsilon} & & \downarrow{\scriptstyle N\epsilon} \\ & & NAA & \xrightarrow{\mu A} & NA \\ & & \downarrow{\scriptstyle Nm} & & \downarrow{\scriptstyle \mu} \\ & & NA & \xrightarrow{\mu} & N \end{array}$$

If $m \circ A\epsilon \circ t = \eta$, then $\varepsilon_N \circ \theta_N = \mu \circ N\eta = N$.
Conversely, if $\varepsilon \circ \theta = \mathrm{Id}_{\mathcal{C}_A}$, then $\varepsilon_A \circ \theta_A = A$, and we find from the commutativity of the diagram that $\eta = m \circ Am \circ AA\epsilon \circ At \circ \eta = m \circ mA \circ \eta AA \circ A\epsilon \circ t = m \circ A\epsilon \circ t$. □

Coseparable coalgebras were introduced by Larson in [28]. This notion can be generalized to coalgebras in (strict) monoidal categories. Remark that a coalgebra C is a C-bicomodule, with left and right C-coaction induced by comultiplication.

Definition 6.2. A coalgebra C is coseparable if it is a relative injective C-bicomodule in $\mathcal{C}$, which comes down to the following property. If $i: M \to N$ in ${}^C\mathcal{C}^C$ has a left inverse $p: M \to N$ in $\mathcal{C}$, then every $f: M \to C$ in ${}^C\mathcal{C}^C$ factors through i in ${}^C\mathcal{C}^C$: there exists a C-bicolinear morphism $g: N \to C$ such that $g \circ i = f$.

Proposition 6.3. *For a coalgebra C in a (strict) monoidal category $\mathcal{C}$, the following assertions are equivalent.*

(i) *C is coseparable;*
(ii) *the comultiplication Δ has a C-bicolinear left inverse $\gamma: CC \to C$;*
(iii) *there exists a morphism $B: CC \to \underline{1}$ in $\mathcal{C}$ such that*

$$B \circ \Delta = \varepsilon \text{ and } CB \circ \Delta C = BC \circ C\Delta. \tag{6.1}$$

Proof. (sketch) (i) ⇒ (ii). $C\varepsilon: CC \to C$ is a left inverse of Δ, so the identity $C \to C$ factors through Δ in ${}^C\mathcal{C}^C$, which means that Δ has a C-bicolinear left inverse.
(ii) ⇒ (iii). Let $B = \varepsilon \circ \gamma: CC \to \underline{1}$. $B \circ \Delta = \varepsilon$ follows immediately from $\gamma \circ \Delta = C$. The left C-colinearity of γ means that $C\gamma \circ \Delta C = \Delta \circ \gamma$, and this implies that $CB \circ \Delta C = C\varepsilon \circ C\gamma \circ \Delta C = C\varepsilon \circ \Delta \circ \gamma = \gamma$. The right C-colinearity entails that $BC \circ C\delta = \gamma$, and the second formula in (6.1) follows.
(iii) ⇒ (ii). $\gamma = CB \circ \Delta C = BC \circ C\Delta$ is a C-bicolinear left inverse of Δ.
(ii) ⇒ (i). Let $i,\ p,\ f$ be as in Definition 6.2. Then $g = \gamma \circ C(\varepsilon \circ f \circ p)C \circ CN\rho \circ \lambda$ is C-bicolinear and $g \circ i = f$. λ and ρ are the left and right C-coaction on N. □

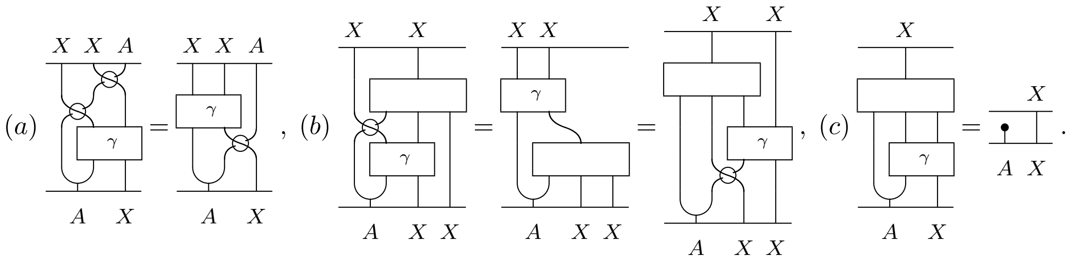
A morphism $B: CC \to \underline{1}$ satisfying the second condition in (6.1) is a Casimir morphism for C, see Definition 4.2. A Casimir morphism is called normalized if it also satisfies the first condition in (6.1). A coseparable coalgebra is coalgebra together with a normalized Casimir morphism.

Proposition 6.4. *For a cowreath (A, X, ψ) in $\mathcal{C}$, the following assertions are equivalent.*

(i) *(X, ψ) is a coseparable coalgebra in $\mathcal{T}_A^{\#}$;*

(ii) *there exists a morphism* $\gamma : XX \to AX$ *in* $\mathcal{C}$ *such that*

(6.2)



(iii) *there exists a Casimir morphism* B *for the coalgebra* (X, ψ) *in* $\mathcal{T}_A^{\#}$ *such that* $m \circ AB \circ \delta = \epsilon$.

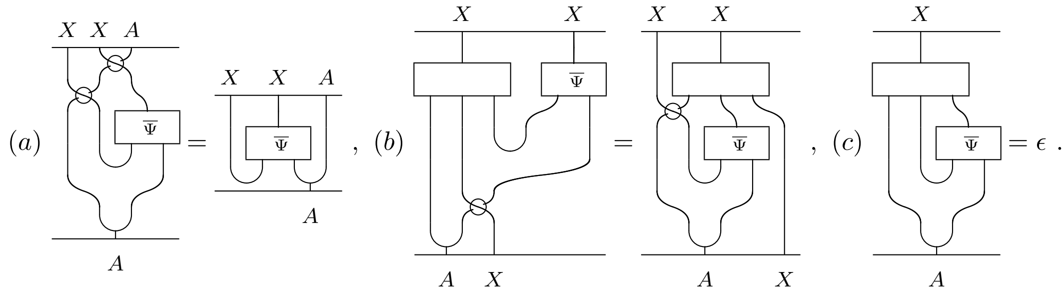
If $X \dashv Y$ *in* $\mathcal{C}$, *then these conditions are equivalent to*

(iv) *there exists a left* A*-linear* $\Psi : AX \to YA$ *in* $\mathcal{C}(\psi)_A^X$ *such that*

$$m \circ AdA \circ AX\Psi \circ AX\eta X \circ \delta = \epsilon; \tag{6.3}$$

(v) *there exists a morphism* $\overline{\Psi} : X \to YA$ *in* $\mathcal{C}$ *satisfying the equations:*

(6.4)



Proof. (i) ⇔ (ii) ⇔ (iii). (6.2.a) says that γ is a morphism in $\mathcal{T}_A^{\#}$, (6.2.b) that γ is an (X, ψ)-bicolinear morphism in $\mathcal{T}_A^{\#}$ and (6.2.c) that γ is a left inverse of the comultiplication δ of the coalgebra (X, ψ) in $\mathcal{T}_A^{\#}$, so condition (ii) is condition (ii) from Proposition 6.3 in the special case where $\mathcal{C} = \mathcal{T}_A^{\#}$. A similar observation holds for condition (iii), and the equivalence of (i), (ii) and (iii) follows.

(iii) ⇔ (iv). We have seen in the proof of the equivalence $(v) \Leftrightarrow (viii)$ in Theorem 5.6 that there is a bijective correspondence between left A-linear morphisms $\Psi : AX \to YA$ in $\mathcal{C}(\psi)_A^X$ and Casimir morphisms B for the coalgebra (X, ψ) in $\mathcal{T}_A^{\#}$. Moreover, it is easy to show that Ψ satisfies (6.3) if and only if the corresponding B has the property that $m \circ AB \circ \delta = \epsilon$.

(iv) ⇔ (v). It follows from Lemma 1.2 that we have an isomorphism $\alpha : {}_A\mathcal{C}(AX, AY) \to \mathcal{C}(X, AY)$, given by $\alpha(\Psi) = \Psi \circ \eta A$ and $\alpha^{-1}(\overline{\Psi}) = \nu_{YA} \circ A\overline{\Psi} = Ym \circ YAdA \circ Y\psi AYA \circ bAYA \circ A\overline{\Psi}$. It is left to the reader to check that Ψ is a left A-linear morphism in $\mathcal{C}(\psi)_A^X$ if and only if $\alpha(\Psi) = \overline{\Psi}$ satisfies (6.4.a, b). Finally, (6.3) is equivalent to (6.4.c). □

Our next result is a generalization of [17, Theorem 2.3].

Theorem 6.5. *Assume that* $\underline{1}$ *is a left* $\otimes$*-generator for* $\mathcal{C}$. *For a cowreath* (A, X, ψ) *in* $\mathcal{C}$, *the following statements are equivalent:*

(i) *The forgetful functor* $F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ *is separable;*

(ii) (X, ψ) *is a coseparable coalgebra in* $\mathcal{T}_A^{\#}$.

If $\mathcal{C}$ *has coequalizers and* A *and* X *are left coflat in* $\mathcal{C}$, *these statements are also equivalent to*

(iii) AX *is a coseparable* A*-coring in* $\mathcal{C}$, *that is a coseparable coalgebra in the monoidal category* ${}_A^!\mathcal{C}_A$;

(iv) *the forgetful functor* $U : \mathcal{C}^{AX} \to \mathcal{C}_A$ *is separable.*

Proof. $\underline{(i) \Leftrightarrow (ii)}$. Follows from Proposition 6.1 and the equivalence $(i) \Leftrightarrow (iii)$ in Proposition 6.4.
$\underline{(ii) \Leftrightarrow (iii)}$. We proceed as in the proof of Theorem 5.2. As before we identify $AX \bullet AX = AXX$. Applying Lemma 1.2, we obtain an isomorphism $\alpha : {}_A\mathcal{C}(AXX, AX) \to \mathcal{C}(XX, AX)$. A direct verification shows that $\Omega \in {}_A\mathcal{C}(AXX, AX)$ is right A-linear if and only if $\alpha(\Omega) = \gamma$ satisfies (6.2.a). In this situation, Ω is left and right AX-colinear if and only if γ satisfies (6.2.b). Finally $\Delta \circ \Omega = AX$ if and only if (6.2.c) holds.
$\underline{(i) \Leftrightarrow (iv)}$. The categories $\mathcal{C}^{AX}$ and $\mathcal{C}(\psi)_A^X$ are isomorphic, see [10, Theorem 4.8], and this implies immediately that the separability of F and U is equivalent. □

More equivalent conditions for the coseparability of a coalgebra (X, ψ) in $\mathcal{T}_A^{\#}$ can be given under the assumption that $X \dashv Y$. For the definition of a separable algebra extension in a monoidal category, we refer to [15, Def. 4.5 (ii)].

Proposition 6.6. *Let* $\mathcal{C}$ *be a monoidal category with coequalizers, and assume that every object of* $\mathcal{C}$ *is flat. Let* (A, X, ψ) *be a cowreath in* $\mathcal{C}$. *If* $X \dashv Y$ *in* $\mathcal{C}$ *and every left* A*-module is robust, then the following assertions are equivalent:*

(i) (X, ψ) *is a coseparable coalgebra in* $\mathcal{T}_A^{\#}$;
(ii) (Y, φ) *is a separable algebra in* ${}_A^{\#}\mathcal{T}$, *where* $\overline{\psi}$ *is defined in (3.1);*
(iii) *The smash product* YA *is a separable algebra extension of* A *in* $\mathcal{C}$;
(iv) YA *is a separable* A*-ring, that is a separable algebra in* ${}_A\mathcal{C}_A$.

If $\underline{1}$ *is a left* $\otimes$*-generator in* $\mathcal{C}$ *then (i)-(iv) are also equivalent to*

(v) *The restriction of scalars functor* $F' : \mathcal{C}_{YA} \to \mathcal{C}_A$ *is separable.*

Proof. This is an immediate consequence of Theorems 3.4 and 6.5, and [15, Cor. 8.9] applied to the wreath (Y, A, φ). □

Our final result is a Maschke type Theorem for entwined modules. It generalizes [17, Theorem 2.7] and [4, Theorem 4.2].

Theorem 6.7. *Let* (X, ψ) *be a coseparable coalgebra in* $\mathcal{T}_A^{\#}$.

(i) *If a morphism in* $\mathcal{C}(\psi)_A^X$ *has a section (resp. a retraction) in* $\mathcal{C}_A$ *then it has a section (resp. a retraction) in* $\mathcal{C}(\psi)_A^X$;
(ii) *If an object in* $\mathcal{C}(\psi)_A^X$ *is semisimple (resp. projective, injective) as a right* A*-module then it is semisimple (resp. projective, injective) as an entwined module over* (A, X, ψ).
(iii) *Every* $M \in \mathcal{C}(\psi)_A^X$ *is relative injective (see Definition 6.2 for the definition of relative injectivity).*

Proof. The forgetful functor $F : \mathcal{C}(\psi)_A^X \to \mathcal{C}_A$ is separable since (X, ψ) is a coseparable coalgebra in $\mathcal{T}_A^{\#}$. The three assertions then follow immediately from [18, Prop. 47 and 48, Cor.7]. □

7. Application to Doi-Hopf modules over quasi-bialgebras

In Sections 7, 8 and 9, we apply our results to cowreaths that appear in the context of quasi-Hopf algebras. This provides examples of cowreaths (or generalized entwining structures) that are not distributive laws (or entwining structures) in the classical sense.
We work over a field k. We begin with an auxiliary result. It can be restated as follows: if (X, ψ) is a Frobenius coalgebra in $\mathcal{T}(\mathcal{M}_k)_A^{\#}$, then X has a dual in $\mathcal{M}_k$, that is, X is finite dimensional. We do not know whether this result holds in other monoidal categories $\mathcal{C}$.

Proposition 7.1. *Let* k *be a field. If* $(X, \psi) \in \mathcal{T}(\mathcal{M}_k)_A^{\#}$ *has a left or right dual, then* X *is finite dimensional. Consequently, if* (X, ψ) *is a Frobenius coalgebra in* $\mathcal{T}(\mathcal{M}_k)_A^{\#}$, *then* X *is finite dimensional.*

Proof. Assume that we have an adjunction $(X,\psi) \dashv (Y,\varphi)$ in $\mathcal{T}(\mathcal{M}_k)^{\#}_A$, with unit $b:\ k \to AYX$ and counit $d:\ XY \to A$. Applying the definition of the monoidal structure of $\mathcal{T}(\mathcal{M}_k)^{\#}_A$ (see Section 2), we find that the second formula in (1.1) takes the form

$$mX \circ AdX \circ \psi YX \circ Xb = \eta X.$$

Let $b(1) = a_i \otimes y_i \otimes x_i \in A \otimes Y \otimes X$, and use the following notation for ψ:

$$\psi:\ X \otimes A \to A \otimes X,\ \psi(x \otimes a) = a_\psi \otimes x^\psi, \tag{7.1}$$

where in the both cases the summation is implicitly understood. Then (7.1) can be rewritten as

$$a_{i\psi} d(x^\psi \otimes y_i) \otimes x_i = f_i(x) \otimes x_i = 1_A \otimes x,$$

for all $x \in X$, where $f_i:\ X \to A$, $f_i(x) = a_{i\psi} d(x^\psi \otimes y_i)$. Take a complement V of the subspace $k1_A \subset A$, so that we have $A = V \oplus k1_A$ as vector spaces. Let $p:\ A \to V$ and $g:\ A \to k1_A$ be the projections of A onto V and $k1_A$. Then $a = p(a) + \langle g, a\rangle 1_A$, for all $a \in A$, and we find that

$$(p \circ f_i)(x) \otimes x_i + 1_A \otimes \langle g \circ f_i, x\rangle x_i = 1_A \otimes x$$

in $A \otimes X = V \otimes X \oplus k1_A \otimes X$. Taking the projection of both sides onto $k1_A \otimes x$, we find that

$$1_A \otimes \langle g \circ f_i, x\rangle x_i = 1_A \otimes x,$$

and $x = \langle g \circ f_i, x\rangle x_i$, proving that X is generated by x_i's.
The proof in the case where (X,ψ) has a left dual is similar; the second statement follows from the fact that (X,ψ) is selfdual in $\mathcal{T}^{\#}_A(\mathcal{M}_k)$, see Remark 4.3. □

7.1. **Doi-Hopf modules.** Doi-Hopf modules over a quasi-bialgebra have been introduced in [7]. Several types of modules, for example two-sided Hopf modules and Yetter-Drinfeld modules, are Doi-Hopf modules, see [7, 11]. The category of Doi-Hopf modules appears as the category of entwined modules over an appropriate generalized entwining structure in the category of vector spaces $\mathcal{M}_k$. Since the groundfield k is an $\otimes$-generator of $\mathcal{M}_k$, we can apply Theorems 4.8 and 6.5, to find necessary and sufficient conditions for the separability and Frobenius property of the forgetful functor from Doi-Hopf modules to modules over the underlying algebra. These conditions can be rephrased in the special situation where the underlying algebra A is the quasi-bialgebra H.
Recall from [22] that a right H-comodule algebra is a unital associative algebra A together with an algebra morphism $\rho:\ A \to A \otimes H$, $\rho(a) = a_{\langle 0\rangle} \otimes a_{\langle 1\rangle}$, and an invertible element $\Phi_\rho \in A \otimes H \otimes H$ such that:

$$\Phi_\rho(\rho \otimes H)(\rho(a)) = (A \otimes \Delta)(\rho(a))\Phi_\rho,\ \text{for all } a \in A, \tag{7.2}$$

$$(1_A \otimes \Phi)(A \otimes \Delta \otimes H)(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (A \otimes H \otimes \Delta)(\Phi_\rho)(\rho \otimes H \otimes H)(\Phi_\rho), \tag{7.3}$$

$$(A \otimes \varepsilon) \circ \rho = A, \tag{7.4}$$

$$(A \otimes \varepsilon \otimes H)(\Phi_\rho) = (A \otimes H \otimes \varepsilon)(\Phi_\rho) = 1_A \otimes 1_H. \tag{7.5}$$

In a similar way we can define left comodule algebras $(B, \lambda, \Phi_\lambda)$ over H. In this situation we will denote $\lambda(b) = b_{[-1]} \otimes b_{[0]} \in H \otimes B$, for all $b \in B$, and

$$\Phi_\lambda = \tilde{X}^1_\lambda \otimes \tilde{X}^2_\lambda \otimes \tilde{X}^3_\lambda = \cdots\ ;\ \Phi_\lambda^{-1} = \tilde{x}^1_\lambda \otimes \tilde{x}^2_\lambda \otimes \tilde{x}^3_\lambda = \cdots .$$

A coalgebra in the monoidal category $\mathcal{M}_H$ is called a right H-module coalgebra. In general, it is not coassociative as a coalgebra in $\mathcal{M}_k$.
Let A be a right H-comodule algebra and let C be a right H-module coalgebra. The comultiplication $\Delta_C:\ C \to C \otimes C$ is denoted by $\Delta_C(c) = c_{\underline{1}} \otimes c_{\underline{2}}$. Let $\psi:\ C \otimes A \to A \otimes C$, $\psi(c \otimes a) = a_{\langle 0\rangle} \otimes c \cdot a_{\langle 1\rangle}$. Then $(C,\psi) \in \mathcal{T}^{\#}_A$, and (C,ψ) is a coalgebra in $\mathcal{T}^{\#}_A$, with comultiplication and counit given by the formulas

$$\delta:\ C \to A \otimes C \otimes C,\ \delta(c) = \tilde{X}^1_\rho \otimes c_{\underline{1}} \cdot \tilde{X}^2_\rho \otimes c_{\underline{2}} \cdot \tilde{X}^3_\rho\ ;\ \epsilon: C \to A,\ \epsilon(c) = \varepsilon_C(c)1_A.$$

The category $\mathcal{M}_k(\psi)_A^C$ is isomorphic to the category of Doi-Hopf modules $\mathcal{M}(H)_A^C$, as introduced in in [7, 11]. A Doi-Hopf module is a right A-module with a k-linear map $\rho: M \to M \otimes C$, $\rho(m) = m_{(0)} \otimes m_{(1)}$ such that

$$\begin{aligned} (\rho \otimes M)(\rho(m)) &= (M \otimes \Delta_C)(\rho(m)) \cdot \Phi_\rho, \\ \rho(m \cdot a) &= m_{(0)} \cdot a_{\langle 0\rangle} \otimes m_{(1)} \cdot a_{\langle 1\rangle}. \end{aligned}$$

A morphism in $\mathcal{M}(H)_A^C$ is a k-linear map that is right A-linear and right C-colinear.

7.2. **Frobenius properties for Doi-Hopf modules.** Let H be a quasi-bialgebra, let A be a right H-comodule algebra and let C be a right H-module coalgebra. The forgetful functor $F: \mathcal{M}(H)_A^C \to \mathcal{M}_A$ is Frobenius if and only if one of the seven equivalent conditions in Theorem 5.6 is satisfied. Condition (vii) states that there exists $t = a_i \otimes c_i \in A \otimes C$ (summation understood) such that

$$aa_i \otimes c_i = a_i a_{\langle 0\rangle} \otimes c_i \cdot a_{\langle 1\rangle}, \tag{7.6}$$

for all $a \in A$, and the map

$${}^*C \otimes A \to A \otimes C,\ {}^*c \otimes a \mapsto \langle {}^*c, (c_i)_{\underline{2}} \cdot \tilde{X}_\rho^3\rangle\ a_i \tilde{X}_\rho^1 a_{\langle 0\rangle} \otimes (c_i)_{\underline{1}} \cdot \tilde{X}_\rho^2 a_{\langle 1\rangle} \tag{7.7}$$

is an isomorphism, where ${}^*C = \mathrm{Hom}(C, k)$ is the right dual of C in $\mathcal{M}_k$.
Let H be a quasi-Hopf algebra. A simple computation tells us that

$${}^*C \otimes A \to A \otimes {}^*C,\ {}^*c \otimes a \mapsto \tilde{X}_\rho^1 a_{\langle 0\rangle} \otimes S(\tilde{X}_\rho^2 a_{\langle 1\rangle})\alpha \tilde{X}_\rho^3 \cdot {}^*c \tag{7.8}$$

is an isomorphism with inverse

$$A \otimes {}^*C \to {}^*C \otimes A,\ a \otimes {}^*c \mapsto a_{\langle 1\rangle}\tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \cdot {}^*c \otimes a_{\langle 0\rangle}\tilde{x}_\rho^1, \tag{7.9}$$

where $\langle h \cdot {}^*c, c\rangle = \langle {}^*c, c \cdot h\rangle$, for $h \in H$, $c \in C$ and ${}^*c \in {}^*C$. We therefore obtain the following result.

Corollary 7.2. *Let H be a quasi-Hopf algebra, A a right H-comodule algebra and C a right H-module coalgebra. Then the forgetful functor $F: \mathcal{M}(H)_A^C \to \mathcal{M}_A$ is Frobenius if and only if there exists $t = a_i \otimes c_i \in A \otimes C$ satisfying (7.6) and such that*

$$\kappa:\ A \otimes {}^*C \to A \otimes C,\ a \otimes {}^*c \mapsto \langle {}^*c, (c_i)_{\underline{2}} \cdot (\tilde{x}_\rho^2)_2 p^2 S(\tilde{x}_\rho^3)\rangle\ aa_i\tilde{x}_\rho^1 \otimes (c_i)_{\underline{1}} \cdot (\tilde{x}_\rho^2)_1 p^1 \tag{7.10}$$

is an isomorphism. Here $p^1 \otimes p^2 = x^1 \otimes x^2\beta S(x^3) \in H \otimes H$.

Proof. This follows from the fact that the morphism (7.10) is the composition of the morphism (7.7) and the isomorphism (7.9). We leave the verification of this detail to the reader. □

Our next result is that Corollary 7.2 can be applied in the case where C is a Frobenius coalgebra in $\mathcal{M}_H$. First we need the following characterisation of Frobenius coalgebras in $\mathcal{M}_H$.

Proposition 7.3. *Let H be a quasi-Hopf algebra. A coalgebra C in $\mathcal{M}_H$ is Frobenius if and only if C is finite dimensional and there exists $t \in C$ obeying*

$$t \cdot h = \varepsilon(h)t, \tag{7.11}$$

for all $h \in H$, and such that

$$\chi: {}^*C \to C,\ \chi({}^*c) = \langle {}^*c, t_{\underline{2}} \cdot p^2\rangle t_{\underline{1}} \cdot p^1 \tag{7.12}$$

is a k-linear isomorphism. Here $p^1 \otimes p^2 = x^1 \otimes x^2\beta S(x^3) \in H \otimes H$.

Proof. Step 1. A Frobenius coalgebra in $\mathcal{M}_H$ is finite dimensional.
If C is a Frobenius coalgebra in $\mathcal{M}_H$, then there exists $t \in C$ and $B: C \otimes C \to k$ such that $t \cdot h = \varepsilon(h)t$, $B(c \cdot h_1 \otimes c' \cdot h_2) = \varepsilon(h)B(c \otimes c')$ and

$$B(c_{\underline{2}} \cdot x^2 \otimes c' \cdot x^3)c_{\underline{1}} \cdot x^1 = B(c \cdot X^1 \otimes c'_{\underline{1}} \cdot X^2)c'_{\underline{2}} \cdot X^3,\ B(t \otimes c) = \varepsilon_C(c) = B(c \otimes t),$$

for all $h \in H$ and $c, c' \in C$, see Definition 4.2. In this situation, C is finite dimensional since

$$c = \varepsilon_C(c_{\underline{2}})c_{\underline{1}} = B(c_{\underline{2}} \otimes t)c_{\underline{1}} = B(c_{\underline{2}} \cdot x^2 \otimes t \cdot x^3)c_{\underline{1}} \cdot x^1 = B(c \cdot X^1 \otimes t_{\underline{1}} \cdot X^2)t_{\underline{2}} \cdot X^3,$$

for all $c \in C$.
Step 2. A finite dimensional coalgebra C in $\mathcal{M}_H$ has a right dual in $\mathcal{M}_H$. ${}^*C = \mathrm{Hom}(C,k)$, with right H-action $\langle {}^*c \cdot h, c\rangle = \langle {}^*c, c \cdot S(h)\rangle$, is a right dual of C in $\mathcal{M}_H$. The evaluation map d and the coevaluation map b are given by the formulas

$$d(c \otimes {}^*c) = \langle {}^*c, c \cdot \beta\rangle \text{ and } b(1) = c^j \otimes c_j \cdot \alpha,$$

where $c^j \otimes c_j \in {}^*C \otimes C$ is the finite dual basis of C as a k-vector space.
Step 3 If C is a finite dimensional coalgebra in $\mathcal{M}_H$, then ${}^*C$ is an algebra in $\mathcal{M}_H$ and C is a right ${}^*C$-module, see (1.3-1.4). The unit of ${}^*C$ is ε_C, and the multiplication is given by

$$\langle {}^*c \diamond {}^*d, c\rangle = \langle {}^*c, c_{\underline{2}} \cdot g^2\rangle \langle {}^*d, c_{\underline{1}} \cdot g^1\rangle,$$

where $g^1 \otimes g^2 = f^{-1} \in H \otimes H$ is the inverse of the Drinfeld twist f. The right ${}^*C$-action on C is given by the formula

$$c \leftharpoonup {}^*c = \langle {}^*c, c_{\underline{2}} \cdot p^2\rangle \; c_{\underline{1}} \cdot p^1,$$

where $p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3) \in H \otimes H$.
Applying Remark 4.3, we obtain that C is Frobenius if and only if C and ${}^*C$ are isomorphic as right ${}^*C$-modules in $\mathcal{M}_H$, which means that there exists a right H-linear isomorphism $\chi : {}^*C \to C$ satisfying

$$\chi({}^*c \diamond {}^*d) = \chi({}^*c) \leftharpoonup {}^*d,$$

for all ${}^*c, {}^*d \in {}^*C$. $t = \chi(\varepsilon_C)$ satisfies (7.11-7.12). Conversely, if t satisfies (7.11-7.12), then χ as defined in (7.12) is a right ${}^*C$-linear isomorphism in $\mathcal{M}_H$. Further detail is left to the reader. □

Proposition 7.4. *Let H be a quasi-Hopf algebra, A a right H-comodule algebra and C a right H-module coalgebra. If C is a Frobenius coalgebra in $\mathcal{M}_H$ then the forgetful functor $F : \mathcal{M}(H)_A^C \to \mathcal{M}_A$ is Frobenius.*

Proof. Proposition 7.3 produces t satisfying (7.11-7.12). It is easy to see that $1_A \otimes t$ satisfies (7.6). It follows from (7.11) and the fact that $\varepsilon(\tilde{x}^2_\rho)\tilde{x}^1_\rho \otimes \tilde{x}^3_\rho = 1_A \otimes 1_H$ that the morphism κ defined in (7.10) takes the form

$$\kappa(a \otimes {}^*c) a \otimes \langle {}^*c, t_{\underline{2}} \cdot p^2\rangle \; t_{\underline{1}} \cdot p^1 = a \otimes \chi({}^*c).$$

κ is an isomorphism since χ is an isomorphism, and we conclude from Corollary 7.2 that F is a Frobenius functor. □

H is a right H-comodule algebra, so we can consider the category of relative Hopf modules $\mathcal{M}(H)_H^C$, also denoted as $\mathcal{M}_H^C$, see [13]. Theorem 7.5 generalizes [20, Theorem 3.5].

Theorem 7.5. *Let H be a quasi-Hopf algebra, let A be a right H-comodule algebra and C a right H-module coalgebra, and assume that the forgetful functor $F : \mathcal{M}(H)_A^C \to \mathcal{M}_A$ is Frobenius. Consider an algebra map $\zeta : A \to k$, and let $\mathfrak{t} = \zeta(a_i)c_i$, where $t = a_i \otimes c_i \in A \otimes C$ satisfies (7.6). Then*

$$\gamma : {}^*C \to C,\; \gamma({}^*c) = \langle \zeta, \tilde{x}^1_\rho\rangle \; \langle {}^*c, \mathfrak{t}_{\underline{2}} \cdot (\tilde{x}^2_\rho)_2 p^2 S(\tilde{x}^3_\rho)\rangle \; \mathfrak{t}_{\underline{1}} \cdot (\tilde{x}^2_\rho)_1 p^1 \tag{7.13}$$

is an isomorphism. If $\tilde{\zeta} = (\zeta \otimes H) \circ \rho : A \to H$ is surjective, then C is a Frobenius coalgebra in $\mathcal{M}_H$. Consequently, the forgetful functor $F : \mathcal{M}(H)_H^C \to \mathcal{M}_H$ is Frobenius if and only if C is a Frobenius coalgebra in $\mathcal{M}_H$.

Proof. Assume that F is Frobenius. It follows from Corollary 7.2 that $\kappa : A \otimes {}^*C \to A \otimes C$, given by (7.10), is an isomorphism. κ is left A-linear, so it follows that $M \otimes {}^*C \cong M \otimes_A (A \otimes {}^*C) \cong M \otimes_A (A \otimes C) \cong M \otimes C$ are isomorphic as vector spaces, for all $M \in \mathcal{M}_A$. Since $k \in \mathcal{M}_A$ by restriction of scalars via ζ, it follows that we have an isomorphism between ${}^*C$ and C. This isomorphism is the map γ given in (7.13).
Assume that $\tilde{\zeta} : A \to H$ is surjective. For all $a \in A$, we have that $\zeta(a_{\langle 0\rangle})\mathfrak{t} \cdot a_{\langle 1\rangle} = \zeta(a)\mathfrak{t}$, and it follows that $\mathfrak{t} \cdot h = \varepsilon(h)t$, for all $h \in H$. Then we obtain that $\gamma({}^*c) = \langle {}^*c, \mathfrak{t}_{\underline{2}} \cdot p^2\rangle \mathfrak{t}_{\underline{1}} \cdot p^1$, hence γ

coincides with the map κ from Proposition 7.3, and it follows that C is a Frobenius coalgebra in $\mathcal{M}_H$.
Take $A = H$, and $\zeta = \varepsilon$. Then $\tilde{\zeta} = H$ is surjective, so C is a Frobenius coalgebra in $\mathcal{M}_H$. Conversely, if C is a Frobenius coalgebra in $\mathcal{M}_H$, then F is Frobenius by Proposition 7.4. □

7.3. **Separability.** We will now study the separability of the functor F. Let C be coalgebra (in the category of vector spaces, or in the category of H-(bi)modules over a quasi-bialgebra H). The convolution product $\langle {}^*c^*d, c\rangle = ({}^*c \otimes {}^*d)\Delta_C(c)$ defines a (possibly non-associative) multiplication on ${}^*C$.

Corollary 7.6. *Let H be a quasi-bialgebra, A a right H-comodule algebra and C a right H-module coalgebra. The forgetful functor $F : \mathcal{M}(H)_A^C \to \mathcal{M}_A$ is separable if and only if C is a coseparable coalgebra in $\mathcal{T}_A^{\#}$. The coseparability of C is equivalent to conditions (i), (ii) and (iii). Under the assumption that C is finite dimensional, these conditions are also equivalent to (iv), (v), (vi) and (vii).*

(i) There exists a k-linear map $\xi : C \otimes C \to A \otimes C$, $\xi(c \otimes c') = \xi^1(c, c') \otimes \xi^2(c, c')$, such that

$$\begin{aligned}
&a_{\langle 0,0\rangle}\xi^1(c\cdot a_{\langle 0,1\rangle}, c'\cdot a_{\langle 1\rangle}) \otimes \xi^2(c\cdot a_{\langle 0,1\rangle}, c'\cdot a_{\langle 1\rangle}) = \xi^1(c,c')a_{\langle 0\rangle} \otimes \xi^2(c,c')\cdot a_{\langle 1\rangle};\\
&(\tilde{X}^1_\rho)_{\langle 0\rangle}\xi^1(c\cdot(\tilde{X}^1_\rho)_{\langle 1\rangle}, c'_{\underline{1}}\cdot\tilde{X}^2_\rho) \otimes \xi^2(c\cdot(\tilde{X}^1_\rho)_{\langle 1\rangle}, c'_{\underline{1}}\cdot\tilde{X}^2_\rho) \otimes c'_{\underline{2}}\cdot\tilde{X}^3_\rho\\
&\qquad = \xi^1(c,c')\tilde{X}^1_\rho \otimes \xi^2(c,c')_{\underline{1}}\cdot\tilde{X}^2_\rho \otimes \xi^2(c,c')_{\underline{2}}\cdot\tilde{X}^3_\rho\\
&\qquad = \tilde{X}^1_\rho\xi^1(c_{\underline{2}}\cdot\tilde{X}^3_\rho, c')_{\langle 0\rangle} \otimes c_{\underline{1}}\cdot\tilde{X}^2_\rho\xi^1(c_{\underline{2}}\cdot\tilde{X}^3_\rho, c')_{\langle 1\rangle} \otimes \xi^2(c_{\underline{2}}\cdot\tilde{X}^3_\rho, c');\\
&\tilde{X}^1_\rho\xi^1(c_{\underline{1}}\cdot\tilde{X}^2_\rho, c_{\underline{2}}\cdot\tilde{X}^3_\rho) \otimes \xi^2(c_{\underline{1}}\cdot\tilde{X}^2_\rho, c_{\underline{2}}\cdot\tilde{X}^3_\rho) = 1_A \otimes c,
\end{aligned}$$

for all $a \in A$ and $c, c' \in C$.

(ii) There exists a k-linear map $\mathbf{B} : C \otimes C \to A$ such that

$$a_{\langle 0,0\rangle}\mathbf{B}(c\cdot a_{\langle 0,1\rangle} \otimes c'\cdot a_{\langle 1\rangle}) = \mathbf{B}(c\otimes c')a, \tag{7.14}$$

$$\begin{aligned}
&\tilde{X}^1_\rho\mathbf{B}(c_{\underline{2}}\cdot\tilde{X}^3_\rho \otimes c')_{\langle 0\rangle} \otimes c_{\underline{1}}\cdot\tilde{X}^2_\rho\mathbf{B}(c_{\underline{2}}\cdot\tilde{X}^3_\rho \otimes c')_{\langle 1\rangle}\\
&\qquad = (\tilde{X}^1_\rho)_{\langle 0\rangle}\mathbf{B}(c\cdot(\tilde{X}^1_\rho)_{\langle 1\rangle} \otimes c'_{\underline{1}}\cdot\tilde{X}^2_\rho) \otimes c'_{\underline{2}}\cdot\tilde{X}^3_\rho,
\end{aligned} \tag{7.15}$$

$$\tilde{X}^1_\rho\mathbf{B}(c_{\underline{1}}\cdot\tilde{X}^2_\rho \otimes c_{\underline{2}}\cdot\tilde{X}^3_\rho) = \varepsilon_C(c)1_A, \tag{7.16}$$

for all $a \in A$ and $c, c' \in C$.

(iii) There exists a k-linear map $T : C \to \mathrm{Hom}_k(C, A)$ such that

$$\begin{aligned}
&T(c)(c')a = a_{\langle 0,0\rangle}T(c\cdot a_{\langle 1\rangle})(c'\cdot a_{\langle 0,1\rangle}),\\
&\tilde{X}^1_\rho T(c')(c_{\underline{2}}\cdot\tilde{X}^3_\rho)_{\langle 0\rangle} \otimes c_{\underline{1}}\cdot\tilde{X}^2_\rho T(c')(c_{\underline{2}}\cdot\tilde{X}^3_\rho)_{\langle 1\rangle}\\
&\qquad = (\tilde{X}^1_\rho)_{\langle 0\rangle}T(c'_{\underline{1}}\cdot\tilde{X}^2_\rho)(c\cdot(\tilde{X}^1_\rho)_{\langle 1\rangle}) \otimes c'_{\underline{2}}\cdot\tilde{X}^3_\rho,\\
&\tilde{X}^1_\rho T(c_{\underline{2}}\cdot\tilde{X}^3_\rho)(c_{\underline{1}}\cdot\tilde{X}^2_\rho) = \varepsilon_C(c)1_A,
\end{aligned}$$

for all $a \in A$ and $c, c' \in C$.

(iv) There exists a left and right A-linear, C-colinear map $\Psi : A \otimes C \to {}^*C \otimes A$, $\Psi(a \otimes c) = \Psi^1(a \otimes c) \otimes \Psi^2(a \otimes c)$, such that

$$\Psi^1(1_A \otimes c_{\underline{2}}\cdot\tilde{X}^3_\rho)(c_{\underline{1}}\cdot\tilde{X}^2_\rho)\tilde{X}^1_\rho\Psi^2(1_A \otimes c_{\underline{2}}\cdot\tilde{X}^3_\rho) = \varepsilon_C(c)1_A, \tag{7.17}$$

for all $c \in C$.

(v) There exists a left and right A-linear, C-colinear map $\Lambda : A\otimes C\to A\otimes {}^*C$, $\Lambda(c\otimes c') = \Lambda^1(c\otimes c')\otimes\Lambda^2(c\otimes c')$, *such that*

$$\Lambda^2(1_A\otimes c_{\underline{2}}\cdot\tilde{X}^3_\rho)(c_{\underline{1}}\cdot\tilde{X}^2_\rho\Lambda^1(1_A\otimes c_{\underline{2}}\cdot\tilde{X}^3_\rho)_{\langle 1\rangle}\tilde{p}^2_\rho)\tilde{X}^1_\rho\Lambda^1(1_A\otimes c_{\underline{2}}\cdot\tilde{X}^3_\rho)_{\langle 0\rangle}\tilde{p}^1_\rho = \varepsilon_C(c)1_A, \tag{7.18}$$

for all $c\in C$. *Here* $\tilde{p}^1_\rho\otimes\tilde{p}^2_\rho = \tilde{x}^1_\rho\otimes S(\tilde{x}^2_\rho)\alpha\tilde{x}^3_\rho\in A\otimes H$. *The left and right A-action and the right C-coaction on* $A\otimes{}^*C$ *are given by the formulas*

$$a\cdot(a'\otimes{}^*c) = aa'\otimes{}^*c\ ;\ (a\otimes{}^*c)\cdot a' = aa'_{\langle 0\rangle}\otimes{}^*c\cdot a'_{\langle 1\rangle}, \tag{7.19}$$

$$a\otimes{}^*c\mapsto a\tilde{X}^1_\rho\otimes(c^j\cdot S^{-1}(\tilde{X}^3_\rho)q^2(\tilde{X}^2_\rho)_2S^{-1}(g^1))({}^*c\cdot q^1(\tilde{X}^2_\rho)_1S^{-1}(g^2))\otimes c_j, \tag{7.20}$$

where $c_j\otimes c^j\in C\otimes{}^*C$ *is the finite dual basis for* C, *and* $q^1\otimes q^2 = X^1\otimes S^{-1}(\alpha X^3)X^2\in H\otimes H$.

(vi) There exists a k-linear map $\overline{\Psi}: C\to{}^*C\otimes A$, $\overline{\Psi}(c) = \overline{\Psi}^1(c)\otimes\overline{\Psi}^2(c)$, *such that*

$$\begin{aligned}
&\overline{\Psi}^1(c'\cdot a_{\langle 1\rangle})(c\cdot a_{\langle 0,1\rangle})a_{\langle 0,0\rangle}\overline{\Psi}^2(c'\cdot a_{\langle 1\rangle}) = \overline{\Psi}^1(c')(c)\overline{\Psi}^2(c')a;\\
&\overline{\Psi}^1(c')(c_{\underline{2}}\cdot\tilde{X}^3_\rho)\tilde{X}^1_\rho\overline{\Psi}^2(c')_{\langle 0\rangle}\otimes c_{\underline{1}}\cdot\tilde{X}^2_\rho\overline{\Psi}^2(c')_{\langle 1\rangle}\\
&\qquad = \overline{\Psi}^1(c'_{\underline{1}}\cdot\tilde{X}^2_\rho)(c\cdot(\tilde{X}^1_\rho)_{\langle 1\rangle})(\tilde{X}^1_\rho)_{\langle 0\rangle}\overline{\Psi}^2(c'_{\underline{1}}\cdot\tilde{X}^2_\rho)\otimes c'_{\underline{2}}\cdot\tilde{X}^3_\rho;\\
&\overline{\Psi}^1(c_{\underline{2}}\cdot\tilde{X}^3_\rho)(c_{\underline{1}}\cdot\tilde{X}^2_\rho)\tilde{X}^1_\rho\overline{\Psi}^2(c_{\underline{2}}\cdot\tilde{X}^3_\rho) = \varepsilon_C(c)1_A,
\end{aligned}$$

for all $a\in A$ *and* $c,c'\in C$.

(vii) There exists a k-linear map $\overline{\Lambda}: C\to A\otimes{}^*C$, $\overline{\Lambda}(c) = \overline{\Lambda}^1(c)\otimes\overline{\Lambda}^2(c)$, *such that*

$$\begin{aligned}
&\overline{\Lambda}^1(c)a_{\langle 0\rangle}\otimes S(a_{\langle 1\rangle})\cdot\overline{\Lambda}^2(c) = a_{\langle 0\rangle}\overline{\Lambda}^1(c\cdot a_{\langle 1\rangle})\otimes\overline{\Lambda}^2(c\cdot a_{\langle 1\rangle}),\\
&\tilde{X}^1_\rho\overline{\Lambda}^1(c_{\underline{1}}\cdot\tilde{X}^2_\rho)\otimes\overline{\Lambda}^2(c_{\underline{1}}\cdot\tilde{X}^2_\rho)\otimes c_{\underline{2}}\cdot\tilde{X}^3_\rho\\
&\qquad = \overline{\Lambda}^1(c)\tilde{X}^1_\rho\otimes\left(g^1S(q^2(\tilde{X}^2_\rho)_2)\tilde{X}^3_\rho\cdot c^j\right)\left(g^2S(q^1(\tilde{X}^2_\rho)_1)\cdot\overline{\Lambda}^2(c)\right)\otimes c_j,\\
&\overline{\Lambda}^2(c_{\underline{2}}\cdot\tilde{X}^3_\rho)(c_{\underline{1}}\cdot\tilde{X}^2_\rho\overline{\Lambda}^1(c_{\underline{2}}\cdot\tilde{X}^3_\rho)_{\langle 1\rangle}\tilde{p}^2_\rho)\tilde{X}^1_\rho\overline{\Lambda}^1(c_{\underline{2}}\cdot\tilde{X}^3_\rho)_{\langle 0\rangle}\tilde{p}^1_\rho = \varepsilon_C(c)1_A,
\end{aligned}$$

for all $c\in C$ *and* $a\in A$. *Here* $g^1\otimes g^2$ *is the inverse of the Drinfeld's twist* f, $\tilde{p}^1_\rho\otimes\tilde{p}^2_\rho = \tilde{x}^1_\rho\otimes\tilde{x}^2_\rho\beta S(\tilde{x}^3_\rho)$ *and* $c_j\otimes c^j\in C\otimes{}^*C$ *is the finite dual basis for* C. *For simplicity, we considered* ${}^*C$ *as a left H-module via* $h\cdot{}^*c = {}^*c\cdot S^{-1}(h)$, *for all* $h\in H$ *and* ${}^*c\in{}^*C$.

Proof. The equivalence between (i), (ii), (iv) and (vi) follow from Proposition 6.4 and Theorem 6.5.
<u>*(ii)* ⇔ *(iii)*</u>. Follows from the fact that $C^*\otimes A$ and $\mathrm{Hom}_k(C,A)$ are isomorphic vector spaces if C is finite dimensional.
<u>*(iv)* ⇔ *(v)*</u>. ${}^*C\otimes A$ and $A\otimes{}^*C$ are isomorphic as vector spaces, see (7.8-7.9). The left and right A-action and the right C-coaction on ${}^*C\otimes A$ can be transported on $A\otimes{}^*C$. A technical but straightforward computation shows that these structure maps are precisely the ones stated in (7.19) and (7.20), we leave the verification of these details to the reader. Furthermore, since Ψ can be recovered from Λ as

$$\Psi^1(a\otimes c)\otimes\Psi^2(a\otimes c) = \Lambda^2(a\otimes c)\cdot S^{-1}(\Lambda^1(a\otimes c)_{\langle 1\rangle}\tilde{p}^2_\rho)\otimes\Lambda^1(a\otimes c)_{\langle 0\rangle}\tilde{p}^1_\rho,$$

we find that Ψ satisfies (7.17) if and only if Λ satisfies the third condition in (vii).
The proof of $(v)\Leftrightarrow(vii)$ is similar to the proof of $(iv)\Leftrightarrow(vi)$. □

Proposition 7.7 can be viewed as a separable analog of Proposition 7.4.

Proposition 7.7. *Let H be a quasi-bialgebra, let A be a right H-comodule algebra and let C be a right H-module coalgebra. If C is a coseparable coalgebra in* $\mathcal{M}_H$ *then* $F:\mathcal{M}(H)^C_A\to\mathcal{M}_A$ *is a separable functor.*

Proof. Applying Proposition 6.3 to the case where $\mathcal{C} = \mathcal{M}_H$, we obtain that C is coseparable if and only if there exists a k-linear map $B : C \otimes C \to k$ such that

$$B(c \cdot h_1 \otimes c' \cdot h_2) = \varepsilon(h)B(c \otimes c')\ ;\ B(c_{\underline{1}} \otimes c_{\underline{2}}) = \varepsilon_C(c)\ ; \tag{7.21}$$

$$B(c \cdot X^1 \otimes c'_{\underline{1}} \cdot X^2)c'_{\underline{2}} \cdot X^3 = B(c_{\underline{2}} \cdot x^2 \otimes c' \cdot x^3)c_{\underline{1}} \cdot x^1, \tag{7.22}$$

for all $c, c' \in C$ and $h \in H$. With the help of B we construct a Casimir morphism $\mathbf{B} : C \otimes C \to A$ for the coalgebra (C, ψ) in $\mathcal{T}_A^\#$ as follows:

$$\mathbf{B}(c \otimes c') = B(c \cdot \tilde{x}_\rho^2 \otimes c' \cdot \tilde{x}_\rho^3)\tilde{x}_\rho^1,$$

for all $c,\ c' \in C$. $\mathbf{B}$ is a morphism in $\mathcal{T}_A^\#$ since

$$\begin{aligned} a_{\langle 0,0\rangle}\mathbf{B}(c \cdot a_{\langle 0,1\rangle} \otimes c' \cdot a_{\langle 1\rangle}) &= B(c \cdot a_{\langle 0,1\rangle}\tilde{x}_\rho^2 \otimes c' \cdot a_{\langle 1\rangle}\tilde{x}_\rho^3)a_{\langle 0,0\rangle}\tilde{x}_\rho^1 \\ &\overset{(7.2)}{=} B(c \cdot \tilde{x}_\rho^2 a_{\langle 1\rangle_1} \otimes c' \cdot \tilde{x}_\rho^3 a_{\langle 1\rangle_2})\tilde{x}_\rho^1 a_{\langle 0\rangle} = \varepsilon(a_{\langle 1\rangle})B(c \cdot \tilde{x}_\rho^2 \otimes c' \cdot \tilde{x}_\rho^3)\tilde{x}_\rho^1 a_{\langle 0\rangle} \overset{(7.4)}{=} \mathbf{B}(c \otimes c')a, \end{aligned}$$

for all $a \in A$ and $c, c' \in C$. $\mathbf{B}$ satisfies (7.15) since

$$\begin{aligned} &(\tilde{X}_\rho^1)_{\langle 0\rangle}\mathbf{B}(c \cdot (\tilde{X}_\rho^1)_{\langle 1\rangle} \otimes c'_{\underline{1}} \cdot \tilde{X}_\rho^2) \otimes c'_{\underline{2}} \cdot \tilde{X}_\rho^3 \\ &= B(c \cdot (\tilde{X}_\rho^1)_{\langle 1\rangle}\tilde{x}_\rho^2 \otimes c'_{\underline{1}} \cdot \tilde{X}_\rho^2\tilde{x}_\rho^3)(\tilde{X}_\rho^1)_{\langle 0\rangle}\tilde{x}_\rho^1 \otimes c'_{\underline{2}} \cdot \tilde{X}_\rho^3 \\ &\overset{(7.3)}{=} B\Big(c \cdot \tilde{x}_\rho^2 X^1(\tilde{X}_\rho^2)_1 \otimes c'_{\underline{1}} \cdot (\tilde{x}_\rho^3)_1 X^2(\tilde{X}_\rho^2)_2\Big)\,\tilde{x}_\rho^1\tilde{X}^1 \otimes c'_{\underline{2}} \cdot (\tilde{x}_\rho^3)_2 X^3\tilde{X}_\rho^3 \\ &\overset{(7.5)}{=} B\Big(c \cdot \tilde{x}_\rho^2 X^1 \otimes c'_{\underline{1}} \cdot (\tilde{x}_\rho^3)_1 X^2\Big)\,\tilde{x}_\rho^1 \otimes c'_{\underline{2}} \cdot (\tilde{x}_\rho^3)_2 X^3 \\ &= B(c_{\underline{2}} \cdot (\tilde{x}_\rho^2)_2 x^2 \otimes c' \cdot \tilde{x}_\rho^3 x^3)\tilde{x}_\rho^1 \otimes c_{\underline{1}} \cdot (\tilde{x}_\rho^2)_1 x^1 \\ &\overset{(7.3)}{=} B\Big(c_{\underline{2}} \cdot \tilde{Y}_\rho^3\tilde{y}_\rho^2(\tilde{x}_\rho^3)_1 \otimes c' \cdot \tilde{y}^3(\tilde{x}_\rho^3)_2\Big)\,\tilde{Y}_\rho^1(\tilde{y}_\rho^1)_{\langle 0\rangle}\tilde{x}_\rho^1 \otimes c_{\underline{1}} \cdot \tilde{Y}_\rho^2(\tilde{y}_\rho^1)_{\langle 1\rangle}\tilde{x}_\rho^2 \\ &\overset{(7.5)}{=} B\Big(c_{\underline{2}} \cdot \tilde{Y}_\rho^3\tilde{y}_\rho^2 \otimes c' \cdot \tilde{y}^3\Big)\,\tilde{Y}_\rho^1(\tilde{y}_\rho^1)_{\langle 0\rangle} \otimes c_{\underline{1}} \cdot \tilde{Y}_\rho^2(\tilde{y}_\rho^1)_{\langle 1\rangle} \\ &= \tilde{Y}_\rho^1\mathbf{B}(c_{\underline{2}} \cdot \tilde{Y}_\rho^3 \otimes c')_{\langle 0\rangle} \otimes c_{\underline{1}} \cdot \tilde{Y}_\rho^2\mathbf{B}(c_{\underline{2}} \cdot \tilde{Y}_\rho^3 \otimes c')_{\langle 1\rangle}. \end{aligned}$$

Finally, $\mathbf{B}$ satisfies the normalizing condition (7.16), since

$$\tilde{X}_\rho^1\mathbf{B}(c_{\underline{1}} \cdot \tilde{X}_\rho^2 \otimes c_{\underline{2}} \cdot \tilde{X}_\rho^3) = B(c_{\underline{1}} \otimes c_{\underline{2}})1_A = \varepsilon_C(c)1_A,$$

for all $c \in C$. □

We have seen in Theorem 7.5 that a right H-module coalgebra C is Frobenius in $\mathcal{M}_H$ if the forgetful functor $F :\ \mathcal{M}_H^C \to \mathcal{M}_H$ is Frobenius. A similar property does not hold in general for separability. In order to conclude that C is a coseparable coalgebra from the fact that the forgetful functor is separable, an additional condition on the Casimir morphism $\mathbf{B} : C \otimes C \to H$ associated to F is needed.

Theorem 7.8. *Let H be a quasi-Hopf algebra with bijective antipode and let C be a right H-module coalgebra. There is a bijective correspondence between the set of normalized Casimir morphisms for C in $\mathcal{M}_H$ and the set of normalized Casimir morphisms $\mathbf{B}$ for (C, ψ) in $\mathcal{T}_H^\#$ satisfying the condition*

$$\mathbf{B}(c \otimes c') = \varepsilon\mathbf{B}(c \cdot x^2 \otimes c' \cdot x^3)x^1, \tag{7.23}$$

for all $c, c' \in C$.

Proof. A normalized Casimir morphism for C in $\mathcal{M}_H$ is a map $B :\ C \otimes C \to k$ satisfying (7.21-7.22). We have seen in the proof of Proposition 7.7 that $\mathbf{B} :\ C \otimes C \to H$, $\mathbf{B}(c \otimes c') = B(c \cdot x^2 \otimes c' \cdot x^3)x^1$ is a normalized Casimir morphism for (C, ψ) in $\mathcal{T}_H^\#$, which means that

$$h_{(1,1)}\mathbf{B}(c \cdot h_{(1,2)} \otimes c' \cdot h_2) = \mathbf{B}(c \otimes c')h, \tag{7.24}$$

$$X^1\mathbf{B}(c_{\underline{2}} \cdot X^3 \otimes c')_1 \otimes c_{\underline{1}} \cdot X^2\mathbf{B}(c_{\underline{2}} \cdot X^3 \otimes c')_2 = X_1^1\mathbf{B}(c \cdot X_2^1 \otimes c'_{\underline{1}} \cdot X^2) \otimes c'_{\underline{2}} \cdot X^3, \tag{7.25}$$

$$X^1\mathbf{B}(c_{\underline{1}}\cdot X^2\otimes c_{\underline{2}}\cdot X^3)=\varepsilon_C(c)1_H. \tag{7.26}$$

From the definition of $\mathbf{B}$, it follows easily that $\mathbf{B}$ satisfies (7.23).

Conversely, assume that $\mathbf{B}:\ C\otimes C\to A$ satisfies (7.23-7.26). Then $B=\varepsilon\mathbf{B}$ is a normalized Casimir morphisms for C in $\mathcal{M}_H$. It follows from (7.24) that B is a morphism in $\mathcal{M}_H$, and it follows from (7.26) that B is normalized, that is, $B(c_{\underline{1}}\otimes c_{\underline{2}})=\varepsilon_C(c)$, for all $c\in C$. The most difficult part is to prove that B satisfies (7.22). First observe that $q_L=\mathfrak{q}^1\otimes\mathfrak{q}^2=S(x^1)\alpha x^2\otimes x^3\in H\otimes H$ has the properties

$$\mathfrak{q}^2_1\mathfrak{p}^1S^{-1}(\mathfrak{q}^1)\otimes\mathfrak{q}^2_2\mathfrak{p}^2=1\otimes 1 \text{ and } S(h_1)\mathfrak{q}^1h_{(2,1)}\otimes\mathfrak{q}^2h_{(2,2)}=\mathfrak{q}^1\otimes h\mathfrak{q}^2, \tag{7.27}$$

for all $h\in H$. Here $p_L=\mathfrak{p}^1\otimes\mathfrak{p}^2=X^2S^{-1}(X^1\beta)\otimes X^3\in H\otimes H$. For later use, we record that

$$S(\mathfrak{p}^1)\mathfrak{q}^1\mathfrak{p}^2_1\otimes\mathfrak{q}^2\mathfrak{p}^2_2=1\otimes 1 \text{ and } h_{(2,1)}\mathfrak{p}^1S^{-1}(h_1)\otimes h_{(2,2)}\mathfrak{p}^2=\mathfrak{p}^1\otimes\mathfrak{p}^2h, \tag{7.28}$$

for all $h\in H$. We will compute

$$X=c'_{\underline{2}}\cdot Y^3y^3S^{-1}(\mathfrak{q}^1X^1\mathbf{B}(c\cdot Y^1\mathfrak{q}^2_1X^2\otimes c'_{\underline{1}}\cdot Y^2\mathfrak{q}^2_2X^3)y^2\beta)y^1$$

in two different ways.

$$\begin{aligned} X &\overset{(7.23)}{=} B(c\cdot Y^1\mathfrak{q}^2_1\otimes c'_{\underline{1}}\cdot Y^2\mathfrak{q}^2_2)c'_{\underline{2}}\cdot Y^3y^3S^{-1}(\mathfrak{q}^1y^2\beta)y^1\\ &\overset{(7.21)}{=} B(c\cdot Y^1\otimes c'_{\underline{1}}\cdot Y^2)c'_{\underline{2}}\cdot Y^3y^3S^{-1}(\alpha y^2\beta)y^1\overset{(1.20)}{=}B(c\cdot Y^1\otimes c'_{\underline{1}}\cdot Y^2)c'_{\underline{2}}\cdot Y^3.\end{aligned}$$

Making use of the formula $\mathfrak{q}^1X^1\otimes\mathfrak{q}^2_1X^2\otimes\mathfrak{q}^2_2X^3=S(z^1)\mathfrak{q}^1z^2_1\otimes\mathfrak{q}^2z^2_2\otimes z^3$, we also find that

$$\begin{aligned} X &= c'_{\underline{2}}\cdot Y^3y^3S^{-1}(\mathfrak{q}^1z^2_1\mathbf{B}(c\cdot Y^1\mathfrak{q}^2z^2_2\otimes c'_{\underline{1}}\cdot Y^2z^3)y^2\beta)z^1y^1\\ &\overset{(7.24,7.27)}{=} c'_{\underline{2}}\cdot Y^3y^3S^{-1}(\mathfrak{q}^1(Y^1_2z^2y^2_1)_1\mathbf{B}(c\cdot\mathfrak{q}^2(Y^1_2z^2y^2_1)_2\otimes c'_{\underline{1}}\cdot Y^2z^3y^2_2)\beta)Y^1_1z^1y^1\\ &\overset{(1.17)}{=} ((c'\cdot x^3)_{\underline{2}}\cdot X^3)\cdot S^{-1}(\mathfrak{q}^1x^2_1X^1_1\mathbf{B}((c\cdot\mathfrak{q}^2x^2_2)\cdot X^1_2\otimes(c'\cdot x^3)_{\underline{1}}\cdot X^2)\beta)x^1\\ &\overset{(7.25)}{=} c_{\underline{1}}\cdot\mathfrak{q}^2_1x^2_{(2,1)}X^2\mathbf{B}(c_{\underline{2}}\cdot\mathfrak{q}^2_2x^2_{(2,2)}X^3\otimes c'\cdot x^3)_2S^{-1}(\mathfrak{q}^1x^2_1X^1\mathbf{B}(c_{\underline{2}}\cdot\mathfrak{q}^2_2x^2_{(2,2)}X^3\otimes c'\cdot x^3)_1\beta)x^1\\ &\overset{(1.19)}{=} B(c_{\underline{2}}\cdot\mathfrak{q}^2_2x^2_{(2,2)}\mathfrak{p}^2\otimes c'\cdot x^3)c_{\underline{1}}\cdot\mathfrak{q}^2_1x^2_{(2,1)}\mathfrak{p}^1S^{-1}(\mathfrak{q}^1x^2_1)x^1\\ &\overset{(7.28)}{=} B(c_{\underline{2}}\cdot\mathfrak{q}^2_2\mathfrak{p}^2x^2\otimes c'\cdot x^3)c_{\underline{1}}\cdot\mathfrak{q}^2_1\mathfrak{p}^1S^{-1}(\mathfrak{q}^1)x^1\\ &\overset{(7.27)}{=} B(c_{\underline{2}}\cdot x^2\otimes c'\cdot x^3)c_{\underline{1}}\cdot x^1.\end{aligned}$$

It follows that $B(c\cdot Y^1\otimes c'_{\underline{1}}\cdot Y^2)c'_{\underline{2}}\cdot Y^3=B(c_{\underline{2}}\cdot x^2\otimes c'\cdot x^3)c_{\underline{1}}\cdot x^1$, for all $c,c'\in C$, which means that B is a normalized Casimir morphism for the coalgebra C in $\mathcal{M}_H$.
It can be easily checked that the two correspondences defined above are inverse each other. □

8. Application to two-sided Hopf modules over quasi-bialgebras

8.1. **Two-sided Hopf modules.** If H is a quasi-bialgebra with reassociator Φ then $H^{\rm op}$ is also a quasi-bialgebra with reassociator Φ^{-1}. If H is a quasi-Hopf algebra with bijective antipode then $H^{\rm op}$ is also a quasi-Hopf algebra, with antipode $S_{\rm op}=S^{-1}$, and distinguished elements $\alpha_{\rm op}=S^{-1}(\beta)$ and $\beta_{\rm op}=S^{-1}(\alpha)$, where $\alpha,\beta\in H$ are as in (1.19) and (1.20).
The category of H-bimodules ${}_H\mathcal{M}_H$ is isomorphic to $\mathcal{M}_{H\otimes H^{\rm op}}$. Since $H\otimes H^{\rm op}$ is a quasi-bialgebra, $\mathcal{M}_{H\otimes H^{\rm op}}$ and ${}_H\mathcal{M}_H$ are monoidal categories. A coalgebra in ${}_H\mathcal{M}_H$ is called an H-bimodule coalgebra.
Consider a quasi-bialgebra H, a right H-comodule algebra A and an H-bimodule coalgebra C. A two-sided (H,A)-bimodule over C is an (H,A)-bimodule M together with a k-linear map ρ_M : $M\to M\otimes C$, $\rho_M(m)=m_{(0)}\otimes m_{(1)}$ satisfying

$$(M\otimes\varepsilon)\rho_M=M\ ;\ \Phi\cdot(\rho_M\otimes{\rm Id}_H)(\rho_M(m))=(M\otimes\Delta)(\rho_M(m))\cdot\Phi_\rho;$$

$$\rho_M(h\succ m)=h_1\succ m_{(0)}\otimes h_2\cdot m_{(1)}\ ;\ \rho_M(m\prec a)=m_{(0)}\prec a_{\langle 0\rangle}\otimes m_{(1)}\cdot a_{\langle 1\rangle},$$

for all $m \in M$, $h \in H$ and $a \in A$. ${}_H\mathcal{M}_A^C$ is the category of two-sided (H, A)-bimodule over C, and left H-linear, right A-linear, C-colinear maps. It was shown in [10, Prop. 6.3] that ${}_H\mathcal{M}_A^C$ is isomorphic to a category of entwined modules over a cowreath in ${}_H\mathcal{M}$.

Proposition 8.1. *Let H be a quasi-bialgebra, let A be a right H-comodule algebra and let C be an H-bimodule coalgebra. $A \in {}_H\mathcal{M}$ by restriction of scalars via ε. Consider the map*

$$\psi : C \otimes A \to A \otimes C, \ \psi(c \otimes a) = a_{\langle 0\rangle} \otimes c \cdot a_{\langle 1\rangle}.$$

Then $(C, \psi) \in \mathcal{T}({}_H\mathcal{M})_A^{\#}$. (C, ψ) is a coalgebra in $\mathcal{T}({}_H\mathcal{M})_A^{\#}$, with comultiplication and counit

$$\delta : C \to A \otimes C \otimes C, \ \delta(c) = \tilde{X}_\rho^1 \otimes c_{\underline{1}} \cdot \tilde{X}_\rho^2 \otimes c_{\underline{2}} \cdot \tilde{X}_\rho^3 \ ; \ \epsilon = \eta_A \circ \varepsilon_C : C \to A.$$

The categories $({}_H\mathcal{M})(\psi)_A^C$ and ${}_H\mathcal{M}_A^C$ are isomorphic.

As before, $\Delta_C(c) = c_{\underline{1}} \otimes c_{\underline{2}}$ is our Sweedler type notation for the comultiplication of a coalgebra C within the monoidal category of H-(bi)modules.

For our purposes, this description of ${}_H\mathcal{M}_A^C$ as a category of entwined modules is not very useful, since the unit object k of ${}_H\mathcal{M}$ is not $\otimes$-generator for ${}_H\mathcal{M}$, a condition that is needed in the more important results from Sections 4 and 6, for example Theorem 4.8 and Proposition 6.1. This is why we provide an alternative description of ${}_H\mathcal{M}_A^C$ as a category of entwined modules, this time over a cowreath structure in ${}_k\mathcal{M}$, which will enable us to discuss when the forgetful functor $F : {}_H\mathcal{M}_A^C \to {}_H\mathcal{M}_A$ is Frobenius or separable.

Proposition 8.2. *Let H be a quasi-bialgebra, let A be a right H-comodule algebra and let C be an H-bimodule coalgebra. Then C is right $H \otimes H^{\rm op}$-module coalgebra and $A \otimes H^{\rm op}$ is a right $H \otimes H^{\rm op}$-comodule algebra. In particular, C is a coalgebra in the category $\mathcal{T}^{\#}_{A\otimes H^{\rm op}}$. Finally the category of Doi-Hopf modules $\mathcal{M}(H \otimes H^{\rm op})^C_{A\otimes H^{\rm op}}$ is isomorphic to the category ${}_H\mathcal{M}_A^C$ of two-sided (H, A)-bimodules over C.*

Proof. We have seen above that ${}_H\mathcal{M}_H$ and $\mathcal{M}_{H\otimes H^{\rm op}}$ are isomorphic as monoidal categories, so C is a right $H \otimes H^{\rm op}$-module coalgebra with right $H \otimes H^{\rm op}$-action $c \cdot (h \otimes h') = h' \cdot c \cdot h$, for all $c \in C$ and $h, \ h' \in H$. It is easy to see that $A \otimes H^{\rm op}$ is a right $H \otimes H^{\rm op}$-comodule algebra with

$$\Phi_{A\otimes H^{\rm op}} = (\tilde{X}_\rho^1 \otimes x^1) \otimes (\tilde{X}_\rho^2 \otimes x^2) \otimes (\tilde{X}_\rho^3 \otimes x^3) \in (A \otimes H^{\rm op}) \otimes (H \otimes H^{\rm op}) \otimes (H \otimes H^{\rm op}),$$

and

$$\rho_{A\otimes H^{\rm op}} : A \otimes H^{\rm op} \to (A \otimes H^{\rm op}) \otimes (H \otimes H^{\rm op}), \ \rho_{A\otimes H^{\rm op}}(a \otimes h) = (a_{\langle 0\rangle} \otimes h_1) \otimes (a_{\langle 1\rangle} \otimes h_2).$$

It was shown in Section 7.1 that C is a coalgebra in the category $\mathcal{T}^{\#}_{A\otimes H^{\rm op}}$. It is now easy to verify that the categories $\mathcal{M}(H \otimes H^{\rm op})^C_{A\otimes H^{\rm op}}$ and ${}_H\mathcal{M}_A^C$ are isomorphic. □

8.2. **Frobenius properties for the category of two-sided Hopf modules.**

Proposition 8.3. *Let H be a quasi-bialgebra, A a right H-comodule algebra and C an H-bimodule coalgebra. The forgetful functor $F : {}_H\mathcal{M}_A^C \to {}_H\mathcal{M}_A$ is Frobenius if and only if there exists $t = a_i \otimes h_i \otimes c_i \in A \otimes H \otimes C$ such that*

$$aa_i \otimes h_i h \otimes c_i = a_i a_{\langle 0\rangle} \otimes h_i h_1 \otimes h_2 \cdot c_i \cdot a_{\langle 1\rangle}, \tag{8.1}$$

for all $a \in A$ and $h \in H$, and the k-linear map $\kappa : {}^*C \otimes A \otimes H \to A \otimes H \otimes C$,

$$\kappa({}^*c \otimes a \otimes h) = \langle {}^*c, \ x^3 \cdot (c_i)_{\underline{2}} \cdot \tilde{X}_\rho^3\rangle a_i \tilde{X}_\rho^1 a_{\langle 0\rangle} \otimes h_1 x^1 h_i \otimes h_2 x^2 \cdot (c_i)_{\underline{1}} \cdot \tilde{X}_\rho^2 a_{\langle 1\rangle}, \tag{8.2}$$

is an isomorphism. If H is a quasi-Hopf algebra with bijective antipode then F is Frobenius if and only if there exists $t = a_i \otimes h_i \otimes c_i \in A \otimes H \otimes C$ satisfying (8.1) and such that $\gamma : A \otimes H \otimes {}^*C \to A \otimes H \otimes C$ given by the formula

$$\gamma(a \otimes h \otimes {}^*c) = \langle {}^*c, \ (S^{-1}(X^3) q^2 X_2^2 \cdot (c_i)_{\underline{2}} \cdot (\tilde{x}_\rho^2)_2 p^2 S(\tilde{x}_\rho^3)\rangle \ aa_i \otimes h_i h \otimes q^1 X_1^2 \cdot (c_i)_{\underline{1}} \cdot (\tilde{x}_\rho^2)_1 p^1 \tag{8.3}$$

is an isomorphism. Here $q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3)X^2 \in H \otimes H$.

Proof. The conditions (8.1) and (8.2) are the conditions (7.6) and (7.7) specialized to the right $H \otimes H^{\rm op}$-comodule algebra $A \otimes H^{\rm op}$. The second assertion follows from Corollary 7.2, applied to the Doi-Hopf datum in Proposition 8.2. Note that $p_R^{H^{\rm op}} = q_R$ and $q_R^{H^{\rm op}} = p_R$, so that $p_R^{H\otimes H^{\rm op}} = (p^1 \otimes q^1) \otimes (p^2 \otimes q^2)$. □

Recall that a left integral in a quasi-bialgebra H is an element $t \in H$ such that $ht = \varepsilon(h)t$, for all $h \in H$. A quasi-Hopf algebra H with bijective antipode contains a non-zero left integral if and only if H is finite dimensional, and in this case, the space of left integrals in H has dimension one, see [6]. Then there exists $\mu : H \to k$ such that $th = \langle \mu, h\rangle t$, for every $h \in H$. μ is called the modular element of H (in the dual space $H^* = {}^*H$). For all $h \in H$, we have that

$$\langle \mu, h_1\rangle\langle \mu, S(h_2)\rangle = \langle \mu, h_1\rangle\langle \mu, S^{-1}(h_2)\rangle = \varepsilon(h),$$

see (1.20-1.19), so it follows that μ is convolution invertible with inverse $\mu^{-1} = \mu \circ S = \mu \circ S^{-1}$. It can be easily verified that μ is an algebra map.
A right integral in H is a left integral in $H^{\rm op}$. We call H unimodular if there exists a non-zero left integral in H that is also a right integral, or, equivalently, if $\mu = \varepsilon$.

Theorem 8.4. *Let H be a quasi-Hopf algebra with bijective antipode, A a right H-comodule algebra and C an H-bimodule coalgebra. Let $F : {}_H\mathcal{M}_A^C \to {}_H\mathcal{M}_A$ be the forgetful functor. Then the following assertions hold:*

(i) *If C is a Frobenius coalgebra in ${}_H\mathcal{M}_H$, F is Frobenius.*
(ii) *Assume that F is Frobenius. Then for every algebra morphism $\zeta :\ A \to k$, there exists $t \in C$ such that $h \cdot t = \varepsilon(h)t$, for all $h \in H$, $\zeta(a_{\langle 0\rangle})t \cdot a_{\langle 1\rangle} = \zeta(a)t$, for all $a \in A$, and*

$$\kappa :\ {}^*C \to C,\ \kappa({}^*c) = \zeta(\tilde{x}^1_\rho)\langle {}^*c, q^2 \cdot t_{\underline{2}} \cdot (\tilde{x}^2_\rho)_2 p^2 S(\tilde{x}^3_\rho)\rangle\ q^1 \cdot t_{\underline{1}} \cdot (\tilde{x}^2_\rho)_1 p^1$$

is an isomorphism of vector spaces. Consequently, if there exists an algebra morphism $\zeta : A \to k$ such that $\tilde{\zeta} = (\zeta \otimes H)\rho$ is surjective then C is a Frobenius coalgebra in ${}_H\mathcal{M}_H$.
(iii) *$F :\ {}_H\mathcal{M}_H^C \to {}_H\mathcal{M}_H$ is Frobenius if and only if C is a Frobenius coalgebra in ${}_H\mathcal{M}_H$.*
(iv) *$F :\ {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}_H$ is Frobenius if and only if H is finite dimensional and unimodular.*

Proof. (i) follows from Proposition 7.4 and Proposition 8.2.
(ii) If F is a Frobenius functor then there exists $t = a_i \otimes h_i \otimes c_i \in A \otimes H \otimes C$ such that (8.1-8.3) hold. If $\zeta : A \to k$ is an algebra map then $\zeta \otimes \varepsilon : A \otimes H^{\rm op} \to k$ is also an algebra map. Furthermore, we have that $\varepsilon(h_i h)\zeta(aa_i)c_i = \zeta(a_i a_{\langle 0\rangle})h \cdot c_i \cdot a_{\langle 1\rangle}$, for all $h \in H$ and $a \in A$, and $t = \varepsilon(h_i)\zeta(a_i)c_i \in C$ is such that $h \cdot t = \varepsilon(h)t$, for all $h \in H$, and $\zeta(a_{\langle 0\rangle})t \cdot a_{\langle 1\rangle} = \zeta(a)t$, for all $a \in A$. The remaining assertions follow from Theorem 7.5 and Proposition 8.2.
(iii) also follows from Theorem 7.5 and Proposition 8.2.
(iv) in view of (iii), it suffices to show that $H \in {}_H\mathcal{M}_H$ is Frobenius if and only H is finite dimensional and unimodular.
If $H \in {}_H\mathcal{M}_H$ is a Frobenius coalgebra, then F is a Frobenius functor, and it follows from Proposition 8.2 and Proposition 7.1 that H is finite dimensional. H is a Frobenius coalgebra in $\mathcal{M}_{H\otimes H^{\rm op}}$, so there exists $t \in H$ such that $ht = th = \varepsilon(h)t$, for all $h \in H$, and

$$\overline{\theta} : {}^*H \to H,\ \overline{\theta}({}^*h) = \langle {}^*h, q^2 t_2 p^2\rangle q^1 t_1 p^1, \tag{8.4}$$

is an isomorphism of vector spaces, see the proof of Proposition 7.4. It follows that t is a non-zero left and right integral in H, and so H is unimodular.
Conversely, assume that H is finite dimensional and unimodular. Let t be a non-zero left and right integral in H. Since t is a non-zero integral in H by [6, Remarks 2.6(ii)] we know that $\overline{\theta}$ defined in (8.4) is a left H-linear isomorphism, where ${}^*H$ is considered as a left H-module via $\langle h \cdot {}^*h, h'\rangle = \langle {}^*h, S^{-1}(h)h'\rangle$, for all $h, h' \in H$. But this is part of the structure of the right dual

of H in ${}_H\mathcal{M}_H$, in the sense that ${}^*H$ is a right dual of H in ${}_H\mathcal{M}_H$. ${}^*H$ is an H-bimodule via $\langle h\cdot {}^*h\cdot h', k\rangle = \langle h^*, S^{-1}(h)kS(h')\rangle$, for all $h, h', k\in H$. The evaluation and coevaluation are given by the formulas

$$d:\ H\otimes {}^*H\to k,\ d(h\otimes {}^*h) = \langle {}^*h, S^{-1}(\alpha)h\beta\rangle \text{ and } b:\ k\to {}^*H\otimes H,\ b(1) = l^i\otimes S^{-1}(\beta)l_i\alpha,$$

where $l_i\otimes l^i\in H\otimes {}^*H$ is the finite dual basis for H.
t is also a right integral in H. Using the formula $t_1p^1\otimes t_2p^2S(h) = t_1p^1h\otimes t_2p^2$, for all $h\in H$, we can prove that $\overline{\theta}$ is right H-linear. Therefore $\overline{\theta}$ is an isomorphism between H and ${}^*H$ in ${}_H\mathcal{M}_H$. Looking now again at the isomorphism between ${}_H\mathcal{M}_H$ and $\mathcal{M}_{H\otimes H^{\rm op}}$ and taking into account the formulas (7.11-7.12) applied to the quasi-Hopf algebra $H\otimes H^{\rm op}$, we conclude that H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$. □

8.3. **Separability for the category of two-sided Hopf modules.** Let H be a quasi-Hopf algebra with bijective antipode, A a right H-comodule algebra and C an H-bimodule coalgebra. Then C is a coalgebra in three different monoidal categories:

- C is a coalgebra in the monoidal category ${}_H\mathcal{M}_H\cong \mathcal{M}_{H\otimes H^{\rm op}}$, by assumption. Let $\mathcal{W}$ be the set of normalized Casimir morphisms for C in ${}_H\mathcal{M}_H$.
- C is a coalgebra in the monoidal category $\mathcal{T}^{\#}_{A\otimes H^{\rm op}}$, see Proposition 8.2. Let $\mathcal{W}^{\#}$ be the set of normalized Casimir morphisms for C in $\mathcal{T}^{\#}_{A\otimes H^{\rm op}}$.
- C is a coalgebra in the monoidal category $\mathcal{T}({}_H\mathcal{M})^{\#}_A$, see Proposition 8.1. Let ${}_H\mathcal{W}^{\#}$ be the set of normalized Casimir morphisms for C in $\mathcal{T}^{\#}_{A\otimes H^{\rm op}}$.

Recall from Proposition 6.3 that a coseparable coalgebra is a coalgebra together with a normalized Casimir morphism.
It follows from Proposition 7.7 that coseparability of C as a coalgebra in ${}_H\mathcal{M}_H\cong\mathcal{M}_{H\otimes H^{\rm op}}$ implies coseparability of C as a coalgebra in $\mathcal{T}^{\#}_{A\otimes H^{\rm op}}$. This implies that we have a map $w:\ \mathcal{W}\to\mathcal{W}^{\#}$.
If C is coseparable as a coalgebra in $\mathcal{T}({}_H\mathcal{M})^{\#}_A$, then the forgetful functor $F:\ ({}_H\mathcal{M})(\psi)^C_A\cong {}_H\mathcal{M}^C_A\cong \mathcal{M}(H\otimes H^{\rm op})^C_{A\otimes H^{\rm op}}\to {}_H\mathcal{M}_A\cong\mathcal{M}_{A\otimes H^{\rm op}}$ is separable, which is equivalent to the coseparability of C as a coalgebra in $\mathcal{T}^{\#}_{A\otimes H^{\rm op}}$. This implies that we have a map $w^{\#}:\ {}_H\mathcal{W}^{\#}\to\mathcal{W}^{\#}$.
The aim of Theorem 8.5 is to study w and $w^{\#}$. We first give an explicit description of the elements of $\mathcal{W}$, $\mathcal{W}^{\#}$ and ${}_H\mathcal{W}^{\#}$.

$\mathcal{W}$ consists of k-linear maps $\Sigma:\ C\otimes C\to k$ such that

$$\Sigma(h_1\cdot c\cdot h'_1\otimes h_2\cdot c'\cdot h'_2) = \varepsilon(h)\varepsilon(h')\Sigma(c\otimes c'), \tag{8.5}$$

$$\Sigma(X^2\cdot c_{\underline{2}}\cdot x^2\otimes X^3\cdot c'\cdot x^3)X^1\cdot c_{\underline{1}}\cdot x^1 = \Sigma(x^1\cdot c\cdot X^1\otimes x^2\cdot c'_{\underline{1}}\cdot X^2)x^3\cdot c'_{\underline{2}}\cdot X^3, \tag{8.6}$$

$$\Sigma(c_{\underline{1}}\otimes c_{\underline{2}}) = \varepsilon_C(c), \tag{8.7}$$

for all $c, c'\in C$ and $h\in H$.

$\mathcal{W}^{\#}$ consists of k-linear maps $B:\ C\otimes C\to A\otimes H$, $B(c\otimes c') = B^A(c\otimes c')\otimes B^H(c\otimes c')\in A\otimes H$, such that

$$\begin{aligned} &a_{\langle 0,0\rangle}B^A(h_{(1,2)}\cdot c\cdot a_{\langle 0,1\rangle}\otimes h_2\cdot c'\cdot a_{\langle 1\rangle})\otimes B^H(h_{(1,2)}\cdot c\cdot a_{\langle 0,1\rangle}\otimes h_2\cdot c'\cdot a_{\langle 1\rangle})h_{(1,1)}\\ &= \ B^A(c\otimes c')a\otimes hB^H(c\otimes c');\end{aligned} \tag{8.8}$$

$$\begin{aligned} &\tilde{X}^1_\rho B^A(x^3\cdot c_{\underline{2}}\cdot\tilde{X}^3_\rho\otimes c')_{\langle 0\rangle}\otimes B^H(x^3\cdot c_{\underline{2}}\cdot\tilde{X}^3_\rho\otimes c')_1x^1\otimes B^H(x^3\cdot c_{\underline{2}}\cdot\tilde{X}^3_\rho\otimes c')_2x^2\cdot c_{\underline{1}}\\ &\quad\cdot\tilde{X}^2_\rho B^A(x^3\cdot c_{\underline{2}}\cdot\tilde{X}^3_\rho\otimes c')_{\langle 1\rangle} = (\tilde{X}^1_\rho)_{\langle 0\rangle}B^A(x^1_2\cdot c\cdot(\tilde{X}^1_\rho)_{\langle 1\rangle}\otimes x^2\cdot c'_{\underline{1}}\cdot\tilde{X}^2_\rho)\\ &\quad\otimes B^H(x^1_2\cdot c\cdot(\tilde{X}^1_\rho)_{\langle 1\rangle}\otimes x^2\cdot c'_{\underline{1}}\cdot\tilde{X}^2_\rho)x^1_1\otimes x^3\cdot c'_{\underline{2}}\cdot\tilde{X}^3_\rho;\end{aligned} \tag{8.9}$$

$$\tilde{X}^1_\rho B^A(x^2\cdot c_{\underline{1}}\cdot\tilde{X}^2_\rho\otimes x^3\cdot c_{\underline{2}}\cdot\tilde{X}^3_\rho)\otimes B^H(x^2\cdot c_{\underline{1}}\cdot\tilde{X}^2_\rho\otimes x^3\cdot c_{\underline{2}}\cdot\tilde{X}^3_\rho)x^1 = \varepsilon_C(c)1_A\otimes 1_H, \tag{8.10}$$

for all $c, c' \in C$, $a \in A$ and $h \in H$.

${}_H\mathcal{W}^\#$ consists of k-linear maps $\mathbb{B} : C \otimes C \to A$ such that

$$a_{\langle 0,0\rangle}\mathbb{B}(h_1 \cdot c \cdot a_{\langle 0,1\rangle} \otimes h_2 \cdot c' \cdot a_{\langle 1\rangle}) = \varepsilon(h)\mathbb{B}(c \otimes c')a, \tag{8.11}$$

$$(\tilde{X}^1_\rho)_{\langle 0\rangle}\mathbb{B}(x^1 \cdot c \cdot (\tilde{X}^1_\rho)_{\langle 1\rangle} \otimes x^2 \cdot c'_{\underline{1}} \cdot \tilde{X}^2_\rho) \otimes x^3 \cdot c'_{\underline{2}} \cdot \tilde{X}^3_\rho$$
$$= \tilde{X}^1_\rho\mathbb{B}(X^2 \cdot c_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes X^3 \cdot c')_{\langle 0\rangle} \otimes X^1 \cdot c_{\underline{1}} \cdot \mathbb{B}(X^2 \cdot c_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes X^3 \cdot c')_{\langle 1\rangle} \tag{8.12}$$

$$\tilde{X}^1_\rho\mathbb{B}(c_{\underline{1}} \cdot \tilde{X}^2_\rho \otimes c_{\underline{2}} \cdot \tilde{X}^3_\rho) = \varepsilon_C(c)1_A, \tag{8.13}$$

for all $c, c' \in C$, $a \in A$ and $h \in H$.

Theorem 8.5. *Let H be a quasi-Hopf algebra with bijective antipode, A a right H-comodule algebra and C an H-bimodule coalgebra.*

(i) *We have a map $w : \mathcal{W} \to \mathcal{W}^\#$,*

$$w(\Sigma)(c \otimes c') = \Sigma(X^2 \cdot c \cdot \tilde{x}^2_\rho \otimes X^3 \cdot c' \cdot \tilde{x}^3_\rho)\tilde{x}^1_\rho \otimes X^1.$$

In the case where $A = H$, w corestricts to a bijection between $\mathcal{W}$ and the subset $\underline{\mathcal{W}}^\# \subset \mathcal{W}^\#$ consisting of $B \in \mathcal{W}^\#$ satisfying

$$B(c \otimes c') = (\varepsilon \otimes \varepsilon)B(X^2 \cdot c \cdot x^2 \otimes X^3 \cdot c' \cdot x^3)x^1 \otimes X^1,$$

for all $c, c' \in C$. The inverse of w is $w^{-1}(B) = (\varepsilon \otimes \varepsilon)B$.

(ii) *We have a map $w^\# : {}_H\mathcal{W}^\# \to \mathcal{W}^\#$,*

$$w^\#(\mathbb{B})(c \otimes c') = \mathbb{B}(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1.$$

$w^\#$ corestricts to an isomorphism between ${}_H\mathcal{W}^\#$ and the subset $\underline{\underline{\mathcal{W}}}^\# \subset \mathcal{W}^\#$ consisting of $B \in \mathcal{W}^\#$ satisfying

$$B(c \otimes c') = (A \otimes \varepsilon)B(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1,$$

for all $c, c' \in C$. The inverse of $w^\#$ is $(w^\#)^{-1}(B) = (A \otimes \varepsilon) \circ B$.

(iii) *If $A = H$, then $\underline{\mathcal{W}}^\# \subset \underline{\underline{\mathcal{W}}}^\#$. Consequently, if C is a coseparable coalgebra in ${}_H\mathcal{M}_H$, then it is a coseparable coalgebra in $\mathcal{T}({}_H\mathcal{M})^\#_H$, too.*

Proof. (i) Follows easily from Theorem 7.8 applied to the data in Proposition 8.2.
(ii) We first show that $B = w(\Sigma) \in \mathcal{W}^\#$. B satisfies (8.9) since

$$\begin{aligned}
&\tilde{X}^1_\rho\mathbb{B}(X^2x^3 \cdot c_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes X^3 \cdot c')_{\langle 0\rangle} \otimes X^1_1x^1 \otimes X^1_2x^2 \cdot c_{\underline{1}} \cdot \mathbb{B}(X^2x^3 \cdot c_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes X^3 \cdot c')_{\langle 1\rangle}\\
\overset{(1.17)}{=}\;& \tilde{X}^1_\rho\mathbb{B}(x^3_1X^2Y^2_2 \cdot c_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes x^3_2X^3Y^3 \cdot c')_{\langle 0\rangle} \otimes x^1Y^1\\
&\otimes x^2X^1Y^2_1 \cdot c_{\underline{1}} \cdot \tilde{X}^2_\rho\mathbb{B}(x^3_1X^2Y^2_2 \cdot c_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes x^3_2X^3Y^3 \cdot c')_{\langle 1\rangle}\\
\overset{(8.11)}{=}\;& \tilde{X}^1_\rho\mathbb{B}(X^2 \cdot (Y^2 \cdot c)_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes X^3 \cdot (Y^3 \cdot c'))_{\langle 0\rangle} \otimes Y^1\\
&\otimes X^1 \cdot (Y^2 \cdot c)_{\underline{1}} \cdot \tilde{X}^2_\rho\mathbb{B}(X^2 \cdot (Y^2 \cdot c)_{\underline{2}} \cdot \tilde{X}^3_\rho \otimes X^3 \cdot (Y^3 \cdot c'))_{\langle 1\rangle}\\
\overset{(8.12)}{=}\;& (\tilde{X}^1_\rho)_{\langle 0\rangle}\mathbb{B}(x^1Y^2 \cdot c \cdot (\tilde{X}^1_\rho)_{\langle 1\rangle} \otimes x^2Y^3_1 \cdot c'_{\underline{1}} \cdot \tilde{X}^2_\rho) \otimes Y^1 \otimes x^3Y^3_2 \cdot c'_{\underline{2}} \cdot \tilde{X}^3_\rho\\
\overset{(8.11)}{=}\;& (\tilde{X}^1_\rho)_{\langle 0\rangle}\mathbb{B}(y^2_1x^1Y^2 \cdot c \cdot (\tilde{X}^1_\rho)_{\langle 1\rangle} \otimes y^2_2x^2Y^3_1 \cdot c'_{\underline{1}} \cdot \tilde{X}^2_\rho) \otimes y^1Y^1 \otimes y^3x^3Y^3_2 \cdot c'_{\underline{2}} \cdot \tilde{X}^3_\rho\\
\overset{(1.17)}{=}\;& (\tilde{X}^1_\rho)_{\langle 0\rangle}\mathbb{B}(X^2x^1_2 \cdot c \cdot (\tilde{X}^1_\rho)_{\langle 1\rangle} \otimes X^3x^2 \cdot c'_{\underline{1}} \cdot \tilde{X}^2_\rho) \otimes X^1x^1_1 \otimes x^3 \cdot c'_{\underline{2}} \cdot \tilde{X}^3_\rho.
\end{aligned}$$

It is left to the reader to show that (8.11,1.16) imply (8.8), and that (8.13) implies (8.10). Moreover, $B(c \otimes c') = (A \otimes \varepsilon)B(X^2 \cdot c \otimes X^3 \cdot c') \otimes X^1$, so that $\mathrm{Im}(w^\#) \subset \underline{\underline{\mathcal{W}}}^\#$.
Conversely, if $B \in \underline{\underline{\mathcal{W}}}^\#$, then $\mathbb{B} = (A \otimes \varepsilon) \circ B \in {}_H\mathcal{W}^\#$. Applying $A \otimes \varepsilon \otimes C$ to (8.9) we find that

$$(\tilde{X}^1_\rho)_{\langle 0\rangle}\mathbb{B}(x^1 \cdot c \cdot (\tilde{X}^1_\rho)_{\langle 1\rangle} \otimes x^2 \cdot c'_{\underline{1}} \cdot \tilde{X}^2_\rho) \otimes x^3 \cdot c'_{\underline{2}} \cdot \tilde{X}^3_\rho$$

$$
\begin{aligned}
&= \tilde{X}^1_\rho B^A(c_{\underline{2}}\cdot \tilde{X}^3_\rho \otimes c')_{\langle 0\rangle} \otimes B^H(c_{\underline{2}}\cdot \tilde{X}^3_\rho \otimes c')\cdot c_{\underline{1}}\cdot \tilde{X}^2_\rho B^A(c_{\underline{2}}\cdot \tilde{X}^3_\rho \otimes c')_{\langle 1\rangle}\\
&\overset{(*)}{=} \tilde{X}^1_\rho \mathbb{B}(X^2\cdot c_{\underline{2}}\cdot \tilde{X}^3_\rho \otimes c')_{\langle 0\rangle} \otimes X^1\cdot c_{\underline{1}}\cdot \tilde{X}^2_\rho \mathbb{B}(X^2\cdot c_{\underline{2}}\cdot \tilde{X}^3_\rho \otimes c')_{\langle 1\rangle},
\end{aligned}
$$

as needed. At $(*)$, we used the formula

$$B(c\otimes c') = (A\otimes\varepsilon)B(X^2\cdot c\otimes X^3\cdot c')\otimes X^1 = \mathbb{B}(X^2\cdot c\otimes X^3\cdot c')\otimes X^1,$$

for all $c, c' \in C$. Applying $A\otimes\varepsilon$ to (8.8) and (8.10), we obtain (8.11) and (8.13), proving that $\mathbb{B}\in {}_H\mathcal{W}^\#$. It is straightforward to see that both constructions are inverses.
(iii) If $B\in \underline{\mathcal{W}}^\#$ then $B\in \underline{\underline{\mathcal{W}}}^\#$ since

$$
\begin{aligned}
&(H\otimes\varepsilon)B(X^2\cdot c\otimes X^3\cdot c')\otimes X^1\\
&\quad= (\varepsilon\otimes\varepsilon)B(Y^2X^2\cdot c\cdot x^2\otimes Y^3X^3\cdot c'\cdot x^3)(H\otimes\varepsilon)(x^1\otimes Y^1)\otimes X^1\\
&\quad= (\varepsilon\otimes\varepsilon)B(X^2\cdot c\cdot x^2\otimes X^3\cdot c'\cdot x^3)x^1\otimes X^1 = B(c\otimes c'),
\end{aligned}
$$

for all $c, c' \in C$. □

We will now focus on the situation where $A = C = H$. The coseparability of H as a coalgebra in ${}_H\mathcal{M}_H$ has been studied in [23, Sec. 7]. Following [23], a biinvariant form is a morphism $\Xi : H\otimes H\to k$ in ${}_H\mathcal{M}_H$. Ξ is called cocentral if

$$\Xi(X^2h_2x^2\otimes X^3h'x^3)X^1h_1x^1 = \Xi(x^1hX^1\otimes x^2h'_1X^2)x^3h'_2X^3,$$

for all $h, h'\in H$. Thus a biinvariant cocentral form in the sense of [23] is a Casimir morphism for H as a coalgebra in ${}_H\mathcal{M}_H$.
The comments following [23, Cor. 7.5] entail that the existence of a non-zero biinvariant cocentral form $\Xi : H\otimes H\to k$ for a quasi-Hopf algebra H with bijective antipode is equivalent to the unimodularity of H. In this situation, there is a bijective correspondence between biinvariant cocentral forms and left cointegrals λ on H, these are functionals $\lambda : H\to k$ satisfying

$$\lambda(V^2h_2U^2)V^1h_1U^1 = \mu(x^1)\lambda(hS(x^2))x^3, \tag{8.14}$$

for all $h\in H$, where μ is the modular element of H and

$$U = U^1\otimes U^2 = g^1S(q^2)\otimes g^2S(q^1) \text{ and } V = V^1\otimes V^2 = S^{-1}(f^2p^2)\otimes S^{-1}(f^1p^1). \tag{8.15}$$

$f = f^1\otimes f^2$ is the Drinfeld twist with inverse $f^{-1} = g^1\otimes g^2$, $q^1\otimes q^2 = X^1\otimes S^{-1}(\alpha X^3)X^2$ and $p^1\otimes p^2 = x^1\otimes x^2\beta S(x^3)$.
If H is finite dimensional then the the space $\mathcal{L}$ of left cointegrals is of dimension one. H is cosemisimple if and only if there exists $\lambda\in\mathcal{L}$ such that $\lambda(S^{-1}(\alpha)\beta) = 1$, see [23]. This leads to the following result.

Proposition 8.6. *Let H be a finite dimensional quasi-Hopf algebra. H is coseparable as a coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and cosemisimple.*

Proof. We have seen above that there exists a Casimir morphism for the coalgebra H in ${}_H\mathcal{M}_H$ if and only if H is unimodular. In this case, we have a bijection between $\mathcal{L}$ and the set of Casimir morphisms for H as a coalgebra in ${}_H\mathcal{M}_H$, see [23, Cor. 7.4]. The Casimir morphism Ξ corresponding to $\lambda\in\mathcal{L}$ is given by the formula

$$\Xi(h\otimes h') = \lambda(S^{-1}(\alpha)h\beta S(h')).$$

Thus H is a coseparable coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and there exists $\lambda\in\mathcal{L}$ such that $\lambda(S^{-1}(\alpha)h_1\beta S(h_2)) = \varepsilon(h)$, for all $h\in H$, or, equivalently, $\lambda(S^{-1}(\alpha)\beta) = 1$, that is, H is cosemisimple. □

We will now study the separability of the forgetful functor $F : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}_H$ in the case where H is a finite dimensional quasi-Hopf algebra.

Theorem 8.7. *For a finite dimensional quasi-Hopf algebra H, the following assertions are equivalent:*

(i) *the forgetful functor $F : {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}_H$ is separable;*
(ii) *H is unimodular;*
(iii) *F is a Frobenius functor.*

Proof. (i) $\Rightarrow$ (ii). Suppose that F is separable. Applying Corollary 7.6 (v) and Proposition 8.2 we obtain a morphism $\Lambda : H\otimes H^{\rm op}\otimes H \to H\otimes H^{\rm op}\otimes {}^*H$ in $\mathcal{M}(H\otimes H^{\rm op})^H_{H\otimes H^{\rm op}}$ which is left $H\otimes H^{\rm op}$-linear and satisfies a certain normalizing condition. Then Λ is left $H\otimes H^{\rm op}$-linear and a morphism in ${}_H\mathcal{M}_H^H$. As in the proof of Theorem 7.5 we deduce that for any right $H\otimes H^{\rm op}$-module M (or, equivalently, for any H-bimodule M), there exists a morphism $M\otimes H\to M\otimes {}^*H$ in ${}_H\mathcal{M}_H^H$. Applying Corollary 7.6 (v) to the Doi-Hopf datum described in Proposition 8.2, we find that $M\otimes H,\ M\otimes {}^*H \in {}_H\mathcal{M}_H^H$. The structure maps on $M\otimes H$ and $M\otimes {}^*H$ are given by the formulas

$$
\begin{aligned}
&h\cdot(m\otimes \hbar)\cdot h' = h_1\cdot m\cdot h'_2\otimes h_2\hbar h'_2;\\
&\rho(m\otimes \hbar) = x^1\cdot m\cdot X^1\otimes x^2h_1X^2\otimes x^3h_2X^3;\\
&h\cdot(m\otimes {}^*h)\cdot h' = h_1\cdot m\cdot h'_1\otimes S(h'_2)\rightharpoonup {}^*h\leftharpoonup S^{-1}(h_2);\\
&\rho(m\otimes {}^*h) = x^1\cdot m\cdot X^1\otimes \big(S(X^2)_1U^1X^3\rightharpoonup h^i\leftharpoonup x^3V^1S^{-1}(x^2)_1\big)\\
&\qquad\qquad \big(S(X^2)_2U^2\rightharpoonup {}^*h\leftharpoonup V^2S^{-1}(x^2)_2\big)\otimes h_i,
\end{aligned}
$$

for all $h,\hbar,h'\in H$, ${}^*h\in {}^*H$ and $m\in M$. Here $h_i\otimes h^i\in H\otimes {}^*H$ is the finite dual basis of H, U and V are the elements defined in (8.15) and $\rightharpoonup$ and $\leftharpoonup$ are the well-known canonical left and right actions of H on ${}^*H$, given by the formula $\langle h\rightharpoonup {}^*h\leftharpoonup h', h''\rangle = \langle {}^*h, h''h'h\rangle$.
Applying this to $M=k\in {}_H\mathcal{M}$ by restriction of scalars via ε, we obtain a morphism $H\to {}^*H$ in ${}_H\mathcal{M}_H^H$. $H\in {}_H\mathcal{M}_H^H$ via the multiplication and comultiplication, and the structure on ${}^*H$ is given by the formulas

$$h\cdot {}^*h\cdot h' = S(h')\rightharpoonup {}^*h\leftharpoonup S^{-1}(h) \text{ and } \rho({}^*h) = \big(U^1\rightharpoonup h^i\leftharpoonup V^1\big)\big(U^2\rightharpoonup {}^*h\leftharpoonup V^2\big)\otimes h_i, \tag{8.16}$$

for all $h,h'\in H$ and ${}^*h\in {}^*H$. Applying [23, Theorem 7.3], we find a Casimir morphism for H as a coalgebra in ${}_H\mathcal{M}_H$, and therefore H is unimodular.

(ii) $\Rightarrow$ (i). If H is unimodular, then there exists a non-zero left and right integral t in H and a non-zero left cointegral λ on H such that $\langle\lambda, S^{-1}(t)\rangle = 1$. By [9, Prop. 4.1] we have that

$$\lambda(q^2t_2p^2)q^1t_1p^1 = \lambda(S^{-1}(q^1t_1p^1))q^2t_2p^2 = 1. \tag{8.17}$$

In the sequel we will need the following formula, see [6, Lemma 3.3]:

$$\lambda(S^{-1}(h)h') = \mu(h_1)\lambda(h'S(h_2)), \tag{8.18}$$

for all $h,h'\in H$. So when H is unimodular we have $\lambda(S^{-1}(h)h') = \lambda(h'S(h))$, or, equivalently,

$$\lambda\leftharpoonup S^{-1}(h) = S(h)\rightharpoonup\lambda, \tag{8.19}$$

for all $h\in H$.
It follows from Corollary 7.6 (vii) and Proposition 8.2 that the separability of F is equivalent to the existence of a k-linear map $\overline{\Lambda} : H\to H\otimes H\otimes {}^*H$, $\overline{\Lambda}(h) = \overline{\Lambda}^1(h)\otimes\overline{\Lambda}^2(h)\otimes\overline{\Lambda}^3(h)$ such that:

$$
\begin{aligned}
&\overline{\Lambda}^1(\hbar)h_1\otimes h'_1\overline{\Lambda}^2(\hbar)\otimes S(h_2)\rightharpoonup\overline{\Lambda}^3(\hbar)\leftharpoonup S^{-1}(h'_2)\\
&\quad = h_1\overline{\Lambda}^1(h'_2\hbar h_2)\otimes\overline{\Lambda}^2(h'_2\hbar h_2)h'_1\otimes\overline{\Lambda}^3(h'_2\hbar h_2),
\end{aligned} \tag{8.20}
$$

$$
\begin{aligned}
&\overline{\Lambda}^3(x^2h_1X^2)(S^{-1}(Y^2)h'S(y^2))X^1\overline{\Lambda}^1(x^2h_1X^2)y^1\otimes Y^1\overline{\Lambda}^2(x^2h_1X^2)x^1\otimes Y^3x^3h_2X^3y^3\\
&\quad = \overline{\Lambda}^3(h)(V^2h'_2U^2)\overline{\Lambda}^1(h)\otimes\overline{\Lambda}^2(h)\otimes V^1h'_1U^1,
\end{aligned} \tag{8.21}
$$

$$\overline{\Lambda}^3(x^3h_2X^3)(q^2\overline{\Lambda}^2(x^3h_2X^3)_2x^2h_1X^2\overline{\Lambda}^1(x^3h_2X^3)_2p^2)X^1\overline{\Lambda}^1(x^3h_2X^3)_1p^1$$
$$\otimes\quad q^1\overline{\Lambda}^2(x^3h_2X^3)_1x^1 \;=\; \varepsilon(h)1\otimes 1, \tag{8.22}$$

for all $h, \hbar, h' \in H$. We will show that

$$\overline{\Lambda}:\ H\to H\otimes H\otimes {}^*H,\ \overline{\Lambda}(h)=1\otimes t\otimes S(h)\rightharpoonup\lambda$$

satisfies (8.20-8.22). (8.20) is satisfied since

$$\begin{aligned}
&\overline{\Lambda}^1(\hbar)h_1\otimes h_1'\overline{\Lambda}^2(\hbar)\otimes S(h_2)\rightharpoonup\overline{\Lambda}^3(\hbar)\leftharpoonup S^{-1}(h_2')\\
&=\ h_1\otimes h_1't\otimes S(\hbar h_2)\rightharpoonup\lambda\leftharpoonup S^{-1}(h_2')\overset{(8.19)}{=}h_1\otimes t\otimes S(h'\hbar h_2)\rightharpoonup\lambda\\
&=\ h_1\otimes th_1'\otimes S(h_2'\hbar h_2)\rightharpoonup\lambda = h_1\overline{\Lambda}^1(h_2'\hbar h_2)\otimes\overline{\Lambda}^2(h_2'\hbar h_2)h_1'\otimes\overline{\Lambda}^3(h_2'\hbar h_2),
\end{aligned}$$

for all $h, \hbar, h' \in H$. To prove (8.21), we first compute that

$$\begin{aligned}
&\overline{\Lambda}^3(x^2h_1X^2)(S^{-1}(Y^2)h'S(y^2))X^1\overline{\Lambda}^1(x^2h_1X^2)y^1\otimes Y^1\overline{\Lambda}^2(x^2h_1X^2)x^1\otimes Y^3x^3h_2X^3y^3\\
&=\ (S(x^2h_1X^2)\rightharpoonup\lambda)(S^{-1}(Y^2)h'S(y^2))X^1y^1\otimes Y^1tx^1\otimes Y^3x^3h_2X^3y^3\\
&=\ \lambda(h'S(h_1X^2y^2))X^1y^1\otimes t\otimes h_2X^3y^3=\lambda(h'S(h_1))1\otimes t\otimes h_2,
\end{aligned}$$

for all $h, h' \in H$. From [23, Lemma 3.13], we recall that the formula

$$U[1\otimes S(h)]=\Delta(S(h_1))U[h_2\otimes 1], \tag{8.23}$$

holds for all $h \in H$. This allows us to compute that

$$\begin{aligned}
&\overline{\Lambda}^3(h)(V^2h_2'U^2)\overline{\Lambda}^1(h)\otimes\overline{\Lambda}^2(h)\otimes V^1h_1'U^1=(S(h)\rightharpoonup\lambda)(V^2h_2'U^2)1\otimes t\otimes V^1h_1'U^1\\
&=\ \lambda\left(V^2(h'S(h_1))_2U^2\right)1\otimes t\otimes V^1(h'S(h_1))_1U^1h_2\overset{(8.14)}{=}\lambda(h'S(h_1))1\otimes t\otimes h_2,
\end{aligned}$$

for all $h, h' \in H$, and (8.21) follows. Finally

$$\begin{aligned}
&\overline{\Lambda}^3(x^3h_2X^3)(q^2\overline{\Lambda}^2(x^3h_2X^3)_2x^2h_1X^2\overline{\Lambda}^1(x^3h_2X^3)_2p^2)X^1\overline{\Lambda}^1(x^3h_2X^3)_1p^1\otimes q^1\overline{\Lambda}^2(x^3h_2X^3)_1x^1\\
&=\ (S(x^3h_2X^3)\rightharpoonup\lambda)(q^2t_2x^2h_1X^2p^2)X^1p^1\otimes q^1t_1x^1\\
&=\ \lambda(q^2t_2x^2h_1X^2p^2S(x^3h_2X^3))X^1p^1\otimes q^1t_1x^1=\lambda(q^2t_2x^2h_1\beta S(x^3h_2))1\otimes q^1t_1x^1\\
&=\ \varepsilon(h)\lambda(q^2t_2p^2)1\otimes q^1t_1p^1\overset{(8.17)}{=}\varepsilon(h)1\otimes 1,
\end{aligned}$$

for all $h \in H$, so that (8.22) holds.

□

Our conclusion is that the forgetful functor $F: {}_H\mathcal{M}_H^H\to {}_H\mathcal{M}_H$ is Frobenius if and only if H is a Frobenius coalgebra within ${}_H\mathcal{M}_H$, that is, if and only if H is finite dimensional and unimodular. But in the separable case we have a completely different situation, provided that H is a finite dimensional quasi-Hopf algebra: H is a coseparable coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular and cosemisimple while F is separable if and only if H is unimodular; so we might have F separable although H is not a coseparable coalgebra in ${}_H\mathcal{M}_H$. However the remarkable thing is that in the finite dimensional case F is separable if and only if it is Frobenius, since both properties reduce at the unimodularity property of H.

9. Application to Yetter-Drinfeld modules over quasi-Hopf algebras

The aim of this Section is to apply our results to the category of Yetter-Drinfeld modules over a quasi-Hopf algebra. In particular, we will be able to characterize when the algebra extension $H\to D(H)$, from a finite dimensional quasi-Hopf algebra H to its Drinfeld double $D(H)$ is Frobenius or separable.

We recall the notion of bicomodule algebra, as it was first introduced by Hausser and Nill in [22] under the name "quasi-commuting pair of H-coactions".

Definition 9.1. Let H be a quasi-bialgebra. An H-bicomodule algebra A is a sixtuple $(A,\lambda,\rho,\Phi_\lambda,\Phi_\rho,\Phi_{\lambda,\rho})$ such that (A,λ,Φ_λ) is a left H-comodule algebra, (A,ρ,Φ_ρ) is a right H-comodule algebra, and $\Phi_{\lambda,\rho}\in H\otimes A\otimes H$ is an invertible element such that the following compatibility relations hold, for all $u\in A$:

$$\begin{aligned}\Phi_{\lambda,\rho}(\lambda\otimes H)(\rho(u)) &= (H\otimes\rho)(\lambda(u))\Phi_{\lambda,\rho};\\ (1_H\otimes\Phi_{\lambda,\rho})(H\otimes\lambda\otimes H)(\Phi_{\lambda,\rho})(\Phi_\lambda\otimes 1_H) &= (H\otimes H\otimes\rho)(\Phi_\lambda)(\Delta\otimes \mathrm{Id}_{\mathbb{A}}\otimes\mathrm{Id}_H)(\Phi_{\lambda,\rho});\\ (1_H\otimes\Phi_\rho)(H\otimes\rho\otimes H)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho}\otimes 1_H) &= (H\otimes S\otimes\Delta)(\Phi_{\lambda,\rho})(\lambda\otimes H\otimes H)(\Phi_\rho).\end{aligned}$$

It is shown in [22] that the following additional relations hold in an H-bicomodule algebra A:

$$(H\otimes A\otimes\varepsilon)(\Phi_{\lambda,\rho})=1_H\otimes 1_A,\ \ (\varepsilon\otimes A\otimes H)(\Phi_{\lambda,\rho})=1_A\otimes 1_H.$$

$(H,\Delta,\Delta,\Phi,\Phi,\Phi)$ is an example of an H-bicomodule algebra. If A is an H-bicomodule algebra, then $(A^{\rm op},\lambda,\rho,\Phi_\lambda^{-1},\Phi_\rho^{-1},\Phi_{\lambda,\rho}^{-1})$ is an $H^{\rm op}$-bicomodule algebra.
We will use the following notation:

$$\Phi_{\lambda,\rho}=\Theta^1\otimes\Theta^2\otimes\Theta^3=\overline{\Theta}^1\otimes\overline{\Theta}^2\otimes\overline{\Theta}^3\ ;\ \Phi_{\lambda,\rho}^{-1}=\theta^1\otimes\theta^2\otimes\theta^3=\overline{\theta}^1\otimes\overline{\theta}^2\otimes\overline{\theta}^3.$$

Following [12] we introduce the notion of right Yetter-Drinfeld module over a quasi-bialgebra.

Definition 9.2. Let H be a quasi-bialgebra, let C be an H-bimodule coalgebra and let A an H-bicomodule algebra. A right (H,A,C)-Yetter-Drinfeld module is a right A-module M together with a k-linear map $\rho_M : M\to M\otimes C$, $\rho_M(m)=m_{(0)}\otimes m_{(1)}$, called the right C-coaction on M, such that $\varepsilon(m_{(1)})m_{(0)}=m$ and

$$\begin{aligned}&(m_{(0)}\cdot\Theta^2)_{(0)}\otimes\Theta^1\cdot(m_{(0)}\cdot\Theta^2)_{(1)}\otimes m_{(1)}\cdot\Theta^3\\ &\qquad= (m\cdot\tilde{X}^3_\lambda)_{(0)}\cdot\tilde{X}^1_\rho\otimes\tilde{X}^1_\lambda\cdot(m\cdot\tilde{X}^3_\lambda)_{(1)_{\underline{1}}}\cdot\tilde{X}^2_\rho\otimes\tilde{X}^2_\lambda\cdot(m\cdot\tilde{X}^3_\lambda)_{(1)_{\underline{2}}}\cdot\tilde{X}^3_\rho;\\ &m_{(0)}\cdot u_{<0>}\otimes m_{(1)}\cdot u_{\langle 1\rangle}=(m\cdot u_{[0]})_{(0)}\otimes u_{[-1]}\cdot(m\cdot u_{[0]})_{(1)},\end{aligned}$$

for all $m\in M$ and $u\in A$. $\mathcal{YD}(H)^C_A$ will be the category of right (H,A,C)-Yetter-Drinfeld modules and A-linear maps preserving the C-coaction.

The category of left-right Yetter-Drinfeld modules is isomorphic to a certain category of left-right Doi-Hopf modules, see [11]. A similar result for the category of right Yetter-Drinfeld modules can easily be deduced from this. For a right H-comodule algebra A, and a left H-module coalgebra C, we have an isomorphism of categories ${}_A\mathcal{M}(H)^C\cong\mathcal{M}(H^{\rm op})^C_{A^{\rm op}}$. For a right Yetter-Drinfeld datum (H,A,C) as in Definition 9.2, we have an isomorphism of categories

$$\mathcal{YD}(H)^C_A\cong{}_{A^{\rm op}}\mathcal{YD}(H)^C. \tag{9.1}$$

Combining these properties with [11, Theorem 3.8], we obtain the following isomorphisms of categories:

$$\mathcal{YD}(H)^C_A\cong{}_{A^{\rm op}}\mathcal{YD}(H^{\rm op})^C\cong{}_{A^{\rm op2}}\mathcal{M}(H\otimes H^{\rm op})^C\cong\mathcal{M}(H^{\rm op}\otimes H)^C_{A^{\rm op2op}},$$

where $A^{\rm op2}$ is the right $H\otimes H^{\rm op}$-comodule algebra associated to the $H^{\rm op}$-bicomodule algebra $A^{\rm op}$ as in [11, Prop. 3.3]; the H-bimodule coalgebra C is viewed as a right $H^{\rm op}\otimes H$-module coalgebra through the monoidal isomorphism of categories identification ${}_H\mathcal{M}_H\cong\mathcal{M}_{H^{\rm op}\otimes H}$. More precisely, if we denote $\underline{A}^2=A^{\rm op2op}$ then $\underline{A}^2=A$ as a k-algebra and it is a right $H^{\rm op}\otimes H$-comodule algebra with coaction

$$\underline{\rho}^2 : A\to A\otimes(H^{\rm op}\otimes H),\ \underline{\rho}^2(u)=u_{[0]_{\langle 0\rangle}}\otimes\Big(S(u_{[-1]})\otimes u_{[0]_{\langle 1\rangle}}\Big);$$

and

$$\Phi_{\underline{\rho}^2}=(\tilde{x}^3_\lambda)_{\langle 0\rangle}\tilde{X}^1_\rho\Theta^2_{\langle 0\rangle}\otimes\Big(S(\tilde{x}^2_\lambda\Theta)^1f^1\otimes(\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\tilde{X}^2_\rho\Theta^2_{\langle 1\rangle}\Big)\otimes\Big(S(\tilde{x}^1_\lambda)f^2\otimes(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3\Big).$$

C is a right $H^{\rm op}\otimes H$-module coalgebra, with right $H^{\rm op}\otimes H$-action given by $c\cdot(h\otimes h')=h\cdot c\cdot h'$, for all $h,h'\in H$ and $c\in C$.

By the opposite versions of [11, Lemmas 3.6 and 3.7], we have that the category isomorphism $F:\ \mathcal{YD}(H)_A^C\to\mathcal{M}(H^{\rm op}\otimes H)_{\underline{A}^2}^C$ is given by the following formulas. $F(M)=M$ as a right A-module, and the right C-coaction is given by

$$\rho'_M(m)=m_{(0')}\otimes m_{(1')}=(m\cdot\tilde{q}^2_\lambda)_{(0)}\otimes\tilde{q}^1_\lambda\cdot(m\cdot\tilde{q}^2_\lambda)_{(1)},$$

for all $m\in M$. Here $\tilde{q}_\lambda=\tilde{q}^1_\lambda\otimes\tilde{q}^2_\lambda=S(\tilde{x}^1_\lambda)\alpha\tilde{x}^2_\lambda\otimes\tilde{x}^3_\lambda\in H\otimes A$.

We present a description of the inverse G of F. For a right $(H^{\rm op}\otimes H,\underline{A}^2,C)$-Hopf module M, with right A-action $\cdot$ and right C-coaction ρ'_M, $\rho'_M(m)=m_{(0')}\otimes m_{(1')}\in M\otimes C$, we define $G(M)=M$ as a right A-module, with right C-coaction $\overline{\rho}_M:\ M\to M\otimes C$, given by the formula

$$\overline{\rho}_M(m)=m_{\overline{(0)}}\otimes m_{\overline{(1)}}=m_{(0')}\cdot(\tilde{p}^2_\lambda)_{\langle 0\rangle}\otimes S(\tilde{p}^1_\lambda)\cdot m_{(1')}\cdot(\tilde{p}^2_\lambda)_{\langle 1\rangle},\tag{9.2}$$

for all $m\in M$. Here $\tilde{p}_\lambda=\tilde{p}^1_\lambda\otimes\tilde{p}^2_\lambda=\tilde{X}^2_\lambda S^{-1}(\tilde{X}^1_\lambda\beta)\otimes\tilde{X}^3_\lambda\in H\otimes A$. G is the identity on morphisms.

The study of the Frobenius and separable properties of the forgetful functor $F:\mathcal{YD}(H)_A^C\to\mathcal{M}_A$ reduces to the study of the Frobenius and separable properties of the forgetful functor

$$F:\mathcal{M}(H^{\rm op}\otimes H)_{\underline{A}^2}^C\to\mathcal{M}_A.$$

Thus necessary and sufficient conditions for the Frobenius property or the separability of F can be obtained by applying Corollaries 7.2 and 7.6 to the Doi-Hopf datum $(H^{\rm op}\otimes H,\underline{A}^2,C)$.

Corollary 9.3. *Let H be a quasi-Hopf algebra, let C be an H-bimodule coalgebra and let A be an H-bicomodule algebra. The forgetful functor $F:\mathcal{YD}(H)_A^C\to\mathcal{M}_A$ is Frobenius if and only if there exists $t=u_i\otimes c_i\in A\otimes C$ such that*

$$uu_i\otimes c_i=u_iu_{[0]_{\langle 0\rangle}}\otimes S(u_{[-1]})c_iu_{[0]_{\langle 1\rangle}},$$

for all $u\in A$, and the map $\kappa:\ A\otimes{}^*C\to A\otimes C$,

$$\begin{aligned}\kappa(u\otimes{}^*c)=&\langle{}^*c,\tilde{X}^1_\lambda S(\theta^1_1(\tilde{X}^2_\lambda)_1\mathfrak{p}^1)f^2\cdot(c_i)_{\underline{2}}\cdot\theta^2_{\langle 1\rangle_2}(\tilde{x}^2_\rho)_2p^2S(\theta^3\tilde{x}^3_\rho)\rangle\\ &uu_i\theta^2_{\langle 0\rangle}\tilde{x}^1_\rho(\tilde{X}^3_\lambda)_{\langle 0\rangle}\otimes S(\theta^1_2(\tilde{X}^2_\lambda)_2\mathfrak{p}^2)f^1\cdot(c_i)_{\underline{1}}\cdot\theta^2_{\langle 1\rangle_1}(\tilde{x}^2_\rho)_1p^1(\tilde{X}^3_\lambda)_{\langle 1\rangle},\end{aligned}$$

is an isomorphism. As before, $f=f^1\otimes f^2$ is the Drinfeld's twist, $p_L=\mathfrak{p}^1\otimes\mathfrak{p}^2=X^2S^{-1}(X^1\beta)\otimes X^3$ and $p_R=p^1\otimes p^2=x^1\otimes x^2\beta S(x^3)$. Consequently, F is a Frobenius functor if C is a Frobenius coalgebra in ${}_H\mathcal{M}_H$.

Proof. Recall that $p_R^{H^{\rm op}}=q_R^H=q^1\otimes q^2=X^1\otimes S^{-1}(\alpha X^3)X^2$. Therefore $p_R^{H^{\rm op}\otimes H}=(q^1\otimes p^1)\otimes(q^2\otimes p^2)$. Moreover, p_R and q_R satisfy the property

$$p^1h\otimes p^2=h_{(1,1)}p^1\otimes h_{(1,2)}p^2S(h_2)\ ,\ \ hq^1\otimes q^2=q^1h_{(1,1)}\otimes S^{-1}(h_2)q^2h_{(1,2)},\tag{9.3}$$

for all $h\in H$. This allows us to compute that

$$\begin{aligned}&\tilde{x}^1_{\underline{\rho}^2}\otimes(\tilde{x}^2_{\underline{\rho}^2})_1(q^1\otimes p^1)\otimes(\tilde{x}^2_{\underline{\rho}^2})_2(q^2\otimes p^2)(S^{-1}\otimes S)(\tilde{x}^3_{\underline{\rho}^2})\\ =\ &\theta^2_{\langle 0\rangle}\tilde{x}^1_\rho(\tilde{X}^3_\lambda)_{\langle 0\rangle}\otimes\Big(q^1g^1_1S(\theta^1\tilde{X}^2_\lambda)_1\otimes\theta^2_{\langle 1\rangle_1}(\tilde{x}^2_\rho)_1(\tilde{X}^3_\lambda)_{\langle 1\rangle_{(1,1)}}p^1\Big)\\ &\otimes\Big(\tilde{X}^1_\lambda S^{-1}(g^2)q^2g^1_2S(\theta^1\tilde{X}^2_\lambda)_2\otimes\theta^2_{\langle 1\rangle_2}(\tilde{x}^2_\rho)_2(\tilde{X}^3_\lambda)_{\langle 1\rangle_{(1,2)}}p^2S(\theta^3\tilde{x}^3_\rho(\tilde{X}^3_\lambda)_{\langle 1\rangle_2})\Big)\\ \overset{(9.3)}{\underset{(1.21)}{=}}\ &\theta^2_{\langle 0\rangle}\tilde{x}^1_\rho(\tilde{X}^3_\lambda)_{\langle 0\rangle}\otimes\Big(q^1g^1_1G^1S(\theta^1_2(\tilde{X}^2_\lambda)_2)f^1\otimes\theta^2_{\langle 1\rangle_1}(\tilde{x}^2_\rho)_1p^1(\tilde{X}^3_\lambda)_{\langle 1\rangle}\Big)\\ &\otimes\Big(\tilde{X}^1_\lambda S^{-1}(g^2)q^2g^1_2G^2S(\theta^1_1(\tilde{X}^2_\lambda)_1)f^2\otimes\theta^2_{\langle 1\rangle_2}(\tilde{x}^2_\rho)_2p^2S(\theta^3\tilde{x}^3_\rho)\Big)\\ \overset{(*)}{=}\ &\theta^2_{\langle 0\rangle}\tilde{x}^1_\rho(\tilde{X}^3_\lambda)_{\langle 0\rangle}\otimes\Big(S(\theta^1_2(\tilde{X}^2_\lambda)_2\mathfrak{p}^2)f^1\otimes\theta^2_{\langle 1\rangle_1}(\tilde{x}^2_\rho)_1p^1(\tilde{X}^3_\lambda)_{\langle 1\rangle}\Big)\end{aligned}$$

$$\otimes\left(\tilde{X}^1_\lambda S(\theta^1_1(\tilde{X}^2_\lambda)_1\mathfrak{p}^1)f^2 \otimes \theta^2_{\langle 1\rangle_2}(\tilde{x}^2_\rho)_2 p^2 S(\theta^3\tilde{x}^3_\rho)\right),$$

in $A\otimes(H^{\rm op}\otimes H)^{\otimes 2}$. Here $f^{-1}=G^1\otimes G^2$ is a second copy of f^{-1}. At $(*)$, we used the equality

$$q^1g^1_1G^1\otimes S^{-1}(g^2)q^2g^1_2G^2 = S(\mathfrak{p}^2)\otimes S(\mathfrak{p}^1), \tag{9.4}$$

see [9, (4.13)]. The first part in the statement is an immediate consequence of Corollary 7.2, applied to $(H^{\rm op}\otimes H, \underline{A}^2, C)$. The second part can be deduced easily from Proposition 7.4, using the monoidal category isomorphism ${}_H\mathcal{M}_H\cong\mathcal{M}_{H^{\rm op}\otimes H}$. □

The natural question arises whether the converse of the final statement in Corollary 9.3 holds: is C a Frobenius coalgebra in ${}_H\mathcal{M}_H$ if F is Frobenius? Theorem 7.5 provides an answer to questions of this type, but, unfortunately, it cannot be applied in our situation. We have an algebra map $\varepsilon:\ H\to k$, but the associated map $\tilde{\varepsilon}:\ H\to H^{\rm op}\otimes H$, $\tilde{\varepsilon}(h)=S(h_1)\otimes h_2$, is not surjective. However, in the case where $C=H$, we obtain an affirmative answer to the question, using the structure theorem for quasi-Hopf algebras. We need some preliminary results first.

Lemma 9.4. *Let H be a finite dimensional quasi-Hopf algebra and let μ be its modular element. H_μ is the vector space equipped with left and right H-action and right H-coaction ρ given by the formulas*

$$h\cdot\hbar\cdot h' = \mu(h_1)h_2\hbar h'\ ;\ \rho(\hbar)=\mu(x^1)x^2\hbar_1\otimes x^3\hbar_2.$$

H_μ is a quasi-Hopf H-bimodule. If λ is a non-zero left cointegral on H then

$$\zeta:\ H_\mu\to {}^*H,\ \xi(\hbar)=S(\hbar)\rightharpoonup\lambda$$

is an isomorphism in ${}_H\mathcal{M}^H_H$, with inverse given by the formula

$$\zeta^{-1}({}^*h)=\langle {}^*h, S^{-1}(q^1t_1p^1)\rangle S^{-1}(q^2t_2p^2),$$

where t is a left integral in H such that $\lambda(S^{-1}(t))=1$.

Proof. Recall from (8.16) that ${}^*H\in {}_H\mathcal{M}^H_H$. It follows from the comments made before Definition 5.3 in [23] that H_μ is a quasi-Hopf H-bimodule and that ζ is an isomorphism in ${}_H\mathcal{M}^H_H$. Our present contribution is the explicit description of ζ^{-1}. Using (9.3) and (8.17), one can verify that ζ and ζ^{-1} are inverses. □

Let H be a quasi-bialgebra. $\mathcal{YD}^H_H$ will be a short notation for $\mathcal{YD}(H)^H_H$.

Lemma 9.5. *Let H be a quasi-Hopf algebra. If the forgetful functor $F:\mathcal{YD}^H_H\to\mathcal{M}_H$ is Frobenius then H is finite dimensional and there exists an element $t\in H$ such that*

$$\mu(h_{(2,1)})S(h_1)th_{(2,2)}=\mu(h)t, \tag{9.5}$$

for all $h\in H$, where μ is the modular element of H. Furthermore, the map $\Upsilon: {}^*H\to H$,

$$\Upsilon({}^*h)=\langle\mu, y^2_1x^1X^3_1\rangle\ \langle {}^*h, X^1S(y^1_1X^2_1\mathfrak{p}^1)f^2t_2y^2_{(2,2)}x^2_2p^2S(y^3x^3)\rangle\ S(y^1_2X^2_2\mathfrak{p}^2)f^1t_1y^2_{(2,1)}x^2_1p^1X^3_2$$

is an isomorphism and satisfies

$$\langle\mu, h_{(2,1)}\rangle S(h_1)\Upsilon({}^*h)h_{(2,2)} = \langle\mu, h_{(2,1)}\rangle\Upsilon(S(h_{(2,2)})\rightharpoonup{}^*h\leftharpoonup h_1), \tag{9.6}$$

for all ${}^*h\in{}^*H$ and $h\in H$.

Proof. It follows from Proposition 7.1 that H is finite dimensional if F is Frobenius. Then the forgetful functor $\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}\to\mathcal{M}_H$ is also Frobenius. Applying the first part of Theorem 7.5 to the algebra map $\mu: H\to k$, we find $t\in H$ obeying $\mu(h_{(2,1)})S(h_1)th_{(2,2)}=\mu(h)t$, for all $h\in H$, and such that $\Upsilon:\ {}^*H\to H$,

$$\Upsilon({}^*h)=\langle\mu, y^2_1x^1X^3_1\rangle\ \langle {}^*h, X^1S(y^1_1X^2_1\mathfrak{p}^1)f^2t_2y^2_{(2,2)}x^2_2p^2S(y^3x^3)\rangle S(y^1_2X^2_2\mathfrak{p}^2)f^1t_1y^2_{(2,1)}x^2_1p^1X^3_2$$

is an isomorphism. Note that we made also use of the computation performed in the proof of Corollary 9.3, applied to the Doi-Hopf datum $(H^{\rm op}\otimes H, \underline{H}^2, H)$.
In order to prove (9.6), we compute that

$$\begin{aligned}
\langle \mu, h_{(2,1)}\rangle\ S(h_1)\Upsilon(^*h)h_{(2,2)} &= \langle \mu, y_1^2x^1(X^3h_2)_1)\rangle\ \langle ^*h, X^1S(y_1^1X_1^2\mathfrak{p}^1)f^2t_2y^2_{(2,2)}x_2^2p^2S(y^3x^3)\rangle\\
&\qquad S(y_2^1X_2^2\mathfrak{p}^2h_1)f^1t_1y^2_{(2,1)}x_1^2p^1(X^3h_2)_2\\
&\overset{(7.28,1.16)}{=} \langle \mu, y_1^2x^1h_{(2,2)_1}X_1^3\rangle\ \langle ^*h, h_1X^1S((y^1h_{(2,1)})_1X_1^2\mathfrak{p}^1)f^2t_2y^2_{(2,2)}x_2^2p^2S(y^3x^3)\rangle\\
&\qquad S((y^1h_{(2,1)})_2X_2^2\mathfrak{p}^2)f^1t_1y^2_{(2,1)}x_1^2p^1h_{(2,2)_2}X_2^3\\
&\overset{(9.3,1.16)}{=} \langle ^*h, h_1X^1S((y^1(h_2)_1)_1X_1^2\mathfrak{p}^1)f^2t_2(y^2(h_2)_{(2,1)})_{(2,2)}x_2^2p^2S(y^3(h_2)_{(2,2)}x^3)\rangle\\
&\qquad \mu((y^2(h_2)_{(2,1)})_1x^1X_1^3)S((y^1(h_2)_1)_2X_2^2\mathfrak{p}^2)f^1t_1(y^2(h_2)_{(2,1)})_{(2,1)}x_1^2p^1X_2^3\\
&\overset{(1.16,1.21)}{=} \langle ^*h, h_1X^1S(y_1^1X_1^2\mathfrak{p}^1)f^2(S(h_{(2,1)_1})th_{(2,1)_{(2,2)}})_2y^2_{(2,2)}x_2^2p^2S(h_{(2,2)}y^3x^3)\rangle\\
&\qquad \langle \mu, h_{(2,1)_{(2,1)}}y_1^2x^1X_1^3\rangle S(y_2^1X_2^2\mathfrak{p}^2)f^1(S(h_{(2,1)_1})th_{(2,1)_{(2,2)}})_1y^2_{(2,1)}x_1^2p^1X_2^3\\
&\overset{(9.5)}{=} \langle \mu, h_{(2,1)}y_1^2x^1X_1^3\rangle\langle ^*h, h_1X^1S(y_1^1X_1^2\mathfrak{p}^1)f^2t_2y^2_{(2,2)}x_2^2p^2S(h_{(2,2)}y^3x^3)\rangle\\
&\qquad S(y_2^1X_2^2\mathfrak{p}^2)f^1t_1y^2_{(2,1)}x_1^2p^1X_2^3\\
&= \langle \mu, h_{(2,1)}\rangle\Upsilon(S(h_{(2,2)})\rightharpoonup {}^*h\leftharpoonup h_1),
\end{aligned}$$

for all $h\in H$ and $^*h\in {}^*H$, finishing the proof of the Lemma. □

We will need the following formulas, see [23], and [9, (4.14)].

$$(9.7)\qquad [1\otimes S^{-1}(h)]V = [h_2\otimes 1]V\Delta(S^{-1}(h_1)),$$

$$(9.8)\qquad q_R = [\tilde{q}^2\otimes 1]V\Delta(S^{-1}(\tilde{q}^1)),$$

$$(9.9)\qquad p_R = \Delta(S(\tilde{p}^1))U[\tilde{p}^2\otimes 1],\ \text{and}$$

$$(9.10)\qquad S(g^1)\mathfrak{q}^1g_1^2\otimes \mathfrak{q}^2g_2^2 = S(p^2)f^1\otimes S(p^1)f^2.$$

By the definitions of p_R and p_L and using (1.17) we obtain that

$$(9.11)\qquad X_1^1p^1\otimes X_2^1p^2S(X^2)\otimes X^3 = x^1\otimes x^2S(x_1^3\mathfrak{p}^1)\otimes x_2^3\mathfrak{p}^2\ \text{and}$$

$$(9.12)\qquad y^1\mathfrak{p}^1\otimes y^2\mathfrak{p}_1^2\otimes y^3\mathfrak{p}_2^2 = X_1^2\mathfrak{p}^1S^{-1}(X^1)\otimes X_2^2\mathfrak{p}^2\otimes X^3.$$

Theorem 9.6 generalizes [20, Theorem 4.2] to the quasi-Hopf algebra setting. Note that our approach is different from the one in [20].

Theorem 9.6. *For a quasi-Hopf algebra H, the following assertions are equivalent:*
- (i) *The forgetful functor $F:\mathcal{YD}_H^H\to\mathcal{M}_H$ is Frobenius;*
- (ii) *H is finite dimensional and unimodular;*
- (iii) *H is finite dimensional and Frobenius as a coalgebra in ${}_H\mathcal{M}_H$.*

Proof. <u>*(ii) ⇔ (iii)*</u>. If H is finite dimensional then H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$ if and only if H is unimodular, see Proposition 8.6.
<u>*(i) ⇒ (ii)*</u>. It follows from Lemma 9.5 that H is finite dimensional, and that there exists $t\in H$ satisfying (9.5) and such that $\Upsilon:\ {}^*H\to H$ defined in (9.6) is an isomorphism. For all $^*h\in{}^*H$, we have that

$$\begin{aligned}
\Upsilon(^*h) &\overset{(1.21,9.3)}{=} \langle \mu, y_1^2x^1X_1^3\rangle\ \langle ^*h, X^1S(\mathfrak{p}^1)f^2(S(y^1X^2)\mathfrak{t}y_2^2x^2X^3_{(2,1)})_2p^2S(y^3x^3X^3_{(2,2)})\rangle\\
&\qquad S(\mathfrak{p}^2)f^1(S(y^1X^2)\mathfrak{t}y_2^2x^2X^3_{(2,1)})_1p^1\\
&\overset{(9.4,1.16)}{=} \langle \mu, y_1^2X^3_{(1,1)}x^1\rangle\ \langle ^*h, S^{-1}(g^2S(X^1))q^2(g^1S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2)_2p^2S(y^3X_2^3x^3)\rangle\\
&\qquad q^1(g^1S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2)_1p^1,
\end{aligned}$$

Let $\zeta:\ H\to {}^*H$ be the isomorphism defined in Lemma 9.4. For all $h\in H$, we have that

$$
\begin{aligned}
(\Upsilon\circ\zeta)(h) = {}& \langle\mu, y_1^2X^3_{(1,1)}x^1\rangle\rangle\ \langle\lambda, S^{-1}(g^2S(X^1))q^2(g^1S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2)_2p^2S(hy^3X_2^3x^3)\rangle\\
& q^1(g^1S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2)_1p^1\\
\overset{(9.8,9.9)}{=}{}& \langle\lambda, S^{-1}(g^2S(X^1))V^2(S^{-1}(\mathfrak{q}^1)g^1S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2S(\mathfrak{p}^1))_2U^2S(hy^3X_2^3x^3)\rangle\\
& \langle\mu, y_1^2X^3_{(1,1)}x^1\rangle\ \mathfrak{q}^2V^1(S^{-1}(\mathfrak{q}^1)g^1S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2S(\mathfrak{p}^1))_1U^1\mathfrak{p}^2\\
\underset{(8.14)}{\overset{(8.23,9.7)}{=}}{}& \langle\lambda, S^{-1}(\mathfrak{q}^1g_1^2S(X^1)_1)g^1S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2S(z^2h_1y_1^3X^3_{(2,1)}x_1^3\mathfrak{p}^1))\rangle\\
& \langle\mu, z^1y_1^2X^3_{(1,1)}x^1\rangle\ \mathfrak{q}^2g_2^2S(X^1)_2z^3h_2y_2^3X^3_{(2,2)}x_2^3\mathfrak{p}^2\\
\overset{(1.21,9.10)}{=}{}& \langle\lambda, S^{-1}(f^1)X_2^1p^2S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2S(z^2h_1y_1^3X^3_{(2,1)}x_1^3\mathfrak{p}^1))\rangle\\
& \langle\mu, z^1y_1^2X^3_{(1,1)}x^1\rangle\ S(X_1^1p^1)f^2z^3h_2y_2^3X^3_{(2,2)}x_2^3\mathfrak{p}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\mu,h_1\rangle(\Upsilon\circ\zeta)(h_2) \overset{(1.16)}{=}{}& \langle\lambda, S^{-1}(f^1)X_2^1p^2S(y^1X^2)\mathfrak{t}y_2^2X^3_{(1,2)}x^2S(h_{(1,2)}z^2y_1^3X^3_{(2,1)}x_1^3\mathfrak{p}^1))\rangle\\
& \langle\mu, z^1h_{(1,1)}y_1^2X^3_{(1,1)}x^1\rangle\ S(X_1^1p^1)f^2h_2z^3y_2^3X^3_{(2,2)}x_2^3\mathfrak{p}^2\\
\overset{(8.18,9.5)}{=}{}& \langle\lambda, S^{-1}(f^1h_1)X_2^1p^2S(z_1^1y^1X^2)\mathfrak{t}(z_2^1y^2)_2X^3_{(1,2)}x^2S(z^2y_1^3X^3_{(2,1)}x_1^3\mathfrak{p}^1))\rangle\\
& \langle\mu, (z_2^1y^2)_1X^3_{(1,1)}x^1\rangle\ S(X_1^1p^1)f^2h_2z^3y_2^3X^3_{(2,2)}x_2^3\mathfrak{p}^2\\
\overset{(1.17,1.16)}{=}{}& \langle\lambda, S^{-1}(f^1h_1)X_2^1p^2S(z^1y^1X^2)\mathfrak{t}z_2^2y^2_{(1,2)}X^3_{(1,1)_2}w_2^1x^2S(z^3y_2^2X^3_{(1,2)}w^2x_1^3\mathfrak{p}^1))\rangle\\
& \langle\mu, z_1^2y^2_{(1,1)}X^3_{(1,1)_1}w_1^1x^1\rangle\ S(X_1^1p^1)f^2h_2y^3X_2^3w^3x_2^3\mathfrak{p}^2\\
\overset{(9.11)}{=}{}& \langle\lambda, S^{-1}(f^1h_1)X_2^1p^2S(z^1y^1X^2)\mathfrak{t}z_2^2(y^2X_1^3)_{(1,2)}P^2S(z^3(y^2X_1^3)_2)\rangle\\
& \langle\mu, z_1^2(y^2X_1^3)_{(1,1)}P^1\rangle S(X_1^1p^1)f^2h_2y^3X_2^3\\
\overset{(9.3)}{=}{}& \langle\lambda, S^{-1}(f^1h_1)X_2^1p^2S(z^1y^1X^2)\mathfrak{t}z_2^2P^2S(z^3)\rangle\\
& \langle\mu, z_1^2P^1y^2X_1^3\rangle\ S(X_1^1p^1)f^2h_2y^3X_2^3\\
\overset{(9.11,1.16)}{=}{}& \langle\lambda, S^{-1}(f^1h_1)x^2S(z^1x^3_{(1,1)}y^1\mathfrak{p}^1)\mathfrak{t}z_2^2P^2S(z^3)\rangle\\
& \langle\mu, z_1^2P^1x^3_{(1,2)}y^2\mathfrak{p}_1^2\rangle\ S(x^1)f^2h_2x_2^3y^3\mathfrak{p}_2^2,
\end{aligned}
$$

where $p_R=P^1\otimes P^2$ a second copy of p_R. Taking $h=\alpha$, we find that

$$
\begin{aligned}
\langle\mu,\alpha_1\rangle(\Upsilon\circ\zeta)(\alpha_2) \overset{(1.24)}{=}{}& \langle\lambda, S^{-1}(\gamma^1)x^2S(z^1x^3_{(1,1)}y^1\mathfrak{p}^1)\mathfrak{t}z_2^2P^2S(z^3)\rangle\ \langle\mu, z_1^2P^1x^3_{(1,2)}y^2\mathfrak{p}_1^2\rangle\ S(x^1)\gamma^2x_2^3y^3\mathfrak{p}_2^2\\
\overset{(1.22,8.18)}{=}{}& \langle\lambda, S^{-1}(\alpha)w^1X^2x^2S(z^1x^3_{(1,1)}y^1\mathfrak{p}^1)\mathfrak{t}z_2^2P^2S((w^2X_1^3)_2z^3)\rangle\\
& \langle\mu, (w^2X_1^3)_1z_1^2P^1x^3_{(1,2)}y^2\mathfrak{p}_1^2\rangle\ S(X^1x^1)\alpha w^3X_2^3x_2^3y^3\mathfrak{p}_2^2\\
\overset{(9.5,1.16)}{=}{}& \langle\lambda, S^{-1}(\alpha)w^1X^2x^2S(z^1(w^2X_1^3)_1x^3_{(1,1)}y^1\mathfrak{p}^1)\mathfrak{t}z_2^2(w^2X_1^3)_{(2,1)_2}P^2S(z^3(w^2X_1^3)_{(2,2)})\rangle\\
& \langle\mu, z_1^2(w^2X_1^3)_{(2,1)_1}P^1x^3_{(1,2)}y^2\mathfrak{p}_1^2\rangle S(X^1x^1)\alpha w^3X_2^3x_2^3y^3\mathfrak{p}_2^2\\
\overset{(9.3)}{=}{}& \langle\lambda, S^{-1}(\alpha)w^1S(z^1w_1^2y^1\mathfrak{p}^1)\mathfrak{t}z_2^2P^2S(z^3))\rangle\ \langle\mu, z_1^2P^1w_2^2y^2\mathfrak{p}_1^2\rangle\ \alpha w^3y^3\mathfrak{p}_2^2\\
\overset{(9.12)}{=}{}& \langle\mu, z_1^2P^1\mathfrak{p}^2\rangle\ \langle\lambda, S^{-1}(\alpha)S(z^1\mathfrak{p}^1)\mathfrak{t}z_2^2P^2S(z^3)\rangle\alpha = x\alpha,
\end{aligned}
$$

with

$$x=\langle\mu, z_1^2P^1\mathfrak{p}^2\rangle\ \langle\lambda(S^{-1}(\alpha)S(z^1\mathfrak{p}^1)\mathfrak{t}z_2^2P^2S(z^3)\rangle\in k. \tag{9.13}$$

Let $\hbar=(\Upsilon\circ\zeta)^{-1}(\alpha)$. Since $\mu(\alpha_1)(\Upsilon\circ\zeta)(\alpha_2)=x\alpha=x\Upsilon\zeta(\hbar)$, we have that $\mu(\alpha_1)\alpha_2=x\hbar$, and therefore $0\neq\mu(\alpha)=x\varepsilon(\hbar)$, and $x\neq 0$. We have now all the ingredients to prove that H is unimodular, that is, $\mu=\varepsilon$. Observe that

$$(\varepsilon\circ\Upsilon)({}^*h) \quad = \quad \langle\mu, y_1^2x^1X^3\rangle\ \langle{}^*h, X^1S(y^1X^2S^{-1}(\beta))\mathfrak{t}y_2^2x^2\beta S(y^3x^3)\rangle$$

$$= \langle \mu, y_1^2 p^1 \mathfrak{p}^2\rangle \ \langle {}^*h, S(y^1\mathfrak{p}^1)\mathfrak{t}y_2^2p^2S(y^3)\rangle,$$

for all ${}^*h \in {}^*H$. Thus $(\varepsilon\circ\Upsilon)(\lambda\leftharpoonup S^{-1}(\alpha)) = x$. Applying ε to (9.6) we obtain that

$$\begin{aligned}\langle\mu, h\rangle(\varepsilon\circ\Upsilon)({}^*h) &= \langle\mu, h_{(2,1)}\rangle(\varepsilon\circ\Upsilon)(S(h_{(2,2)})\rightharpoonup{}^*h\leftharpoonup h_1)\\ &= \langle\mu, h_{(2,1)}y_1^2p^1\mathfrak{p}^2\rangle\ \langle{}^*h, h_1S(y^1\mathfrak{p}^1)\mathfrak{t}y_2^2p^2S(h_{(2,2)}y^3)\rangle\end{aligned}$$

for all ${}^*h\in{}^*H$ and $h\in H$. Taking ${}^*h = \lambda\leftharpoonup S^{-1}(\alpha) = \langle\mu,\alpha_1\rangle S(\alpha_2)\rightharpoonup\lambda$, we conclude that

$$\begin{aligned}\langle\mu,h\rangle x &= \langle\mu,h\rangle(\varepsilon\circ\Upsilon)(\lambda\leftharpoonup S^{-1}(\alpha)) = \langle\mu,(\alpha h_2)_1y_1^2p^1\mathfrak{p}^2\rangle\langle\lambda, h_1S(y^1\mathfrak{p}^1)\mathfrak{t}y_2^2p^2S((\alpha h_2)_2y^3)\rangle\\ &\overset{(8.18)}{=} \langle\mu, y_1^2p^1\mathfrak{p}^2)\rangle\ \langle\lambda, S^{-1}(\alpha h_2)h_1S(y^1\mathfrak{p}^1)\mathfrak{t}y_2^2p^2S(y^3)\rangle = \varepsilon(h)x,\end{aligned}$$

for all $h\in H$, and it follows that $\mu=\varepsilon$ since $x\neq 0$.
<u>*(ii)* $\Rightarrow$ *(i)*</u>. If H is finite dimensional and unimodular, then H is a Frobenius coalgebra in ${}_H\mathcal{M}_H$, see Proposition 8.6. It follows from Proposition 7.4 that the forgetful functor $\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}\to\mathcal{M}_H$ is Frobenius, and therefore F is a Frobenius functor as well since $\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}$ and $\mathcal{YD}^H_H$ are isomorphic. □

We now focus on the separability of the forgetful functor $F:\ \mathcal{YD}(H)^C_A\to\mathcal{M}_A$. Applying Corollary 7.6 and using the isomorphism between the categories $\mathcal{YD}(H)^C_A$ and $\mathcal{M}(H^{\rm op}\otimes H)^C_{\underline{A}^2}$, we obtain necessary and sufficient conditions for the separability of F.

Proposition 9.7. *Let H be a quasi-Hopf algebra, let A be an H-bicomodule algebra, and let C be a finite dimensional H-bimodule coalgebra with dual basis $c_j\otimes c^j\in C\otimes{}^*C$. Then the forgetful functor $F:\mathcal{YD}(H)^C_A\to\mathcal{M}_A$ is separable if and only if there exists a k-linear map $\overline{\Lambda}: C\to A\otimes{}^*C$, $\overline{\Lambda}(c)=\overline{\Lambda}^1(c)\otimes\overline{\Lambda}^2(c)\in A\otimes{}^*C$, such that*

$$\begin{aligned}&\overline{\Lambda}^1(c)u_{[0]_{\langle 0\rangle}}\otimes S(u_{[0]_{\langle 1\rangle}})\rightharpoonup\overline{\Lambda}^2(c)\leftharpoonup u_{[-1]}\\ &\quad= u_{[0]_{\langle 0\rangle}}\overline{\Lambda}^1(S(u_{[-1]})\cdot c\cdot u_{[0]_{\langle 1\rangle}})\otimes\overline{\Lambda}^2(S(u_{[-1]})\cdot c\cdot u_{[0]_{\langle 1\rangle}}),\\ &(\tilde{x}^3_\lambda)_{\langle 0\rangle}\tilde{X}^1_\rho\Theta^2_{\langle 0\rangle}\overline{\Lambda}^1(S(\tilde{x}^2_\lambda\Theta^1)f^1\cdot c_{\underline{1}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\tilde{X}^2_\rho\Theta^2_{\langle 1\rangle})\otimes\overline{\Lambda}^2(S(\tilde{x}^2_\lambda\Theta^1)f^1\cdot c_{\underline{1}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\tilde{X}^2_\rho\Theta^2_{\langle 1\rangle})\\ &\quad\otimes\ S(\tilde{x}^1_\lambda)f^2\cdot c_{\underline{2}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3 = \overline{\Lambda}^1(c)(\tilde{x}^3_\lambda\tilde{q}^1_\rho\Theta^2_{\langle 0\rangle})_{\langle 0\rangle}\tilde{x}^1_\rho\otimes\Big(g^1S(\Theta^2_{\langle 1\rangle}\tilde{x}^3_\rho)\tilde{q}^2_\rho\Theta^3\rightharpoonup c^j\\ &\qquad\leftharpoonup S(\tilde{x}^1_\lambda)\mathfrak{q}^1(\tilde{x}^2_\lambda)_1\Theta^1_1\Big)\Big(g^2S((\tilde{x}^3_\lambda\tilde{q}^1_\rho\Theta^2_{\langle 0\rangle})_{\langle 1\rangle}\tilde{x}^2_\rho)\rightharpoonup\overline{\Lambda}^2(c)\leftharpoonup\mathfrak{q}^2(\tilde{x}^2_\lambda)_2\Theta^1_2\Big)\otimes c_i,\\ &\overline{\Lambda}^2(S(\tilde{x}^1_\lambda)f^2\cdot c_{\underline{2}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3)\Big(S(\tilde{x}^2_\lambda)\Theta^1\overline{\Lambda}^1(S(\tilde{x}^1_\lambda)f^2\cdot c_{\underline{2}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3)_{[-1]}\theta^1\tilde{p}^1_\lambda)f^1\\ &\quad\cdot c_{\underline{1}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\tilde{X}^2_\rho\Theta^2_{\langle 1\rangle}\overline{\Lambda}^1(S(\tilde{x}^1_\lambda)f^2\cdot c_{\underline{2}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3)_{[0]_{\langle 1\rangle}}\theta^2_{\langle 1\rangle}\tilde{p}^2_\rho S(\theta^3)\Big)\\ &\qquad(\tilde{x}^3_\lambda)_{\langle 0\rangle}\tilde{X}^1_\rho\Theta^2_{\langle 0\rangle}\overline{\Lambda}^1(S(\tilde{x}^1_\lambda)f^2\cdot c_{\underline{2}}\cdot(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3)_{[0]_{\langle 0\rangle}}\theta^2_{\langle 0\rangle}\tilde{p}^1_\rho\tilde{p}^2_\lambda = \varepsilon_C(c)1_A,\end{aligned}$$

for all $c\in C$ and $u\in A$. $\rightharpoonup$ and $\leftharpoonup$ are the left and right H-actions on ${}^*C$ induced by the H-bimodule structure of C, namely $\langle h\rightharpoonup{}^*c\leftharpoonup h', c\rangle = \langle{}^*c, h'\cdot c\cdot h\rangle$.

Proof. The follows from Corollary 7.6 (vii) after we show that

$$\begin{aligned}&(\tilde{x}^3_\lambda)_{\langle 0\rangle}\tilde{X}^1_\rho\Theta^2_{\langle 0\rangle}\otimes(S^{-1}(F^2)\otimes g^1)\left(S^{-1}\otimes S\right)\Big((p^2\otimes q^2)(S(\tilde{x}^2_\lambda\Theta^1)_2f^1_2\otimes(\tilde{x}^3_\lambda)_{\langle 1\rangle_{(1,2)}}(\tilde{X}^2_\rho)_2\Theta^2_{\langle 1\rangle_2})\\ &\quad(S(\tilde{x}^1_\lambda)f^2\otimes(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3)\Big)\otimes(S^{-1}(F^2)\otimes g^2)\left(S^{-1}\otimes S\right)\Big((p^1\otimes q^1)\\ &\quad(S(\tilde{x}^2_\lambda\Theta^1)_1f^1_1\otimes(\tilde{x}^3_\lambda)_{\langle 1\rangle_{(1,1)}}(\tilde{X}^2_\rho)_1\Theta^2_{\langle 1\rangle_1})\Big) = (\tilde{x}^3_\lambda\tilde{q}^1_\rho\Theta^2_{\langle 0\rangle})_{\langle 0\rangle}\tilde{x}^1_\rho\otimes\Big(S(\tilde{x}^1_\lambda)\mathfrak{q}^1(\tilde{x}^2_\lambda)_1\Theta^1_1\\ &\quad\otimes g^1S(\Theta^2_{\langle 1\rangle}\tilde{x}^3_\rho)\tilde{q}^2_\rho\Theta^3\Big)\otimes\Big(\mathfrak{q}^2(\tilde{x}^2_\lambda)_2\Theta^1_2\otimes g^2S((\tilde{x}^3_\lambda\tilde{q}^1_\rho\Theta^2_{\langle 0\rangle})_{\langle 1\rangle}\tilde{x}^2_\rho)\Big)\end{aligned}$$

in $A\otimes(H^{\rm op}\otimes H)^{\otimes 2}$. Note that the Drinfeld twist in $H^{\rm op}$ is $S^{-1}(g^2)\otimes S^{-1}(g^1)$, where $g^1\otimes g^2$ is the inverse of the Drinfeld twist f in H. The above equality follows from a straightforward computation using (1.21), the axioms of a quasi-Hopf algebra and of a bicomodule algebra over a quasi-bialgebra, and the formula

$$f^2S^{-1}(F^2f_2^1p^2)\otimes F^1f_1^1p^1=\mathfrak{q}^1\otimes S(\mathfrak{q}^2), \tag{9.14}$$

which is a consequence of (1.25) and of the fact that $f^1\beta S(f^2)=S(\alpha)$. We leave the verification of the details to the reader. □

The conditions in Proposition 9.7 are rather complicated. However they are fulfilled if C is a coseparable coalgebra in ${}_H\mathcal{M}_H$, see Proposition 7.7. We will show that the converse is also true in the particular case when $C=A=H$. It will turn out that the structure theorem for quasi-Hopf algebras plays a crucial role in the proof, indicating that it is not possible to make a generalization for arbitrary H-bimodule coalgebras C. We also emphasise the fact that this result is also new in the case where H is a classical Hopf algebra. We first need some preparatory results.

Lemma 9.8. *Let H be a finite dimensional quasi-Hopf algebra and let λ be a left cointegral on H. Then the following relations hold, for all $h,h'\in H$,*

$$\langle\lambda,\mathfrak{q}^2h_2U^2\rangle\mathfrak{q}^1h_1U^1 = \langle\mu,x^1\rangle\ \langle\lambda,S^{-1}(f^1)hS(x^2)\rangle\ f^2x^3, \tag{9.15}$$

$$\langle\lambda,h'h_2U^2\rangle h_1U^1 = \langle\mu,x^1\rangle\ \langle\lambda,S^{-1}(f^1)h'_2\mathfrak{p}^2hS(x^2)\rangle\ S(h'_1\mathfrak{p}^1)f^2x^3. \tag{9.16}$$

Proof.

$$\begin{aligned}
\langle\lambda,\mathfrak{q}^2h_2U^2\rangle\mathfrak{q}^1h_1U^1 &\overset{(9.14)}{=} \langle\lambda,S^{-1}(F^1f_1^1p^1)h_2U^2\rangle\ f^2S^{-1}(F^2f_2^1p^2)h_1U^1\\
&\overset{(1.21)}{=} \langle\lambda,V^2(S^{-1}(f^1)h)_2U^2\rangle\ f^2V^1(S^{-1}(f^1)h)_1U^1\\
&\overset{(8.14)}{=} \langle\mu,x^1\rangle\ \langle\lambda,S^{-1}(f^1)hS(x^2)\rangle f^2x^3;\\
\langle\lambda,h'h_2U^2\rangle\ h_1U^1 &\overset{(7.28)}{=} \langle\lambda,h'\mathfrak{q}^2\mathfrak{p}_2^2h_2U^2\rangle\ S(\mathfrak{p}^1)\mathfrak{q}^1\mathfrak{p}_1^2h_1U^1\\
&\overset{(7.27)}{=} \langle\lambda,\mathfrak{q}^2(h'_2\mathfrak{p}^2h)_2U^2\rangle\ S(h'_1\mathfrak{p}^1)\mathfrak{q}^2(h'_2\mathfrak{p}^2h)_1U^1\\
&\overset{(9.15)}{=} \langle\mu,x^1\rangle\ \langle\lambda,S^{-1}(f^1)h'_2\mathfrak{p}^2hS(x^2)\rangle\ S(h'_1\mathfrak{p}^1)f^2x^3,
\end{aligned}$$

as stated. □

Lemma 9.9. *Let H be a finite dimensional quasi-Hopf algebra and let μ be the modular element of H. Consider the isomorphism $\zeta:\ H_\mu\to {}^*H$ from Lemma 9.4. We can define Yetter-Drinfeld module structures on H_μ and ${}^*H$ such that ζ is an isomorphism of Yetter-Drinfeld modules. These structures are given by* (9.20, 9.21) *and* (9.17, 9.19).

Proof. The first aim is to make ${}^*H$ into a Yetter-Drinfeld module. This goes in several steps.
1) By Corollary 7.6 (v) we know that $H\otimes {}^*H\in\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}$ via the structure given by $(\hbar\otimes {}^*h)\cdot h=\hbar h_{(2,1)}\otimes S(h_{(2,2)})\rightharpoonup {}^*h\leftharpoonup h_1$ and

$$\begin{aligned}
\rho(\hbar\otimes {}^*h) &= \hbar x_1^3X^1Y_1^2\otimes\Big(g^1S(q^2(x^3_{(2,1)}X^2Y_2^2)_2)x^3_{(2,2)}X^3Y^3\rightharpoonup h^i\leftharpoonup S(x^1)f^2S^{-1}(F^2S(x^2Y^1)_2f_2^1p^2)\Big)\\
&\qquad\Big(g^2S(q^1(x^3_{(2,1)}X^2Y_2^2)_1)\rightharpoonup {}^*h\leftharpoonup S^{-1}(F^1S(x^2Y^1)_1f_1^1p^1)\Big)\otimes h_i\\
&\overset{(1.21)}{\underset{(9.14)}{=}} \hbar x_1^3X^1Y_1^2\otimes\Big(g^1S(q^2X_2^2Y^2_{(2,2)})X^3Y^3\rightharpoonup h^i\leftharpoonup S(x^1)\mathfrak{q}^1x_1^2Y_1^1\Big)\\
&\qquad\Big(g^2S(x_2^3q^1X_1^2Y^2_{(2,1)})\rightharpoonup {}^*h\leftharpoonup\mathfrak{q}^2x_2^2Y_2^1\Big)\otimes h_i,
\end{aligned}$$

for all $h,\hbar\in H$ and ${}^*h\in {}^*H$.
2) $H\otimes {}^*H$ is a left H-module via $h\cdot(\hbar\otimes {}^*h)=h\hbar\otimes {}^*h$. Consequently $M\otimes_H(H\otimes {}^*H)\in\mathcal{M}(H^{\rm op}\otimes$

$H)^H_{\underline{H}^2}$, for every $M \in \mathcal{M}_H$, and the fact that $M \otimes {}^*H$ is naturally isomorphic to $M \otimes_H (H \otimes {}^*H)$, entails that $M \otimes {}^*H \in \mathcal{M}(H^{\rm op} \otimes H)^H_{\underline{H}^2}$. The structure maps are the following:

$$\begin{aligned}(m \otimes {}^*h) \cdot h &= mh_{(2,1)} \otimes S(h_{(2,2)}) \rightharpoonup {}^*h \leftharpoonup h_1,\\ \rho(m \otimes {}^*h) &= mx_1^3X^1Y_1^2 \otimes \Big(g^1S(q^2X_2^2Y^2_{(2,2)})X^3Y^3 \rightharpoonup h^i \leftharpoonup S(x^1)\mathfrak{q}^1x_1^2Y_1^1\Big)\\ &\qquad \Big(g^2S(x_2^3q^1X_1^2Y^2_{(2,1)}) \rightharpoonup {}^*h \leftharpoonup \mathfrak{q}^2x_2^2Y_2^1\Big) \otimes h_i,\end{aligned}$$

for all $m \in M$, $h \in H$ and ${}^*h \in {}^*H$.
3) $k \in \mathcal{M}_H$ by restriction of scalars via ε, hence $k \otimes {}^*H \cong {}^*H \in \mathcal{M}(H^{\rm op} \otimes H)^H_{\underline{H}^2}$, with structure maps

$$ {}^*h \triangleleft h = S(h_2) \rightharpoonup {}^*h \leftharpoonup h_1 \tag{9.17}$$

$$\rho({}^*h) = (g^1S(q^2Y_2^2)Y^3 \rightharpoonup h^i \leftharpoonup S(x^1)\mathfrak{q}^1x_1^2Y_1^1)(g^2S(x^3q^1Y_1^2) \rightharpoonup {}^*h \leftharpoonup \mathfrak{q}^2x_2^2Y_2^1) \otimes h_i, \tag{9.18}$$

for all ${}^*h \in {}^*H$ and $h \in H$.
4) We have an isomorphism of categories $F: \ \mathcal{YD}^H_H \to \mathcal{M}(H^{\rm op} \otimes H)^H_{\underline{H}^2}$, see (9.1), and therefore ${}^*H \in \mathcal{YD}^H_H$.

At this point, the proof is basically finished: ζ can be used to transport the Yetter-Drinfeld structure on ${}^*H$ to H_μ. It remains to compute the explicit structure maps. The right H-action on ${}^*H$ is (9.17). The right H-coaction can be computed from the action and coaction (9.17-9.18) on ${}^*H \in \mathcal{M}(H^{\rm op} \otimes H)^H_{\underline{H}^2}$ using (9.2). Using the fact that ${}^*H$ is an algebra in ${}_H\mathcal{M}_H$ and the equations (1.16), (9.3) and (7.28), we find that this coaction is given by the formula

$$\rho({}^*h) = (g^1S(q^2Y_2^2)Y^3 \rightharpoonup h^i \leftharpoonup X^1Y_1^1)(g^2S(X^3q^1Y_1^2) \rightharpoonup {}^*h \leftharpoonup X^2Y_2^1) \otimes h_i, \tag{9.19}$$

where $h_i \otimes h^i \in H \otimes {}^*H$ is the finite dual basis of H.

Let us describe the structure on H_μ. The H-bimodule structure on H_μ (see Lemma 9.4) induces the following right H-module structure on H_μ:

$$\hbar \triangleleft h = \mu(S(h_1)_1)S(h_1)_2\hbar h_2, \tag{9.20}$$

for all $h,\ \hbar \in H$. The right H-coaction $\rho: \ H_\mu \to H_\mu \otimes H$ is given by the formula

$$\rho(\hbar) = \hbar_{(0)} \otimes \hbar_{(1)} := \mu(S(Y_2^2\mathfrak{p}^2X^1)_1f_1^1x^1)S(Y_2^2\mathfrak{p}^2X^1)_2f_2^1x^2\hbar_1Y_1^3X^2 \otimes Y^1S(Y_1^2\mathfrak{p}^1)f^2x^3\hbar_2Y_2^3X^3, \tag{9.21}$$

for all $\hbar \in H$. The proof is finished after we show that ζ is right H-linear and colinear with respect to the structures (9.20, 9.21) and (9.17, 9.19). Since $\zeta: H_\mu \to {}^*H$ is an H-bimodule map, it follows immediately that it is also right H-linear. For all $h \in H$ we have that

$$\begin{aligned}\rho(\zeta(h)) &\overset{(9.19)}{=} \big(g^1S(q^2Y_2^2)Y^3 \rightharpoonup h^i \leftharpoonup X^1Y_1^1\big)\big(g^2S(hX^3q^1Y_1^2) \rightharpoonup \lambda \leftharpoonup X^2Y_2^1\big) \otimes h_i\\ &\overset{(9.3)}{=} \langle \lambda, X^2(Y^1h_iS(X_1^3Y^2))_2U^2S(h)\rangle\ h^i \otimes X^1(Y^1h_iS(X_1^3Y^2))_1U^2X_2^3Y^3\\ &\overset{(8.23)}{=} \langle \lambda, X^2(Y^1h_iS(h_1X_1^3Y^2))_2U^2\rangle\ h^i \otimes X^1(Y^1h_iS(h_1X_1^3Y^2))_1U^2h_2X_2^3Y^3\\ &\overset{(9.16)}{=} \langle \mu, x^1\rangle\ \langle \lambda, S^{-1}(f^1)X_2^2\mathfrak{p}^2Y^1h_iS(x^2h_1X_1^3Y^2)\rangle\ h^i \otimes X^1S(X_1^2\mathfrak{p}^1)f^2x^3h_2X_2^3Y^3\\ &= \langle \mu, x^1\rangle S(x^2h_1X_1^3Y^2) \rightharpoonup \lambda \leftharpoonup S^{-1}(f^1)X_2^2\mathfrak{p}^2Y^1 \otimes X^1S(X_1^2\mathfrak{p}^1)f^2x^3h_2X_2^3Y^3\\ &\overset{(8.18)}{=} \langle \mu, S(X_2^2\mathfrak{p}^2Y^1)_1f_1^1x^1\rangle\ S(X_2^2\mathfrak{p}^2Y^1)_2f_2^1x^2h_1X_1^3Y^2 \rightharpoonup \lambda \otimes X^1S(X_1^2\mathfrak{p}^1)f^2x^3h_2X_2^3Y^3\\ &\overset{(9.21)}{=} \zeta(h_{(0)}) \otimes h_{(1)},\end{aligned}$$

proving that ζ is right H-colinear. □

Theorem 9.10. *For a finite dimensional quasi-Hopf algebra H, the following assertions are equivalent:*

(i) *The forgetful functor* $F:\mathcal{YD}_H^H\to\mathcal{M}_H$ *is separable;*
(ii) H *is coseparable as a coalgebra in* ${}_H\mathcal{M}_H$*;*
(iii) H *is unimodular and cosemisimple.*

Proof. (ii) ⇒ (i). If H is a coseparable coalgebra in ${}_H\mathcal{M}_H$, then the forgetful functor $\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}\to\mathcal{M}_H$ is separable, by Proposition 7.7. It follows that F is separable since the categories $\mathcal{YD}_H^H$ and $\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}$ are isomorphic.
(i) ⇒ (iii). If $F:\mathcal{YD}_H^H\to\mathcal{M}_H$ is separable, then the forgetful functor $\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}\to\mathcal{M}_H$ is separable, and it follows from Corollary 7.6 (v) that there exists a left H-linear morphism $\Lambda: H\otimes H\to H\otimes {}^*H$, $\Lambda(1\otimes h)=\overline{\Lambda}^1(h)\otimes\overline{\Lambda}^2(h)\in H\otimes {}^*H$, that is also a morphism in $\mathcal{M}(H^{\rm op}\otimes H)^H_{\underline{H}^2}$ and, a fortiori, in $\mathcal{YD}_H^H$, satisfying

$$\begin{aligned}\varepsilon(h)1=\overline{\Lambda}^2(S(x^1)f^2h_2x^3_{(2,2)}X^3Y^3)\Big(S(x^2Y^1\overline{\Lambda}^1(S(x^1)f^2h_2x^3_{(2,2)}X^3Y^3)_1y^1\mathfrak{p}^1)f^1h_1x^3_{(2,1)}X^2Y^2_2\\ \times\overline{\Lambda}^1(S(x^1)f^2h_2x^3_{(2,2)}X^3Y^3)_{(2,2)}y^2_2p^2S(y^3)\Big)\,x^3_1X^1Y^2_1\overline{\Lambda}^1(S(x^1)f^2h_2x^3_{(2,2)}X^3Y^3)_{(2,1)}y^2_1p^1\mathfrak{p}^2,\end{aligned}\tag{9.22}$$

for all $h\in H$. For any $M\in\mathcal{M}_H$, the map $\Lambda_M:M\otimes H\to M\otimes {}^*H$ given by

$$\Lambda_M(m\otimes h)=m\overline{\Lambda}^1(h)\otimes\overline{\Lambda}^2(h),$$

for all $m\in M$ and $h\in H$, is a morphism in $\mathcal{YD}_H^H$. The Yetter-Drinfeld structure on $M\otimes {}^*H$ is described explicitly in the proof Lemma 9.9; the structure on $M\otimes H$ is similar, and is given by the formulas

$$\begin{aligned}(m\otimes\hbar)\cdot h&=m\cdot h_{(2,1)}\otimes S(h_1)\hbar h_{(2,2)};\\ \rho(m\otimes\hbar)&=m\cdot x^3_1X^1Y^2_1(\mathfrak{p}^2_1)_{(2,1)}\otimes S(x^2Y^1(\mathfrak{p}^2_1)_1)f^1\hbar_1x^3_{(2,1)}X^2Y^2_2(\mathfrak{p}^2_1)_{(2,2)}\\ &\qquad\otimes S(x^1\mathfrak{p}^1)f^2\hbar_2x^3_{(2,2)}X^3Y^3\mathfrak{p}^2_2\in M\otimes H\otimes H,\end{aligned}$$

for all $m\in M$ and $\hbar,h\in H$. Taking $M=k$ as a right H-module by restriction of scalars via ε, we obtain a morphism $\Lambda_k:H\to {}^*H$, $\Lambda_k(h)=\varepsilon(\overline{\Lambda}^1(h))\overline{\Lambda}^2(h)$, in $\mathcal{YD}_H^H$. The structure of ${}^*H$ in $\mathcal{YD}_H^H$ is given by the formulas (9.17, 9.19). $H\in\mathcal{YD}_H^H$ with structure

$$\hbar\triangleleft h=S(h_1)\hbar h_2\ ;\ \rho(\hbar)=S(x^2Y^1\mathfrak{p}^2_{(1,1)})f^1\hbar_1x^3_1Y^2\mathfrak{p}^2_{(1,2)}\otimes S(x^1\mathfrak{p}^1)f^2\hbar_2x^3_2Y^3\mathfrak{p}^2_2.$$

Recall from Lemma 9.9 that we have an isomorphism $\zeta:H_\mu\to {}^*H$ in $\mathcal{YD}_H^H$, and consider the composition $\chi=\zeta^{-1}\circ\Lambda_k:H\to H_\mu$ in $\mathcal{YD}_H^H$. The right H-linearity of χ is expressed by the formula

$$\chi(S(h_1)\hbar h_2)=\mu(S(h_1)_1)S(h_1)_2\chi(\hbar)h_2.$$

Taking $\hbar=\alpha$, we find that $\varepsilon(h)\chi(\alpha)=\mu(S(h_1)_1)S(h_1)_2\chi(\alpha)h_2$, for all $h\in H$. Applying ε to both sides of this equation, we find that

$$\varepsilon(h)\vartheta(\alpha)=\mu(S(h))\vartheta(\alpha),\tag{9.23}$$

for all $h\in H$, with $\vartheta=\varepsilon\circ\chi$. The right H-colinearity of χ comes out as

$$\begin{aligned}&\chi(S(x^2Y^1\mathfrak{p}^2_{(1,1)})f^1\hbar_1x^3_1Y^2\mathfrak{p}^2_{(1,2)})\otimes S(x^1\mathfrak{p}^1)f^2\hbar_2x^3_2Y^3\mathfrak{p}^2_2\\ &\quad=\ \mu(S(Y^2_2\mathfrak{p}^2X^1)_1f^1_1x^1)S(Y^2_2\mathfrak{p}^2X^1)_2f^1_2x^2\chi(\hbar)_1Y^3_1X^2\otimes Y^1S(Y^2_1\mathfrak{p}^1)f^2x^3\chi(\hbar)_2Y^3_2X^3,\end{aligned}$$

for all $\hbar\in H$. Applying $\varepsilon\otimes H$ to this equality we obtain that

$$\mu(S(Y^2_2\mathfrak{p}^2)f^1)Y^1S(Y^2_1\mathfrak{p}^1)f^2\chi(\hbar)Y^3=\vartheta(S(x^2Y^1\mathfrak{p}^2_{(1,1)})f^1\hbar_1x^3_1Y^2\mathfrak{p}^2_{(1,2)})S(x^1\mathfrak{p}^1)f^2\hbar_2x^3_2Y^3\mathfrak{p}^2_2,\tag{9.24}$$

for all $\hbar\in H$. Let $\hbar=S(\mathfrak{q}^2_1)\hbar\mathfrak{q}^2_2$ in (9.24), and multiply both sides of it to the left by $\mathfrak{q}^1$. Using the formulas $Y^2_1\mathfrak{p}^1S^{-1}(Y^1)\otimes Y^2_2\mathfrak{p}^2\otimes Y^3=y^1\mathfrak{p}^1\otimes y^2\mathfrak{p}^2_1\otimes y^3\mathfrak{p}^2_2$, (1.16), (1.21) and (7.28) we deduce that

$$\mu(S(y^2)f^1)S(y^1)f^2\chi(\hbar)y^3=\vartheta(S(x^2Y^1)f^1\hbar_1x^3_1Y^2)S(x^1)f^2\hbar_2x^3_2Y^3,$$

for all $\hbar \in H$, or, equivalently,

$$\chi(\hbar) = \mu(g^1 S(Z^2))\vartheta(S(x^2Y^1)f^1\hbar_1 x_1^3 Y^2)g^2 S(x^1Z^1)f^2\hbar_2 x_2^3 Y^3 Z^3.$$

In particular,

$$\chi(\alpha) \overset{(1.24)}{=} \mu(g^1 S(Z^2))\vartheta(S(x^2Y^1)\gamma^1 x_1^3 Y^2)g^2 S(x^1Z^1)\gamma^2 x_2^3 Y^3 Z^3 \overset{(1.22)}{=} \mu(g^1 S(Z^2))g^2 S(Z^1)\alpha Z^3,$$

which implies that $\vartheta(\alpha) = \varepsilon\chi(\alpha) = \varepsilon(\alpha) \neq 0$. Therefore, (9.23) is equivalent to $\varepsilon(h) = \mu(S(h))$, for all $h \in H$, and this is clearly equivalent to $\mu = \varepsilon$. Thus we have shown that H is unimodular, and, in particular, that $\chi(\alpha) = \alpha$. Applying ε to (9.22) we find that

$$\overline{\Lambda}^2(S(x^1)f^2h_2x_2^3Y^3)(\beta S(x^2Y^1\overline{\Lambda}^1(S(x^1)f^2h_2x_2^3Y^3)_1p^1)f^1h_1x_1^3Y^2\overline{\Lambda}^1(S(x^1)f^2h_2x_2^3Y^3)_2p^2) = \varepsilon(h),$$

for all $h \in H$. Take $h = \alpha$; using (1.24) and (1.22), a similar computation shows that

$$\varepsilon(\alpha) = \overline{\Lambda}^2(\alpha)(\beta S(\overline{\Lambda}^1(\alpha)_1p^1)\alpha\overline{\Lambda}^1(\alpha)_2p^2) = \varepsilon(\overline{\Lambda}^1(\alpha))\overline{\Lambda}^2(\alpha)(\beta S(p^1)\alpha p^2) \overset{(1.20)}{=} \Lambda_k(\alpha)(\beta).$$

Now $\Lambda_k = \zeta\chi$ and $\chi(\alpha) = \alpha$, so $0 \neq \varepsilon(\alpha) = \Lambda_k(\alpha)(\beta) = \zeta(\alpha)(\beta) = \lambda(\beta S(\alpha)) \overset{(8.19)}{=} \lambda(S^{-1}(\alpha)\beta)$, which implies that H is cosemisimple.

$\underline{(ii) \Leftrightarrow (iii)}$ follows from Proposition 8.6. □

Remark 9.11. Replacing H by $H^{\rm op}$, we obtain necessary and sufficient conditions for the Frobenius property and separability of the forgetful functor $F : {}_H\mathcal{YD}^H \to {}_H\mathcal{M}$. Unimodularity (resp. unimodularity and cosemisimplicity) of H and $H^{\rm op}$ are equivalent, and therefore $F : {}_H\mathcal{YD}^H \to {}_H\mathcal{M}$ is Frobenius (resp. separable) if and only if H is unimodular (resp. unimodular and cosemisimple). If H is finite dimensional, then ${}_H\mathcal{YD}^H \cong {}_{D(H)}\mathcal{M}$, where $D(H)$ is the quantum double of H, see [22]. So our results imply that the algebra extension $H \hookrightarrow D(H)$ is Frobenius (resp. separable) if and only if H is unimodular (resp. unimodular and cosemisimple).

References

[1] F. Borceux, "Handbook of categorical algebra I: basic category theory", *Encyclopedia Math. Appl.* **50**, Cambridge University Press, Cambridge, 1994.

[2] T. Brzeziński and S. Majid, *Coalgebra bundles*, Comm. Math. Phys. **191** (1998), 467–492.

[3] T. Brzeziński, *On modules associated to coalgebra-Galois extensions*, J. Algebra **215** (1999), 290–317.

[4] T. Brzeziński, *Frobenius properties and Maschke-type theorems for entwined modules*, Proc. Amer. Math. Soc. **128** (1999), 2261–2270.

[5] D. Bulacu, *A Clifford algebra is a weak Hopf algebra in a suitable symmetric monoidal category*, J. Algebra **332** (2011), 244–284.

[6] D. Bulacu and S. Caenepeel, *Integrals for (dual) quasi-Hopf algebras. Applications*, J. Algebra **266** (2003), 552–583.

[7] D. Bulacu and S. Caenepeel, *Two-sided two-cosided Hopf modules and Doi-Hopf modules for quasi-Hopf algebras*, J. Algebra **270** (2003), 55–95.

[8] D. Bulacu and S. Caenepeel, *Corings in monoidal categories*, in "New techniques in Hopf algebras and graded ring theory", 53–78, K. Vlaam. Acad. België Wet. Kunsten (KVAB), Brussels, 2007.

[9] D. Bulacu and S. Caenepeel, *On integrals and cointegrals for quasi-Hopf algebras*, J. Algebra **351** (2012), 390–425.

[10] D. Bulacu and S. Caenepeel, *Monoidal ring and coring structures obtained from wreaths and cowreaths*, Algebr. Represent. Theory **17** (2014), 1035–1082.

[11] D. Bulacu, S. Caenepeel and B. Torrecillas, *Doi-Hopf modules and Yetter-Drinfeld modules for quasi-Hopf algebras*, Comm. Algebra **34** (2006), 3413–3449.

[12] D. Bulacu, F. Panaite and F. Van Oystaeyen, *Generalized diagonal crossed products and smash products for (quasi) Hopf algebras*, Comm. Math. Phys. **266** (2006), 355–399.

[13] D. Bulacu and E. Nauwelaerts, *Relative Hopf modules for (dual) quasi-Hopf algebras*, J. Algebra **229** (2000), 632–659.

[14] D. Bulacu and B. Torrecillas, *On Doi-Hopf modules and Yetter-Drinfeld modules in symmetric monoidal categories*, Bull. Belg. Math. Soc. - Simon Stevin **21** (2014), 89–115.

[15] D. Bulacu and B. Torrecillas, *On Frobenius and separable algebra extensions in monoidal categories. Applications to wreaths*, J. Noncommut. Geom. **9** (2015), 707–774.

[16] S. Caenepeel, M. De Lombaerde, *A categorical approach to Turaev's Hopf group-coalgebras*, Comm. Algebra **34** (2006), 2631–2657.
[17] S. Caenepeel, G. Militaru, B. Ion and S. Zhu, *Separable functors for the category of Doi-Hopf modules, Applications*, Adv. Math. **145** (1999), 239–290.
[18] S. Caenepeel, G. Militaru and S. Zhu, *Crossed modules and Doi-Hopf modules*, Israel J. Math. **100** (1997), 221–247.
[19] S. Caenepeel, G. Militaru, Zhu Shenglin, *A Maschke-type theorem for Doi-Hopf modules. Applications*, J. Algebra **187** (1997), 388–412.
[20] S. Caenepeel, G. Militaru and S. Zhu, *Doi-Hopf modules, Yetter-Drinfel'd modules and Frobenius type properties*, Trans. Amer. Math. Soc. **349** (1997), 4311–4342.
[21] Y. Doi, *Unifying Hopf modules*, J. Algebra **153** (1992), 373–385.
[22] F. Hausser and F. Nill, *Diagonal crossed products by duals of quasi-quantum groups*, Rev. Math. Phys. **11** (1999), 553–629.
[23] F. Hausser, F. Nill, *Integral theory for quasi-Hopf algebras*, preprint math. QA/9904164.
[24] D. Hobst and B. Pareigis, *Double Quantum Groups*, J. Algebra **242** (2001), 460–494.
[25] C. Kassel, "Quantum Groups", *Graduate Texts in Mathematics* **155**, Springer Verlag, Berlin, 1995.
[26] M. Koppinen, *Variations on the smash product with applications to group-graded rings*, J. Pure Appl. Algebra **104** (1995), 61–80.
[27] S. Lack, R. Street, *The formal theory of monads II*, J. Pure Appl. Algebra **175** (2002), 243–265.
[28] R. G. Larson, *Coseparable Hopf algebras*, J. Pure Appl. Algebra **3** (1973), 261–267.
[29] S. Mac Lane, "Categories for the working mathematician", 2nd edition. *Grad. Texts Math.* **5**, Springer, Berlin, 1998.
[30] B. Mesablishvili, *Entwining structures in monoidal categories*, J. Algebra **319** (2008), 2496–2517.
[31] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, *Separable functors applied to graded rings*, J. Algebra **123** (1989), 397–413.
[32] B. Pareigis, *Non-additive ring and module theory V. Projective and coflat objects*, Algebra Ber. **40** (1980).
[33] D. M. Rafael, *Separable functors revisited*, Comm. Algebra **18** (1990), 1445–1459.
[34] P. Schauenburg, *Actions on monoidal categories and generalized Hopf smash products*, J. Algebra **270** (2003), 521–563.
[35] R. Street, *The formal theory of monads*, J. Pure Appl. Algebra **2** (1972), 149–168.
[36] R. Street, *Frobenius monads and pseudomonoids*, J. Math. Phys. **45** (2004), 3930–3948.
[37] D. Tambara, *The coendomorphism bialgebra of an algebra*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990), 425–456.

Faculty of Mathematics and Informatics, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania
E-mail address: daniel.bulacu@fmi.unibuc.ro

Faculty of Engineering, Vrije Universiteit Brussel, B-1050 Brussels, Belgium
E-mail address: scaenepe@vub.ac.be

Department of Algebra and Analysis, Universidad de Almería, E-04071 Almería, Spain
E-mail address: btorreci@ual.es