ARE BISEPARABLE EXTENSIONS FROBENIUS?

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ABSTRACT. In Secion 1 we describe what is known of the extent to which a separable extension of unital associative rings is a Frobenius extension. A problem of this kind is suggested by asking if three algebraic axioms for finite Jones index subfactors are dependent. In Section 2 the problem in the title is formulated in terms of separable bimodules. In Section 3 we specialize the problem to ring extensions, noting that a biseparable extension is a two-sided finitely generated projective, split, separable extension. Some reductions of the problem are discussed and solutions in special cases are provided. In Section 4 various examples are provided of projective separable extensions that are neither finitely generated nor Frobenius and which give obstructions to weakening the hypotheses of the question in the title. We show in Section 5 that characterizations of the separable extensions among the Frobenius extensions in [HS, K, K99] are special cases of a result for adjoint functors.

1. INTRODUCTION

An old problem is the extent to which separable algebras are Frobenius algebras. By a Frobenius algebra we mean a finite dimensional algebra A with a non-degenerate linear functional, which induces an A-module isomorphism $A \cong A^*$; symmetric algebra if this isomorphism is an A-bimodule map. Eilenberg and Nakayama observed in [EN] that the (reduced) trace of a central simple algebra over a field is non-degenerate, which implies that a finite dimensional semisimple algebra is symmetric. Passing to a commutative ground ring k, Hattori [H] and DeMeyer [D] showed that a k-projective separable k-algebra A is symmetric as well if the Hattori-Stallings rank of A over its center C is an invertible element in C. Endo and Watanabe essentially extended this result to k-projective separable faithful k-algebras by using the Auslander-Goldman Galois theory for commutative rings to define a more general notion of reduced trace [EW].

The main theorem in [EW] led to several general results by Sugano [S70] for when separable extensions [HS] are Frobenius extensions [K60]. These are noncommutative ring extensions and are natural objects for study from the point of view of induced representations [Hoch]. Sugano shows that a centrally projective separable extension R/S is Frobenius since it satisfies $R \cong S \otimes_{Z(S)} C_R(S)$ where the centralizer $C_R(S)$ is faithfully projective and separable over the center $Z(S)$, whence Frobenius. Somewhat similarly, it is shown that a split one-sided finite projective H-separable extension R/S is Frobenius, since in this case the endomorphism ring $\mathrm{End}(R_S) \cong R \otimes_{Z(R)} C_R(S)$ with $C_R(S)$ again separable, the result following from the endomorphism ring theorem as developed in [K60, Mu64, M67]: if R_S

1991 *Mathematics Subject Classification.* 16L60, 16H05.

The authors thank G. Bergman, O. Kerner, O.A. Laudal, V. Mazurchuk, D. Nikshych, P. Seibt, A.A. Stolin, and R. Wisbauer for discussions related to this paper.

is generator module, R/S is Frobenius iff $\mathrm{End}(R_S)/R$ is Frobenius. However, it is implicit in the literature that there are several cautionary examples showing separable extensions are not always Frobenius extensions in the ordinary untwisted sense [K99]: in Section 4 we show that a non-finite ring leads to an example of split, separable, two-sided projective extension which is not finitely generated, whence not Frobenius.

As an independent line of inquiry, algebraic axioms for finite Jones index subfactors have been investigated in [K95, K96, NK]. If we simplify the discussion somewhat, we may start with an irreducible subfactor M/N of finite index: from the Pimsner-Popa orthonormal base, the natural modules M_N and ${}_N M$ are finite projective [JS], and the algebra extension M/N is (1) split, (2) separable, and (3) Frobenius. A ring extension M/N is said to be *split* if there is a bimodule projection $E : R \to S$. At the same time, Axiom (2) yields a Casimir element $e = \sum_i x_i \otimes y_i$ such that $\sum_i x_i y_i = 1$. A problem in the independence of the axioms above becomes whether Axioms (1) and (2) imply Axiom (3) in the presence of the assumption of two-sided finite projectivity of the ring extension: i.e., whether a bimodule map $E : A \to S$ and Casimir element $e \in A \otimes_S A$ may be chosen such that $\sum_i E(x_i) y_i = 1 = \sum_i x_i E(y_i)$. Equivalently, can a bimodule map E be found such that the E-multiplication on $R \otimes_S R$ [J] is unital? That Axiom (3) in combination with (1) or (2) does not imply the other are easy examples disposed of in Section 3. Many other algebraic examples of split separable Frobenius can be found [K95] but no one has so far observed a finite projective split separable extension that is not Frobenius.

In this paper, we will formulate the problem of independence of axioms for subfactors in several algebraic ways. In Section 2 we first formulate the problem using separable bimodules [S71], a theory in which separable extension and split extension become dual notions [K96, K99]. In analogy with "bialgebra," we will baptise finite projective split separable extensions as *biseparable extensions.* Posed in the negative, our question then becomes if biseparable extensions are Frobenius. This question will be formulated in several other ways in Section 3, with one special case being answered in the affirmative. We point out here that the problem has many interesting sub-problems if restrictions are placed on the rings (e.g., "finite dimensional algebras," "Hopf algebras," etc.).

In Section 5 we discuss a type of converse to the considerations above. We find a common feature of the theorems in [HS, K, K99] on when a Frobenius extension or bimodule is separable: in each case, it is a specific example of a folkloric theorem on adjoint functors, which we expose in this last section.

Preliminaries. A ring R will mean a unital associative ring. A ring homomorphism sends 1 into 1. A right module M_R or left module ${}_R M$ is always unitary. Bimodules are associative with respect to the left and right actions.

An R-S-bimodule M is denoted by ${}_R M_S$. Its right dual is defined by $M^* := \mathrm{Hom}(M_S, S_S)$, an S-R-bimodule where $sfr(m) := sf(rm)$. The left dual of M is ${}^*M := \mathrm{Hom}({}_R M, {}_R R)$ is also S-R-bimodule where $(m)(sfr) := [(ms)f]r$. Both $M \mapsto M^*$ and $M \mapsto {}^*M$ are contravariant functors of bimodule categories, sending ${}_R\mathcal{M}_S \to {}_S\mathcal{M}_R$.

If $R = S$ in the last paragraph, denote $\hat{M} := \mathrm{Hom}_{S-S}(M, S)$. Define the group of S-central or Casimir elements by $M^S := \{m \in M |\ ms = sm,\ \forall s \in S\}$. Note that $(M^*)^S = ({}^*M)^S = \hat{M}$.

If ${}_RM_S$, ${}_RN_T$, ${}_TQ_S$ and ${}_SP_T$ are bimodules, then $M \otimes_S P$ receives the natural R-T bimodule structure indicated by $r(m \otimes n)t := rm \otimes nt$, and the group of right module homomorphisms $\mathrm{Hom}_S(M_S, Q_S)$ receives the natural T-R-bimodule structure on $\mathrm{Hom}_S(M_S, Q_S)$ indicated by $(tfr)(m) := t(f(rm))$. The group of left module homomorphisms $\mathrm{Hom}_R(M, N)$ receives the natural S-T bimodule structure indicated by $(m)(sft) = ((ms)f)t$. All bimodules arising from Hom and tensor in this paper are the natural ones unless otherwise indicated.

A *ring extension* R/S is a ring homomorphism $S \stackrel{\iota}{\to} R$. A ring extension is an *algebra* if S is commutative and ι factors into $S \to Z(R) \hookrightarrow R$ where $Z(R) := R^R$ is the center of R. A ring extension is *proper* if ι is 1-1, in which case identification is made.

The natural bimodule ${}_SR_S$ is given by $s \cdot a \cdot s' := \iota(s)a\iota(s')$. In particular, we consider the natural modules R_S and ${}_SR$. An adjective, such as right projective or projective, for the ring extension R/S refers to the same adjective for one or both of these natural modules. The structure map ι is usually suppressed.

Separable extensions are studied in [HS, K95, K, RAFAEL, S67, S70] among others. A ring extension R/S is *separable* if the natural (multiplication) map $R \otimes_S R \to R$ is a split epimorphism of R-bimodules. Examples are abundant among finite dimensional algebras since a separable algebra is a separable extension of any of its subalgebras. The next proposition, whose proof follows Sugano [S82, Prop. 1], is important to keep in mind when finding examples of separable extensions from the class of finite dimensional algebras.

Proposition 1.1. *If $0 \to J \to A \stackrel{\pi}{\to} S \to 0$ is a split exact sequence of algebras and A/S is a separable extension, then J is an idempotent ideal (i.e., $J^2 = J$).*

Proof. We assume with no loss of generality that $S \subseteq A$ and $\pi|_S = \mathrm{id}_S$. Let $\sum_i x_i \otimes y_i$ be a separability element in $A \otimes_S A$. Let $e = \sum_i \pi(x_i)y_i$. Then e satisfies $ex = \pi(x)e$ for all $x \in A$. Since $\pi(e) = 1$, it follows that e is idempotent. Similarly, $f = \sum_i x_i \pi(y_i)$ satisfies $xf = f\pi(x)$, $\pi(f) = 1$ and is idempotent. Then

$$e = \pi(f)e = ef = f.$$

Then $xe = e\pi(x)e = ex$ and e is central. Then $J = (1 - e)A$, whence J is idempotent. □

An example of a ring epi splitting in the next corollary would be the one implicit in the Wedderburn Principal Theorem for finite dimensional algebras.

Corollary 1.2. If A/S is a split, separable extension with splitting map $\pi : A \to S$ a ring epimorphism with nilpotent kernel J, then $J = 0$ and $A = S$.

Indeed a separable finitely generated (f.g.) extension of a separable algebra is itself separable [HS]. The next proposition builds new separable extensions from old using multiplicative bimodules [P].

Proposition 1.3. *Suppose R/S is a separable extension and I is a multiplicative R-bimodule. Then $A = R \oplus I$ is a separable extension of $T = S \oplus I$.*

Proof. Let $f = \sum_i x_i \otimes y_i \in R \otimes_S R$ be a separability element for R/S. Let e be its image in $A \otimes_T A$ induced by $R \hookrightarrow A$. If $x \in I$, then:

$$xe = \sum_i 1 \otimes x x_i y_i = \sum_i x_i y_i x \otimes 1 = ex.$$

We easily conclude that e is a separability element for A/T. □

As our final preliminary topic we recall Frobenius and QF extensions.[1] A ring extension R/S is *Frobenius* if R_S is f.g. projective and $R \cong R^*$ as S-R-bimodules: note that this extends the notion of Frobenius algebra. We recall also the Morita characterization of Frobenius extensions [M65]: an extension R/S is Frobenius iff induction and co-induction (of S-modules to R-modules) are naturally isomorphic (cf. [K]).

A ring extension R/S is a left *Quasi-Frobenius* (QF) extension if ${}_S R$ is finitely generated projective and ${}_R R_S$ is isomorphic to a direct summand of a finite direct product of ${}^*_R R_S$ with itself. Equivalently, R_S and ${}_S R$ are finitely generated projectives and ${}_S R^*_R$ is a direct summand of a finite direct sum of copies of ${}_S R_R$. We similarly define right QF extensions [Mu64, Mu65]. There is no published example of a right QF extension that is not left QF.

2. Biseparable Bimodules

In this section we pose our question in the more general terms of bimodules rather than ring extensions. There are two reasons for this. First, the problem has a more attractive symmetrical formulation in terms of bimodules. Second, Morita has shown in [M67] how to generate interesting examples of ring extensions from bimodules via the endomorphism ring.

Let R and T be rings. Given a bimodule ${}_T M_R$, there is a natural T-bimodule homomorphism,

$$\mu_M : \ M \otimes_R {}^*M \longrightarrow T, \quad m \otimes f \mapsto (m)f.$$

We next recall the definition of a separable bimodule [S71].

Definition 2.1. M is separable, or T is M-separable over R, if μ_M is a split T-epimorphism.

It follows trivially that M_R is a generator module [AF]. By applying a splitting map to 1_T, we note that M is separable iff there is an element

$$e = \sum_i m_i \otimes f_i \in M \otimes_R {}^*M,$$

called an *M-separability element*, which satisfies $\mu_M(e) = 1_T$ and $te = et$ for all $t \in T$. As is the case with separability elements and idempotents [P, DMI], M-separability elements are usually not unique.

Retaining this notation, we recall a useful proposition and its proof [S71, Proposition 3]. But first a lemma which does not require M to be separable.

Lemma 2.2. *If M_R is finitely generated projective, then $\alpha_M : M \otimes_R {}^*M \to Hom({}_R M^*, {}^*_R M)$ given by*

$$m \otimes f \longmapsto (g \mapsto g(m)f)$$

is a T-bimodule isomorphism. Similarly, ${}_T M$ f.g. projective implies that

$$\eta_M : M^* \otimes_T M \to Hom({}^*M_T, M^*_T), \quad f \otimes m \longmapsto (h \mapsto f[(m)h])$$

is an R-bimodule isomorphism.

[1][K60, K61, O, Mu64, Mu65, M65, M67, FMS, K]

Proof. The map α_M is clearly a well-defined T-bimodule homomorphism. If $x_i \in M$ and $f_i \in M^*$ give a dual base for M_R, it is easy to check that the mapping $\mathrm{Hom}_R(M^*, {}^*M) \to M \otimes_R {}^*M$ given by

$$G \mapsto \sum_i x_i \otimes (f_i)G$$

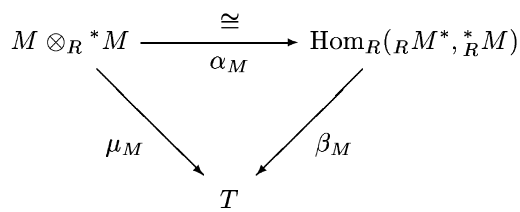
is an inverse. The second statement is proven similarly. □

Now define $\beta_M : \mathrm{Hom}({}_RM^*, {}^*_RM) \to T$ by

$$\beta_M(G) = \sum_i (x_i)[(f_i)G].$$

We arrive at the proposition below by applying the inverse mapping in the proof of the lemma.

Proposition 2.3. *We have $\mu_M = \beta_M \circ \alpha_M$; i.e., the diagram below is commutative. Whence M is separable iff there is an R-T-bimodule homomorphism $\gamma_M : M^* \to {}^*M$ such that $(x_i)[(f_i)\gamma_M] = 1_T$.*

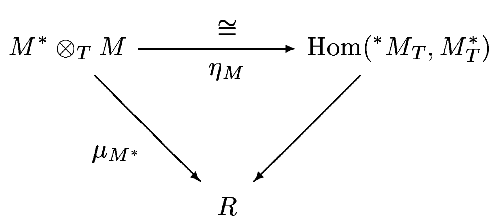


Proof. ¿From $\mathrm{Hom}_R(M^*, {}^*M)^T = \mathrm{Hom}_{R-T}(M^*, {}^*M)$ we note that γ_M corresponds under α_M to an M-separability element. □

Definition 2.4. A bimodule ${}_TM_R$ is said to be biseparable if M and M^* are separable and ${}_TM$, M_R are finite projective modules.

We derive some consequences of assuming ${}_TM$ f.g. projective and M^* separable. First, since M_R is reflexive, it follows that ${}^*(M^*) \cong M$ via the "evaluation mapping" from M to $\mathrm{Hom}_R(M^*, R)$. It follows that $\mu_{M^*} : M^* \otimes_T M \to R$ under this identification is the evaluation map given by $f \otimes m \mapsto f(m)$. Let $\{g_j\} \subset {}^*M$, $\{y_j\} \subset M$ be finite dual bases for the f.g. projective module ${}_TM$. We have the following easy analog of Proposition 2.3, whose proof we leave to the reader.

Proposition 2.5. *The triangle below is commutative. Whence if M^* is separable there is a R-T-bimodule homomorphism $\rho_M : {}^*M \to M^*$ such that $\sum_j \rho_M(g_j)(y_j) = 1_R$.*



The downward map to the right is given by $G \mapsto \sum_j G(g_j)(y_j)$.

It follows that a biseparable bimodule M has a nontrivial bimodule arrow $M^* \to {}^*M$ and another in the reverse direction ${}^*M \to M^*$. We may eliminate the seeming chirality in the defintion of biseparability by noting the following lemma, a simple consequence of the two propositions directly above.

Lemma 2.6. *M is biseparable iff ${}_TM$, M_R are f.g. projective modules and ${}^*M$, M are separable bimodules*

For example, a bimodule ${}_TM_R$ yielding a Morita equivalence of T and R is biseparable, since ${}^*M \cong M^*$ as R-T bimodules [M67] while μ_M and μ_{M^*} are isomorphisms. There have been various studies of properties shared by rings R and T related by a bimodule ${}_TM_R$ in a Morita context and generalizations of this [AF, C, M, M65]. A precursor of these studies is the theorem of D.G. Higman [Hi] that a finite group has finite representation type (f.r.t.) in characteristic p iff its Sylow p-subgroup is cyclic, which later became a corollary of the theorem of J.P. Jans [Ja] that for Artinian algebras $R \subseteq T$ in a split separable extension, R has f.r.t. iff T has f.r.t. (cf. [P]). It is in this spirit that the next theorem offers a sample of shared properties of R and T linked by a biseparable bimodule ${}_TM_R$.

Theorem 2.7. *If ${}_TM_R$ is biseparable, then*

1. *T is a QF ring if and only if R is a QF ring;*
2. *T is semisimple if and only if R is semisimple;*
3. *weak global dimension $D(R) = D(T)$.*

Proof. (QF). Assume T is QF and P_R is injective. By the Faith-Walker theorem, it will suffice to show that P_R is projective. Since ${}_TM$ is projective, then flat, we note that $H_T := \mathrm{Hom}_R(M, P)$ is injective, then projective. Since M_R is projective, we note that $H \otimes_T M_R$ is projective. But the evaluation mapping

$$ev: \ H \otimes_T M_R \to P_R, \quad f \otimes m \mapsto f(m)$$

is a split epi, for if $\sum_j f_j \otimes m_j$ is an M^*-separability element, then $p \mapsto pf_j \otimes m_j$ defines a splitting R-monic, where $pf_j : m \mapsto pf_j(m)$. Hence P_R is isomorphic to a direct summand in $H \otimes_T M_R$ and is projective.

Assuming that R is QF, and ${}_TQ$ is injective, we argue similarly that ${}_RH' := \mathrm{Hom}_T(M, Q)$ is injective-projective and that Q is isomorphic to direct summand in the projective T-module $M \otimes_R H'$.

(SEMISIMPLICITY). Suppose T is semisimple and P_R is a module. It suffices to note that P_R is projective. Since $H_T := \mathrm{Hom}_R(M, P)$ is projective and the map ev defined as above is a split R-epimorphism, it follows that P_R is isomorphic to a direct summand of the projective module $H \otimes_R M_R$. Similarly, we argue that given R semisimple and module ${}_TQ$, ${}_TQ$ is projective.

(WEAK GLOBAL DIMENSION.) If $X_. \to N_T$ is a projective resolution of N_T, then $X_. \otimes_T M_R \to N \otimes_T M_R$ is a projective resolution as well since ${}_TM$ is flat and M_R is projective. Recall that $\mathrm{Tor}_n^T(N, Q)$ is the n'th homology group of the chain complex $X_. \otimes_T Q$ for each non-negative integer n, so $\mathrm{Tor}_n^R(N \otimes_T M, {}^*M \otimes_T Q)$ is the n'th homology group of $X_. \otimes_T M \otimes_R {}^*M \otimes_T Q$. At the level of chain complex, there is a split epi

$$X_n \otimes_T M \otimes_R {}^*M \otimes_T Q \longmapsto X_n \otimes_T Q, \quad x \otimes m \otimes g \otimes q \mapsto x(mg) \otimes q$$

which implies that $\mathrm{Tor}_n^T(N, Q)$ is isomorphic to a direct summand in $\mathrm{Tor}_n^R(N \otimes_T M, {}^*M \otimes_T Q)$ for each n. This shows that $D(R) \geq D(T)$. A similar argument with left modules shows that $D(T) \geq D(R)$. $\square$

We remark that in trying to prove other shared homological properties of biseparable ${}_T M_R$, particularly one-sided notions, one may run into the following complications: although the modules ${}^*M_T$ and ${}_R M^*$ are (quite easily seen to be) f.g. projective, one should avoid assuming the same of ${}^*_R M$ and M^*_T.

We next recall the definition of Frobenius bimodule [AF, K].

Definition 2.8. A bimodule ${}_T M_R$ is Frobenius if M_R, ${}_T M$ are f.g. projective and ${}^*M \cong M^*$ as R-T-bimodules.

Based on the many examples in [K95] and elsewhere, we propose the following problem, which turns out to be almost equivalent to the ring extension formulation in the title:

Problem 2.9. Is a biseparable bimodule ${}_T M_R$ a Frobenius bimodule?

For example, can we choose γ_M and ρ_M such that they are inverses to one another? The problem above subsumes many interesting questions in various restricted cases. For example, what can be said for the problem above if T and R are finite dimensional algebras? There is an affirmative answer in the next section if one algebra is separable.

A generalization of Frobenius bimodule is a *twisted Frobenius bimodule* ${}_\alpha M_\beta$ where $\alpha : T \to T$ and $\beta : R \to R$ are ring automorphisms, and the bimodule structure is now given by $t \cdot m \cdot r := \alpha(t) m \beta(r)$ (for the definition see [K99]). We might ask more widely

Problem 2.10. Is a biseparable bimodule a twisted Frobenius bimodule?

However, this problem is the same as the previous one if the following question has an affirmative answer:

Problem 2.11. If a twisted Frobenius bimodule ${}_\alpha M_\beta$ is biseparable, does this imply that ${}_\alpha M_\beta \cong {}_T M_R$?

We say that a twisted Frobenius bimodule is nontrivial if it not isomorphic to an untwisted Frobenius bimodule; for a β-Frobenius extension R/S nontriviality means that $\beta : S \to S$ is not an *extended inner automorphism* in the sense that there is a unit $u \in R$ such that β is conjugation by u [NT]. We pose the last question since we have never observed a nontrivial β-Frobenius extension (e.g. in [FMS, NT]) which was simultaneously split and separable (cf. next section). In this more limited setting, which covers Hopf subalgebras of finite dimensional Hopf algebras, the question becomes:

Problem 2.12. If a β-Frobenius extension is split and separable, is β an extended inner automorphism?

We will return to a discussion of this problem in the next section.

3. Biseparable Extensions

Suppose R/S is a ring extension. Letting $M = {}_R R_S$ in the definition of separable bimodule, we observe the following lemma [S71].

Lemma 3.1. *R/S is a separable extension iff ${}_R R_S$ is a separable bimodule.*

Dually, we let $M = {}^*({}_R R_S) \cong {}_S R_R$ and observe the following lemma [K96].

Lemma 3.2. *R/S is a split extension iff ${}_S R_R$ is a separable bimodule.*

The M-separability element in this case is a *bimodule projection* $E : {}_S R_S \to {}_S S_S$, which implies R/S is a proper extension. E is also called a *conditional expectation* if it satisfies additional properties in subfactor theory. ¿From the last two lemmas and Lemma 2.6, it follows that:

Lemma 3.3. *R/S is a split, separable, two-sided finite projective extension iff $_R R_S$ is a biseparable bimodule.*

We call R/S a *biseparable extension* if $_R R_S$ is a biseparable bimodule. Additionally, we have the following lemma [K].

Lemma 3.4. *R/S is a Frobenius extension iff $_R R_S$ or $_S R_R$ is a Frobenius bimodule.*

The last two lemmas lead to the seemingly restricted formulation of Problem 2.9, also the title of this article.

Problem 3.5. Are biseparable extensions Frobenius?

Surprisingly, this problem is almost equivalent to Problem 2.9 because of the endomorphism ring theorems for Frobenius bimodules [K, Chapter 2]. Suppose we knew an affirmative answer to the somewhat weaker problem where biseparable extension includes *left* f.g. projective, split separable extensions. Given a biseparable bimodule $_T M_R$, we know from Sugano that $\mathcal{E} = \mathrm{End}(M_R)$ is a left projective, split, separable extension of T (whose elements are identified in $\mathcal{E}$ with left multiplication operators) [S71, Theorem 1, Prop. 2] (cf. [K99, Theorem 3.1]). Then $\mathcal{E}/T$ is a Frobenius extension by our affirmative answer to the weak Problem 3.5. Since $_\mathcal{E} M_R$ is faithfully balanced by Morita's Lemma, it follows from [M67, Theorem 1.1] that $^*_\mathcal{E} M_R \cong M^*$ and then from the endomorphism ring theorem-converse [K, Theorem 2.8] that $_T M_R$ is a Frobenius bimodule.

What evidence do we have then for proposing Problem 3.5? First, if R/S is an S-algebra, we are in the situation of a faithfully projective separable algebra, which is Frobenius by the Endo-Watanabe Theorem [EW] discussed in the introduction. This implies by elementary considerations that k-algebra extensions of the form $R \otimes_k A$ over R are Frobenius if A is faithfully projective k-separable.

Second, there are the many examples of split, separable, Frobenius extensions [K95, K96] and apparently none that contradict in the literature for noncommutative rings and ring extensions [JS, L, M67, NT]. We recall from [K95, K96] that some of the examples of R/S split, separable Frobenius are the following:

(1) Let $R = E$ a field of characteristic p (zero or prime) and $S = F$ a subfield such that E/F is a finite separable extension where p does not divide $[E : F]$. The classical trace map from E into F is a Frobenius homomorphism in this example.
(2) Let R be the group algebra $k[G]$ for a discrete group G, k a field, and $S = k[H]$, where H is a subgroup of G with finite index not divisible by the characteristic of k (e.g., char $k = p$ and H a Sylow p-subgroup). Note that if the characteristic of k divides $[G : H]$ we have an example of a split Frobenius extension which is *not* separable. By [S71] the endomorphism ring extension $\mathrm{End}(R_S)/R$ is a separable Frobenius extension which is *not* split.
(3) Let R and S be algebras over a commutative ring k such that R/S is an Hopf-Galois extension over the f.g. projective Hopf k-algebra H which is separable and coseparable over k.
(4) Let R be a type II_1 factor, S a subfactor of R of finite Jones index, as discussed in the introduction.

Third, Sugano's result [S70] for when H-separable extensions are Frobenius is evidence for biseparable implies Frobenius. This is because an H-separable extension is a strong type of separable extension [H]: see Section 4 for a separable extension which is not H-separable. Thus the result that a one-sided projective split H-separable extension is a (symmetric) Frobenius extension is a particular case of a biseparable extension which is Frobenius (cf. [K, Section 2.6][2]).

The next proposition shows that a biseparable extension is almost a two-sided Quasi-Frobenius (QF) extension in a certain sense. If a module M_R is isomorphic to a direct summand in another module P_R, we denote this by $M_R <_\oplus P_R$.

Proposition 3.6. *Suppose R/S is a biseparable extension. Then all R-modules are S-relative injective and S-relative projective; moreover, there are positive integers n and m such that*

$$R_R <_\oplus R^{*n}_R \quad \text{and} \quad {}_RR <_\oplus {}^*_RR^m$$

Proof. The first statement follows from the fact that a separable extension is both right and left semisimple extension and properties of these [HS].

For the second statement, we first establish an interesting isomorphism below involving R and its dual R^*. On the one hand, since R/S is a separable extension, $\mathcal{E} = \mathrm{Hom}(R_S, R_S)$ is a split extension of R, for if $\sum_i x_i \otimes y_i$ is a separability element we define a bimodule projection by $\mathcal{E} \to R$ by $f \mapsto \sum_i f(x_i)y_i$ (cf. [Mu65]). Then as R-bimodules, $\mathcal{E} \cong R \oplus M$ for some M: moreover, by restriction this is true as S-R-bimodules. On the other hand, since R/S is split, it follows that for some S-bimodule N, which is left and right projective S-module, $R \cong S \oplus N$ as S-bimodules; whence

$$\mathcal{E} \cong R^* \oplus \mathrm{Hom}(R_S, N_S)$$

as S-R-bimodules. Putting together the two isomorphisms for $\mathcal{E}$, we obtain

$$R \oplus M \cong \mathrm{Hom}(R_S, S_S) \oplus \mathrm{Hom}(R_S, N_S). \tag{1}$$

Since N_S is f.g. projective, there is a module P_S such that $N_S \oplus P_S \cong S^r_S$. Then applying $\mathrm{Hom}_S(R_S, -)$ to this:

$$\mathrm{Hom}(R_S, N_S) \oplus \mathrm{Hom}(R_S, P_S) \cong \mathrm{Hom}(R_S, S_S)^r.$$

Combining this with Eq. (1), we obtain

$$R_R \oplus M_R \oplus \mathrm{Hom}(R_S, P_S) \cong R^{*\,r+1}_R. \tag{2}$$

This establishes that $R_R <_\oplus R^{*\,r+1}_R$.

We similarly conclude ${}_RR <_\oplus {}^*_RR^{t+1}$ by combining the split extension $\mathcal{E}' := \mathrm{Hom}({}_SR, {}_SR)/R$ with the S-bimodule isomorphism $R \cong S \oplus N$ and the existence of ${}_SQ$ such that ${}_SQ \oplus {}_SN \cong {}_SS^t$. □

¿From the proof just completed, we obtain a corollary worth noting for its relatively easy proof. ${}_AM_A$ is said to be *centrally projective* if ${}_AM_A <_\oplus {}_AA^n_A$ for some positive integer n.

Corollary 3.7. (Cf. [S70, Theorem 2]) If R/S is centrally projective biseparable extension, then it is a QF extension.

[2][K, Theorem2.25] should read "Suppose A/S is a right progenerator split H-separable..." as the proof clearly shows.

Proof. Since there is a bimodule ${}_SP'_S$ such that ${}_SR_S \oplus {}_SP'_S \cong {}_SS^r_S$, we combine this with ${}_SR_S \cong {}_SS_S \oplus {}_SN_S$ to see that Eq. (2) is an S-R-isomorphism, whence R/S is a right QF extension [Mu64]. Similarly, we show R/S to be a left QF extension. □

The proposition and corollary above lead naturally to the problem below, a weakening of Problem 3.5.

Problem 3.8. Are biseparable extensions QF?

The next theorem provides a solution of Problem 3.5 in case A or S is a separable algebra. We assume our algebras to be faithful.

Theorem 3.9. *Suppose A/S is a biseparable extension of k-algebras with k a commutative ring. If either S is a k-projective separable k-algebra, or A is a separable k-algebra with k a field, then A/S is a Frobenius extension.*

Proof. The proof does not make use of ${}_SA$ being f.g. projective. Suppose k is a field and A is k-separable. Let $D(R)$ denote the right global dimension of a ring R and $d(M)$ denote the projective dimension of a module M_R. Then $D(A) = 0$ since A is finite dimensional semisimple. By Cohen's Theorem [Kap],

$$D(S) \leq D(A) + d(A_S)$$

whence $D(S) = 0$ and S is semisimple [Kap]. Then A and S are finite dimensional semisimple algebras. It follows from [EN] that S and A are symmetric algebras.

Similarly, we arrive at symmetric algebras S and A via [EW] under the assumption that S is k-projectively k-separable with no restriction on k. For then A is k-projective and k-separable by transitivity for projectivity and separability.

Now we compute using the *bimodule isomorphisms* $A \cong A^*$ and $S \cong S^*$ and the hom-tensor adjunction:

$$\begin{aligned} {}_SA_A &\cong {}_S\mathrm{Hom}_k(A,k)_A \\ &\cong \mathrm{Hom}_k(A \otimes_S S_S, k)_A \\ &\cong {}_S\mathrm{Hom}_S(A_S, \mathrm{Hom}_k(S,k)_S)_A \\ &\cong {}_S\mathrm{Hom}_S(A_S, S_S)_A. \end{aligned}$$

Then, since A_S is f.g. projective, A/S is a Frobenius extension. □

Part of the theorem is true without the hypothesis of biseparable extension for a finite-dimensional *Hopf subalgebra* pair $H \supseteq K$: if H is semisimple, then K is semisimple [M, 2.2.2], and H/K is a Frobenius extension (cf. [FMS, 1.8]).

Finally, Problem (3.5) can be widened to twisted extensions, as Problem (2.9) was widened to twisted Frobenius bimodules in Problem (2.10).

Problem 3.10. Are biseparable extensions α-β-Frobenius?

We refer the reader to [M65, K99] for the definition of these twisted extensions, which are more general than the usual β-Frobenius extensions. Also, Problem (3.8) has a twisted enlargement.

4. Examples and Counterexamples

In this section, we consider weakening the definition of biseparability in various ways, and find examples of non-Frobenius extension for each such case. We will see an example of non-finitely generated projective separable extension, which in itself is an obstruction to extending Villamayor's theorem [P, Prop. 10.3] and Tominaga's theorem [T].

Lemma 4.1. *Suppose k is a commutative ring and R is a k-algebra with $xy = 1$ but $yx \neq 1$. Then R is a separable extension over $S = k1 + yRx$.*

Proof. We note that $e = yx$ is a nontrivial idempotent in S. Consider $f = x \otimes y \in R \otimes_S R$. Of course, $\mu(f) = 1$. We compute with $r \in R$:

$$rx \otimes y = xyrx \otimes y = x \otimes yrxy = x \otimes yr. \quad \square$$

Now if R is a *finitely* generated, projective k-algebra, it is well-known that $xy = 1$ implies $yx = 1$. So we let V be a countably infinite rank free k-module and

$$R = \mathrm{End}_k(V) = \{ \begin{pmatrix} a & \underline{b} \\ \underline{c}^t & D \end{pmatrix} \mid a \in k, D \in M_\infty(k), \underline{b}, \underline{c}^t \in M_{1\times\infty}(k) \},$$

where $X \mapsto X^t$ denotes transpose. R is a ring satisfying the hypothesis in the lemma with elements $x, y \in R$ given in terms of the matrix units $e_{i,j}$ by

$$x = \sum_{n=1}^{\infty} e_{n,n+1}, \quad y = \sum_{n=1}^{\infty} e_{n+1,n}.$$

Clearly, $xy = 1$ but

$$e = yx = \begin{pmatrix} 0 & \underline{0} \\ \underline{0}^t & I \end{pmatrix}$$

and the k-subalgebra,

$$S = k1_R + yRx = \{ \begin{pmatrix} a & \underline{0} \\ \underline{0}^t & D \end{pmatrix} \mid a \in k, D \in M_\infty(k) \}$$

Proposition 4.2. *R/S is a split, separable, projective and non-finitely generated extension.*

Proof. We have seen in the lemma that R/S is separable. It is split since we easily check that $E : R \to S$ below is a bimodule projection:

$$E \begin{pmatrix} a & \underline{b} \\ \underline{c}^t & D \end{pmatrix} = \begin{pmatrix} a & \underline{0} \\ \underline{0}^t & D \end{pmatrix}$$

R_S is countably generated projective, since $f_n : R_S \to S_S$ defined by

$$f_1 \begin{pmatrix} a & \underline{b} \\ \underline{c}^t & D \end{pmatrix} = \begin{pmatrix} a & \underline{0} \\ \underline{0}^t & \begin{pmatrix} \underline{b} \\ 0 \end{pmatrix} \end{pmatrix} \quad f_2 \begin{pmatrix} a & \underline{b} \\ \underline{c}^t & D \end{pmatrix} = \begin{pmatrix} 0 & \underline{0} \\ \underline{0}^t & D \end{pmatrix}$$

and $f_{2+n} \begin{pmatrix} a & \underline{b} \\ \underline{c}^t & D \end{pmatrix} = \begin{pmatrix} c_n & \underline{0} \\ \underline{0}^t & 0 \end{pmatrix}$, where $\underline{c} = (c_1, c_2, \ldots)$, which satisfies the dual base equation,

$$A = \begin{pmatrix} a & \underline{b} \\ \underline{c}^t & D \end{pmatrix} = \begin{pmatrix} 1 & (1\ \underline{0}) \\ \underline{0}^t & 0 \end{pmatrix} f_1(A) + \begin{pmatrix} 0 & \underline{0} \\ \underline{0}^t & I \end{pmatrix} f_2(A) + \sum_{n=1}^{\infty} e_{n+1,1} f_{2+n}(A)$$

By using the transpose, we similarly find a countable projective base for ${}_S R$. The rest of the proof is now clear. $\square$

Corollary 4.3. R/S is not Frobenius and not H-separable.

Proof. For R/S to be a Frobenius extension, we must have R_S finitely generated from the very start. A right projective H-separable extension is right f.g. by [T]. $\square$

Finally, we consider various weakenings of Problem 3.5 and note that they all have known counterexamples. There are easy examples of separable extensions which are not Frobenius, such as the rationals $\mathcal{Q}$ extending the integers $\mathcal{Z}$. There are even f.g. free separable extensions that are not Frobenius in the ordinary sense, but are β-Frobenius [K99]. As for a f.g. split, separable extension that is not Frobenius, here is one that is not flat, whence not projective: consider $\mathcal{Z} \oplus \mathcal{Z}_2$ as a $\mathcal{Z}$-algebra, which has projection onto its first factor as splitting bimodule projection, is a direct sum of separable algebras – whence separable – and its second factor is of course not flat over $\mathcal{Z}$.

There are clearly many examples of split extensions that are not Frobenius, let alone f.g. Asking for a split, f.g. projective extension that fails to be Frobenius is not hard: for example, let R be the upper triangular $n \times n$ matrix algebra with splitting $S \oplus I$ where S is the subalgebra of diagonal matrices and $I^n = 0$. It is well-known that R is not a QF-algebra, and certainly not Frobenius [L], but S is semisimple, so R_S and ${}_SR$ are f.g. projective; moreover, R/S cannot be Frobenius by the transitivity property of Frobenius extension (cf. [SK2]). This is an example too of $\mathrm{Hom}(R_S, S_S)$ not being a projective right R-module.

As a last cautionary example, we consider $R = \mathcal{Z}_2 \oplus \mathcal{Z}_2$ and $S = \mathcal{Z}_2$. It is easy to check that R/S is split, separable and Frobenius, even f.g. free. But there are only two bimodule projections $E : R \to S$, neither of which is a Frobenius homomorphism, i.e. in possession of dual bases [K]. The Frobenius homomorphism in this example is unique, since the group of units in $C_R(S)$ consists only of the identity [K].

5. Categorical interpretation

Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. F induces a natural transformation $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet))$, $\mathcal{F}_{C,C'}(f) = F(f)$. F is called a separable functor [NVO] if $\mathcal{F}$ splits, i.e. there exists a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \to \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$ such that $\mathcal{P} \circ \mathcal{F}$ is the identity natural transformation on $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$.

Proposition 5.1. [RAFAEL] *Assume that F has a right adjoint G, and let $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{D}}$ be the unit and counit of the adjunction.*
F is separable if and only if there exists a natural transformation $\nu : GF \to 1_{\mathcal{C}}$ such that $\nu \circ \eta$ is the identity natural transformation on $\mathcal{C}$.
G is separable if and only if there exists a natural transformation $\zeta : 1_{\mathcal{D}} \to FG$ such that $\varepsilon \circ \zeta$ is the identity natural transformation on $\mathcal{D}$.

The terminology stems from the fact that, for a ring homomorphism $i : R \to S$, the restriction of scalars functor is separable if and only if S/R is separable (see [NVO], [RAFAEL]); Separable functors satisfy the following version of Maschke's Theorem: if a morphism f in $\mathcal{C}$ is such that $F(f)$ has a left or right inverse in $\mathcal{D}$, then f has a left or right inverse in $\mathcal{C}$.
The functor F is called Frobenius if F has a right adjoint G that is at the same time a right adjoint. We will then say that (F, G) is a Frobenius pair. Now the terminology is inspired by the property that a ring homomorphism $i : R \to S$ is Frobenius if and only if the restriction of scalars functor is Frobenius. Frobenius pairs were introduced in [M65], and studied more recently in [CMZ97] and [CGN99]. For more details and examples of separable functors and Frobenius functors, we refer the reader to [CMZ].

Suppose we know that (F, G) is a Frobenius pair. Then Rafael's Theorem can be simplified: we can give an easier criterium for F or G to be separable. First we need a result on adjoint functors, which is folklore. Since we did not find an appropriate reference in the literature, we include an outline of the proof. Let ε and η be the counit and unit of an adjunction (F, G). Recall that

$$\varepsilon_{F(C)} \circ F(\eta_C) = I_{F(C)} \text{ and } G(\varepsilon_D) \circ \eta_{G(D)} = I_{G(D)} \tag{3}$$

for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

Lemma 5.2. *Let (F, G) be an adjoint pair functors, then we have isomorphisms*

$$\underline{\underline{\mathrm{Nat}}}(F, F) \cong \underline{\underline{\mathrm{Nat}}}(G, G) \cong \underline{\underline{\mathrm{Nat}}}(1_{\mathcal{C}}, GF) \cong \underline{\underline{\mathrm{Nat}}}(FG, 1_{\mathcal{D}})$$

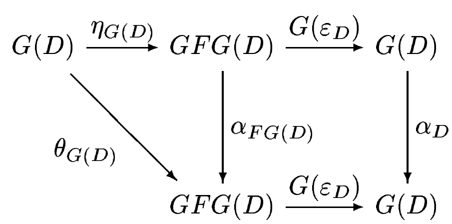
Proof. We will show that $\underline{\underline{\mathrm{Nat}}}(G, G) \cong \underline{\underline{\mathrm{Nat}}}(1_{\mathcal{C}}, GF)$, the proof of the other assertions is left to the reader. For a natural transformation $\theta : 1_{\mathcal{C}} \to GF$, we define $\alpha = X(\theta) : G \to G$ by

$$\alpha_D = G(\varepsilon_D) \circ \theta_{G(D)} \tag{4}$$

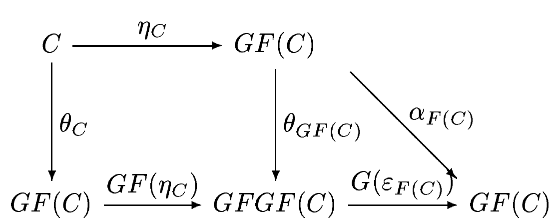
Conversely, for $\alpha : G \to G$, $\theta = X^{-1}(\alpha) : 1_{\mathcal{C}} \to GF$ is defined by

$$\theta_C = \alpha_{F(C)} \circ \eta_C \tag{5}$$

We are done if we can show that X and X^{-1} are each others inverses. First take $\alpha : G \to G$, and $\theta = X^{-1}(\alpha)$. The diagram



commutes: the triangle is commutative because of (5), and the square commutes because α is natural. ¿From (3), it follows that the composition of the two maps in the top row is $I_{G(N)}$, and then we see from the diagram that $\alpha = X(\theta)$.
Conversely, take $\theta : 1_{\mathcal{C}} \to GF$, and let $\alpha = X(\theta)$. Then $\theta = X^{-1}(\alpha)$ because the following diagram commutes:



□

Corollary 5.3. Let (F, G) be a Frobenius pair of functors, then we have isomorphisms

$$\begin{aligned} \underline{\underline{\mathrm{Nat}}}(F, F) \cong \underline{\underline{\mathrm{Nat}}}(G, G) &\cong \underline{\underline{\mathrm{Nat}}}(1_{\mathcal{C}}, GF) \cong \underline{\underline{\mathrm{Nat}}}(FG, 1_{\mathcal{D}}) \\ &\cong \underline{\underline{\mathrm{Nat}}}(GF, 1_{\mathcal{C}}) \cong \underline{\underline{\mathrm{Nat}}}(1_{\mathcal{D}}, FG) \end{aligned}$$

For a Frobenius pair (F,G), we will write $\nu:\ GF\to 1_{\mathcal{C}}$ and $\zeta:\ 1_{\mathcal{D}}\to FG$ for the counit and unit of the adjunction (G,F). For all $C\in\mathcal{C}$ and $D\in\mathcal{D}$, we then have

(6) $$F(\nu_C)\circ\zeta_{F(C)}=I_{F(C)} \text{ and } \nu_{G(D)}\circ G(\zeta_D)=I_{G(D)}$$

Proposition 5.4. *Let (F,G) be a Frobenius pair, and let η, ε, ν and ζ be as above. The following statements are equivalent:*

- *F is separable;*
- *$\exists\alpha\in\underline{\underline{\mathrm{Nat}}}(F,F):\ \nu_C\circ G(\alpha_C)\circ\eta_C=I_C$ for all $C\in\mathcal{C}$;*
- *$\exists\beta\in\underline{\underline{\mathrm{Nat}}}(G,G):\ \nu_C\circ\beta_{F(C)}\circ\eta_C=I_C$ for all $C\in\mathcal{C}$.*

We have a similar characterization for the separability of G: the statements

- *G is separable;*
- *$\exists\alpha\in\underline{\underline{\mathrm{Nat}}}(F,F):\ \varepsilon_D\circ\alpha_{G(D)}\circ\zeta_D=I_D$ for all $D\in\mathcal{D}$;*
- *$\exists\beta\in\underline{\underline{\mathrm{Nat}}}(G,G):\ \varepsilon_D\circ F(\beta_D)\circ\zeta_D=I_D$ for all $D\in\mathcal{D}$.*

are equivalent.

Proof. Assume that F is separable. By Rafael's Theorem, there exists $\tilde{\nu}\in\underline{\underline{\mathrm{Nat}}}(GF,1_{\mathcal{C}})$ such that $\tilde{\nu}_C\circ\eta_C=I_C$ for all $C\in\mathcal{C}$. Let $\alpha:\ F\to F$ be the corresponding natural transformation of Corollary 5.3, i.e. $\alpha_C=F(\tilde{\nu}_C)\circ\zeta_{F(C)}$, and $\tilde{\nu}_C=\nu_C\circ G(\alpha_C)$, and the first implication of the Proposition follows. The converse follows trivially from Rafael's Theorem. All the other equivalences can be proved in a similar way. □

Application to bimodules. We use the notation of Section 2: let R and T be rings, and M a (T,R)-bimodule. We have already seen that $M^*=\mathrm{Hom}(M_R,R_R)$ and ${}^*M=\mathrm{Hom}({}_TM,{}_TT)$ are (R,T)-bimodules. We also have that

$$\mathrm{Hom}({}_TM,{}_TM) \text{ and } \mathrm{Hom}(M_R,M_R)$$

are respectively an (R,R)-bimodule and a (T,T)-bimodule, via

$$(m)(r\varphi s)=((mr)\varphi)s \text{ and } (t\psi u)(m)=t\psi(um)$$

for all $r,s\in R$, $t,u\in T$, $\varphi\in\mathrm{Hom}({}_TM,{}_TM)$ and $\psi\in\mathrm{Hom}(M_R,M_R)$. Furthermore, the induction functor

$$F=M\otimes_R\bullet:\ {}_R\mathcal{M}\to{}_T\mathcal{M}$$

has a right adjoint

$$G=\mathrm{Hom}({}_TM,{}_T\bullet):\ {}_T\mathcal{M}\to{}_R\mathcal{M}$$

For $Q\in{}_T\mathcal{M}$, $\mathrm{Hom}({}_TM,{}_TQ)\in{}_R\mathcal{M}$ via $(m)(rf)=(mr)(f)$. For $\kappa:\ Q\to Q'$ in ${}_T\mathcal{M}$, we put $G(\kappa)(f)=\kappa\circ f$. The unit and counit of the adjunction are given by

$$\eta_P:\ P\to GF(P)=\mathrm{Hom}({}_TM,{}_TM\otimes_R P),\ (m)\eta_P(p)=m\otimes_R p$$

$$\varepsilon_Q:\ FG(Q)=M\otimes_R\mathrm{Hom}({}_TM,{}_TQ)\to Q,\ \varepsilon_Q(m\otimes f)=(m)f$$

We also have a functor

$$G'={}^*M\otimes_T\bullet:\ {}_T\mathcal{M}\to{}_R\mathcal{M}$$

and a natural transformation $\gamma:\ G'\to G$ given by

$$\gamma_Q:\ {}^*M\otimes_T Q\to\mathrm{Hom}({}_TM,{}_TQ),\ (m)(\gamma_Q(f\otimes_T q))=(m)fq$$

We will also write $\gamma_Q(f\otimes_T q)=f\cdot q$. γ_Q is well-defined on the tensor product, and left R-linear, so

(7) $$(rft)\cdot q=r(f\cdot(tq))$$

for all $r \in R$ and $t \in T$. If Q is a (T, R)-bimodule (for example, $T = M$), then γ_Q is also right R-linear, and we have

$$f \cdot (qr) = (f \cdot q)r \tag{8}$$

Now assume that ${}_TM$ is finitely generated and projective, and consider a dual basis $\{n_j \in M,\ g_j \in {}^*M\}$, i.e.

$$m = \sum_j (m) g_j n_j$$

for all $m \in M$, or $I_M = \sum_j \gamma_M(g_j \otimes_T n_j)$. Then $\gamma :\ G \to G'$ is a natural isomorphism, and for all left T-linear $f :\ M \to Q$, we have

$$\gamma_Q^{-1}(f) = \sum_j g_j \otimes_T f(n_j)$$

In order to decide whether F or G is separable, or whether (F, G) is a Frobenius pair, we have to investigate natural transformations $GF \to 1_{{}_R\mathcal{M}}$ and $1_{{}_T\mathcal{M}} \to FG$. This is done in the next two Propositions.

Proposition 5.5. *Let F and G be as above, and consider*

$$V = \underline{\underline{\text{Nat}}}(GF, 1_{{}_R\mathcal{M}}),\ \ V_1 = Hom({}_R Hom({}_TM, {}_TM)_R, {}_RR_R)$$

$$V_2 = Hom({}_R({}^*M)_T, {}_R(M^*)_T)$$

Then we have maps

$$V \xrightarrow{\ \alpha\ } V_1 \xrightarrow{\ \alpha_1\ } V_2$$

which are isomorphisms if M is finitely generated and projective as a left T-module.

Proof. For $\nu \in V$, we put $\alpha(\nu) = \nu_R$. By definition, ν_R is left R-linear. Left R-linearity follows from the naturality of ν: for any $s \in R$, we consider the left R-linear map $m_s :\ R \to R$, $m_s(r) = rs$. We have a commutative diagram

$$\begin{array}{ccc}
\text{Hom}({}_TM, {}_TM) & \xrightarrow{\ \nu_R\ } & R \\
\Big\downarrow{\scriptstyle GF(m_s)} & & \Big\downarrow{\scriptstyle m_s} \\
\text{Hom}({}_TM, {}_TM) & \xrightarrow{\ \nu_R\ } & R
\end{array}$$

We easily compute that $GF(m_s)(f) = fs$, and the diagram tells us that $\nu_R(f)s = \nu_R(fs)$.
$\alpha_1 :\ V_1 \to V_2$ is given by $\alpha_1(\overline{\nu}) = \overline{\phi}$, with

$$\overline{\phi}(f)(m) = \overline{\nu}(\gamma_M(f \otimes m)) = \overline{\nu}(f \cdot m)$$

Using (7) and (8), we easily deduce that $\overline{\phi}(f)$ is right R-linear, and that $\overline{\phi}$ is left R-linear and right T-linear.
Assume that ${}_TM$ is finitely generated projective, and, as above, assume that $\{n_j \in M,\ g_j \in {}^*M\}$ is a dual basis. We can then define the inverse α^{-1} of α as follows. We view $\overline{\nu} \in V_1$ as a map $\overline{\nu} :\ {}^*M \otimes_T M \to R$, and we identify G and G'. We then define $\nu \in V$ by

$$\nu_P = \overline{\nu} \otimes_R P :\ GF(P) \cong {}^*M \otimes_T M \otimes_R P \to R \otimes_R P \cong P$$

It is clear that ν is natural and that α and α^{-1} are each others inverses.
For $\overline{\phi} \in V_2$, we define $\alpha_1^{-1}(\overline{\phi}) = \overline{\nu}$ by

$$\overline{\nu}(\varphi) = \sum_j \overline{\phi}(g_j)((n_j)\varphi)$$

for all $\varphi :\ M \to M$ in ${}_T\mathcal{M}$. Straightforward computations yield that α_1^{-1} is well-defined, and is indeed an inverse of α_1. □

In a similar fashion, we have:

Proposition 5.6. *Let F and G be as above, and consider*

$$W = \underline{\underline{\mathrm{Nat}}}(1_{{}_T\mathcal{M}}, FG),\ W_1 = \{e \in M \otimes_R {}^*M \mid te = et,\ \text{for all } t \in T\}$$

$$W_2 = Hom({}_R(M^*)_T, {}_R({}^*M)_T)$$

We have maps

$$W \xrightarrow{\beta} W_1 \xrightarrow{\beta_1} W_2$$

β is an isomorphism, and β_1 is an isomorphism if M is finitely generated as a right R-module.

Proof. For a natural transformation $\zeta :\ 1_{{}_T\mathcal{M}} \to FG$, we define $e = \beta(\zeta) = \zeta_T(1) \in FG(T) = M \otimes_R {}^*M$. The fact that $et = te$ follows from the naturality of ζ: for any $t \in T$, we consider the left T-linear map $m_t :\ T \to T$, $m_t(u) = ut$. We easily find that $FG(m_t) :\ M \otimes_R {}^*M \to M \otimes_R {}^*M$ is given by $FG(m_t)(m \otimes f) = m \otimes ft$. From the commutativity of the diagram

$$\begin{array}{ccc} T & \xrightarrow{\zeta_T} & M \otimes_R {}^*M \\ \downarrow{\scriptstyle m_t} & & \downarrow{\scriptstyle FG(m_t)} \\ T & \xrightarrow{\zeta_T} & M \otimes_R {}^*M \end{array}$$

and the left T-linearity of ζ_T, we deduce that

$$\zeta_T(1)t = \zeta_T(t) = t\zeta_T(1)$$

Conversely, take $e = \sum_i m_i \otimes_R f_i \in W_1$. $\zeta \in W$ is defined as follows: for all $Q \in {}_T\mathcal{M}$, we put

$$\zeta_Q :\ Q \to M \otimes_R \mathrm{Hom}({}_T M, {}_T, Q),\ \zeta_Q(q) = \sum_i m_i \otimes_R f_i \cdot q$$

ζ_Q is left T-linear, since

$$\begin{aligned} \zeta_Q(tq) &= \sum_i m_i \otimes_R f_i \cdot (tq) \\ &= \sum_i m_i \otimes_R (f_i t) \cdot q = \sum_i t m_i \otimes_R f_i \cdot q \end{aligned}$$

Let us check that ζ is natural. For a left T-linear map $\kappa: Q \to Q'$, we have to show that the following diagram commutes

$$\begin{array}{ccc} Q & \xrightarrow{\zeta_Q} & M \otimes_R \operatorname{Hom}({}_TM, {}_TQ) \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle FG(\kappa)} \\ Q' & \xrightarrow{\zeta_{Q'}} & M \otimes_R \operatorname{Hom}({}_TM, {}_TQ') \end{array}$$

We first compute that

$$FG(\kappa)(\zeta_Q(q)) = \sum_i m_i \otimes_R \kappa \circ (f_i \cdot q) = \sum_i m_i \otimes_R f_i \cdot \kappa(q)$$

using

$$(m)(f_i \cdot \kappa(q)) = ((m)f_i)\kappa(q) = \kappa((m)f_i \cdot q) = (m)(\kappa \circ (f_i \cdot q))$$

and the commutativity of the diagram follows easily.
Now we define $\beta_1: W_1 \to W_2$. For $e = \sum_i m_i \otimes_R f_i \in W_1$, we let

$$\beta_1(e) = \phi: M^* \to {}^*M$$

be given by

$$\phi(h) = \sum_i h(m_i)f_i \text{ or } (m)\phi(h) = \sum_i (mh(m_i))f_i$$

Straightforward computations show that $\phi(h) \in {}^*M$, and that ϕ is (R,T)-bilinear.
Conversely, suppose that M_R is finitely generated projective. Let $\{h_k \in M^*, p_k \in M\}$ be a finite dual basis:

$$m = \sum_k p_k h_k(m) \text{ and } h = \sum_k h(p_k)h_k$$

for all $m \in M$, $h \in M^*$. We define $\beta_1^{-1}: W_2 \to W_1$ by

$$\beta_1^{-1}(\phi) = e = \sum_k p_k \otimes_R \phi(h_k)$$

We leave it to the reader to verify that $e \in W_1$, and that β_1 and β_1^{-1} are each others inverses. □

The two previous results can be used to decide when the induction functor F and its adjoint G are separable.

Corollary 5.7. Let M be a (T,R)-bimodule, and assume that M is finitely generated projective as a left T-module, with finite dual basis $\{n_j, g_j\}$. Then the following assertions are equivalent:

- $F = M \otimes_R \bullet$ is a separable functor;
- there exists $\overline{\nu} \in V_1$ such that $\overline{\nu}(I_M) = 1_R$;
- there exists $\overline{\phi} \in V_2$ such that $\sum_j \overline{\phi}(g_j)(n_j) = 1_R$.

Corollary 5.8. Let M be a (T,R)-bimodule. The functor $G = \operatorname{Hom}({}_TM, {}_T\bullet)$ is separable if and only if T is M-separable over R, in the sense of Definition 2.1. If M is finitely generated and projective as a left R-module, with finite dual basis

$\{h_k, p_k\}$, then this is also equivalent to the existence of $\phi: M^* \to {}^*M$ in W_2 such that

$$\sum_k (p_k)\phi(h_k) = 1$$

Theorem 5.9. *Let M be a (T,R)-bimodule, and consider the functors $F = M\otimes_R \bullet$ and $G = Hom({}_TM, {}_T\bullet)$. The following assertions are equivalent.*

1. *(F,G) is a Frobenius pair, and ${}_TM$ is finitely generated and projective;*
2. *${}_TM$ is finitely generated and projective, and there exist $e = \sum_i m_i \otimes_R f_i \in W_1$ and $\overline{\nu} \in V_1$ such that*

$$m = \sum_i m_i \overline{\nu}(f_i \cdot m) \tag{9}$$

$$f = \sum_i \overline{\nu}(f \cdot m_i) f_i \tag{10}$$

for all $m \in M$ and $f \in {}^*M$;

3. *${}_TM_R$ is Frobenius in the sense of Definition 2.8.*

Proof. 1) $\Rightarrow$ 2). Recall that (F,G) is Frobenius if and only if there exist natural transformations $\nu \in V$, $\zeta \in W$ such that

$$F(\nu_P)\circ\zeta_{F(P)} = I_{F(P)} \text{ and } \nu_{G(Q)}\circ G(\zeta_Q) = I_{G(Q)} \tag{11}$$

for all $P \in {}_R\mathcal{M}$ and $Q \in {}_T\mathcal{M}$. Let $\overline{\nu} = \nu_R \in V_1$ and $e = \sum_i m_i \otimes_R f_i = \zeta_T(1) \in W_1$. Putting $P = R$ in (11), we find (9). Then take $Q = T$ in (11). Making the identification $G' = G$ (${}_TM$ is finitely generated projective), we find

$$G(\zeta_T):\ {}^*M \to {}^*M \otimes_T M \otimes_R {}^*M,\ G(\zeta_T)(f) = \sum_i f \otimes_T m_i \otimes_R f_i$$

and $\nu_{G(T)}(G(\zeta_T)(f)) = \sum_i \overline{\nu}(f\cdot m_i) f_i$, so (10) follows.
1) $\Rightarrow$ 2). Let $\nu = \alpha^{-1}(\overline{\nu})$ and $\zeta = \beta^{-1}(e)$. ν and ζ satisfy (11), so (F,G) is Frobenius.
2) $\Rightarrow$ 3). (9) implies that M_R is finitely generated projective. Let $\overline{\phi} = \alpha_1(\overline{\nu})$ and $\phi = \beta_1(e)$. We easily compute that $\overline{\phi}$ and ϕ are each others inverses. Indeed, for all $f \in {}^*M$, $h \in M^*$ and $m \in M$, we have

$$\begin{aligned}
\overline{\phi}(\phi(h))(m) &= \overline{\nu}(\phi(h)\cdot m) = \sum_i \overline{\nu}(h(m_i)f_i \cdot m) \\
&= \sum_i \overline{\nu}(h(m_i)(f_i\cdot m)) = \sum_i h(m_i)\overline{\nu}(f_i \cdot m) \\
&= \sum_i h(m_i\overline{\nu}(f_i\cdot m)) = h(m)
\end{aligned}$$

and

$$\phi(\overline{\phi}(f)) = \sum_i \overline{\phi}(f)(m_i) f_i = \sum_i \overline{\nu}(f\cdot m_i) f_i = f$$

3) $\Rightarrow$ 2). If ${}^*M$ and M^* are isomorphic as (R,T)-bimodules, then there exist $\overline{\phi} \in V_2$ and $\phi \in W_2$ that are each others inverses. Put $\overline{\nu} = \alpha_1^{-1}(\overline{\phi})$ and $e = \beta_1^{-1}(\phi)$. Straightforward computations show that $\overline{\nu}$ and e satisfy (9-10). □

Obviously our results also hold for functors between categories of right modules. As before, let M be a (T,R)-bimodule, and consider the functors

$$\widetilde{F} = \bullet \otimes_T M : \ \mathcal{M}_T \to \mathcal{M}_R \text{ and } \widetilde{G} = \mathrm{Hom}(M_R, \bullet_R) : \ \mathcal{M}_R \to \mathcal{M}_T$$

Then $(\widetilde{F}, \widetilde{G})$ is an adjoint pair. We have a natural transformation

$$\widetilde{\gamma} : \ \widetilde{G'} = \bullet \otimes_R M^* \to G$$

For all $Q \in \mathcal{M}_R$, $\widetilde{\gamma}_Q : \ Q \otimes_R M^* \to \mathrm{Hom}(M_R, Q_R)$ is given by

$$\gamma_Q(q \otimes h)(m) = qh(m)$$

We denote $\gamma_Q(q \otimes h) = q \cdot h$. The analogs of Propositions 5.5 and 5.6 are the following:

Proposition 5.10. *With notation as above, we have maps*

$$\widetilde{V} = \underline{\underline{\mathrm{Nat}}}(\widetilde{G}\widetilde{F}, 1_{\mathcal{M}_T}) \xrightarrow{\widetilde{\alpha}} \widetilde{V}_1 = Hom({}_T Hom(M_R, M_R)_T, {}_T T_T) \xrightarrow{\widetilde{\alpha}_1} W_2$$

and

$$\widetilde{W} = \underline{\underline{\mathrm{Nat}}}(1_{\mathcal{M}_R}, \widetilde{F}\widetilde{G}) \xrightarrow{\widetilde{\beta}} \widetilde{W}_1 \xrightarrow{\widetilde{\beta}_1} V_2$$

where $\widetilde{W}_1 = \{e \in M^* \otimes_T M \mid re = er \text{ for all } r \in R\}$. $\widetilde{\beta}$ *is always an isomorphism,* $\widetilde{\alpha}$ *and* $\widetilde{\alpha}_1$ *are isomorphisms if* M_R *is finitely generated projective, and* $\widetilde{\beta}_1$ *is an isomorphism if* ${}_T M$ *is finitely generated projective.*

Proof. Completely similar to the proof of Propositions 5.5 and 5.6. Let us mention that

$$\alpha_1(\overline{\nu}) = \phi \text{ with } (m)\phi(h) = \overline{\nu}(\gamma_M(m \otimes h))$$
$$\beta_1(\sum_i k_i \otimes m_i) = \overline{\phi} \text{ with } \overline{\phi}(f) = \sum_i k_i((m_i)f) \quad \square$$

As a consequence, we obtain relations between the separability and Frobenius properties of F, G, $\widetilde{F}$ and $\widetilde{G}$.

Corollary 5.11. Let M be a (T,R)-bimodule, and assume that M_R is finitely generated projective. Then the following assertions are equivalent:

- $\widetilde{F}$ is separable;
- there exists $\overline{\nu} \in \widetilde{V}_1$: $\overline{\nu}(I_M) = 1_T$;
- there exists $\phi \in W_2$: $\sum_k (p_k)\phi(h_k) = 1$;
- G is separable.

Here $\{p_k, h_k\}$ is a finite dual basis of M as a right R-module.

Proof. The equivalence of the first three statements is obtained in exactly the same way as Corollary 5.7. The equivalence of the third and the fourth statement is of one of the equivalences in Corollary 5.8. $\square$

Corollary 5.12. Let M be a (T,R)-bimodule. The following statements are equivalent:

- $\widetilde{G}$ is separable;
- there exists $e = \sum_i k_i \otimes_T m_i \in \widetilde{W}_1$ such that $\sum_i k_i(m_i) = 1_R$.

If ${}_T M$ is finitely generated projective, then they are also equivalent to

- There exists $\overline{\phi} \in V_2$ such that $\sum_j \overline{\phi}(g_j)n_j = 1_R$;
- F is separable.

Here $\{n_j, g_j\}$ is a finite dual basis of M as a left T-module.

Corollary 5.13. Let M be a (T, R)-bimodule. The following statements are equivalent:

- $(\widetilde{F}, \widetilde{G})$ is a Frobenius pair and M_R is finitely generated projective;
- M_R is finitely generated projective and there exist $e = \sum_i k_i \otimes_T m_i \in \widetilde{W}_1$ and $\overline{\nu} \in \widetilde{V}_1$ such that

$$m = \sum_i \overline{\nu}(q \cdot k_i)m_i \text{ and } h = \sum_i k_i\overline{\nu}(m_i \cdot f) \tag{12}$$

  for all $m \in M$ and $f \in M^*$;
- ${}_TM_R$ is a Frobenius bimodule.

We now address the following problem: assume that we know ${}_TM_R$ to be a Frobenius bimodule. When is T M-separable over R? In view of the above considerations, it suffices to apply Proposition 5.4. We first need a Lemma.

Lemma 5.14. *Let M be a (T, R)-bimodule, and $F = M \otimes_R \bullet$. Then*

$$\underline{\underline{\mathrm{Nat}}}(F, F) \cong \mathit{Hom}({}_TM_R, {}_TM_R)$$

Proof. Let $\alpha : F \to F$ be a natural transformation. Then $\alpha_R : M \to M$ is left T-linear. Right R-linearity follows from the naturality: for $s \in R$, we consider the left R-linear map $m_s : R \to R$, $m_s(r) = rs$. We have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha_R} & M \\ \Big\downarrow{\scriptstyle F(m_s)} & & \Big\downarrow{\scriptstyle F(m_s)} \\ M & \xrightarrow{\alpha_R} & M \end{array}$$

$F(m_s) : M \to M$ is given by $F(m_s)(m) = ms$, and left R-linearity follows from the commutativity of the diagram.
Given a (T, R)-linear map $\alpha_R : M \to M$, we define a natural transformation $\alpha : F \to F$ as follows: $\alpha_P = \alpha_R \otimes I_P$. □

Proposition 5.15. *Assume that M is a Frobenius (T, R)-bimodule, and let $e = \sum_i m_i \otimes_R f_i \in W_1$ and $\overline{\nu} \in V_1$ be as in the second statement of Theorem 5.9. Then T is M-separable over R (i.e. G is separable) if and only if there exists a (T, R)-linear map $\overline{\alpha} : M \to M$ such that*

$$\sum_i (\overline{\alpha}(m_i))f_i = 1_T$$

This proposition recovers [K, Theorem 2.15]. One can similarly recover [K99, Theorem 4.1].

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