

COVARIANT STRONG MORITA THEORY OF STAR PRODUCT ALGEBRAS

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Abstract

In this note we recall some recent progress in understanding the representation theory of $*$ -algebras over rings $C = R(i)$ where R is ordered and $i^2 = -1$. The representation spaces are modules over auxiliary $*$ -algebras with inner products taking values in this auxiliary $*$ -algebra. The ring ordering allows to implement positivity requirements for the inner products. Then the representations are required to be compatible with the inner product. Moreover, one can add the notion of symmetry in form of Hopf algebra actions. For all these notions of representations there is a well-established Morita theory which we review. The core of each version of Morita theory is the corresponding Picard groupoid for which we give tools to compute and determine both the orbits and the isotropy groups.

1 Introduction

Deformation quantization [2] has proven to be a physically reasonable and mathematically rich approach to quantization of finite-dimensional classical mechanical systems modelled on phase spaces being symplectic or even general Poisson manifolds. Many questions like existence and classification of star products have been successfully answered and non-trivial applications to other branches of mathematics like index theorems have been found. To name a few highlights see [17, 19, 11, 10, 4] and confer to [27] for a gentle introduction with further and detailed references.

While the structure of the deformed algebras, playing the role of observables in deformation quantization, is by now very well understood, this is not yet enough for the original physical purpose: One also has to develop a physically reasonable notion for states allowing for an implementation of the superposition principle. The solution of this problem is to proceed analogously as in the theory of C^* -algebras: states are identified with positive linear functionals, the superposition principle is then implemented in the GNS representations of the star product algebra arising from such positive functionals. This program has been pursued successfully in a sequence of articles, see in particular [9, 25, 23, 5] as well as [27, Chap. 7] for an overview and more references.

The aim of this note is now to review recent developments in studying the representation theory of star product algebras in the presence of a *symmetry*. The physical relevance of this question should be clear.

The results presented in this review are mainly based on the work [15], see also [13] and the forthcoming project [14]. We start explaining the basic ingredients in Section 2. Then we define the adapted notions of representations in Section 3 and introduce the relevant tensor products for

Morita theory in Section 4. Based on these tensor products we explain the induced notions of Picard (bi-)groupoids in Section 5 and comment on some of the Morita invariants arising from actions of the Picard groupoids. Finally, we conclude this review with some further remarks in Section 6.

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2 The set-up

The framework we discuss here is slightly more general than actually needed for star product algebras. First we consider an *ordered ring* R which in the examples will mainly be either \mathbb{R} or $\mathbb{R}[[\lambda]]$ both with their usual ordering structure. The positivity in R will then be used to induce notions of positivity for linear functionals, algebra elements, inner products, etc. Next, we take the ring extension $\mathbb{C} = R(i)$ by a square root of -1 , i.e. $i^2 = -1$ to have a replacement for the complex numbers. In the main two examples this results in \mathbb{C} and $\mathbb{C}[[\lambda]]$, respectively.

The algebras we consider in this note are **-algebras* \mathcal{A} over \mathbb{C} , i.e. associative algebras over \mathbb{C} equipped with a antilinear involutive antiautomorphism, the **-involution*, written as $a \mapsto a^*$ for $a \in \mathcal{A}$. The notion of *symmetry* we are interested is based on a Hopf algebra H which we require to be a Hopf **-algebra*, i.e. a Hopf algebra which is a **-algebra* such that the comultiplication Δ and the counit ε are **-homomorphisms* and such that $S^*S^* = \text{id}$ for the antipode S . In fact, the requirement $S^*S^* = \text{id}$ is superfluous and follows already from the remaining properties of a Hopf **-algebra*, see e.g. [16] for more details on Hopf algebras with **-involutions*. Then all **-algebras* \mathcal{A} are thought to carry a *left *-action* of the Hopf **-algebra*, i.e. a left H -module structure such that

$$g \triangleright (ab) = (g_{(1)} \triangleright a)(g_{(2)} \triangleright b), \quad g \triangleright \mathbb{1}_{\mathcal{A}} = \varepsilon(g) \mathbb{1}_{\mathcal{A}}, \quad \text{and} \quad (g \triangleright a)^* = S(g)^* \triangleright a^*, \quad (1)$$

where $\Delta(g) = g_{(1)} \otimes g_{(2)}$ is the usual Sweedler notation and $g \in H$ and $a, b \in \mathcal{A}$. The main examples of interest in deformation quantization are the group algebras $H = \mathbb{C}[G]$ with **-involution* specified by $g^* = g^{-1}$ and the usual Hopf algebra structure as well as the complexified universal enveloping algebras $U(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}$ of a Lie algebra \mathfrak{g} over \mathbb{R} with **-involution* determined by $\xi^* = -\xi$ for $\xi \in \mathfrak{g}$ and the usual Hopf algebra structure.

We call such a **-action* *inner* if there is a **-homomorphism*, the *momentum map* $J : H \longrightarrow \mathcal{A}$, such that

$$g \triangleright a = J(g_{(1)})aJ(S(g_{(2)})). \quad (2)$$

In general, it is a quite strong requirement to have a momentum map and many interesting cases do *not* allow for such a **-action*. In particular, if the action is non-trivial and \mathcal{A} is commutative, then there can be no momentum map. However, there is one important exception in deformation quantization: assume that we have a real Lie algebra \mathfrak{g} acting on a manifold M by vector fields. Then we have an action by derivations on $C^\infty(M)$ via the Lie derivative. Assume now that we have a star product \star which has a quantum momentum map. This is a linear map $J : \mathfrak{g} \longrightarrow C^\infty(M)[[\lambda]]$ such that

$$J_\xi \star J_\eta - J_\eta \star J_\xi = i\lambda J_{[\xi, \eta]} \quad (3)$$

for all $\xi, \eta \in \mathfrak{g}$. Then we consider the ‘rescaled’ universal enveloping algebra $U(\mathfrak{g}_{i\lambda})$ where we rescale the Lie bracket of \mathfrak{g} by $i\lambda$ and view it as Lie algebra over $\mathbb{C}[[\lambda]]$. Then J extends to a momentum map for the Lie algebra action induced by the \star -commutator with J_ξ . This deforms

the classical action in the following sense: the lowest order terms in the star product commutator are

$$[J_\xi, f]_\star = i\lambda \{J_\xi, f\} + \cdot = -i\lambda \xi_M f + \dots, \quad (4)$$

where ξ_M is the fundamental vector field on M corresponding to ξ . Clearly, the action $f \mapsto [J_\xi, f]_\star$ is inner by the very construction. Note however, that the classical action $f \mapsto \xi_M f$ is not at all inner (unless it is trivial).

Star products with such quantum momentum maps have been excessively studied, see e.g.[12, 18]. However, if we pass to a Lie group action with a Lie group acting on the underlying manifold by diffeomorphisms, such an action can never be inner neither classically nor for a star product. This is easy to see as the right hand side of (2) for star products is local in the algebra element a while the action of a diffeomorphism via pull-backs certainly not.

3 The representation theories

As we consider algebras \mathcal{A} with quite a lot additional structures the purely algebraic framework of modules will not be the appropriate category for interesting representations. Thus we look for representation spaces with extra structure reflecting the presence of a \ast -involution and symmetry of the algebras. The following definition turns out to be very useful:

Definition 3.1 (Inner Product Module). An *inner product module* $\mathcal{E}_\mathcal{A}$ over \mathcal{A} is a right \mathcal{A} -module with inner product

$$\langle \cdot, \cdot \rangle_\mathcal{A} : \mathcal{E}_\mathcal{A} \times \mathcal{E}_\mathcal{A} \longrightarrow \mathcal{A}$$

such that for all $x, y \in \mathcal{E}_\mathcal{A}$ and $a \in \mathcal{A}$

1. $\langle \cdot, \cdot \rangle_\mathcal{A}$ is \mathbb{C} -linear in second argument,
2. $\langle x, y \rangle_\mathcal{A} = \langle y, x \rangle_\mathcal{A}^\ast$,
3. $\langle x, y \cdot a \rangle_\mathcal{A} = \langle x, y \rangle_\mathcal{A} a$,
4. $\langle \cdot, \cdot \rangle_\mathcal{A}$ is non-degenerate.

In addition, $\mathcal{E}_\mathcal{A}$ is called *H-covariant* if it carries a H -action with

$$g \triangleright \langle x, y \rangle_\mathcal{A} = \langle S(g_{(1)})^\ast \triangleright x, g_{(2)} \triangleright y \rangle_\mathcal{A}. \quad (5)$$

It follows from the non-degeneracy that $g \triangleright (x \cdot a) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright a)$.

Having an inner product module one can consider the following operators on it. We call a map $A : \mathcal{E}_\mathcal{A} \longrightarrow \mathcal{E}'_\mathcal{A}$ from one inner product right \mathcal{A} -module to another *adjointable* if there exists a map $A^\ast : \mathcal{E}'_\mathcal{A} \longrightarrow \mathcal{E}_\mathcal{A}$ with

$$\langle x', Ax \rangle_\mathcal{A}^{\mathcal{E}'_\mathcal{A}} = \langle A^\ast x', x \rangle_\mathcal{A}^\mathcal{E} \quad (6)$$

for all $x \in \mathcal{E}_\mathcal{A}$ and $x' \in \mathcal{E}'_\mathcal{A}$. It is then an easy check that the maps A and A^\ast are necessarily right \mathcal{A} -linear, A^\ast is uniquely determined by A and A^\ast is adjointable as well with $A^{\ast\ast} = A$. Denoting the set of all adjointable operators by $\mathfrak{B}(\mathcal{E}_\mathcal{A}, \mathcal{E}'_\mathcal{A})$ we have the usual properties: linear combinations and composition of adjointable operators are again adjointable with adjoints given in the usual way. In particular, $\mathfrak{B}(\mathcal{E}_\mathcal{A}) = \mathfrak{B}(\mathcal{E}_\mathcal{A}, \mathcal{E}_\mathcal{A})$ becomes a unital \ast -algebra itself. If $\mathcal{E}_\mathcal{A}$ is H -covariant then $\mathfrak{B}(\mathcal{E}_\mathcal{A})$ carries a \ast -action of H itself, given explicitly by

$$(g \triangleright A)x = g_{(1)} \triangleright (AS(g_{(2)}) \triangleright x) \quad (7)$$

for $x \in \mathcal{E}_{\mathcal{A}}$ and $A \in \mathfrak{B}(\mathcal{E}_{\mathcal{A}})$. Note however that this is *not* an inner action. The reason is that the maps $x \mapsto g \triangleright x$ are *not* adjointable in general and hence not given by elements of $\mathfrak{B}(\mathcal{E}_{\mathcal{A}})$. This follows immediately from (5): only if the action $g \triangleright \langle x, y \rangle_{\mathcal{A}}^{\mathcal{E}}$ would be the trivial action, the map $x \mapsto g \triangleright x$ is adjointable with adjoint given by $x \mapsto g^* \triangleright x$.

Having an inner product right \mathcal{A} -module $\mathcal{E}_{\mathcal{A}}$ we can define $*$ -representations of a $*$ -algebra \mathcal{B} on $\mathcal{E}_{\mathcal{A}}$ as follows:

Definition 3.2 ($*$ -Representation). A $*$ -representation of \mathcal{B} on $\mathcal{E}_{\mathcal{A}}$ is a $*$ -homomorphism $\mathcal{B} \longrightarrow \mathfrak{B}(\mathcal{E}_{\mathcal{A}})$. It is called H -covariant if in addition

$$g \triangleright (b \cdot x) = (g_{(1)} \triangleright b) \cdot (g_{(2)} \triangleright x)$$

An *intertwiner* is an adjointable \mathcal{B} -module map $T : \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}'_{\mathcal{A}}$. A H -covariant *intertwiner* is an intertwiner which is H -equivariant.

In other words, a H -covariant $*$ -representation is a $*$ -homomorphism which intertwines the $*$ -action of H on \mathcal{B} with the canonical $*$ -action on $\mathfrak{B}(\mathcal{E}_{\mathcal{A}})$.

Definition 3.3 ($*$ -Representation Theory). Let \mathcal{A}, \mathcal{B} be $*$ -algebras over \mathbb{C} .

- The category of $*$ -representations of \mathcal{B} on inner product right \mathcal{A} -modules with intertwiners as morphisms is denoted by $*$ -Mod $_{\mathcal{A}}(\mathcal{B})$.
- The category of H -covariant $*$ -representations of \mathcal{B} on inner product right \mathcal{A} -modules with H -covariant intertwiners as morphisms is denoted by $*$ -Mod $_{H, \mathcal{A}}(\mathcal{B})$.

It is clear that we indeed obtain categories this way as the composition of intertwiners is again an intertwiner in both situations.

Remark 3.4. To be more precise, we should add that the above definitions require *unital* $*$ -algebras and modules where the algebra unit always acts as identity. For reasons to become clear later on, the case of non-unital $*$ -algebras is much more technical and one should add the condition that $\mathcal{B} \cdot {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ in the above definition. However, we shall mainly discuss unital algebras in the sequel.

Up to now, we have not yet used the positivity available from the ordering of \mathbb{R} at all. To incorporate this physically most important feature one first defines positive functionals as follows: a linear functional $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ is called *positive* if

$$\omega(a^*a) \geq 0 \tag{8}$$

for all $a \in \mathcal{A}$. It is easy to see that ω satisfies a Cauchy-Schwarz inequality and the reality condition $\omega(a^*b) = \overline{\omega(b^*a)}$ which gives $\omega(a^*) = \overline{\omega(a)}$ if \mathcal{A} is e.g. unital. Having positive functionals one defines *positive algebra elements* to be those elements $a \in \mathcal{A}$ with $\omega(a) \geq 0$ for all positive functionals ω . They form a convex cone in \mathcal{A} stable under the operations $a \mapsto b^*ab$ for arbitrary $b \in \mathcal{A}$. The cone of positive elements is denoted by \mathcal{A}^+ while the convex cone elements of the form $\sum_i \lambda_i a_i^* a_i$ with $\lambda_i > 0$ and $a_i \in \mathcal{A}$ is denoted by \mathcal{A}^{++} . We have $\mathcal{A}^{++} \subseteq \mathcal{A}^+$ and in general the inclusion is strict. A remarkable exception are the C^* -algebras where $\mathcal{A}^{++} = \mathcal{A}^+$ and in fact every positive a can uniquely be written as $a = b^2$ with a positive b , the square root of a . More sophisticated notions of positivity are discussed in [22] based on particular choices of sub-cones of positive functionals, see [24] for a comparison. However, here we only use the above definition.

Analogously to the situation of Hilbert modules over C^* -algebras one defines in our completely algebraic framework pre-Hilbert modules:

Definition 3.5 (Pre-Hilbert module). An inner product right \mathcal{A} -module $\mathcal{E}_{\mathcal{A}}$ is called *pre-Hilbert module* if for all n and $x_1, \dots, x_n \in \mathcal{E}_{\mathcal{A}}$

$$\left(\langle x_i, x_j \rangle_{\mathcal{A}} \right) \in M_n(\mathcal{A})^+ \quad (9)$$

i.e. $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is *completely positive*.

In fact, for a C^* -algebra, a positive inner product is always completely positive. In general, we have to use the above definition in order to have good behaviour under tensor products. Restricting to pre-Hilbert modules instead of general inner product modules gives more specific notions for $*$ -representations:

- The sub-category of $*$ -representations of \mathcal{B} on pre-Hilbert modules over \mathcal{A} is denoted by ${}^* \text{-Rep}_{\mathcal{A}}(\mathcal{B})$.
- The sub-category of H -covariant $*$ -representations of \mathcal{B} on pre-Hilbert modules over \mathcal{A} is denoted by ${}^* \text{-Rep}_{H, \mathcal{A}}(\mathcal{B})$.

The most important case will be when the auxiliary $*$ -algebra \mathcal{A} is just the ring of scalars \mathbb{C} (with trivial action of H).

In principle, for a given $*$ -algebra \mathcal{B} , one would like to understand the category ${}^* \text{-Mod}_{\mathcal{A}}(\mathcal{B})$ for any coefficient algebra \mathcal{A} or at least for $\mathcal{A} = \mathbb{C}$ in some detail: basic questions would be to understand the “irreducible” representations, a notion for which it is not even clear what is appropriate, the decomposition of a given representation into irreducible ones, etc. Of course, in this generality and even for $*$ -algebras like $\mathcal{B} = (C^\infty(M)[[\lambda]], \star)$ such a program is much too hard to be attacked successfully. Instead, one has to be more modest and try to find “interesting” representations, e.g. by a GNS construction out of “interesting” positive functionals. In the case of star products “interesting” could mean that the positive functionals have some concrete geometric and hence classical interpretation. Here many results have been found, see e.g. the overview in [25].

4 Tensor products and Morita theory

In this section we shall now proceed with our investigation of the representation theories, but from a different point of view: even though it is hard or even impossible to describe ${}^* \text{-Mod}_{\mathcal{A}}(\mathcal{B})$ or ${}^* \text{-Rep}_{\mathcal{A}}(\mathcal{B})$ in a reasonably “explicit” way, it might well be possible to *compare* the representation theories of different $*$ -algebras and determine whether they are “the same”.

To be more precisely, we are interested in finding functors from one category of representations into another with particular interest in *equivalences* of categories. In the framework of rings and modules this is the realm of Morita theory. Thus we want to adapt the well-known ring-theoretic notions of Morita theory to our more specific categories of modules.

The first step is to find an appropriate notion of tensor products. Here we can rely on the ideas of Rieffel [20, 21] from C^* -algebra theory. In fact, the following definition also makes sense in our much more algebraic framework: Given ${}_{\mathcal{C}} \mathcal{F}_{\mathcal{B}} \in {}^* \text{-Mod}_{\mathcal{B}}(\mathcal{C})$ and ${}_{\mathcal{B}} \mathcal{E}_{\mathcal{A}} \in {}^* \text{-Mod}_{\mathcal{A}}(\mathcal{B})$ we can define *Rieffel’s inner product* on their \mathcal{B} -tensor product by

$$\langle \phi \otimes x, \psi \otimes y \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}} = \langle x, \langle \phi, \psi \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot y \rangle_{\mathcal{A}}^{\mathcal{E}} \quad (10)$$

for factorizing tensors and extend this to a \mathcal{A} -valued inner product on $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}$. Since (10) obviously has the correct linearity properties, this extension is possible. However, the inner

product may still be degenerate. In fact, for C^* -algebras this can not happen, but in general, there are examples where one has degeneracy. Thus we have to divide by the degeneracy space

$${}_c\mathcal{F}_B \widehat{\otimes}_B \mathcal{E}_A = ({}_c\mathcal{F}_B \otimes_B \mathcal{E}_A) / ({}_c\mathcal{F}_B \otimes_B \mathcal{E}_A)^\perp, \quad (11)$$

and arrive at a $(\mathcal{C}, \mathcal{A})$ -bimodule with non-degenerate \mathcal{A} -valued inner product. Since \mathcal{C} acts by adjointable operators, the left \mathcal{C} -module structure survives the quotient (11). It is then an easy check to verify that (11) indeed is a $*$ -representation of \mathcal{C} on an inner product right \mathcal{A} -module. We call $\widehat{\otimes}_B$ the internal tensor product (over B).

As long as we do not take care of positivity we are done with the above construction yielding a tensor product $\widehat{\otimes}$ with good functorial properties, see [1]. Taking positivity into account, it becomes more tricky: here the internal tensor product of positive inner products is not behaving well. Instead, one needs *completely* positive inner products. Then one can show that their tensor product (10) gives again a completely positive inner product [6, 9].

Finally, we have to consider the H -covariant situation: this is again easy, one just computes that if both inner products on ${}_c\mathcal{F}_B$ and ${}_B\mathcal{E}_A$ satisfy (5) then the Rieffel inner product is H -covariant again with respect to the canonical H -action on the tensor product, see [15, 13]. Collecting all these results together with the afore mentioned functoriality properties eventually gives the following theorem [1, 9, 6, 15, 13]:

Theorem 4.1 (Internal Tensor Product). *The internal tensor product $\widehat{\otimes}$ gives functors*

$$\widehat{\otimes} : {}^*\text{-Mod}_B(\mathcal{C}) \times {}^*\text{-Mod}_A(B) \longrightarrow {}^*\text{-Mod}_A(\mathcal{C}) \quad (12)$$

$$\widehat{\otimes} : {}^*\text{-Rep}_B(\mathcal{C}) \times {}^*\text{-Rep}_A(B) \longrightarrow {}^*\text{-Rep}_A(\mathcal{C}) \quad (13)$$

$$\widehat{\otimes} : {}^*\text{-Mod}_{H,B}(\mathcal{C}) \times {}^*\text{-Mod}_{H,A}(B) \longrightarrow {}^*\text{-Mod}_{H,A}(\mathcal{C}) \quad (14)$$

$$\widehat{\otimes} : {}^*\text{-Rep}_{H,B}(\mathcal{C}) \times {}^*\text{-Rep}_{H,A}(B) \longrightarrow {}^*\text{-Rep}_{H,A}(\mathcal{C}) \quad (15)$$

Moreover, $\widehat{\otimes}$ is associative up to the usual unitary intertwiners.

The theorem can now be used to enhance the category of $*$ -algebras in the following way: instead of taking $*$ -homomorphisms as morphisms between $*$ -algebras we take isometric isomorphism classes of $*$ -representations ${}_B\mathcal{E}_A \in {}^*\text{-Mod}_A(B)$ as morphisms from \mathcal{A} to \mathcal{B} . The composition of such bimodules is then the internal tensor product $\widehat{\otimes}$. Then unit arrow will be the canonical $*$ -representation ${}_A\mathcal{A}_A$ with inner product $\langle a, b \rangle = a^*b$. Since the tensor product $\widehat{\otimes}$ is associative up to unitary intertwiners we get indeed an associative composition law on the level of isomorphism classes. The only difficulty is that the bimodule ${}_A\mathcal{A}_A$ might not act as unit at all. The problem is that if \mathcal{A} does not have a unit element, then tensoring with ${}_A\mathcal{A}_A$ might not give a result isomorphic to the representation we started with. Examples are easily found. The way out is to restrict either to unital $*$ -algebras, or, more generally, to $*$ -algebras which are non-degenerate and idempotent: non-degenerate means that $ab = 0$ for all a implies $b = 0$ and idempotent means that products ab span already \mathcal{A} . In the following, we shall stick to unital $*$ -algebras for convenience, the details for the non-degenerate and idempotent case can be found in [9].

With these considerations, the following definition makes sense: for \mathcal{A}, \mathcal{B} we define the class

$$\text{Bimod}^*(\mathcal{B}, \mathcal{A}) = \{\text{isomorphism classes of } {}_B\mathcal{E}_A \in {}^*\text{-Mod}_A(B)\} \quad (16)$$

and denote by Bimod^* the category whose objects are unital $*$ -algebras over \mathbb{C} with the isomorphism classes $\text{Bimod}^*(\mathcal{B}, \mathcal{A})$ of bimodules as morphisms from \mathcal{A} to \mathcal{B} , the internal tensor

product $\widehat{\otimes}$ as composition law, and the isomorphism class of ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ as unit arrow. It is then clear from the functoriality of $\widehat{\otimes}$ that $\widehat{\otimes}$ is well-defined on isomorphism classes. Therefore we end up with a category of bimodules, completely analogous to the well-known ring-theoretic case. In a completely analogous fashion we can define the category $\text{Bimod}_{\mathcal{H}}^*$ of $*$ -algebras with isomorphism classes of H -covariant bimodules as morphisms, the category $\text{Bimod}^{\text{str}}$ of $*$ -algebras with isomorphism classes of bimodules with completely positive inner products as morphisms, and finally, the category $\text{Bimod}_{\mathcal{H}}^{\text{str}}$ of $*$ -algebras with isomorphism classes of H -covariant bimodules with completely positive inner products as morphisms.

Remark 4.2. In fact, identifying bimodules up to isomorphisms might not be a good idea after all. Instead, one can equally well take *all* bimodules (of the particular, specific types). Then the tensor product $\widehat{\otimes}$ fails to be associative and ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ fails to be a unit arrow. However, the failure is encoded in a functorial way whence we end up with a *bicategory* in each of the above four cases. The 1-morphisms are the bimodules and the 2-morphisms, i.e. the arrows between the arrows, are the intertwiners, always in the correct version respecting the inner products and the H -covariance. A detailed study of these bicategorical aspects can be found in [28] as well as in [13]. Its ring-theoretic version goes back to Benabou [3] and was probably the motivation to study bicategories at all.

Using either the bicategorical approach or the one presented above, Morita equivalence is now simply isomorphism in the enhanced categories:

Definition 4.3 (Morita equivalence). Two (unital) $*$ -algebras \mathcal{A} and \mathcal{B} are called

- *$*$ -Morita equivalent* if they are isomorphic in Bimod^* .
- *Strongly Morita equivalent* if they are isomorphic in $\text{Bimod}^{\text{str}}$.
- *H -covariantly $*$ -Morita equivalent* if they are isomorphic in $\text{Bimod}_{\mathcal{H}}^*$.
- *H -covariantly strongly Morita equivalent* if they are isomorphic in $\text{Bimod}_{\mathcal{H}}^{\text{str}}$.

A bimodule representing such isomorphism is called *equivalence bimodule*.

The case of C^* -algebras was first studied by Rieffel in [21] and serves as motivation for all the generalizations. In fact, he coined the name strong Morita equivalence. Ara discussed the notion of $*$ -Morita equivalence in [1] while in [6, 9] we extended Rieffel's notion to arbitrary $*$ -algebras now taking into account the complete positivity of inner product compared to the approach of Ara. Finally, the H -covariant situation was discussed in detail in [15, 13] based on earlier formulations in C^* -algebra theory where the symmetry was implemented by a strongly continuous group action of some suitable topological group.

While the above definition of Morita equivalence is very elegant it needs further work to get a formulation suitable for practical purposes. In fact, one has to find criteria whether a given bimodule is “invertible” or not. In the ring-theoretic version the equivalence bimodules are the full projective modules over \mathcal{A} such that \mathcal{B} is isomorphic to the right \mathcal{A} -linear endomorphisms. However, in this classical formulation the fact that one deals with *unital* rings is crucial. The following formulation based on Rieffel's pioneering work avoids the usage of the unit and extends to non-degenerate and idempotent $*$ -algebras as well. The $*$ -Morita version is due to Ara, the strong version is from [9]:

Theorem 4.4. *The following statements are equivalent:*

- ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in \text{Bimod}^*(\mathcal{B}, \mathcal{A})$ is an equivalence bimodule.
- On ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^*\text{-Mod}_{\mathcal{A}}(\mathcal{B})$ there is a \mathcal{B} -valued inner product ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ such that
 1. $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ is full, i.e. $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathcal{A}}^{\mathcal{E}} = \mathcal{A}$.
 2. ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ is full.
 3. ${}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} \cdot z = x \cdot \langle y, z \rangle_{\mathcal{A}}^{\mathcal{E}}$ for all $x, y, z \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$.

The analogous statement holds for strong Morita equivalence. Here, in addition, ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ is completely positive.

The unital version of this theorem is a rather straightforward adaption of the ring-theoretic Morita theorem. To include non-degenerate and idempotent * -algebras one has to put in some more effort.

Not surprising, there is also a characterization of the H -covariant equivalence bimodules, both for the * -equivalence and the strong equivalence bimodules [15]:

Theorem 4.5. *The following statements are equivalent:*

- ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in \text{Bimod}_H^*(\mathcal{B}, \mathcal{A})$ is an equivalence bimodule.
- On ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^*\text{-Mod}_{H, \mathcal{A}}(\mathcal{B})$ there is a \mathcal{B} -valued inner product ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ such that
 1. ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ is compatible with H -action.
 2. $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ is full, i.e. $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathcal{A}}^{\mathcal{E}} = \mathcal{A}$.
 3. ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ is full.
 4. ${}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} \cdot z = x \cdot \langle y, z \rangle_{\mathcal{A}}^{\mathcal{E}}$.

Again, there is an analogous statement for H -covariantly strong Morita equivalence. Here ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ is in addition completely positive.

Having Theorem 4.4, the results in Theorem 4.5 are obtained rather easily by observing that the H -action ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is necessarily compatible with the inner product ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ thanks to the compatibility of the two inner products ${}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} \cdot z = x \cdot \langle y, z \rangle_{\mathcal{A}}^{\mathcal{E}}$.

Remark 4.6 (Finite rank operators). For a general inner product right \mathcal{A} -module $\mathcal{E}_{\mathcal{A}}$ one defines the rank-one operator $\Theta_{x,y} : \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}_{\mathcal{A}}$ by

$$\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_{\mathcal{A}}^{\mathcal{E}}, \quad (17)$$

where $x, y, z \in \mathcal{E}_{\mathcal{A}}$. Then it is easy to see that $\Theta_{x,y}$ is adjointable with $\Theta_{x,y}^* = \Theta_{y,x}$. The linear span of all rank-one operators is denoted by $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ and called the *finite rank operators* on $\mathcal{E}_{\mathcal{A}}$. Clearly, $\mathfrak{F}(\mathcal{E}_{\mathcal{A}}) \subseteq \mathfrak{B}(\mathcal{E}_{\mathcal{A}})$ is a * -ideal. One can now show that for a Morita equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ we have $\mathcal{B} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ via the left action. Moreover, the \mathcal{B} -valued inner product ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ corresponds to $\Theta_{\cdot, \cdot}$ under this * -isomorphism. Finally, in the H -covariant situation, the H -action on \mathcal{B} corresponds to the canonical action (7) which is easily shown to preserve $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$. In the unital case we have $\mathcal{B} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{A}}) = \mathfrak{B}(\mathcal{E}_{\mathcal{A}})$.

5 Picard groupoids

Having realized Morita equivalence as the notion of isomorphism in the enhanced categories Bimod^* , $\text{Bimod}^{\text{str}}$, Bimod_H^* , and $\text{Bimod}_H^{\text{str}}$, respectively, it is of course not only interesting to ask *whether* two $*$ -algebras are Morita equivalent, but also *in who many ways*. Answers to both questions are encoded in the Picard groupoid, for which we again have four different flavors:

- Definition 5.1** (Picard groupoids). • The (large) groupoid of invertible arrows in Bimod^* is called the $*$ -Picard groupoid Pic^* . The isotropy group at \mathcal{A} is called the $*$ -Picard group $\text{Pic}^*(\mathcal{A})$ of \mathcal{A} .
- Analogously, one defines the *strong Picard groupoid* Pic^{str} , the *H-covariant $*$ -Picard groupoid* Pic_H^* , and the *H-covariant strong Picard groupoid* $\text{Pic}_H^{\text{str}}$ together with the Picard groups $\text{Pic}^{\text{str}}(\mathcal{A})$, $\text{Pic}_H^*(\mathcal{A})$, and $\text{Pic}_H^{\text{str}}(\mathcal{A})$, respectively.

From this point of view, Morita theory consists in the following two principle tasks:

1. Determine the *orbits* of the Picard groupoid. These are precisely the Morita equivalence classes of $*$ -algebras.
2. Determine the *isotropy groups* of the Picard groupoid. They encode in how many ways a $*$ -algebra can be Morita equivalent to itself.

In particular, by groupoid abstract non-sense isotropy groups are always isomorphic along the orbits whence the Picard groups are isomorphic for Morita equivalent $*$ -algebras. Needless to say, the above program should be carried through in all four versions of Morita theory. Moreover, as unital $*$ -algebras are particular types of rings, we also have the ring-theoretic version of Morita theory at hand, as well as a ring-theoretic version of H -covariant Morita theory. Thus we also can study the Picard groupoids Pic and Pic_H , respectively.

One strategy to learn something about the various Picard group(oids) in general is to use the following groupoid morphisms which simply forget the additional structure successively. This gives a commuting diagram of groupoid morphisms

$$\begin{array}{ccccc}
 \text{Pic}_H^{\text{str}} & \xrightarrow{\quad} & & \xrightarrow{\quad} & \text{Pic}_H^* \\
 \downarrow & \searrow & & \swarrow & \downarrow \\
 & & \text{Pic}_H & & \\
 \text{Pic}^{\text{str}} & \xrightarrow{\quad} & & \xrightarrow{\quad} & \text{Pic}^* \\
 & \searrow & & \swarrow & \\
 & & \text{Pic} & &
 \end{array} \tag{18}$$

for each of which one would like to know kernel and image. In [9] the arrow $\text{Pic}^{\text{str}} \rightarrow \text{Pic}$ was studied in detail. The arrows from and to the $*$ -versions encode how many inner products with *different signature* than the completely positive one can have. From that point of view, they are the simplest to understand.

In the following, we concentrate on the arrow

$$\text{Pic}_H^{\text{str}} \longrightarrow \text{Pic}^{\text{str}}, \tag{19}$$

following [15]. To this end, we restrict to unital $*$ -algebras in the sequel. The non-unital case is much more mysterious. In order to determine the kernel of this groupoid morphism we need

some preparation. First recall that the linear maps $\text{Hom}_{\mathbb{C}}(H, \mathcal{A})$ form an associative algebra over \mathbb{C} with respect to the *convolution* product

$$(\mathbf{a} * \mathbf{b})(g) = \mathbf{a}(g_{(1)})\mathbf{b}(g_{(2)}) \quad (20)$$

for $\mathbf{a}, \mathbf{b} \in \text{Hom}_{\mathbb{C}}(H, \mathcal{A})$ and with unit given by $\mathbf{e}(g) = \varepsilon(g)\mathbb{1}_{\mathcal{A}}$. We now consider the following particular linear maps:

Definition 5.2. Define $\text{GL}(H, \mathcal{A}) \subseteq \text{Hom}_{\mathbb{C}}(H, \mathcal{A})$ to be the subset of those \mathbf{a} with

- $\mathbf{a}(\mathbb{1}_H) = \mathbb{1}_{\mathcal{A}}$,
- $\mathbf{a}(gh) = \mathbf{a}(g_{(1)})(g_{(2)} \triangleright \mathbf{a}(h))$,
- $(g_{(1)} \triangleright \mathbf{b})\mathbf{a}(g_{(2)}) = \mathbf{a}(g_{(1)})(g_{(2)} \triangleright \mathbf{b})$,

for all $g, h \in H$ and $b \in \mathcal{A}$. Furthermore, define $\text{U}(H, \mathcal{A}) \subseteq \text{GL}(H, \mathcal{A})$ by the additional condition

- $\mathbf{a}(g_{(1)})(\mathbf{a}(S(g_{(2)}))^*)^* = \varepsilon(g)\mathbb{1}_{\mathcal{A}}$.

The following result is a rather straightforward verification:

Proposition 5.3. $\text{GL}(H, \mathcal{A})$ is a group with respect to the convolution product of $\text{Hom}_{\mathbb{C}}(H, \mathcal{A})$ and $\text{U}(H, \mathcal{A})$ is a subgroup.

Let $c \in \text{U}(\mathcal{Z}(\mathcal{A}))$ be unitary and central then

$$\hat{c}(g) = (g \triangleright c^{-1})c \quad (21)$$

defines an element $\hat{c} \in \text{U}(H, \mathcal{A})$ as a simple computation shows. It is easy to see that this gives in fact a group homomorphism whose image is central.

Proposition 5.4. *There is an exact sequence of groups*

$$1 \longrightarrow \text{U}(\mathcal{Z}(\mathcal{A}))^H \longrightarrow \text{U}(\mathcal{Z}(\mathcal{A})) \xrightarrow{\widehat{\quad}} \text{U}(H, \mathcal{A}), \quad (22)$$

and the image of $\text{U}(\mathcal{Z}(\mathcal{A}))$ is central in $\text{U}(H, \mathcal{A})$

Note that in general the group $\text{U}(H, \mathcal{A})$ is far from being abelian. Nevertheless, the above proposition allows to consider the following quotient group

$$\text{U}_0(H, \mathcal{A}) = \text{U}(H, \mathcal{A}) / \widehat{\text{U}(\mathcal{Z}(\mathcal{A}))}. \quad (23)$$

Note that the same construction also works without the “unitarity” requirement and yields a group $\text{GL}_0(H, \mathcal{A}) = \text{GL}(H, \mathcal{A}) / \widehat{\text{GL}(\mathcal{Z}(\mathcal{A}))}$, where now we also allow for arbitrary invertible central elements instead of unitary ones.

In order to understand the groupoid morphism $\text{Pic}_H^{\text{str}} \longrightarrow \text{Pic}^{\text{str}}$ we assume to have a strong equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ between \mathcal{A} and \mathcal{B} . Then the question is whether we can find a compatible action of H on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ turning it into a H -covariant equivalence bimodule. Thus the *surjectivity* is a *lifting problem*. As already simple examples show, the computation of the image will be a very difficult task. Moreover, the result will depend strongly on the two *-algebras \mathcal{A} and \mathcal{B} :

Example 5.5 (Lifting of group actions). Let $\mathcal{A} = \mathcal{B} = C^\infty(M)$ be the smooth functions on a manifold and assume that the symmetry is given by a Lie algebra action of \mathfrak{g} . The interesting self-equivalence bimodules are then known to be the sections $\Gamma^\infty(L)$ of complex line bundles $L \rightarrow M$, endowed with their usual bimodule structure. Thus the question whether we can endow the bimodule $\Gamma^\infty(L)$ with a compatible symmetry of \mathfrak{g} is equivalent to the question whether we can lift the Lie algebra action to L . In general, this is a difficult and of course classical question in differential geometry whose answer is known to depend very much on the example. Analogous statements are of course valid for Lie group actions instead of Lie algebra actions.

While the image is hard to determine, we can say something about the kernel. So we assume that we have at least one lifting, i.e. there is at least one compatible action \triangleright of H on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. Then we have to understand how many compatible actions we can have. The idea is to parameterize to possible actions starting from the given \triangleright by $\mathbf{a} \in U(H, \mathcal{B})$. This yields in fact a bijection. However, it may happen that some of the actions yield isomorphic equivalence bimodules. It turns out that they are precisely parameterized by those \mathbf{a} coming from $U(\mathcal{Z}(\mathcal{A}))$ via (21). This eventually yields the following result, see [15]:

Theorem 5.6. *Assume $\text{Pic}^{\text{str}}(\mathcal{B}, \mathcal{A})$ is non-empty. Then one has the alternatives:*

- $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) = \emptyset$.
- $\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) \rightarrow \text{im}(\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}))$ is a principal $U_0(H, \mathcal{B})$ -bundle over the image, i.e. $U_0(H, \mathcal{B})$ acts freely and transitively on the fibers.

The same holds for the $$ -Picard groupoids. Moreover, analogous statements hold in the ring-theoretic setting with $\text{GL}_0(H, \mathcal{B})$ instead of $U_0(H, \mathcal{B})$.*

By symmetry of Morita equivalence it is immediately clear that the groups $U_0(H, \mathcal{B})$ are *invariant* under H -covariant $*$ -equivalence while $\text{GL}_0(H, \mathcal{A})$ is invariant under H -covariant ring-theoretic Morita invariance. In fact, one can show that the whole exact sequence (22) is a *Morita invariant*.

Back in our geometric context, one can actually compute the group $U_0(H, \mathcal{A})$ in terms of classical geometric data:

Example 5.7. Let $\mathcal{A} = C^\infty(M)$ and let $H = U_{\mathbb{C}}(\mathfrak{g})$ be the complexified universal enveloping algebra of a real Lie algebra \mathfrak{g} acting on M by vector fields. Then one can show [26]

$$U_0(H, \mathcal{A}) = H_{CE}^1(\mathfrak{g}, C^\infty(M, i\mathbb{R})) / H_{dR}^1(\widehat{M}, 2\pi i\mathbb{Z}), \quad (24)$$

where $H_{dR}^1(M, 2\pi i\mathbb{Z})$ denotes the $2\pi i$ -integral deRham cohomology of M and the map

$$\widehat{} : H_{dR}^1(M, 2\pi i\mathbb{Z}) \rightarrow H_{CE}^1(\mathfrak{g}, C^\infty(M, i\mathbb{R})) \quad (25)$$

is obtained from a “logarithmic derivative” induced by (21). Finally, $H_{CE}^1(\mathfrak{g}, C^\infty(M, i\mathbb{R}))$ denotes the Chevalley-Eilenberg cohomology of \mathfrak{g} with values in the imaginary-valued functions on M . Thus we obtain a linear space divided by some integer lattice for the group $U_0(H, \mathcal{A})$.

Remark 5.8. We note that the above result generalizes to other algebras \mathcal{A} which have a sort of “exponential function”, see [26] for details. We also note that the above statement gives a classification of how many inequivalent lifts of the Lie algebra action on M one has for a complex line bundle. This way, one can re-produce well-known results from differential geometry (the lifting problem is of course classical) by Morita theoretic considerations.

6 Further remarks

We have seen that the surjectivity of (19) is quite hard to understand in general. However, under the additional assumption of *inner actions* of H on \mathcal{A} and \mathcal{B} one can always lift: indeed, assume ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a strong Morita equivalence bimodule then

$$g \triangleright x = J_{\mathcal{B}}(g_{(1)}) \cdot x \cdot J_{\mathcal{A}}(S(g_{(2)})) \quad (26)$$

defines a compatible action of H on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. In fact, one even has the following statement, see the forthcoming project [14]:

Theorem 6.1. *Assume $\text{Pic}^{\text{str}}(\mathcal{B}, \mathcal{A}) \neq \emptyset$ and let both algebras have inner actions. Then*

$$\text{Pic}_H^{\text{str}}(\mathcal{B}, \mathcal{A}) \longrightarrow \text{Pic}^{\text{str}}(\mathcal{B}, \mathcal{A}) \quad (27)$$

is surjective. Moreover,

$$1 \longrightarrow \text{U}_0(H, \mathcal{A}) \longrightarrow \text{Pic}_H^{\text{str}}(\mathcal{A}) \longrightarrow \text{Pic}^{\text{str}}(\mathcal{A}) \longrightarrow 1$$

splits via the momentum map.

Thus we are in some sense in an optimal situation: we have determined the H -covariant Picard groupoid completely in terms of the groups (23) and the usual Picard groupoid. Of course, having inner actions is a severe restriction in general. Nevertheless, there are important examples as discussed in Section 2.

In [14], we investigate the particular case of *symplectic* star product algebras where the existence of a quantum momentum map is well-understood. In particular, one can use these results to obtain lifting results this way.

At last, we remark that the study of the Picard groupoids for star product algebras is understood best if one uses the classical limit functor, where the formal parameter λ is set to zero. Then one can study how the Picard groupoid behaves under formal deformations of the underlying algebras. As shown in [8] this allows to explicitly compute the ring-theoretic Picard groups of certain star product algebras as well as the orbits [7]. Thus it remains as a challenge to use these results and extend them to the H -covariant as well as to the $*$ -equivalence and strong equivalence setting.

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