

# SURVEY ON DYNAMICAL YANG-BAXTER MAPS

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## Abstract

In this survey, we introduce dynamical Yang-Baxter maps by means of the tensor category.

## Introduction

The Yang-Baxter equation is at a crossing of mathematics and physics. This equation was first recognized in physics [1, 2, 3, 19, 20], gave birth to the quantum group [4, 5, 9, 11, 17], and still plays an important role in both mathematics and physics.

The dynamical Yang-Baxter map [15, 16] is a set-theoretical solution to a version of the quantum dynamical Yang-Baxter equation [8, 10]; it is a generalization of the Yang-Baxter map [6, 7, 13], a set-theoretical solution to the quantum Yang-Baxter equation. A loop [14, 18] with a group structure produces a dynamical Yang-Baxter map [15]. Moreover, a left quasigroup, together with a ternary operation satisfying some conditions, yields a dynamical Yang-Baxter map satisfying an invariance condition, and vice versa [16].

This article gives a survey on the dynamical Yang-Baxter map satisfying the invariance condition from the viewpoint of the tensor category.

The organization of this article is as follows. In the first section, we construct a tensor category  $\text{Sets}_H$ , which plays a central role in this article.

After introducing the dynamical Yang-Baxter map, the dynamical braiding map, and the braided object of  $\text{Sets}_H$ , we show in Section 2 that the following notions are equivalent:

- (1) the dynamical Yang-Baxter map satisfying the invariance condition (2.4);
- (2) the dynamical braiding map satisfying (2.3);
- (3) the braided object of  $\text{Sets}_H$ .

Section 3 establishes a relation between the braided object and the ternary operation. This section also discusses monoid objects in  $\text{Sets}_H$ .

## 1 Tensor category $\text{Sets}_H$

Let  $H$  be a nonempty set. This section is devoted to defining the tensor category  $\text{Sets}_H$ . We follow the notation of [12, Chapter XI].

An object of  $\text{Sets}_H$  is, by definition, a pair  $(S, \psi)$  such that  $S$  is a nonempty set, together with a map  $\psi : H \times S \rightarrow H$ .  $\text{Ob}(\text{Sets}_H)$  is the class of all objects of  $\text{Sets}_H$ .

Let  $S = (S, \psi_S)$  and  $T = (T, \psi_T)$  be objects of  $\text{Sets}_H$ . A morphism  $f : S \rightarrow T$  of  $\text{Sets}_H$  is a map  $f : H \rightarrow \text{Map}(S, T)$  such that

$$\psi_T(\lambda, f(\lambda)(s)) = \psi_S(\lambda, s) \quad (\forall \lambda \in H, \forall s \in S).$$

Let  $\text{Hom}(\text{Sets}_H)$  denote all morphisms of  $\text{Sets}_H$ . For the morphism  $f : S \rightarrow T$ , the objects  $S$  and  $T$  are called the source  $s(f)$  and the target  $b(f)$ , respectively.

For each object  $S$  of  $\text{Sets}_H$ , define the map  $\text{id}_S : H \rightarrow \text{Map}(S, S)$  by  $\text{id}_S(\lambda)(s) = s$  ( $\lambda \in H, s \in S$ ). Let  $f$  and  $g$  be morphisms of  $\text{Sets}_H$  satisfying  $b(f) = s(g)$ . We define the composition  $g \circ f \in \text{Hom}_{\text{Sets}_H}(s(f), b(g))$  by  $(g \circ f)(\lambda) = g(\lambda) \circ f(\lambda)$  ( $\lambda \in H$ ).

**Proposition 1.1.**  *$\text{Sets}_H$  is a category of  $\text{Ob}(\text{Sets}_H)$  and  $\text{Hom}(\text{Sets}_H)$ , together with the identity  $\text{id}$ , the source  $s$ , the target  $b$ , and the composition  $\circ$ .*

The next task is to explain a tensor product  $\bar{\otimes} : \text{Sets}_H \times \text{Sets}_H \rightarrow \text{Sets}_H$ . Let  $S$  and  $T$  be objects of the category  $\text{Sets}_H$ . Define the set  $S \bar{\otimes} T$  and the map  $\psi_{S \bar{\otimes} T} : H \times (S \bar{\otimes} T) \rightarrow H$  by:

$$\begin{aligned} S \bar{\otimes} T &= S \times T; \\ \psi_{S \bar{\otimes} T}(\lambda, (s, t)) &= \psi_T(\psi_S(\lambda, s), t) \quad (\lambda \in H, s \in S, t \in T). \end{aligned} \quad (1.1)$$

Clearly, the pair  $S \bar{\otimes} T := (S \bar{\otimes} T, \psi_{S \bar{\otimes} T})$  is an object of  $\text{Sets}_H$ .

Let  $f$  and  $g$  be morphisms of the category  $\text{Sets}_H$ . We denote by  $f \bar{\otimes} g$  the following map from  $H$  to  $\text{Map}(s(f) \bar{\otimes} s(g), b(f) \bar{\otimes} b(g))$ .

$$f \bar{\otimes} g(\lambda)(s, t) = (f(\lambda)(s), g(\psi_{s(f)}(\lambda, s))(t)) \quad (\lambda \in H, (s, t) \in s(f) \bar{\otimes} s(g)).$$

**Proposition 1.2.**  *$f \bar{\otimes} g \in \text{Hom}_{\text{Sets}_H}(s(f) \bar{\otimes} s(g), b(f) \bar{\otimes} b(g))$ ; moreover,  $\bar{\otimes} : \text{Sets}_H \times \text{Sets}_H \rightarrow \text{Sets}_H$  is a functor.*

The associativity constraint  $a : \bar{\otimes}(\bar{\otimes} \times \text{id}) \rightarrow \bar{\otimes}(\text{id} \times \bar{\otimes})$  is given by

$$a_{S, T, U}(\lambda)((s, t), u) = (s, (t, u)) \quad (S, T, U \in \text{Ob}(\text{Sets}_H), \lambda \in H, ((s, t), u) \in (S \bar{\otimes} T) \bar{\otimes} U).$$

Let  $I_{\text{Sets}_H}$  denote the set consisting of one element, together with the map  $\psi_{I_{\text{Sets}_H}} : H \times I_{\text{Sets}_H} \rightarrow H$  defined by  $\psi_{I_{\text{Sets}_H}}(\lambda, e) = \lambda$  ( $\lambda \in H, I_{\text{Sets}_H} = \{e\}$ ). The pair  $I_{\text{Sets}_H} := (I_{\text{Sets}_H}, \psi_{I_{\text{Sets}_H}})$  is an object of  $\text{Sets}_H$ .

The left and the right unit constraints  $l : \bar{\otimes}(I_{\text{Sets}_H} \times \text{id}) \rightarrow \text{id}$ ,  $r : \bar{\otimes}(\text{id} \times I_{\text{Sets}_H}) \rightarrow \text{id}$  with respect to  $I_{\text{Sets}_H}$  are given by

$$\begin{aligned} l_S(\lambda)(e, s) &= r_S(\lambda)(s, e) = s \\ (S &= (S, \psi_S) \in \text{Ob}(\text{Sets}_H), \lambda \in H, s \in S, I_{\text{Sets}_H} = \{e\}). \end{aligned}$$

**Proposition 1.3.**  *$(\text{Sets}_H, \bar{\otimes}, I_{\text{Sets}_H}, a, l, r)$  is a tensor category.*

A direct computation shows this proposition.

## 2 Dynamical Yang-Baxter maps and braided objects

In this section, we introduce the notion of a dynamical Yang-Baxter map [15, 16], which is related to a braided object of  $\text{Sets}_H$  through a dynamical braiding map [15, (2.8)].

Let  $H$  and  $X$  be nonempty sets, together with a map  $(\cdot) : H \times X \rightarrow H$ .

**Definition 2.1.** The map  $R(\lambda) : X \times X \rightarrow X \times X$  ( $\lambda \in H$ ) is a dynamical Yang-Baxter map associated with  $H$ ,  $X$ , and  $(\cdot)$ , iff  $R(\lambda)$  satisfies a version of the quantum dynamical Yang-Baxter equation for all  $\lambda \in H$ .

$$R^{23}(\lambda)R^{13}(\lambda \cdot X^{(2)})R^{12}(\lambda) = R^{12}(\lambda \cdot X^{(3)})R^{13}(\lambda)R^{23}(\lambda \cdot X^{(1)}).$$

Here,  $R^{12}(\lambda)$  and  $R^{23}(\lambda \cdot X^{(1)})$  are the following maps from  $X \times X \times X$  to itself: for  $\lambda \in H$  and  $u, v, w \in X$ ,

$$R^{12}(\lambda)(u, v, w) = (R(\lambda)(u, v), w); \quad (2.1)$$

$$R^{23}(\lambda \cdot X^{(1)})(u, v, w) = R^{23}(\lambda \cdot u)(u, v, w) = (u, R(\lambda \cdot u)(v, w)). \quad (2.2)$$

The other maps are similarly defined.

The notion of the dynamical Yang-Baxter map is equivalent to that of a dynamical braiding map, which is more suitable for the tensor category  $\text{Sets}_H$ .

**Definition 2.2.** The map  $\sigma(\lambda) : X \times X \rightarrow X \times X$  ( $\lambda \in H$ ) is a dynamical braiding map associated with  $H$ ,  $X$ , and  $(\cdot)$ , iff  $\sigma(\lambda)$  satisfies the following equation for all  $\lambda \in H$ .

$$\sigma^{12}(\lambda)\sigma^{23}(\lambda \cdot X^{(1)})\sigma^{12}(\lambda) = \sigma^{23}(\lambda \cdot X^{(1)})\sigma^{12}(\lambda)\sigma^{23}(\lambda \cdot X^{(1)}).$$

Here the definitions of the maps  $\sigma^{12}(\lambda)$  and  $\sigma^{23}(\lambda \cdot X^{(1)})$  from  $X \times X \times X$  to itself are similar to (2.1) and (2.2).

We denote by  $P_X$  the map from  $X \times X$  to itself defined by  $P_X(u, v) = (v, u)$  ( $u, v \in X$ ). The following is Proposition 2.1 in [15].

**Proposition 2.3.** *If  $R(\lambda)$  is a dynamical Yang-Baxter map associated with  $H$ ,  $X$ , and  $(\cdot)$ , then  $P_X R(\lambda)$  is a dynamical braiding map associated with  $H$ ,  $X$ , and  $(\cdot)$ ; if  $\sigma(\lambda)$  is a dynamical braiding map associated with  $H$ ,  $X$ , and  $(\cdot)$ , then  $P_X \sigma(\lambda)$  is a dynamical Yang-Baxter map associated with  $H$ ,  $X$ , and  $(\cdot)$ .*

Let  $\sigma(\lambda)$  be a dynamical braiding map associated with  $H$ ,  $X$ , and  $(\cdot)$ , and let  $\sigma(\lambda)(u, v)_1$  and  $\sigma(\lambda)(u, v)_2$  ( $\lambda \in H, u, v \in X$ ) denote the elements of  $X$  defined by  $(\sigma(\lambda)(u, v)_1, \sigma(\lambda)(u, v)_2) := \sigma(\lambda)(u, v)$ .

**Proposition 2.4.** *If the dynamical braiding map  $\sigma(\lambda)$  satisfies*

$$(\lambda \cdot \sigma(\lambda)(u, v)_1) \cdot \sigma(\lambda)(u, v)_2 = (\lambda \cdot u) \cdot v \quad (\forall \lambda \in H, \forall u, v \in X), \quad (2.3)$$

*then  $\sigma \in \text{Hom}_{\text{Sets}_H}(X \bar{\otimes} X, X \bar{\otimes} X)$ .*

We call (2.3) an invariance condition. Similarly, the following (2.4) is also called an invariance condition [16, (3.4)].

$$(\lambda \cdot R(\lambda)(u, v)_2) \cdot R(\lambda)(u, v)_1 = (\lambda \cdot u) \cdot v \quad (\forall \lambda \in H, \forall u, v \in X). \quad (2.4)$$

Here  $(R(\lambda)(u, v)_1, R(\lambda)(u, v)_2) := R(\lambda)(u, v)$ .

**Definition 2.5.** An object  $X = (X, \cdot)$  of  $\text{Sets}_H$ , together with a morphism  $\sigma : X \bar{\otimes} X \rightarrow X \bar{\otimes} X$ , is braided, iff the morphism  $\sigma$  satisfies

$$(\text{id} \bar{\otimes} \sigma)a(\sigma \bar{\otimes} \text{id})a^{-1}(\text{id} \bar{\otimes} \sigma)a = a(\sigma \bar{\otimes} \text{id})a^{-1}(\text{id} \bar{\otimes} \sigma)a(\sigma \bar{\otimes} \text{id}).$$

Here,  $a = a_{X, X, X}$  and  $\text{id} = \text{id}_X$ .

If this morphism  $\sigma$  is an isomorphism, then  $\sigma$  is a Yang-Baxter operator on  $X$  [12, Definition XIII.3.1].

**Proposition 2.6.** *The object  $X = (X, \cdot)$  of  $\text{Sets}_H$  with a morphism  $\sigma : X \bar{\otimes} X \rightarrow X \bar{\otimes} X$  is braided, if and only if  $\sigma(\lambda)$  is a dynamical braiding map associated with  $H$ ,  $X$ , and  $(\cdot)$  satisfying the invariance condition (2.3).*

From Propositions 2.3 and 2.6, the dynamical Yang-Baxter map associated with  $H$ ,  $X$ , and  $(\cdot)$  satisfying the invariance condition (2.4) is exactly a braided object of  $\text{Sets}_H$ .

### 3 Braided objects and ternary systems

Let  $X = (X, \cdot)$  be a braided object of the category  $\text{Sets}_H$  satisfying that the map  $\lambda \cdot : X \ni x \mapsto \lambda \cdot x \in H$  is bijective for every  $\lambda \in H$ . This section clarifies a relation between such a braided object and a ternary operation.

*Remark 3.1.* The set  $X$  is isomorphic to the set  $H$ , because the map  $\lambda \cdot : X \ni x \mapsto \lambda \cdot x \in H$  is bijective. As a result, the map  $(\cdot) : H \times X \rightarrow H$  can be regarded as a binary operation on  $H$ , which satisfies that the map  $\lambda \cdot : H \ni x \mapsto \lambda \cdot x \in H$  is bijective for every  $\lambda \in H$ . The pair  $(H, \cdot)$  is thus a left quasigroup [16, Definition 2.1].

We denote by  $\lambda \setminus$  the inverse of the bijection  $\lambda \cdot : X \rightarrow H$ ; hence,  $\lambda \setminus (\lambda \cdot u) = u$  and  $\lambda \cdot (\lambda \setminus h) = h$  for any  $\lambda, h \in H$  and  $u \in X$ .

A ternary system  $M = (M, \mu)$  is, by definition, a pair of a nonempty set  $M$  and a ternary operation  $\mu : M \times M \times M \rightarrow M$ .

**Theorem 3.2.** *If  $X = (X, \cdot)$  with  $\sigma \in \text{Hom}_{\text{Sets}_H}(X \bar{\otimes} X, X \bar{\otimes} X)$  is braided, then the ternary operation  $\mu$  on  $H$  defined by*

$$\mu(a, b, c) := a \cdot \sigma(a)(a \setminus b, b \setminus c)_1 \quad (a, b, c \in H)$$

satisfies:

$$\mu(a, \mu(a, b, c), \mu(\mu(a, b, c), c, d)) = \mu(a, b, \mu(b, c, d)); \quad (3.1)$$

$$\mu(\mu(a, b, c), c, d) = \mu(\mu(a, b, \mu(b, c, d)), \mu(b, c, d), d) \quad (\forall a, b, c, d \in H). \quad (3.2)$$

Conversely,

**Theorem 3.3.** *A ternary system  $M = (M, \mu)$  satisfying (3.1) and (3.2), together with a bijection  $\pi : H \rightarrow M$ , gives birth to a morphism  $\sigma : X \bar{\otimes} X \rightarrow X \bar{\otimes} X$ , which makes  $X = (X, \cdot)$  a braided object. In fact,*

$$\begin{aligned} \sigma(\lambda)(u, v) &:= (\sigma(\lambda)(u, v)_1, \sigma(\lambda)(u, v)_2); \\ \sigma(\lambda)(u, v)_1 &:= \lambda \setminus \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda \cdot u), \pi((\lambda \cdot u) \cdot v))); \\ \sigma(\lambda)(u, v)_2 &:= (\lambda \cdot \sigma(\lambda)(u, v)_1) \setminus ((\lambda \cdot u) \cdot v). \end{aligned}$$

The proofs of the above theorems are straightforward (cf. [16]).

Several examples of the ternary systems  $M$  in Theorem 3.3 are constructed in [16, Section 6].

**Example 3.4** ( $\mu_1^G$  (6.2) in [16]). Every group  $G = (G, *)$  produces a ternary operation satisfying (3.1) and (3.2). Let  $\mu_G$  denote the following ternary operation on  $G$ .

$$\mu_G(u, v, w) = u * v^{-1} * w \quad (u, v, w \in G).$$

A simple computation shows that the ternary operation  $\mu_G$  satisfies (3.1) and (3.2).

It is worth pointing out that  $X = (X, \cdot)$  is a monoid object of the category  $\text{Sets}_H$ . Define the map  $m(\lambda) : X \times X \rightarrow X$  ( $\lambda \in H$ ) (cf. [15, (3.5)]) by

$$m(\lambda)(u, v) = \lambda \setminus ((\lambda \cdot u) \cdot v) \quad (\lambda \in H, u, v \in X). \quad (3.3)$$

**Proposition 3.5.**  $m \in \text{Hom}_{\text{Sets}_H}(X \bar{\otimes} X, X)$ ; furthermore,

$$m \circ (m \bar{\otimes} \text{id}_X) = m \circ (\text{id}_X \bar{\otimes} m) \circ a_{X, X, X}. \quad (3.4)$$

Let  $\eta(\lambda)$  ( $\lambda \in H$ ) denote the map from  $I_{\text{Sets}_H}$  to  $X$  defined by

$$\eta(\lambda)(e) = \lambda \setminus \lambda \quad (I_{\text{Sets}_H} = \{e\}).$$

**Proposition 3.6.**  $\eta \in \text{Hom}_{\text{Sets}_H}(I_{\text{Sets}_H}, X)$ . Moreover:

$$l = m \circ \eta \bar{\otimes} \text{id}_X; \quad r = m \circ \text{id}_X \bar{\otimes} \eta. \quad (3.5)$$

It follows immediately from Propositions 3.5 and 3.6 that  $X = (X, \cdot)$ , together with the morphisms  $m : X \bar{\otimes} X \rightarrow X$  and  $\eta : I_{\text{Sets}_H} \rightarrow X$ , is a monoid object of  $\text{Sets}_H$ ; that is,  $m$  and  $\eta$  satisfy (3.4) and (3.5).

From (3.3), the invariance condition (2.3) is exactly  $m \circ \sigma = m$ ; hence, the morphism  $\sigma$  is compatible with this monoid structure on  $X$ .

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