SURVEY ON DYNAMICAL YANG-BAXTER MAPS

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Abstract

In this survey, we introduce dynamical Yang-Baxter maps by means of the tensor category.

Introduction

The Yang-Baxter equation is at a crossing of mathematics and physics. This equation was first recognized in physics [1, 2, 3, 19, 20], gave birth to the quantum group [4, 5, 9, 11, 17], and still plays an important role in both mathematics and physics.

The dynamical Yang-Baxter map [15, 16] is a set-theoretical solution to a version of the quantum dynamical Yang-Baxter equation [8, 10]; it is a generalization of the Yang-Baxter map [6, 7, 13], a set-theoretical solution to the quantum Yang-Baxter equation. A loop [14, 18] with a group structure produces a dynamical Yang-Baxter map [15]. Moreover, a left quasigroup, together with a ternary operation satisfying some conditions, yields a dynamical Yang-Baxter map satisfying an invariance condition, and vice versa [16].

This article gives a survey on the dynamical Yang-Baxter map satisfying the invariance condition from the viewpoint of the tensor category.

The organization of this article is as follows. In the first section, we construct a tensor category $\text{Sets}_H$, which plays a central role in this article.

After introducing the dynamical Yang-Baxter map, the dynamical braiding map, and the braided object of $\text{Sets}_H$, we show in Section 2 that the following notions are equivalent:

1. the dynamical Yang-Baxter map satisfying the invariance condition (2.4);
2. the dynamical braiding map satisfying (2.3);
3. the braided object of $\text{Sets}_H$.

Section 3 establishes a relation between the braided object and the ternary operation. This section also discusses monoid objects in $\text{Sets}_H$.

1 Tensor category $\text{Sets}_H$

Let $H$ be a nonempty set. This section is devoted to defining the tensor category $\text{Sets}_H$. We follow the notation of [12, Chapter XI].

An object of $\text{Sets}_H$ is, by definition, a pair $(S, \psi)$ such that $S$ is a nonempty set, together with a map $\psi : H \times S \to H$. $\text{Ob}(\text{Sets}_H)$ is the class of all objects of $\text{Sets}_H$. 

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Let $S = (S, \psi_S)$ and $T = (T, \psi_T)$ be objects of $\text{Sets}_H$. A morphism $f : S \to T$ of $\text{Sets}_H$ is a map $f : H \to \text{Map}(S, T)$ such that

$$\psi_T(\lambda, f(\lambda)(s)) = \psi_S(\lambda, s) \quad (\forall \lambda \in H, \forall s \in S).$$

Let $\text{Hom}(\text{Sets}_H)$ denote all morphisms of $\text{Sets}_H$. For the morphism $f : S \to T$, the objects $S$ and $T$ are called the source $s(f)$ and the target $b(f)$, respectively.

For each object $S$ of $\text{Sets}_H$, define the map $\text{id}_S : H \to \text{Map}(S, S)$ by $\text{id}_S(\lambda)(s) = s (\lambda \in H, s \in S)$. Let $f$ and $g$ be morphisms of $\text{Sets}_H$ satisfying $b(f) = s(g)$. We define the composition $g \circ f \in \text{Hom}_{\text{Sets}_H}(s(f), b(g))$ by $(g \circ f)(\lambda) = g(\lambda) \circ f(\lambda) (\lambda \in H)$.

**Proposition 1.1.** $\text{Sets}_H$ is a category of $\text{Ob}(\text{Sets}_H)$ and $\text{Hom}(\text{Sets}_H)$, together with the identity $\text{id}$, the source $s$, the target $b$, and the composition $\circ$.

The next task is to explain a tensor product $\otimes : \text{Sets}_H \times \text{Sets}_H \to \text{Sets}_H$. Let $S$ and $T$ be objects of the category $\text{Sets}_H$. Define the set $S \otimes T$ and the map $\psi_{S \otimes T} : H \times (S \otimes T) \to H$ by:

$$S \otimes T = S \times T;$$
$$\psi_{S \otimes T}(\lambda, (s, t)) = \psi_T(\psi_S(\lambda, s), t) \quad (\lambda \in H, s \in S, t \in T). \quad (1.1)$$

Clearly, the pair $S \otimes T := (S \otimes T, \psi_{S \otimes T})$ is an object of $\text{Sets}_H$.

Let $f$ and $g$ be morphisms of the category $\text{Sets}_H$. We denote by $f \otimes g$ the following map from $H$ to $\text{Map}(s(f) \otimes s(g), b(f) \otimes b(g))$.

$$f \otimes g(\lambda)(s, t) = (f(\lambda)(s), g(\psi_S(\lambda, s))(t)) \quad (\lambda \in H, (s, t) \in s(f) \otimes s(g)).$$

**Proposition 1.2.** $f \otimes g \in \text{Hom}_{\text{Sets}_H}(s(f) \otimes s(g), b(f) \otimes b(g))$; moreover, $\otimes : \text{Sets}_H \times \text{Sets}_H \to \text{Sets}_H$ is a functor.

The associativity constraint $a : \otimes(\otimes \times \text{id}) \to \otimes(\text{id} \times \otimes)$ is given by

$$a_{S, T, U}(\lambda)((s, t), u) = (s, (t, u)) \quad (S, T, U \in \text{Ob}(\text{Sets}_H), \lambda \in H, ((s, t), u) \in (S \otimes T) \otimes U).$$

Let $I_{\text{Sets}_H}$ denote the set consisting of one element, together with the map $\psi_{I_{\text{Sets}_H}} : H \times I_{\text{Sets}_H} \to H$ defined by $\psi_{I_{\text{Sets}_H}}(\lambda, e) = \lambda (\lambda \in H, I_{\text{Sets}_H} = \{e\})$. The pair $I_{\text{Sets}_H} := (I_{\text{Sets}_H}, \psi_{I_{\text{Sets}_H}})$ is an object of $\text{Sets}_H$.

The left and the right unit constraints $l : \otimes(I_{\text{Sets}_H} \times \text{id}) \to \text{id}$, $r : \otimes(\text{id} \times I_{\text{Sets}_H}) \to \text{id}$ with respect to $I_{\text{Sets}_H}$ are given by

$$l_S(\lambda)(e, s) = r_S(\lambda)(s, e) = s$$
$$\langle S = (S, \psi_S) \in \text{Ob}(\text{Sets}_H), \lambda \in H, s \in S, I_{\text{Sets}_H} = \{e\} \rangle.$$

**Proposition 1.3.** $(\text{Sets}_H, \otimes, I_{\text{Sets}_H}, a, l, r)$ is a tensor category.

A direct computation shows this proposition.
2 Dynamical Yang-Baxter maps and braided objects

In this section, we introduce the notion of a dynamical Yang-Baxter map [15, 16], which is related to a braided object of Sets\(_H\) through a dynamical braiding map [15, (2.8)].

Let \(H\) and \(X\) be nonempty sets, together with a map \((\cdot) : H \times X \rightarrow H\).

**Definition 2.1.** The map \(R(\lambda) : X \times X \rightarrow X \times X (\lambda \in H)\) is a dynamical Yang-Baxter map associated with \(H, X, \) and \((\cdot)\), iff \(R(\lambda)\) satisfies a version of the quantum dynamical Yang-Baxter equation for all \(\lambda \in H\).

\[ R^{23}(\lambda)R^{13}(\lambda \cdot X^{(2)})R^{12}(\lambda) = R^{12}(\lambda \cdot X^{(3)})R^{13}(\lambda)R^{23}(\lambda \cdot X^{(1)}). \]

Here, \(R^{12}(\lambda)\) and \(R^{23}(\lambda \cdot X^{(1)})\) are the following maps from \(X \times X \times X\) to itself: for \(\lambda \in H\) and \(u, v, w \in X\),

\[ R^{12}(\lambda)(u, v, w) = (R(\lambda)(u, v), w); \quad (2.1) \]
\[ R^{23}(\lambda \cdot X^{(1)})(u, v, w) = R^{23}(\lambda \cdot u)(u, v, w) = (u, R(\lambda \cdot u)(v, w)). \quad (2.2) \]

The other maps are similarly defined.

The notion of the dynamical Yang-Baxter map is equivalent to that of a dynamical braiding map, which is more suitable for the tensor category \(\text{Sets}_H\).

**Definition 2.2.** The map \(\sigma(\lambda) : X \times X \rightarrow X \times X (\lambda \in H)\) is a dynamical braiding map associated with \(H, X\), and \((\cdot)\), iff \(\sigma(\lambda)\) satisfies the following equation for all \(\lambda \in H\).

\[ \sigma^{12}(\lambda)\sigma^{23}(\lambda \cdot X^{(1)})\sigma^{12}(\lambda) = \sigma^{23}(\lambda \cdot X^{(1)})\sigma^{12}(\lambda)\sigma^{23}(\lambda \cdot X^{(1)}). \]

Here the definitions of the maps \(\sigma^{12}(\lambda)\) and \(\sigma^{23}(\lambda \cdot X^{(1)})\) from \(X \times X \times X\) to itself are similar to (2.1) and (2.2).

We denote by \(P_X\) the map from \(X \times X\) to itself defined by \(P_X(u, v) = (v, u) (u, v \in X)\). The following is Proposition 2.1 in [15].

**Proposition 2.3.** If \(R(\lambda)\) is a dynamical Yang-Baxter map associated with \(H, X, \) and \((\cdot)\), then \(P_XR(\lambda)\) is a dynamical braiding map associated with \(H, X, \) and \((\cdot)\); if \(\sigma(\lambda)\) is a dynamical braiding map associated with \(H, X, \) and \((\cdot)\), then \(P_X\sigma(\lambda)\) is a dynamical Yang-Baxter map associated with \(H, X, \) and \((\cdot)\).

Let \(\sigma(\lambda)\) be a dynamical braiding map associated with \(H, X, \) and \((\cdot)\), and let \(\sigma(\lambda)(u, v)_1\) and \(\sigma(\lambda)(u, v)_2 (\lambda \in H, u, v \in X)\) denote the elements of \(X\) defined by \((\sigma(\lambda)(u, v)_1, \sigma(\lambda)(u, v)_2) := \sigma(\lambda)(u, v)\).

**Proposition 2.4.** If the dynamical braiding map \(\sigma(\lambda)\) satisfies

\[ (\lambda \cdot \sigma(\lambda)(u, v)_1) \cdot \sigma(\lambda)(u, v)_2 = (\lambda \cdot u) \cdot v \quad (\forall \lambda \in H, \forall u, v \in X), \]

then \(\sigma \in \text{Hom}_{\text{Sets}_H}(X \otimes X, X \otimes X)\).

We call (2.3) an invariance condition. Similarly, the following (2.4) is also called an invariance condition [16, (3.4)].

\[ (\lambda \cdot R(\lambda)(u, v)_2) \cdot R(\lambda)(u, v)_1 = (\lambda \cdot u) \cdot v \quad (\forall \lambda \in H, \forall u, v \in X). \]

Here \((R(\lambda)(u, v)_1, R(\lambda)(u, v)_2) := R(\lambda)(u, v)\).
Definition 2.5. An object $X = (X, \cdot)$ of $\text{Sets}_H$, together with a morphism $\sigma : X \otimes X \to X \otimes X$, is braided, iff the morphism $\sigma$ satisfies

$$(\text{id} \otimes \sigma)(\sigma \otimes \text{id})a^{-1}(\text{id} \otimes \sigma)a = a(\sigma \otimes \text{id})a^{-1}(\text{id} \otimes \sigma)a(\sigma \otimes \text{id}).$$

Here, $a = a_{X,X,X}$ and $\text{id} = \text{id}_X$.

If this morphism $\sigma$ is an isomorphism, then $\sigma$ is a Yang-Baxter operator on $X$ [12, Definition XIII.3.1].

Proposition 2.6. The object $X = (X, \cdot)$ of $\text{Sets}_H$ with a morphism $\sigma : X \otimes X \to X \otimes X$ is braided, if and only if $\sigma(\lambda)$ is a dynamical braiding map associated with $H$, $X$, and $(\cdot)$ satisfying the invariance condition (2.3).

From Propositions 2.3 and 2.6, the dynamical Yang-Baxter map associated with $H$, $X$, and $(\cdot)$ satisfying the invariance condition (2.4) is exactly a braided object of $\text{Sets}_H$.

3 Braided objects and ternary systems

Let $X = (X, \cdot)$ be a braided object of the category $\text{Sets}_H$ satisfying that the map $\lambda \cdot : X \ni x \mapsto \lambda \cdot x \in H$ is bijective for every $\lambda \in H$. This section clarifies a relation between such a braided object and a ternary operation.

Remark 3.1. The set $X$ is isomorphic to the set $H$, because the map $\lambda \cdot : X \ni x \mapsto \lambda \cdot x \in H$ is bijective. As a result, the map $(\cdot) : H \times X \to H$ can be regarded as a binary operation on $H$, which satisfies that the map $\lambda \cdot : H \ni x \mapsto \lambda \cdot x \in H$ is bijective for every $\lambda \in H$. The pair $(H, \cdot)$ is thus a left quasigroup [16, Definition 2.1].

We denote by $\lambda \backslash u$ the inverse of the bijection $\lambda \cdot : X \to H$; hence, $\lambda \backslash (\lambda \cdot u) = u$ and $\lambda \cdot (\lambda \backslash h) = h$ for any $\lambda, h \in H$ and $u \in X$.

A ternary system $M = (M, \mu)$ is, by definition, a pair of a nonempty set $M$ and a ternary operation $\mu : M \times M \times M \to M$.

Theorem 3.2. If $X = (X, \cdot)$ with $\sigma \in \text{Hom}_{\text{Sets}_H}(X \otimes X, X \otimes X)$ is braided, then the ternary operation $\mu$ on $H$ defined by

$$\mu(a, b, c) := a \cdot \sigma(a)(a \cdot b \cdot \sigma^{-1}(a \cdot b \cdot c)) = (a, b, c \in H)$$

satisfies:

$$\mu(a, \mu(a, b, c), \mu(a, b, c, d)) = \mu(a, b, \mu(c, d));$$

$$\mu(\mu(a, b, c), c, d) = \mu(\mu(a, b, c, d), \mu(c, d, d)) \quad (\forall a, b, c, d \in H).$$

Conversely,

Theorem 3.3. A ternary system $M = (M, \mu)$ satisfying (3.1) and (3.2), together with a bijection $\pi : H \to M$, gives birth to a morphism $\sigma : X \otimes X \to X \otimes X$, which makes $X = (X, \cdot)$ a braided object. In fact,

$$\sigma(\lambda)(u, v) := (\sigma(\lambda)(u, v)_1, \sigma(\lambda)(u, v)_2);$$

$$\sigma(\lambda)(u, v)_1 := \lambda \cdot \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda \cdot u), \pi((\lambda \cdot u) \cdot v)));$$

$$\sigma(\lambda)(u, v)_2 := (\lambda \cdot \sigma(\lambda)(u, v)_1) \setminus ((\lambda \cdot u) \cdot v).$$

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The proofs of the above theorems are straightforward (cf. [16]). Several examples of the ternary systems \( M \) in Theorem 3.3 are constructed in [16, Section 6].

**Example 3.4** (\( \mu_G^6 \) (6.2) in [16]). Every group \( G = (G, * ) \) produces a ternary operation satisfying (3.1) and (3.2). Let \( \mu_G \) denote the following ternary operation on \( G \).

\[ \mu_G(u, v, w) = u * v^{-1} * w \quad (u, v, w \in G). \]

A simple computation shows that the ternary operation \( \mu_G \) satisfies (3.1) and (3.2).

It is worth pointing out that \( X = (X, \cdot ) \) is a monoid object of the category \( \text{Sets}_H \). Define the map \( m(\lambda) : X \times X \rightarrow X \) (\( \lambda \in H \)) (cf. [15, (3.5)]) by

\[ m(\lambda)(u, v) = \lambda \langle (\lambda \cdot u) \cdot v \rangle \quad (\lambda \in H, u, v \in X). \quad (3.3) \]

**Proposition 3.5.** \( m \in \text{Hom}_{\text{Sets}_H}(X \bar{\otimes} X, X) \); furthermore,

\[ m \circ (m \bar{\otimes} \text{id}_X) = m \circ (\text{id}_X \bar{\otimes} m) \circ a_{X, X, X}. \quad (3.4) \]

Let \( \eta(\lambda) \) (\( \lambda \in H \)) denote the map from \( I_{\text{Sets}_H} \) to \( X \) defined by

\[ \eta(\lambda)(e) = \lambda \backslash \lambda \quad (I_{\text{Sets}_H} = \{e\}). \]

**Proposition 3.6.** \( \eta \in \text{Hom}_{\text{Sets}_H}(I_{\text{Sets}_H}, X) \). Moreover:

\[ l = m \circ \eta \bar{\otimes} \text{id}_X; \quad r = m \circ \text{id}_X \bar{\otimes} \eta. \quad (3.5) \]

It follows immediately from Propositions 3.5 and 3.6 that \( X = (X, \cdot ) \), together with the morphisms \( m : X \bar{\otimes} X \rightarrow X \) and \( \eta : I_{\text{Sets}_H} \rightarrow X \), is a monoid object of \( \text{Sets}_H \); that is, \( m \) and \( \eta \) satisfy (3.4) and (3.5).

From (3.3), the invariance condition (2.3) is exactly \( m \circ \sigma = m \); hence, the morphism \( \sigma \) is compatible with this monoid structure on \( X \).

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**References**


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