THE RELATIVE PICARD GROUP OF A COMODULE ALGEBRA AND HARRISON COHOMOLOGY

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ABSTRACT. Let A be a commutative comodule algebra over a commutative bialgebra H. The group of invertible relative Hopf modules maps to the Picard group of A, and the kernel is described as a quotient group of the group of invertible grouplike elements of the coring $A \otimes H$, or as a Harrison cohomology group. Our methods are based on elementary K-theory. The Hilbert 90 Theorem follows as a corollary. The part of the Picard group of the coinvariants that becomes trivial after base extension embeds in the Harrison cohomology group, and the image is contained in a well-defined subgroup E. It equals E if H is a cosemisimple Hopf algebra over a field.

INTRODUCTION

Let l be a cyclic Galois field extension of k. The Hilbert 90 Theorem tells us that every cocycle in $Z^1(C_p, l^*)$ is a coboundary. There exist various generalizations of this result. For example, if we have a Galois extension $B \to A$ of commutative rings, with Galois group G, then the cohomology group $H^1(G, \mathbb{G}_m(A))$ is isomorphic to $\mathrm{Pic}(A/B)$, the kernel of the natural map from the Picard group of B to the Picard group of A, see for example [10]. Now we can ask the following question: suppose that G acts on A as a group of isomorphisms. Can we still give an algebraic interpretation of $H^1(G, \mathbb{G}_m(A))$? A second problem is whether there is any relation between $H^1(G, \mathbb{G}_m(A))$ and the Picard group of the ring of invariants $B = A^G$.

In this note, we will discuss these two problems in a more general situation: we will assume that A is a commutative H-comodule algebra, with H an arbitrary commutative bialgebra over a commutative ring k. We then ask for an algebraic interpretation of the first Harrison cohomology group $H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$ (with notation as in [6]). If H is finitely generated and projective, then this Harrison cohomology group is isomorphic to a Sweedler cohomology group $Z^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$, and if $H = \mathbb{Z}G$ with G a finite group, then it reduces to the cohomology group $H^1(G, \mathbb{G}_m(A))$.

We proceed as follows: we introduce the relative Picard group $\mathrm{Pic}^H(A)$ as the Grothendieck group of the category of invertible relative Hopf modules.

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The forgetful functor to the category of invertible A-modules induces a K-theoretic exact sequence, linking the Picard group of A, the relative Picard group, and the groups of unit elements of A and the coinvariants $B = A^{\mathrm{co}H}$; the middle term in the sequence can be computed, and it is the group of invertible grouplike elements of the coring $A \otimes H$. We show also that these grouplike elements are precisely the Harrison cocycles, and it follows from the exactness of the sequence that the first Harrison cohomology group is the kernel of the map $\mathrm{Pic}^H(A) \to \mathrm{Pic}(A)$, answering our first question.
Then we observe that there is a similar exact sequence associated to the induction functor $\underline{\underline{\mathrm{Pic}}}(B) \to \underline{\underline{\mathrm{Pic}}}(A)$, and that the two exact sequences fit into a commutative diagram. If A is a faithfully flat Hopf Galois extension of B, then the categories of B-modules and relative Hopf modules are equivalent, hence $\mathrm{Pic}(B) \cong \mathrm{Pic}^H(A)$, and we recover Hilbert 90. In general, we have an injection $\mathrm{Pic}(A/B) \to H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$, and we can describe a subgroup of $H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m)$ that contains the image of $\mathrm{Pic}(A/B)$. The image is precisely this subgroup if H is a cosemisimple Hopf algebra over a field k.
A special situation is the following: let k be an algebraically closed field, A a finitely generated commutative normal k-algebra, and G a connected algebraic group acting rationally on A. Then A is an H-comodule algebra, with H the affine coordinate ring of G. In this case, our exact sequence was given by Magid in [14], but apparently the author of [14] was not aware of the connection to Harrison cohomology, grouplike elements of corings and the generalized Hilbert 90 Theorem.
In Section 4, we study the Harrison cocycles (or the grouplike elements in $A \otimes H$) in some particular cases. First we look at the situation considered by Magid in [14], and then it turns out that the grouplike elements of $G(A \otimes H)$ are induced by the grouplike elements of H. In the situation where A is a $\mathbb{Z}$-graded commutative k-algebra, the relative Picard group turns out to be the graded Picard group $\mathrm{Pic}_g(A)$ studied by the first author in [5]. If A is reduced, then the grouplike elements of $A \otimes H$ can also be described using the grouplikes of H, according to a result in [5].

1. Preliminary results

1.1. **The language of corings.** Relative Hopf modules can be viewed as comodules over a coring. This will be used in the sequel, and this is why we briefly recall some properties of corings. Recall that an A-coring is a comonoid in the monoidal category ${}_A\mathcal{M}_A$ of A-bimodules. Thus an A-coring $\mathfrak{C}$ is an A-bimodule together with two A-bimodule maps

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C} \quad \text{and} \quad \varepsilon_{\mathfrak{C}} : \mathfrak{C} \to A$$

satisfying the usual coassociativity and counit properties. We refer to [2, 3, 4, 11, 18] for a detailed discussion of corings.

$$G(\mathfrak{C}) = \{X \in \mathfrak{C} \mid \Delta_{\mathfrak{C}}(X) = X \otimes_A X \text{ and } \varepsilon_{\mathfrak{C}}(X) = 1\}$$

is the set of grouplike elements of $\mathfrak{C}$. A right $\mathfrak{C}$-comodule M is a right A-module together with a right A-linear map $\rho_M: M \to M \otimes_A \mathfrak{C}$ satisfying

$$(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho_M = M, \quad \text{and} \quad (M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M.$$

A morphism of right $\mathfrak{C}$-comodules $f: M \to N$ is an A-linear map f such that

$$\rho_N \circ f = (f \otimes_A \mathfrak{C}) \circ \rho_M.$$

$\mathcal{M}^{\mathfrak{C}}$ will be the category of right $\mathfrak{C}$-comodules and comodule morphisms. We have the following interpretation of the grouplike elements of $\mathfrak{C}$.

Lemma 1.1. *Let $\mathfrak{C}$ be an A-coring. Then there is a bijective correspondence between $G(\mathfrak{C})$ and the set of maps $\rho: A \to A \otimes_A \mathfrak{C} = \mathfrak{C}$ making A into a right $\mathfrak{C}$-comodule. The coaction ρ_X corresponding to $X \in G(\mathfrak{C})$ is given by*

$$\rho_X(a) = Xa.$$

With this notation, $A^X = (A, \rho_X) \cong A^Y = (A, \rho_Y)$ as right $\mathfrak{C}$-comodules if and only if there exists an invertible $b \in A$ such that $\rho_Y(b) = Yb = bX$.

Proof. The first part is well-known (and straightforward), see for example [3]. Let $f: A^X \to A^Y$ be a right $\mathfrak{C}$-colinear isomorphism. Then $f(a) = ba$ for some $b \in A$, which is invertible since f is an isomorphism. The fact that f is $\mathfrak{C}$-colinear tells us that

$$Yb = \rho_Y(f(1)) = (f \otimes_A \mathfrak{C})(\rho_X(1)) = bX.$$

The converse property is obvious. □

1.2. **Relative Hopf modules.** Let H be a bialgebra over a commutative ring k, and A a right H-comodule algebra. Throughout this note, we will assume that H and A are commutative. Then A is a commutative algebra and we have a right H-coaction ρ on A such that

$$\rho(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]},$$

for all $a, b \in A$. Here we use the Sweedler-Heyneman notation for the coaction ρ: $\rho(a) = a_{[0]} \otimes a_{[1]}$, with summation implicitely understood. For the comultiplication on H, we use the notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$.
A relative Hopf module M is a k-module, together with a right A-action and a right C-coaction ρ_M such that

$$\rho_M(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]},$$

for all $a \in A$ and $m \in M$. The category of relative Hopf modules and A-linear H-colinear maps will be denoted by $\mathcal{M}_A^H$. The coinvariant submodule $M^{\text{co}H}$ of $M \in \mathcal{M}_A^H$ is defined by

$$M^{\text{co}H} = \{m \in M \mid \rho_M(m) = m \otimes 1\}.$$

$A^{\text{co}H} = B$ is a k-subalgebra of A, and $M^{\text{co}H}$ is a B-module. We obtain a functor $(-)^{\text{co}H}: \mathcal{M}_A^H \to \mathcal{M}_B$, that has a left adjoint $T = - \otimes_B A: \mathcal{M}_B \to \mathcal{M}_A^H$. The right H-coaction on $N \otimes_B A$ is $N \otimes_B \rho$. The unit u and counit

c of the adjunction are given by the following formulas, for $N \in \mathcal{M}_B$ and $M \in \mathcal{M}_A^H$:

$$u_N:\ N \to (N \otimes_B A)^{\mathrm{co}H},\ u_N(n) = n \otimes 1;$$

$$c_M:\ M^{\mathrm{co}H} \otimes_B A \to M,\ c_M(m \otimes a) = ma.$$

A is called a Hopf algebra extension of $B = A^{\mathrm{co}H}$ if the canonical map

$$\mathrm{can}:\ A \otimes_B A \to A \otimes H,\ \mathrm{can}(a \otimes_B b) = ab_{[0]} \otimes b_{[1]}$$

is an isomorphism. If A is a faithfully flat Hopf Galois extension, then the adjunction $(- \otimes_B A, (-)^{\mathrm{co}H})$ is a pair of inverse equivalences. We refer to [9, 15, 17] for a detailed discussion of Hopf algebras and relative Hopf modules.

$\mathfrak{C} = A \otimes H$ is a coring, with structure maps

$$a'(b \otimes h)a = a'ba_{[0]} \otimes ha_{[1]}$$

$$\Delta_{\mathfrak{C}}(a \otimes h) = (a \otimes h_{(1)}) \otimes_A (1 \otimes h_{(2)})$$

$$\varepsilon_{\mathfrak{C}}(a \otimes h) = a\varepsilon(h)$$

The category $\mathcal{M}^{A\otimes H}$ is isomorphic to the category $\mathcal{M}_A^H$ of relative Hopf modules; we refer to [2, 4] for full detail. Note that $X = \sum_i a_i \otimes h_i \in G(A \otimes H)$ if and only if

$$\sum_i (a_i \otimes h_{i(1)} \otimes h_{i(2)}) = \sum_{i,j} (a_i a_{j[0]} \otimes h_i a_{j[1]} \otimes h_j) \quad \text{and} \quad \sum a_i \varepsilon(h_i) = 1. \tag{1}$$

$A \otimes H$ is also a commutative algebra, with multiplication

$$(a \otimes h)(b \otimes k) = ab \otimes hk.$$

The product of two grouplike elements is a grouplike element, and $1_A \otimes 1_H$ is grouplike, hence $G^i(A \otimes H)$, the set of invertible grouplike elements, is an abelian group. Also observe that an invertible grouplike element is precisely a normalized Harrison 1-cocycle (see for example [6, Sec. 9.2] for the definition of the Harrison complex).

Let H be a finitely generated projective cocommutative Hopf algebra, and A a commutative left H-module algebra. Then H^* is a commutative Hopf algebra, and A is a right H^*-comodule algebra. If $\sum_i a_i \otimes f_i \in A \otimes H^*$ is an invertible grouplike element (or a normalized Harrison cocycle), then

$$\phi:\ H \to A,\ \phi(h) = \sum_i a_i f_i(h), \tag{2}$$

is a normalized Sweedler 1-cocycle, this means that $\phi(1_H) = 1_A$, and the cocycle condition

$$\phi(hh') = \sum_i (h_{(1)}.(\phi(h')))\phi(h_{(2)}), \tag{3}$$

is satisfied. This gives a bijective correspondence between Harrison and Sweedler cocycles, see [6, Prop. 9.2.3]. For the definition of the Sweedler complex, see [16] or [6, Sec. 9.1]. In the case where $H = kG$, with G a finite group, Sweedler cohomology reduces to group cohomology.

1.3. **Elementary algebraic K-theory.** Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \otimes, J)$ be skeletally small symmetric monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a cofinal, strong monoidal functor. Then we can consider the Grothendieck and Whitehead groups of $\mathcal{C}$ and $\mathcal{D}$, and we have an exact sequence connecting them (see for example [1, Ch. VII]):

$$K_1\mathcal{C} \xrightarrow{K_1F} K_1\mathcal{D} \xrightarrow{d} K_1\underline{\phi F} \xrightarrow{g} K_0\mathcal{C} \xrightarrow{K_0F} K_0\mathcal{D}. \tag{4}$$

$C \in \mathcal{C}$ is called invertible if there exists $C' \in \mathcal{C}$ such that $C \otimes C' \cong I$. If all elements of $\mathcal{C}$ and $\mathcal{D}$ are invertible, then the description of the five groups in (4) and the connecting maps simplifies (see [6, App. C]). $K_0\mathcal{C}$ is the group of isomorphism classes of objects in $\mathcal{C}$ and $K_1\mathcal{C} \cong \mathrm{Aut}_{\mathcal{C}}(I)$ (which is then an abelian group). Let $\underline{\Psi F}$ be the following category: objects are couples (C, α), with $C \in \mathcal{C}$ and $\alpha : F(C) \to J$ an isomorphism in $\mathcal{D}$. A morphism between (C, α) and (C', α') is an isomorphism $f : C \to C'$ in $\mathcal{C}$ such that $\alpha' = F(f) \circ \alpha$. $\underline{\Psi F}$ is monoidal, every object is invertible and $K_1\underline{\phi F} \xrightarrow{g} \cong K_0\underline{\Psi F}$. The maps d and g are given as follows: $d(\alpha) = [(I, \alpha)]$ and $g[(C, \alpha)] = [C]$.
A typical example is the following: For a commutative ring A, let $\underline{\underline{\mathrm{Pic}}}(A)$ be the category of invertible A-modules. If $i : B \to A$ is a morphism of commutative rings, then we have cofinal strongly monoidal functor

$$G = - \otimes_B A : \underline{\underline{\mathrm{Pic}}}(B) \to \underline{\underline{\mathrm{Pic}}}(A),$$

and (4) takes the form

$$1 \to \mathbb{G}_m(B) \to \mathbb{G}_m(A) \xrightarrow{d'} K_1\underline{\phi G} \xrightarrow{g'} \mathrm{Pic}(B) \to \mathrm{Pic}(A). \tag{5}$$

2. The relative Picard group

If $M, N \in \mathcal{M}_A^H$, then $M \otimes_A N \in \mathcal{M}_A^H$, with right H-coaction

$$\rho_{M \otimes_A N}(m \otimes_A n) = m_{[0]} \otimes_A n_{[0]} \otimes m_{[1]} n_{[1]}.$$

So we have a symmetric monoidal category $(\mathcal{M}_A^H, \otimes_A, A)$. Let $\underline{\underline{\mathrm{Pic}}}^H(A)$ be the full subcategory consisting of invertible objects. $\mathrm{Pic}^H(A) = K_0\underline{\underline{\mathrm{Pic}}}^H(A)$, the group of isomorphism classes of relative Hopf modules, will be called the relative Picard group of A and H. The isomorphism class in $\mathrm{Pic}^H(A)$ represented by an invertible relative Hopf module M will be denoted by $\{M\}$. This new invariant fits into an exact sequence:

Proposition 2.1. *We have an exact sequence*

$$1 \to \mathbb{G}_m(B) \to \mathbb{G}_m(A) \xrightarrow{d} G^i(A \otimes H) \xrightarrow{g} \mathrm{Pic}^H(A) \to \mathrm{Pic}(A). \tag{6}$$

Proof. This result can be proved in two ways: a first possibility is to show that (6) is precisely the exact sequence (4), associated to the functor $\underline{\underline{\mathrm{Pic}}}^H(A) \to \underline{\underline{\mathrm{Pic}}}(A)$ forgetting the H-coaction. Let us present an easy direct proof.
The map $\mathbb{G}_m(B) \to \mathbb{G}_m(A)$ is the natural inclusion. Take $a \in A$ invertible, and let $d(a) = X = a^{-1}a_{[0]} \otimes a_{[1]}$. X is grouplike, since $a^{-1}a_{[0]}\varepsilon(a_{[1]}) = 1$, and

$$\begin{aligned} X \otimes_A X &= (a^{-1}a_{[0]} \otimes a_{[1]}) \otimes_A (b^{-1}b_{[0]} \otimes b_{[1]}) \\ &= a^{-1}a_{[0]}(b^{-1})_{[0]}b_{[0]} \otimes a_{[1]}(b^{-1})_{[1]}b_{[1]} \otimes b_{[2]} \\ &= a^{-1}b_{[0]} \otimes b_{[1]} \otimes b_{[2]} \\ &= (a^{-1}b_{[0]} \otimes b_{[1]}) \otimes_A (1 \otimes b_{[2]}) = \Delta(X), \end{aligned}$$

where we identified $(A \otimes H) \otimes_A (A \otimes H) = A \otimes H \otimes H$ and we wrote $a = b$. The inverse of X is $X^{-1} = a(a^{-1})_{[0]} \otimes (a^{-1})_{[1]}$, so $X \in G^i(A \otimes H)$.
If $d(a) = a^{-1}a_{[0]} \otimes a_{[1]} = 1_A \otimes 1_H$, then $a_{[0]} \otimes a_{[1]} = a \otimes 1_H$, so $a \in B$, and the sequence is exact at $\mathbb{G}_m(A)$.
For $X \in G^i(A \otimes H)$, let $g(X) = A^X$, with notation as in Lemma 1.1. g is multiplicative: take $X = \sum_i a_i \otimes h_i$ and $Y = \sum_j b_j \otimes k_j$ in $G^i(A \otimes H)$, then $A^X \otimes_A A^Y = A$ as an A-bimodule, with comultiplication given by $\rho_{A^X \otimes_A A^Y}(1) = \sum_{i,j} a_i \otimes_A b_j \otimes h_i k_j = XY$, as needed.
If $g(X) = \{A\}$ in $\mathrm{Pic}^H(A)$, then there exists an H-colinear A-linear isomorphism $f : A^X \to A$. Then $f(1) = a$ is invertible in A, and, since f is H-colinear, $a_{[0]} \otimes a_{[1]} = \rho(a) = (f \otimes H)(X) = aX$, so $X = a^{-1}a_{[0]} \otimes a_{[1]} = d(a)$, and the sequence is also exact at $G^i(A \otimes H)$.
The exactness of the sequence at $\mathrm{Pic}^H(A)$ follows from Lemma 1.1. □

Remark 2.2. Let $H = k\mathbb{Z}$, and let A be a commutative $\mathbb{Z}$-graded k-algebra. Then $\mathrm{Pic}^H(A) = \mathrm{Pic}_g(A)$, the graded Picard group of A, as introduced in [5], see also [8]. The exact sequence (6) reduces to the exact sequence in [5, Prop. 2.1].

The map $d : \mathbb{G}_m(A) \xrightarrow{d} G^i(A \otimes H)$ is precisely the map $\mathbb{G}_m(A) \to \mathbb{G}_m(A \otimes H)$ in the Harrison complex. From Proposition 2.1, we therefore obtain immediately:

Corollary 2.3. *With H and A as in Proposition 2.1, we have an isomorphism of abelian groups*

$$\mathrm{Pic}^H(A) \cong H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m).$$

This is the promised algebraic interpretation of the first Harrison cohomology group. Note that there are no flatness or projectivity assumptions on H or A. We have Hilbert 90 as an easy consequence.

Corollary 2.4. (Hilbert 90) *Let H, A, B be as in Proposition 2.1. If A is a faithfully flat H-Galois extension of B, then we have an isomorphism of abelian groups*

$$\mathrm{Pic}(A/B) \cong H^1_{\mathrm{Harr}}(H, A, \mathbb{G}_m).$$

Proof. From the fact that the monoidal categories $\mathcal{M}_B$ and $\mathcal{M}_A^H$ are equivalent, it follows that $\mathrm{Pic}(B) \cong \mathrm{Pic}^H(A)$. □

Take the exact sequences (5) and (6), and observe that they fit into a commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \mathbb{G}_m(B) & \longrightarrow & \mathbb{G}_m(A) & \xrightarrow{d'} & K_1\underline{\phi G} & \xrightarrow{g'} & \mathrm{Pic}(B) & \longrightarrow & \mathrm{Pic}(A) \\
 & & \Big\downarrow{=} & & \Big\downarrow{=} & & & & \Big\downarrow{j} & & \Big\downarrow{=} \\
1 & \longrightarrow & \mathbb{G}_m(B) & \longrightarrow & \mathbb{G}_m(A) & \xrightarrow{d} & G^i(A\otimes H) & \xrightarrow{g} & \mathrm{Pic}^H(A) & \longrightarrow & \mathrm{Pic}(A)
\end{array}$$

The map j maps $[N] \in \mathrm{Pic}(B)$ to $\{N \otimes_B A\} \in \mathrm{Pic}^H(A)$. Using the Five Lemma, we find a map $i:\ K_1\underline{\phi G} \to G^i(A\otimes H)$.

Lemma 2.5. *With notation as above, the maps i and j are injective.*

Proof. From the fact that u is a natural transformation between additive endofunctors of the category of B-modules, and since u_B is an isomorphism, it follows that $u_N:\ N \to (N\otimes_B A)^{\mathrm{co}H}$ is an isomorphism if N is finitely generated and projective as a B-module. So if $N\otimes_B A \cong A$, then $N \cong (N\otimes_B A)^{\mathrm{co}H} \cong A^{\mathrm{co}H} = B$, and j is injective. The injectivity of i then follows from an easy diagram chasing argument. □

Our next aim is to characterize the image of i. This will be the topic of Section 3; it will turn out that we obtain nice results in the case where H is cosemisimple.

3. Coinvariantly generated relative Hopf modules

Some of our results will be more specific if we assume that H is a cosemisimple Hopf algebra over a field k. Recall that H is cosemisimple if there exists a left integral ϕ on H^* such that $\phi(1) = 1$ (see e.g. [18]). In this case, the coinvariants functor $(-)^{\mathrm{co}H}:\ \mathcal{M}_A^H \to \mathcal{M}_B$ is exact, see [15, Lemma 2.4.3].
A relative Hopf module M is called *coinvariantly generated* if c_M is surjective, or, equivalently, if $M = M^{\mathrm{co}H}A$. If M is coinvariantly generated, and finitely generated as an A-module, then we can find a finite set $\{m_1,\cdots,m_n\} \in M^{\mathrm{co}H}$ that generates M.
It follows immediately from the properties of adjoint functors that $N\otimes_B A$ is coinvariantly generated, for every $N \in \mathcal{M}_B$; in particular, A is coinvariantly generated. We also have the following:

Lemma 3.1. *Let $M \in \mathcal{M}_A^H$ and $N \in \mathcal{M}_B$. If M is an epimorphic image of $N\otimes_B A$ in $\mathcal{M}_A^H$, then $M^{\mathrm{co}H} = 0$ implies that $M = 0$.*

Proof. If $M^{\mathrm{co}H} = 0$, then $\mathrm{Hom}_A^H(N\otimes_B A, M) = \mathrm{Hom}_B(N, M^{\mathrm{co}H}) = 0$. But $\mathrm{Hom}_A^H(N\otimes_B A, M)$ contains the epimorphism of relative Hopf modules $N\otimes_B A \to M$, so $M = 0$. □

If N is an epimorphic image of M in $\mathcal{M}_A^H$, and if M is coinvariantly generated, then N is also coinvariantly generated.

Lemma 3.2. *Assume that H is a cosemisimple Hopf algebra over a field k. If $N \in \mathcal{M}_B$ is projective, then $N \otimes_B A$ is projective in $\mathcal{M}_A^H$.*

Proof. See [7, Prop. 2.5]. □

Lemma 3.3. *Let k be a field.*

1. *The forgetful functor $\mathcal{M}_A^H \to \mathcal{M}_A$ preserves projectives;*
2. *if H is cosemisimple, then the forgetful functor also reflects projectivity of finitely generated modules.*

Proof. (1) Take $M \in \mathcal{M}_A^H$ projective, and consider the epimorphism $p : M \otimes A \to M$, $p(m \otimes a) = ma$ in $\mathcal{M}_A^H$. The exact sequence

$$0 \to \operatorname{Ker} p \to M \otimes A \xrightarrow{p} M \to 0$$

splits in $\mathcal{M}_A^H$, since M is a projective object, and a fortiori in $\mathcal{M}_A$. Hence M is a direct factor of $M \otimes A$, which is a projective A-module, so M is also a projective A-module.
(2) Let M, N be a relative Hopf modules, and assume that M is finitely generated and projective in $\mathcal{M}_A$. According to [7, Prop. 4.2], $\operatorname{Hom}_A(M, N) \in \mathcal{M}_A^H$, and it is easy to show that $\operatorname{Hom}_A(M, N)^{\operatorname{co}H} = \operatorname{Hom}_A^H(M, N)$. It follows that the functor $\operatorname{Hom}_A^H(M, -) : \mathcal{M}_A^H \to \mathcal{M}$ is exact, since it is the composition of the exact functors $\operatorname{Hom}_A(M, -) : \mathcal{M}_A^H \to \mathcal{M}^H$ ($M \in \mathcal{M}_A$ is projective) and $(-)^{\operatorname{co}H} : \mathcal{M}^H \to \mathcal{M}$ (H is cosemisimple). □

Lemma 3.4. *Let H be a cosemisimple Hopf algebra over a field k, and take $P, Q \in \mathcal{M}_A^H$ finitely generated as A-modules. Assume that Q is a projective object of $\mathcal{M}_A^H$. Then every epimorphism $f : P \to Q$ in $\mathcal{M}_A^H$ has a right inverse in $\mathcal{M}_A^H$.*

Proof. It is clear that $\operatorname{Hom}_A(Q, P)$ and $\operatorname{Hom}_A(Q, Q)$ are right H-comodules, and the map $f^* = \operatorname{Hom}_A(Q, f) : \operatorname{Hom}_A(Q, P) \to \operatorname{Hom}_A(Q, Q)$ is right H-colinear. It follows from Lemma 3.3 that Q is projective as an A-module, so f^* is surjective. Since f^* is H-colinear, it restricts to a surjection

$$\operatorname{Hom}_A^H(Q, P) = \operatorname{Hom}_A(Q, P)^{\operatorname{co}H} \to \operatorname{Hom}_A^H(Q, Q) = \operatorname{Hom}_A(Q, Q)^{\operatorname{co}H}.$$

Take a preimage $g \in \operatorname{Hom}_A^H(Q, P)$ of the identity map id_Q on Q. Then $f \circ g = \mathrm{id}_Q$, and the result follows. □

For $M \in \mathcal{M}_A$, we will denote the dual module by $M^* = \operatorname{Hom}_A(M, A)$.

Proposition 3.5. *Let H be cosemisimple, and assume that $P \in \mathcal{M}_A^H$ is coinvariantly generated and finitely generated projective as an A-module. Then*

1. *$P^{\operatorname{co}H}$ is a finitely generated projective B-module;*
2. *P^* is coinvariantly generated;*
3. *the map c_P is an isomorphism in $\mathcal{M}_A^H$.*

Proof. (1) As we have seen, there exist $p_1, p_2, \cdots, p_n \in P^{\mathrm{co}H}$ such that $P = \sum_i p_i A$. Set $F = A^n$ and let $f: F \to P$ be the A-linear map given by $f(a_1, a_2, ..., a_n) = \sum_i p_i a_i$. Then $F \in \mathcal{M}_A^H$ and f is an epimorphism in $\mathcal{M}_A^H$. By Lemma 3.4, there exists a monomorphism $g \in \mathrm{Hom}_A(P, F)$ such that $f \circ g = \mathrm{id}_P$. The restriction of g to $P^{\mathrm{co}H}$ is then a B-linear right inverse of the restriction of f to $F^{\mathrm{co}H}$, and $F^{\mathrm{co}H} = B^n$, and we obtain (1).
(2) The map $g^* = \mathrm{Hom}_A(g, A): F^* \to P^*$ is surjective and H-colinear. The fact that F^* is coinvariantly generated then implies that P^* is also coinvariantly generated.
(3) Consider the natural transformation $t: (-)^{\mathrm{co}H} \otimes_B A \to (-)$ given by

$$t_P: P^{\mathrm{co}H} \otimes_B A \to P, \; t_P(p \otimes a) = pa.$$

The map t_A is an isomorphism, so t_F is an isomorphism by additivity. It follows that t_P is an isomorphism, since $F = P \oplus \mathrm{Ker}\, f$ as H-comodules. □

Let $X = \sum_i a_i \otimes h_i \in G(A \otimes H)$, and write

$$A_X = \{a \in A \mid \rho(a) = aX = \sum_i aa_i \otimes h_i\},$$

and

$$A_X^i = \{a \in A_X \mid a \text{ is invertible}\}.$$

Observe that

$$\mathrm{Im}\,(d) = \{X \in G^i(A \otimes H) \mid A_X^i \neq \emptyset\}$$

and

$$A_{1\otimes 1} = A^{\mathrm{co}H}.$$

Furthermore, $A_X A_Y \subset A_{XY}$: take $a \in A_X$ and $b \in A_Y$, then $\rho(a) = aX = \sum_i aa_i \otimes h_i$, $\rho(b) = bY = \sum_j bb_j \otimes k_j$ and $\rho(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]} = \sum_{i,j} aa_i bb_j \otimes h_i k_j = abXY$. Also $A_X^i \cap A_Y^i = \emptyset$ if $X \neq Y$.

Lemma 3.6. *The set*

$$E = \{X \in G^i(A \otimes H) \mid AA_X = A \text{ and } AA_{X^{-1}} = A\}$$

is a subgroup of $G^i(A \otimes H)$ *containing* $\mathrm{Im}\,(d)$.

Proof. If $X \in \mathrm{Im}\,(d)$, then there exists an invertible $a \in A_X$, and then $AA_X = A$. since $X^{-1} \in \mathrm{Im}\,(d)$, we also have $AA_{X^{-1}} = A$, hence $X \in E$. It is clear that $1 \otimes 1 \in E$. If $X, Y \in E$, then $AA_{XY} \supset AA_X A_Y = AA_Y = A$, and, in a similar way, $AA_{(XY)^{-1}} = A$, hence $XY \in E$. Finally, if $X \in E$, then obviously $X^{-1} \in E$. □

Proposition 3.7. *Consider the injective map* $j: \mathrm{Pic}(B) \to \mathrm{Pic}^H(A)$. *If* H *is a cosemisimple Hopf algebra over a field* k, *then*

$$\mathrm{Im}\,(j) = \{\{M\} \in \mathrm{Pic}^H(A) \mid M \textit{ is coinvariantly generated}\}.$$

Proof. $M \otimes_B A$ is coinvariantly generated, so $\mathrm{Im}\,(j)$ is contained in the desired set. If H is cosemisimple, and $\{N\} \in \mathrm{Pic}^H(A)$, with N coinvariantly generated, then $N = (N^{\mathrm{co}H}) \otimes_B A \in \mathrm{Im}\,(j)$, by Proposition 3.5(3). □

Lemma 3.8. *Take $X \in G^i(A \otimes H)$. Then A^X is coinvariantly generated if and only if $AA_{X^{-1}} = A$. If H is cosemisimple, then this is also equivalent to $X \in E$.*

Proof. The first statement follows from the fact that $(A^X)^{\mathrm{co}H} = A_{X^{-1}}$. Indeed, $a \in (A^X)^{\mathrm{co}H}$ if and only if $\rho_X(a) = Xa = a \otimes 1$, if and only if $\rho(a) = (1 \otimes 1)a = X^{-1}(a \otimes 1) = aX^{-1}$, which means that $a \in A_{X^{-1}}$.
Let H be cosemisimple. Note that $(A^X)^* \cong A^{X^{-1}}$ as relative Hopf modules. If A^X is coinvariantly generated, then so is $A^{X^{-1}}$, by Proposition 3.5, and then $X \in E$. □

Now we are able to prove the main result of this Section.

Theorem 3.9. *Consider the monomorphism $i: K_1\underline{\phi G} \to G^i(A \otimes H)$ introduced in Lemma 2.5.*
Then $\mathrm{Im}\,(i) \subset E$ *and* $\mathrm{Im}\,(i) = E$ *if H is a cosemisimple Hopf algebra over a field k. In this situation,* $\mathrm{Pic}(A/B) \cong E$.

Proof. Take $[(M, \alpha)] \in K_0\underline{\psi G}$, and let $i[(M, \alpha)] = X \in G^i(A \otimes H)$. Then $\{A^X\} = j(g'[(M, \alpha)]) = j([M]) = \{M \otimes_B A\}$, hence A^X is coinvariantly generated and $AA_{X^{-1}} = A$, by Lemma 3.8. In a similar way, $i([(M, \alpha)]^{-1}) = X^{-1}$, and $A^{X^{-1}} \cong M^* \otimes_B A$ is coinvariantly generated, so $AA_X = A$, again by Lemma 3.8. This proves that $X \in E$.
Assume now that H is cosemisimple, and take $X \in E$. It follows from Lemma 3.8 that A^X is coinvariantly generated, and from Proposition 3.7 that $A^X = M \otimes_B A$ for some $M \in \underline{\underline{\mathrm{Pic}}}(B)$. Since the image of M in $\mathrm{Pic}(A)$ is trivial, $[M] = g'[(M, \alpha)]$ for some $(M, \alpha) \in \mathcal{C}$. Write $i[(M, \alpha)] = Y$. Then $X = Yd(a)$, for some $a \in \mathbb{G}_m(A)$. Consider the map $\alpha' : M \otimes_B A \to A$, $\alpha'(m \otimes b) = a^{-1}\alpha(m \otimes b)$. Then $i[(M, \alpha')] = X$. □

4. On the grouplike elements

We have an injective map $i: G(H) \to G(A \otimes H)$, $i(g) = 1_A \otimes g$. Everything simplifies if i is an isomorphism. We discuss two situations in which this is (almost) the case.
Recall that a commutative algebra which is an integral domain is called normal if it is integrally closed in its field of fractions.

Proposition 4.1. *Let k be an algebraically closed field, A a finitely generated commutative normal k-algebra and G a connected algebraic group acting rationally on A. Let H be the affine coordinate ring of G, and $\chi(G)$ be the group of characters of G. Then*

$$G(A \otimes H) = \{1 \otimes \phi \mid \phi \in G(H) = \chi(G)\}.$$

Proof. Let $x = \sum_i a_i \otimes f_i \in G(A \otimes H)$. Then we have

$$\sum_i (a_i \otimes f_{i(1)} \otimes f_{i(2)}) = \sum_{i,j} (a_i a_{j[0]} \otimes (f_i * a_{j[1]}) \otimes f_j) \tag{7}$$

and $\sum a_i\varepsilon(f_i) = 1$. The map

$$\alpha:\ A\otimes H \to \mathrm{Hom}(kG, A),\ \ \alpha(a\otimes f)(g) = af(g)$$

is injective. Let $\phi = \alpha(x)$. Using (7), we compute for all $g, g' \in G$ that

$$\begin{aligned}
\phi(gg') &= \sum_i a_i f_i(gg') = \sum_i a_i f_{i(1)}(g) f_{i(2)}(g') \\
&= \sum_{i,j} (a_i a_{j[0]}((f_i * a_{j[1]})(g)) f_j(g') = \sum_{i,j} a_i a_{j[0]} f_i(g) a_{j[1]}(g) f_j(g') \\
&= \sum_{i,j} (g.a_j) f_j(g') a_i f_i(g) = \sum_{i,j} g.(a_j f_j(g')) a_i f_i(g) \\
&= (g.(\phi(g')))\phi(g).
\end{aligned}$$

From the second equality, we have $1 = \sum_i a_i f_i(1_G) = \phi(1_G)$. For every $g \in G$, $\phi(g)$ is invertible in A, with inverse $g.(\phi(g^{-1}))$. By the proof of [13, Prop. 1b, p. 46], $\phi(g) \in k$ for every $g \in G$, so $\phi \in \chi(G)$. Now $\chi(G) = G(H) \subset H$ (see [12, p. 25]), so it follows in particular that $\phi \in H$. For all $g \in G$ we now have that

$$\alpha(1\otimes\phi)(g) = \phi(g) = \sum_i a_i f_i(g) = \alpha(x)(g),$$

hence $x = 1\otimes\phi$, by the injectivity of α. □

Now consider the situation from Remark 2.2: $H = k\mathbb{Z} \cong k[X, X^{-1}]$, and A is a commutative $\mathbb{Z}$-graded algebra. In this situation $A\otimes H = A\otimes k[X, X^{-1}]$. Grouplike elements in $A\otimes H$ can be constructed as follows. Let $1 = e_1 + \cdots + e_n$ with the e_i orthogonal idempotents, and take $d_1, \cdots, d_n \in \mathbb{Z}$. Then $\sum_{i=1}^n e_i\otimes X^{d_i}$ is a grouplike element in $A\otimes k[X, X^{-1}]$. In this way, we have an embedding of $\mathcal{C}(\mathrm{Spec}\,(A), \mathbb{Z})$, the continuous functions from $\mathrm{Spec}\,(A)$ (with the Zariski topology) to $\mathbb{Z}$ (with the discrete topology), into $G(A\otimes k[X, X^{-1}])$. The first author was amazed to see that one of his first results, [5, Theorem 2.3], can be restated in such a way that it becomes a result about corings. Recall that a commutative ring is called reduced if it has no nontrivial nilpotents.

Proposition 4.2. *Let A be a reduced $\mathbb{Z}$-graded commutative k-algebra. Then the map $\mathcal{C}(\mathrm{Spec}\,(A), \mathbb{Z}) \to G(A\otimes k[X, X^{-1}])$ is a bijection.*

Example 4.3. (cf. [5, Example 2.6]) Proposition 4.2 does not hold if A contains nilpotent elements; this is related to the fact that there exist non-homogeneous units in this situation. Let $A = k[x]$, with $x^2 = 0$, and put a $\mathbb{Z}$-grading on A by taking $\deg(x) = 1$. Then $1 + ax \in \mathbb{G}_m(A)$, and $d(1+ax) = (1-ax)\otimes 1 + ax\otimes X$ is a grouplike element in $G(A\otimes k[X, X^{-1}])$ which is not in the image of $\mathcal{C}(\mathrm{Spec}\,(A), \mathbb{Z})$.

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