

# MORITA THEORY FOR CORINGS AND CLEFT ENTWINING STRUCTURES

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ABSTRACT. Using the theory of corings, we generalize and unify Morita contexts introduced by Chase and Sweedler [8], Doi [12], and Cohen, Fischman and Montgomery [11]. We discuss when the contexts are strict. We apply our theory to corings arising from entwining structures, and this leads us to the notion of cleft entwining structure.

## INTRODUCTION

Let  $H$  be a Hopf algebra,  $A$  an  $H$ -comodule algebra, and  $B$  the subring of coinvariants. Generalizing a construction due to Chase and Sweedler [8], Doi [12] gave a Morita context, connecting  $B$  and  $\#(H, A)$ , and applied this to the theory of Hopf Galois extensions. In particular, he introduces the notion of cleft  $H$ -comodule algebra, and shows that a cleft  $H$ -comodule algebra is an  $H$ -Galois extension.

A similar Morita context has been constructed by Cohen, Fischman and Montgomery in [11]. They start from a finite dimensional Hopf algebra  $H$  over a field (or a Frobenius Hopf algebra over a commutative ring, see [10]), an  $H$ -module algebra, and give a Morita context connecting the smash product  $A\#H$  and the ring of invariants.

For a finite dimensional Hopf algebra  $H$ , a left  $H$ -module algebra is the same as a right  $H^*$ -comodule algebra, so it seems obvious that both contexts then coincide. That this is the case has been pointed out by Beattie, Dăscălescu and Raianu [3]. However, it is not just a straightforward application of duality principles, since the connecting bimodules are different in both cases, and since the Cohen-Fischman-Montgomery structure relies heavily on the fact that a finite dimensional Hopf algebra is Frobenius (the actions on the connecting bimodules are defined using the distinguished grouplike).

In this paper, we will generalize both contexts. The advantages of our approach are the following: first, all computations become straightforward and elementary; secondly, the duality relation between the two contexts and the

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connecting bimodules becomes clear, and the rôle of Frobenius type arguments is made clear; in third place, our theory can be applied in some other particular situations, for examples to generalized smash products, and to categories of entwined modules; finally, in the infinite dimensional case, it is clarified why Doi's Morita context is never strict.

Our approach is based on a key observation made by Takeuchi [17], that entwined modules, and, in particular, many kinds of modules such as relative Hopf modules, Yetter-Drinfeld modules, Doi-Hopf modules etc, can be viewed as comodules over a certain coring. Takeuchi's observation has led to a revived interest in the theory of corings, which goes back to Sweedler [16]. It became clear that corings provide a unifying and simplifying framework to various topics, such as Galois theory, descent theory, Frobenius functors and Maschke type Theorems (see [5], [13], [18]). Following this philosophy, we can generalize Doi's results, and associate a Morita context to a coring  $\mathcal{C}$  with a fixed grouplike element  $x$  over a ring  $A$  (Section 3). In Section 2, we will show that there is a dual result, which is even more elementary: to a morphism of rings  $i : A \rightarrow R$ , and a right  $R$ -linear map  $\chi : R \rightarrow A$  with  $\chi(\chi(r)s) = \chi(rs)$  for all  $r, s \in R$ , and  $\chi(1_R) = 1_A$ , we can associate a Morita context, which can in fact be viewed as the Morita context associated to the right  $R$ -module  $A$ , following [2]. This Morita context is a generalization of the Cohen-Fischman-Montgomery context; if  $R/A$  is Frobenius, then the second connecting bimodule in the context is isomorphic to  $A$  (see Theorem 2.7). We can give necessary and sufficient conditions for this Morita context to be strict.

To a coring with a fixed grouplike element, we can now associate two Morita contexts: one to the coring, as mentioned above, and another one to the dual of the coring, which is a ring. There exists a morphism between the two contexts, and we have some sufficient conditions for the two contexts being isomorphic: this is the case when the coring is finitely generated and projective as an  $A$ -module, and also when one of the connecting maps in the Morita context coming from the coring is surjective, cf. Corollary 3.4.

In Section 4, we focus attention to the case where the coring  $\mathcal{C}$  arises from an entwining structure  $(A, C, \psi)$ . We introduce the notion of cleft entwining structure, and show that cleftness is equivalent to  $\mathcal{C}$  being Galois in the sense of [18], and  $A$  being isomorphic to  $A^{\text{co}\mathcal{C}} \otimes C$  as a left  $A^{\text{co}\mathcal{C}}$ -module and a right  $C$ -comodule. The results use the Morita contexts of the previous Sections. Surprisingly, we were not able to generalize the notion of cleftness to arbitrary corings with a fixed grouplike element. In Sections Section 5 and Section 6, we look at factorization structures and the smash product, and introduce the notion of cleft factorization structure.

For a coring that is projective, but not necessarily finitely generated, as an  $A$ -module, we expect that there is a third Morita context, connecting the coinvariants and the rational dual of the coring, generalizing one of the Morita contexts discussed in [3]. This will be discussed in a forthcoming paper.

At the time when the first version of this paper was finished, we found out that some of our results were discovered independently by Abuhlail, see the forthcoming [1].

## 1. PRELIMINARIES

*Corings and comodules.* Let  $A$  be a ring. The category  ${}_A\mathcal{M}_A$  of  $(A, A)$ -bimodules is a monoidal category, and an  $A$ -coring  $\mathcal{C}$  is a coalgebra in  ${}_A\mathcal{M}_A$ , that is an  $(A, A)$ -bimodule together with two  $(A, A)$ -bimodule maps

$$\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C} \text{ and } \varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A$$

satisfying the usual coassociativity and counit properties, namely  $(\Delta_{\mathcal{C}} \otimes_A I_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = (I_{\mathcal{C}} \otimes_A \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}$  and  $(\varepsilon_{\mathcal{C}} \otimes_A I_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = (I_{\mathcal{C}} \otimes_A \varepsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = I_{\mathcal{C}}$ . Corings were introduced by Sweedler, see [16]; a detailed treatment of recent applications of the theory of corings will appear in [6]. A right  $\mathcal{C}$ -comodule is a right  $A$ -module  $M$  together with a right  $A$ -module map  $\rho^r : M \rightarrow M \otimes_A \mathcal{C}$  such that  $(\rho^r \otimes_A I_{\mathcal{C}}) \circ \rho^r = (I_M \otimes_A \Delta_{\mathcal{C}}) \circ \rho^r$  and  $(I_M \otimes_A \varepsilon_{\mathcal{C}}) \circ \rho^r = I_M$ . In a similar way, we can define left  $\mathcal{C}$ -comodules and  $(\mathcal{C}, \mathcal{C})$ -bicomodules. We will use the Sweedler-Heyneman notation for corings and comodules over corings:

$$\Delta_{\mathcal{C}}(c) = c_{(1)} \otimes_A c_{(2)} ; \rho^r(m) = m_{[0]} \otimes_A m_{[1]}$$

etc. A map  $f : M \rightarrow N$  between (right)  $\mathcal{C}$ -comodules is called a  $\mathcal{C}$ -comodule map if  $f$  is a right  $A$ -linear, and  $\rho^r(f(m)) = f(m_{[0]}) \otimes_A m_{[1]}$  for all  $m \in M$ . The category of right  $\mathcal{C}$ -comodules and  $\mathcal{C}$ -comodule maps will be denoted by  $\mathcal{M}^{\mathcal{C}}$ . In a similar way, we introduce the categories  ${}^{\mathcal{C}}\mathcal{M}$ ,  ${}^{\mathcal{C}}\mathcal{M}^{\mathcal{C}}$ ,  ${}_A\mathcal{M}^{\mathcal{C}}$ . For example,  ${}_A\mathcal{M}^{\mathcal{C}}$  is the category of right  $\mathcal{C}$ -comodules that are also  $(A, A)$ -bimodules such that the right  $\mathcal{C}$ -comodule map is left  $A$ -linear. If  $\mathcal{C}$  is an  $A$ -coring, then its left dual  ${}^*\mathcal{C} = {}_A\text{Hom}(\mathcal{C}, A)$  is a ring, with (associative) multiplication given by the formula

$$(1) \quad f \# g = g \circ (I_{\mathcal{C}} \otimes_A f) \circ \Delta_{\mathcal{C}} \text{ or } (f \# g)(c) = g(c_{(1)})f(c_{(2)})$$

for all left  $A$ -linear  $f, g : \mathcal{C} \rightarrow A$  and  $c \in \mathcal{C}$ . The unit is  $\varepsilon_{\mathcal{C}}$ . We have a ring homomorphism  $i : A \rightarrow {}^*\mathcal{C}$ ,  $i(a)(c) = \varepsilon_{\mathcal{C}}(c)a$ . We easily compute that

$$(2) \quad (i(a) \# f)(c) = f(ca) \text{ and } (f \# i(a))(c) = f(c)a,$$

for all  $f \in {}^*\mathcal{C}$ ,  $a \in A$  and  $c \in \mathcal{C}$ . We have a functor  $F : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_{{}^*\mathcal{C}}$ , where  $F(M) = M$  as a right  $A$ -module, with right  ${}^*\mathcal{C}$ -action given by  $m \cdot f = m_{[0]}f(m_{[1]})$ , for all  $m \in M$ ,  $f \in {}^*\mathcal{C}$ . If  $\mathcal{C}$  is finitely generated and projective as a left  $A$ -module, then  $F$  is an isomorphism of categories: given a right  ${}^*\mathcal{C}$ -action on  $M$ , we recover the right  $\mathcal{C}$ -coaction by putting  $\rho(m) = \sum_j (m \cdot f_j) \otimes_A c_j$ , where  $\{(c_j, f_j) \mid j = 1, \dots, n\}$  is a finite dual basis of  $\mathcal{C}$  as a left  $A$ -module.  ${}^*\mathcal{C}$  is a right  $A$ -module, by (2):  $(f \cdot a)(c) = f(c)a$ , and we can consider the double dual  $({}^*\mathcal{C})^* = \text{Hom}_A({}^*\mathcal{C}, A)$ . We have a canonical morphism  $i : \mathcal{C} \rightarrow ({}^*\mathcal{C})^*$ ,  $i(c)(f) = f(c)$ , and we call  $\mathcal{C}$  reflexive (as a left  $A$ -module) if  $i$  is an isomorphism. If  $\mathcal{C}$  is finitely generated projective as a

left  $A$ -module, then  $\mathcal{C}$  is reflexive. For any  $\varphi \in (*\mathcal{C})^*$ , we then have that  $\varphi = i(\sum_j \varphi(f_j)c_j)$ .

*Galois corings and Descent Theory.* Let  $\mathcal{C}$  be an  $A$ -coring. Recall that  $x \in \mathcal{C}$  is called grouplike if  $\Delta_{\mathcal{C}}(x) = x \otimes_A x$  and  $\varepsilon_{\mathcal{C}}(x) = 1$ .  $G(\mathcal{C})$  is the set of all grouplike elements in  $\mathcal{C}$ . We have the following interpretations of  $G(\mathcal{C})$  (see e.g. [7, Sec 4.8], [5]).

$$\begin{aligned} G(\mathcal{C}) &\cong \{ \rho^r : A \rightarrow A \otimes_A \mathcal{C} \cong \mathcal{C} \mid \rho^r \text{ makes } A \text{ into a right } \mathcal{C}\text{-comodule} \} \\ &\cong \{ \rho^l : A \rightarrow \mathcal{C} \otimes_A A \cong \mathcal{C} \mid \rho^l \text{ makes } A \text{ into a left } \mathcal{C}\text{-comodule} \}. \end{aligned}$$

Fix a grouplike element  $x$  in  $\mathcal{C}$ . We will call  $(\mathcal{C}, x)$  a coring with fixed grouplike element. The associated coactions on  $A$  are given by

$$\rho^r(a) = xa ; \rho^l(a) = ax.$$

For a right  $\mathcal{C}$ -comodule  $M$ , we define the submodule of coinvariants

$$M^{\text{co}\mathcal{C}} = \{ m \in M \mid \rho(m) = m \otimes_A x \}.$$

Now let  $B \subset A^{\text{co}\mathcal{C}} = {}^{\text{co}\mathcal{C}}A = \{ b \in A \mid bx = xb \}$ . We have a pair of adjoint functors  $(F, G)$  between the categories  $\mathcal{M}_B$  and  $\mathcal{M}^{\mathcal{C}}$ , namely, for  $N \in \mathcal{M}_B$  and  $M \in \mathcal{M}^{\mathcal{C}}$ ,

$$F(N) = N \otimes_B A \text{ and } G(M) = M^{\text{co}\mathcal{C}}.$$

The unit and counit of the adjunction are

$$\begin{aligned} \eta_N : N &\rightarrow (N \otimes_B A)^{\text{co}\mathcal{C}}, \quad \eta_N(n) = n \otimes_B 1; \\ \zeta_M : M^{\text{co}\mathcal{C}} \otimes_B A &\rightarrow M, \quad \zeta_M(m \otimes_B a) = ma. \end{aligned}$$

We say that  $(\mathcal{C}, x)$  satisfies the *Weak Structure Theorem* if  $\zeta_M$  is an isomorphism for all  $M \in \mathcal{M}^{\mathcal{C}}$ , that is,  $G = \bullet^{\text{co}\mathcal{C}}$  is a fully faithful functor.  $(\mathcal{C}, x)$  satisfies the *Strong Structure Theorem* if, in addition, all  $\eta_N$  are isomorphisms, or  $F$  is fully faithful, and therefore  $(F, G)$  is an equivalence between categories. Notice that the Strong Structure Theorem implies that  $B = A^{\text{co}\mathcal{C}}$ .

Let  $i : B \rightarrow A$  be a ring homomorphism. It can be verified easily that  $\mathcal{D} = A \otimes_B A$ , with structure maps

$\Delta_{\mathcal{D}} : A \otimes_B A \rightarrow (A \otimes_B A) \otimes_A (A \otimes_B A) \cong A \otimes_B A \otimes_B A$  and  $\varepsilon_{\mathcal{D}} : A \otimes_B A \rightarrow A$  given by

$$\Delta_{\mathcal{D}}(a \otimes_B b) = (a \otimes_B 1) \otimes_A (1 \otimes_B b) = a \otimes_B 1 \otimes_B b, \quad \varepsilon_{\mathcal{D}}(a \otimes_B b) = ab$$

is an  $A$ -coring. The element  $1 \otimes_B 1$  is grouplike, and  $(\mathcal{D}, 1 \otimes_B 1)$  is called the *canonical coring* associated to the ring morphism  $i$ . Observe that  $*\mathcal{D} = {}_A\text{Hom}(A \otimes_B A, A) \cong {}_B\text{End}(A)^{\text{op}}$ . If  $A$  is finitely generated projective as a left  $B$ -module, then  $\mathcal{D}$  is reflexive. A right  $\mathcal{D}$ -comodule consists of a right  $A$ -module  $M$  together with a right  $A$ -module map  $\rho_M : M \rightarrow M \otimes_A (A \otimes_B A) \cong M \otimes_B A$  such that

$$(3) \quad m_{[0][0]} \otimes_B m_{[0][1]} \otimes_B m_{[1]} = m_{[0]} \otimes_B 1 \otimes_B m_{[1]} \text{ and } m_{[0]} m_{[1]} = m$$

for all  $m \in M$ . If  $A$  is faithfully flat as a  $B$ -module, then  $(\mathcal{D}, 1 \otimes_B 1)$  satisfies the Strong Structure Theorem (see [9]); in [7, Sec. 4.8], a proof in the coring language is presented. In fact it is the basic result of descent theory: an  $A$ -module  $M$  is isomorphic to  $N \otimes_B A$  for some  $B$ -module  $N$  if and only if we can define a right  $\mathcal{D}$ -coaction on  $M$ . In the situation where  $A$  and  $B$  are commutative, there is an isomorphism between the category of comodules over the canonical coring, and the category of *descent data*, as introduced by Knus and Ojanguren in [14], we refer to [7, Sec. 4.8] for details. An unpublished result by Journal and Tierney states that, in the situation where  $A$  and  $B$  are commutative,  $(\mathcal{D}, 1 \otimes_B 1)$  satisfies the Strong Structure Theorem if and only if  $i : B \rightarrow A$  is pure as a morphism of  $B$ -modules. For a proof, we refer to [15].

Now we return to the general situation, and take an arbitrary coring  $(\mathcal{C}, x)$ , with fixed grouplike element. Let  $B = A^{\text{co}\mathcal{C}}$ , and consider the canonical coring  $\mathcal{D} = A \otimes_B A$ . We have a canonical coring morphism  $\text{can} : \mathcal{D} \rightarrow \mathcal{C}$ ,  $\text{can}(a \otimes_B b) = axb$ . We say that  $(\mathcal{C}, x)$  is a *Galois coring* if  $\text{can}$  is an isomorphism of corings. In this situation, we obviously have an isomorphism between the categories  $\mathcal{M}^{\text{co}\mathcal{C}}$  and  $\mathcal{M}^{\text{co}\mathcal{D}}$ .

**Proposition 1.1.** *Let  $(\mathcal{C}, x)$  be an  $A$ -coring with fixed grouplike element,  $B = A^{\text{co}\mathcal{C}}$ , and  $\mathcal{D} = A \otimes_B A$ . We then have a ring homomorphism*

$$*\text{can} : *\mathcal{C} \rightarrow *\mathcal{D} \cong {}_B\text{End}(A)^{\text{op}}; *\text{can}(f)(a) = f(xa).$$

- 1) *If  $(\mathcal{C}, x)$  is Galois, then  $*\text{can}$  is an isomorphism.*
- 2) *If  $*\text{can}$  is an isomorphism, and  $\mathcal{C}$  and  $\mathcal{D}$  are both reflexive (e.g.  $\mathcal{C}$  and  $A$  are finitely generated and projective, resp. as a left  $A$ -module and a left  $B$ -module), then  $(\mathcal{C}, x)$  is Galois.*
- 3) *If  $(\mathcal{C}, x)$  is Galois, and  $(\mathcal{D}, x)$  satisfies the Strong (resp. Weak) Structure Theorem (e.g.  $A$  is faithfully flat (resp. flat) as a right  $B$ -module), then  $(\mathcal{C}, x)$  also satisfies the Strong (resp. Weak) Structure Theorem.*
- 4) *If  $(\mathcal{C}, x)$  satisfies the Weak Structure Theorem, then  $(\mathcal{C}, x)$  is Galois.*

*Proof.* 1), 2), 3) follow immediately from the observations made above.  $\mathcal{C}$  is a right  $\mathcal{C}$ -comodule, using  $\Delta_{\mathcal{C}}$ , and we have a right  $B$ -module map  $i : A \rightarrow \mathcal{C}^{\text{co}\mathcal{C}}$ ,  $i(a) = ax$ . It is easily verified that the restriction of  $\varepsilon_{\mathcal{C}}$  to  $\mathcal{C}^{\text{co}\mathcal{C}}$  is an inverse for  $i$ , so  $A$  and  $\mathcal{C}^{\text{co}\mathcal{C}}$  are isomorphic in  $\mathcal{M}_B$ . Now  $\zeta_{\mathcal{C}} = \text{can} : A \otimes_B A \rightarrow \mathcal{C}$  is an isomorphism.  $\square$

*Entwined modules.* Let  $k$  be a commutative ring,  $A$  a  $k$ -algebra,  $C$  a  $k$ -coalgebra, and  $\psi : C \otimes A \rightarrow A \otimes C$  a  $k$ -linear map satisfying the following four conditions:

- (4)  $(1_A)_{\psi} \otimes c^{\psi} = 1_A \otimes c$ ;  $\varepsilon_C(c^{\psi})a_{\psi} = \varepsilon_C(c)a$ ;
- (5)  $(ab)_{\psi} \otimes c^{\psi} = a_{\psi}b_{\psi'} \otimes c^{\psi\psi'}$ ;  $a_{\psi} \otimes \Delta_C(c^{\psi}) = a_{\psi\psi'} \otimes c_{(1)}^{\psi'} \otimes c_{(2)}^{\psi}$ .

Here we used the notation (summation implicitly understood)

$$\psi(c \otimes a) = a_{\psi} \otimes c^{\psi} = a_{\psi'} \otimes c^{\psi'}.$$

We then call  $(A, C, \psi)$  a (right-right) entwining structure. To an entwining structure  $(A, C, \psi)$ , we can associate an  $A$ -coring  $\mathcal{C} = A \otimes C$ . The structure maps are given by the formulas

$$a'(b \otimes c)a = a'ba_\psi \otimes c^\psi;$$

$$\Delta_{\mathcal{C}}(a \otimes c) = (a \otimes c_{(1)}) \otimes_A (1 \otimes c_{(2)}); \quad \varepsilon_{\mathcal{C}}(a \otimes c) = a\varepsilon_C(c)$$

An entwined module  $M$  is a  $k$ -module together with a right  $A$ -action and a right  $C$ -coaction, in such a way that  $\rho^r(ma) = m_{[0]}a_\psi \otimes m_{[1]}^\psi$ , for all  $m \in M$  and  $a \in A$ . The category  $\mathcal{M}(\psi)_A^C$  of entwined modules and  $A$ -linear  $C$ -colinear maps is isomorphic to the category of right  $\mathcal{C}$ -comodules.

*Factorization structures and the smash product.* Let  $A$  and  $S$  be  $k$ -algebras, and  $\rho: S \otimes A \rightarrow A \otimes S$  a  $k$ -linear map. We will write

$$\rho(s \otimes a) = a_\rho \otimes s_\rho = a_r \otimes s_r$$

(summation understood).  $A \#_\rho S$  will be the  $k$ -module  $A \otimes S$ , with multiplication

$$(6) \quad (a \# s)(b \# t) = ab_\rho \# s_\rho t.$$

It is straightforward to verify that this multiplication is associative with unit  $1_A \# 1_S$  if and only if

$$(7) \quad \rho(s \otimes 1_A) = 1_A \otimes s; \quad \rho(1_S \otimes a) = a \otimes 1_S;$$

$$(8) \quad \rho(st \otimes a) = a_{\rho r} \otimes s_r t_\rho; \quad \rho(s \otimes ab) = a_\rho b_r \otimes s_{\rho r}.$$

for all  $a, b \in A$  and  $s, t \in S$ . We then call  $(A, S, \rho)$  a factorization structure, and  $A \#_\rho S$  the smash product of  $A$  and  $S$ .

## 2. THE GENERAL MORITA CONTEXT

Let  $A$  and  $R$  be rings, and  $i: A \rightarrow R$  a ring morphism. We also consider a map  $\chi: R \rightarrow A$  satisfying the following three conditions, for all  $r, s \in R$ :

- (1)  $\chi$  is right  $A$ -linear;
- (2)  $\chi(\chi(r)s) = \chi(rs)$ ;
- (3)  $\chi(1_R) = 1_A$ .

It follows from the second condition that  $\chi^2 = \chi$ .  $A$  is a right  $R$ -module, with structure  $a \leftarrow r = \chi(ar)$ . The three conditions on the map  $\chi$  can be explained as follows:  $R$  is an algebra in the monoidal category  ${}_A\mathcal{M}_A$  of  $A$ -bimodules. A map  $\chi: A \otimes_A R = R \rightarrow A$  makes  $A$  into a right module over this algebra  $R$  if and only if it satisfies these three conditions. This is the dual result of the fact that grouplike elements of an  $A$ -coring  $\mathcal{C}$  are in one-to-one correspondence with right (or left)  $\mathcal{C}$ -comodule structures on  $A$ . For any right  $R$ -module  $M$ , we define

$$M^R = \{m \in M \mid m \cdot r = m\chi(r)\} \cong \text{Hom}_R(A, M).$$

Then  $B = A^R = \{b \in A \mid b\chi(r) = \chi(br), \text{ for all } r \in R\}$  is a subring of  $A$ , and  $M^R$  is a right  $B$ -module. In fact we obtain a functor  $G = (\bullet)^R$ :

$\mathcal{M}_R \rightarrow \mathcal{M}_B$ , which is a right adjoint of  $F = \bullet \otimes_B A$ . The unit and counit of the adjunction are  $\eta_N : N \rightarrow (N \otimes_B A)^R$ ,  $\eta_N(n) = n \otimes_B 1$  and  $\zeta_M : M^R \otimes_B A \rightarrow M$ ,  $\zeta_M(m \otimes_B a) = ma$ . Now consider

$$(9) \quad Q = R^R = \{q \in R \mid qr = q\chi(r), \text{ for all } r \in R\}$$

and the map  $\zeta_R = \mu : Q \otimes_B A \rightarrow R$ ,  $\mu(q \otimes_B a) = qa$ .  $\chi(Q) \subset B$  since  $\chi(q)\chi(r) = \chi(q\chi(r)) = \chi(qr) = \chi(\chi(q)r)$  for all  $q \in Q$  and  $r \in R$ . Also recall that  $R^* = \text{Hom}_A(R, A)$  is an  $(A, R)$ -bimodule:

$$(a \cdot f \cdot r)(s) = af(rs)$$

for all  $a \in A$ ,  $r, s \in S$  and  $f \in R^*$ .

**Lemma 2.1.**  $(R^*)^R \cong A$  as a right  $B$ -module, and the counit map  $\text{can} = \zeta_{R^*} : A \otimes_B A \rightarrow R^*$  is given by

$$\text{can}(a \otimes_B a')(r) = a\chi(a'r)$$

for all  $a, a' \in A$  and  $r \in R$ .

*Proof.* First observe that  $f \in (R^*)^R$  if and only if  $f(rs) = f(\chi(r)s)$  for all  $r, s \in R$ . Define

$$j : A \rightarrow (R^*)^R, j(a)(r) = a\chi(r) \text{ and } p : (R^*)^R \rightarrow A : p(f) = f(1).$$

It is clear that  $j(a)$  is right  $A$ -linear. Also

$$j(a)(rs) = a\chi(rs) = a\chi(\chi(r)s) = j(a)(\chi(r)s)$$

so  $j(a) \in (R^*)^R$ .  $j$  and  $p$  are inverses, since  $p(j(a)) = j(a)(1) = a\chi(1) = a$  and  $j(p(f))(r) = f(1)\chi(r) = f(\chi(r)) = f(r)$ . Now we compute

$$(j(a) \cdot a')(r) = j(a)(a'r) = a\chi(a'r).$$

From this formula, it follows that, for  $b \in B$ ,

$$(j(a) \cdot b)(r) = a\chi(br) = ab\chi(r) = j(ab)(r)$$

so  $j$  is right  $B$ -linear.  $\square$

The proof of the following result is now an easy exercise, left to the reader.

**Proposition 2.2.** *With notation as above,  $A \in {}_B\mathcal{M}_R$  and  $Q \in {}_R\mathcal{M}_B$ , and we have a Morita context  $(B, R, A, Q, \tau, \mu)$ . The connecting maps  $\mu = \zeta_R : Q \otimes_B A \rightarrow R$  and  $\tau : A \otimes_R Q \rightarrow B$  are given by*

$$(10) \quad \mu(q \otimes_B a) = qa \text{ and } \tau(a \otimes_R q) = a \leftarrow q = \chi(aq).$$

*Remark 2.3.* Let  $R$  be a ring. Recall from [2, II.4] that we can associate a Morita context to any right  $R$ -module  $P$ . If we consider  $i : A \rightarrow R$  and  $\chi : R \rightarrow A$  as above, then the Morita context associated to the right  $R$ -module  $A$  is isomorphic to the Morita context from Proposition 2.2. It suffices to observe that

$$B \cong \text{End}_R(A) \text{ and } Q \cong \text{Hom}_R(A, R).$$

It is easy to establish when the Morita context is strict. First let us investigate when  $\tau$  is surjective.

**Proposition 2.4.** *With notation as in Proposition 2.2, the following assertions are equivalent:*

- 1)  $\tau$  is surjective (and, a fortiori, injective);
- 2) there exists  $\Lambda \in Q$  such that  $\chi(\Lambda) = 1$ ;
- 3) for all  $M \in \mathcal{M}_R$ , the map  $\omega_M : M \otimes_R Q \rightarrow M^R$ ,  $\omega_M(m \otimes_R q) = m \cdot q$  is an isomorphism; 4)  $A$  is finitely generated and projective as a right  $R$ -module.

*Proof.* 1)  $\Rightarrow$  2). If  $\tau$  is surjective, then there exist  $a_j \in A$  and  $q_j \in Q$  such that  $\tau(\sum_j a_j \otimes_R q_j) = \chi(\sum_j a_j q_j) = 1$ .  $\Lambda = \sum_j a_j q_j \in Q$ , since  $Q$  is a left ideal in  $R$ .

2)  $\Rightarrow$  3). For all  $m \in M$  and  $q \in Q$ , we have that  $(m \cdot q) \cdot r = m \cdot (qr) = m \cdot q\chi(r)$ , for all  $r \in R$ , so  $m \cdot q \in M^R$ . Consider the map

$$\theta_M : M^R \rightarrow M \otimes_R Q, \theta_M(m) = m \otimes_R \Lambda.$$

For all  $m \in M^R$ , we easily compute that  $\omega_M(\theta_M(m)) = m \cdot \Lambda = m\chi(\Lambda) = m$  for all  $m \in M^R$ , and, for all  $m \in M$  and  $q \in Q$ ,  $\theta_M(\omega_M(m \otimes_R q)) = m \cdot q \otimes_R \Lambda = m \otimes_R q\Lambda = m \otimes_R q\chi(\Lambda) = m \otimes_R q$ , so  $\theta_M$  and  $\omega_M$  are inverses.

3)  $\Rightarrow$  1). Observe that  $\omega_A = \tau$ .

1)  $\Leftrightarrow$  4) follows from [2, Prop. II.4.4], taking Remark 2.3 into account; we also give an easy direct proof.

2)  $\Rightarrow$  4) Let  $f \in \text{Hom}_R(A, R)$  be given by  $f(a) = \Lambda a$ . Then for all  $a \in A$ , we have  $a = \chi(\Lambda)a = \chi(\Lambda a) = 1 \leftarrow f(a)$ , hence  $\{(1, f)\}$  is a finite dual basis of  $A$  as a right  $A$ -module.

4)  $\Rightarrow$  1) Let  $\{(a_j, f_j) \mid j = 1, \dots, n\}$  be a finite dual basis of  $A$  as a right  $R$ -module, and take  $f_j(1) = q_j \in Q$ . Then  $1 = \sum_j a_j \leftarrow f_j(1) = \chi(\sum_j a_j q_j) = \tau(\sum_j a_j \otimes_R q_j)$ , hence  $\tau$  is surjective.  $\square$

If  $\tau$  is surjective, then we can apply [2, Theorem II.3.4]. It then also follows that  $F = \bullet \otimes_B A : \mathcal{M}_B \rightarrow \mathcal{M}_R$  is fully faithful (see the proof of [2, Theorem II.3.5]). Moreover, we have the following:

**Proposition 2.5.** *Consider the Morita context from Proposition 2.2, and assume that  $\tau$  is surjective. Take  $\Lambda \in Q$  such that  $\chi(\Lambda) = 1$ . Then we have the following properties:*

- 1)  $\Lambda^2 = \Lambda$  and  $\Lambda R \Lambda = \Lambda B \cong B$ .
- 2)  $B$  is a direct summand of  $A$  as a left  $B$ -module.

*Proof.* 1) Since  $\chi(\Lambda) = 1$  and  $\Lambda \in Q$ , we have  $\Lambda^2 = \Lambda\chi(\Lambda) = \Lambda$ . For all  $r, s \in R$ , we have  $\chi(r\Lambda)\chi(s) = \chi(r\Lambda\chi(s)) = \chi(r\Lambda s) = \chi(\chi(r\Lambda)s)$ , so  $\chi(r\Lambda) \in B$ , and  $\Lambda r \Lambda = \Lambda\chi(r\Lambda) \in \Lambda B$ . It follows that  $\Lambda R \Lambda \subset \Lambda B$ .

Now in the above arguments, take  $r = i(b)$ , with  $b \in B$ . It follows that  $\Lambda b \Lambda = \Lambda\chi(b\Lambda) = \Lambda b\chi(\Lambda) = \Lambda b$ , and  $\Lambda B \subset \Lambda R \Lambda$ . Finally, the right  $B$ -module generated by  $\Lambda$  is free since  $\Lambda b = 0$  implies that  $0 = \chi(\Lambda b) = \chi(\Lambda)b = b$ .

2) We define the map  $\text{Tr} : A \rightarrow B$ ,  $\text{Tr}(a) = \tau(a \otimes_R \Lambda) = \chi(a\Lambda)$ .  $\text{Tr}$  is left  $B$ -linear, because  $\tau$  is left  $B$ -linear.  $\text{Tr}$  is a projection, since  $\text{Tr}(b) = \chi(b\Lambda) = b\chi(\Lambda) = b$ , for all  $b \in B$ .  $\square$

We recall from Morita Theory ([2, II.3.4]) that we have ring morphisms

$$\pi : R \rightarrow {}_B\text{End}(A)^{\text{op}}, \quad \pi(r)(a) = a \leftarrow r = \chi(ar);$$

$$\pi' : R \rightarrow \text{End}_B(Q), \quad \pi'(r)(q) = rq.$$

We also have an  $(R, B)$ -bimodule map

$$\kappa : Q \rightarrow {}_B\text{Hom}(A, B), \quad \kappa(q)(a) = \chi(aq)$$

and a  $(B, R)$ -bimodule map

$$\kappa' : A \rightarrow \text{Hom}_B(Q, B), \quad \kappa'(a)(q) = \chi(aq).$$

If  $\mu$  is surjective, then  $\pi$ ,  $\pi'$ ,  $\kappa$  and  $\kappa'$  are isomorphisms, and  $A$  and  $Q$  are finitely generated and projective as resp. a left and right  $B$ -module, and a generator as resp. a right and left  $R$ -module.

**Proposition 2.6.** *Consider the Morita context from Proposition 2.2. The following assertions are equivalent:*

- 1)  $\mu : Q \otimes_B A \rightarrow R$  is surjective;
- 2) the functor  $G = (\bullet)^R : \mathcal{M}_R \rightarrow \mathcal{M}_B$  is fully faithful, that is, for all  $M \in \mathcal{M}_R$ , the counit map  $\zeta_M : M^R \otimes_B A \rightarrow M$  is an isomorphism;
- 3)  $A$  is a right  $R$ -generator;
- 4)  $A$  is projective as a left  $B$ -module, and  $\pi$  is bijective;
- 5)  $A$  is projective as a left  $B$ -module,  $\pi$  is injective, and  $\kappa$  is surjective;
- 6)  $Q$  is projective as a right  $B$ -module,  $\pi'$  is injective, and  $\kappa'$  is surjective.

*Proof.* 1)  $\Rightarrow$  2). This follows from classical Morita theory. We mention that the inverse of  $\zeta_M$  is the map  $\theta_M : M \rightarrow M^R \otimes_B A$ ,  $\theta_M(m) = \sum_j m q_j \otimes a_j$ .

2)  $\Rightarrow$  1):  $\varepsilon_R = \mu$ .

1)  $\Rightarrow$  3, 4, 5, 6): Morita theory (see above).

1)  $\Leftrightarrow$  3) follows from [2, Prop. II.4.4], taking Remark 2.3 into account.

4)  $\Rightarrow$  1): Let  $\{(a_j, p_j)\}$  be a (not necessarily finite) dual basis of  $A$  as a left  $B$ -module, and put  $q_j = \pi^{-1}(p_j) \in R$ . Then  $\chi(aq_j) = \pi(q_j)(a) = p_j(a) \in B$  and  $\pi(q_j \chi(r))(a) = \chi(aq_j \chi(r)) = \chi(aq_j r) = \pi(q_j r)(a)$ .  $\pi$  is injective, so it follows that  $q_j \chi(r) = q_j r$ , and  $q_j \in Q = R^R$ . Now  $\mu(\sum_j q_j \otimes_B a_j) = \sum_j q_j a_j = 1$  since  $\pi$  is injective and

$$\pi\left(\sum_j q_j a_j\right)(a) = \sum_j \chi(aq_j a_j) = \sum_j \chi(aq_j) a_j = \sum_j p_j(a) a_j = a.$$

It follows that  $\mu$  is surjective.

5)  $\Rightarrow$  1): Let  $\{(a_j, p_j)\}$  be a dual basis of  $A$  as a left  $B$ -module, and take  $q_j \in Q$  such that  $\kappa(q_j) = p_j$ . Then proceed as in 4)  $\Rightarrow$  1).

6)  $\Rightarrow$  1): Let  $\{(q_j, p_j)\}$  be a dual basis of  $Q$  as a right  $B$ -module, and take

$a_j \in A$  such that  $\kappa'(a_j) = p_j$ . then we have for all  $q \in Q$  that  $p_j(q) = \kappa'(a_j)(q) = \chi(a_j q)$  and

$$q = \sum_j q_j p_j(q) = \sum_j q_j \chi(a_j q) = \sum_j q_j a_j q$$

so  $\pi'(\sum_j q_j p_j) = \pi'(1_R)$ , and, since  $\pi'$  is injective,  $\mu(\sum_j q_j \otimes_B a_j) = \sum_j q_j a_j = 1$ . Therefore  $\mu$  is surjective.  $\square$

Recall that the ring extension  $R/A$  is called *Frobenius* if there exists an  $A$ -bimodule map  $\bar{\nu} : R \rightarrow A$  and  $e = e^1 \otimes_A e^2 \in R \otimes_A R$  (summation implicitly understood) such that

$$(11) \quad r e^1 \otimes_A e^2 = e^1 \otimes_A e^2 r$$

for all  $r \in R$ , and

$$(12) \quad \bar{\nu}(e^1) e^2 = e^1 \bar{\nu}(e^2) = 1.$$

This is equivalent to the restrictions of scalars  $\mathcal{M}_R \rightarrow \mathcal{M}_A$  being a Frobenius functor, which means that its left and right adjoints are isomorphic (see [7, Sec. 3.1 and 3.2]).  $(e, \bar{\nu})$  is then called a Frobenius system.

**Theorem 2.7.** *Let  $i : A \rightarrow R$  be a morphism of rings, and  $\chi : R \rightarrow A$  a map satisfying the conditions stated at the beginning of this Section. If  $R/A$  is Frobenius, with Frobenius system  $(e, \bar{\nu})$ , then  $A$  is an  $(R, B)$ -bimodule, with left  $R$ -action*

$$r \cdot a = \bar{\nu}(r a \chi(e^1) e^2)$$

Then  $A \cong Q$  as  $(R, B)$ -bimodules, and we have a Morita context

$$(B, R, A, A, \tau, \mu)$$

with connecting maps

$$\mu : A \otimes_B A \rightarrow R : \mu(a \otimes_B a') = a \chi(e^1) e^2 a';$$

$$\tau : A \otimes_R A \rightarrow B : \tau(a \otimes_R a') = \chi(a a' \chi(e^1) e^2).$$

*Proof.* Define  $\alpha : A \rightarrow Q$  by  $\alpha(a) = a \chi(e^1) e^2$ .  $\alpha(a) \in Q$  since  $\alpha(a)r = a \chi(e^1) e^2 r = a \chi(r e^1) e^2 = a \chi(\chi(r) e^1) e^2 = a \chi(e^1) e^2 \chi(r) = \alpha(a) \chi(r)$ , for all  $r \in R$ . The restriction of  $\bar{\nu}$  to  $Q$  is the inverse of  $\alpha$ :

$$\bar{\nu}(\alpha(a)) = \bar{\nu}(a \chi(e^1) e^2) = a \chi(e^1) \bar{\nu}(e^2) = a \chi(e^1 \bar{\nu}(e^2)) = a \chi(1) = a$$

and

$$\alpha(\bar{\nu}(q)) = \bar{\nu}(q) \chi(e^1) e^2 = \bar{\nu}(q \chi(e^1)) e^2 = \bar{\nu}(q e^1) e^2 = \bar{\nu}(e^1) e^2 q = q$$

for all  $a \in A$  and  $q \in Q$ .  $\alpha$  is right  $B$ -linear, since

$$\alpha(ab) = ab \chi(e^1) e^2 = a \chi(b e^1) e^2 = a \chi(e^1) e^2 b = \alpha(a) b$$

for all  $a \in A$  and  $b \in B$ . It is easy to see that the left  $R$ -action on  $Q$  is transported into the required left  $R$ -action on  $A$ . The rest follows easily from Proposition 2.2.  $\square$

*Remark 2.8.* Another possible approach to Theorem 2.7 is the following: if  $R/A$  is Frobenius, then  $R^* = \text{Hom}_A(R, A)$  and  $R$  are isomorphic as  $(A, R)$ -bimodules (see [7, Theorem 28]). Consequently  $Q = R^R \cong (R^*)^R = A$  as right  $B$ -modules. The isomorphism transports the left  $R$ -action on  $Q$  into a left  $R$ -action on  $A$ .

### 3. A MORITA CONTEXT ASSOCIATED TO A CORING

In this Section,  $A$  is a ring and  $(\mathcal{C}, x)$  is an  $A$ -coring with a fixed grouplike element. Let  $R = {}^*\mathcal{C}$  and consider  $\chi : R \rightarrow A$ ,  $\chi(f) = f(x)$ . Using (2), we can easily compute that  $\chi$  is right  $A$ -linear,  $\chi(i(\chi(f))\#g) = \chi(f\#g)$ , and  $\chi(\varepsilon_{\mathcal{C}}) = 1$ . Any right  $\mathcal{C}$ -comodule  $M$  is also a right  ${}^*\mathcal{C}$ -module (see Section 1), and it is easy to prove that  $M^{\text{co}\mathcal{C}} \subset M^{{}^*\mathcal{C}}$ . If  $\mathcal{C}$  is finitely generated and projective as a left  $A$ -module, then the converse implication also holds, and the coinvariants coincide with the invariants. We put

$$\begin{aligned} B' &= A^{\text{co}\mathcal{C}} = \{b \in A \mid bx = xb\} \\ &\subset B = A^{{}^*\mathcal{C}} = \{a \in A \mid f(xa) = af(x), \text{ for all } f \in {}^*\mathcal{C}\}. \end{aligned}$$

${}_A\text{End}(\mathcal{C})$  is a left  ${}^*\mathcal{C}$ -module: for all  $f \in {}^*\mathcal{C}$  and  $\varphi \in {}_A\text{End}(\mathcal{C})$ , and  $c \in \mathcal{C}$ , we define

$$(f\#\varphi)(c) = \varphi(c_{(1)}f(c_{(2)})).$$

Now let

$$\begin{aligned} Q' &= {}^{\mathcal{C}}\text{Hom}(\mathcal{C}, A) = \{q \in {}^*\mathcal{C} \mid q\#I_{\mathcal{C}} = \rho^l \circ q\} \\ (13) \quad &= \{q \in {}^*\mathcal{C} \mid c_{(1)}q(c_{(2)}) = q(c)x, \text{ for all } c \in \mathcal{C}\}. \end{aligned}$$

Observe that  $Q' \subset Q = ({}^*\mathcal{C})^{{}^*\mathcal{C}}$  and  $Q' = Q$  if  $\mathcal{C}$  is finitely generated and projective as a left  $A$ -module.

Applying the results of the previous Section, we find a Morita context connecting  $B$  and  ${}^*\mathcal{C}$ . We will now show that there is another Morita context connecting  $B'$  and  ${}^*\mathcal{C}$ , and that there is a morphism between the two Morita contexts. We already know that  $A$  is a  $(B, {}^*\mathcal{C})$ -bimodule, and this implies that it is also a  $(B', {}^*\mathcal{C})$ -bimodule. We also have

**Lemma 3.1.**  *$Q'$  is a  $({}^*\mathcal{C}, B')$ -bimodule.*

*Proof.* For all  $f \in {}^*\mathcal{C}$ ,  $q \in Q'$  and  $c \in \mathcal{C}$ , we have

$$\begin{aligned} ((f\#q)\#I_{\mathcal{C}})(c) &= (f\#(q\#I_{\mathcal{C}}))(c) = (f\#(\rho^l \circ q))(c) \\ &= (\rho^l \circ q)(c_{(1)}f(c_{(2)})) = q(c_{(1)}f(c_{(2)}))x = (f\#q)(c)x. \end{aligned}$$

□

Now we define maps

$$(14) \quad \mu' : Q' \otimes_{B'} A \rightarrow {}^*\mathcal{C} ; \mu'(q \otimes_{B'} a) = q\#i(a);$$

$$(15) \quad \tau' : A \otimes_{{}^*\mathcal{C}} Q' \rightarrow B' ; \tau'(a \otimes_{{}^*\mathcal{C}} q) = a \cdot q = q(ax).$$

It is clear that  $\mu'$  is well-defined.  $\tau'$  is also well-defined: for all  $f \in {}^*\mathcal{C}$ ,  $a \in A$  and  $q \in Q'$ , we have

$$\tau'(a \otimes_{{}^*\mathcal{C}} (f \# q)) = a \cdot (f \# q) = (a \cdot f) \cdot q = \tau'(a \cdot f) \otimes_{{}^*\mathcal{C}} q$$

and for all  $q \in Q'$  and  $a \in A$ , we have that  $q(xa) \in B'$ , since  $xq(xa) = q(xa)x$ .

**Theorem 3.2.** *With notation as above,  $(B', {}^*\mathcal{C}, A, Q', \tau', \mu')$  is a Morita context, and we have a morphism of Morita contexts*

$$(B', {}^*\mathcal{C}, A, Q', \tau', \mu') \rightarrow (B, {}^*\mathcal{C}, A, Q, \tau, \mu).$$

*Proof.* We have to show that the following two diagrams are commutative.

$$\begin{array}{ccccc} A \otimes_{{}^*\mathcal{C}} Q' \otimes_{B'} A & \xrightarrow{\tau' \otimes I_A} & B' \otimes_{B'} A & Q' \otimes_{B'} A \otimes_{{}^*\mathcal{C}} Q' & \xrightarrow{\mu' \otimes I_{Q'}} & {}^*\mathcal{C} \otimes_{{}^*\mathcal{C}} Q' \\ \downarrow I_A \otimes \mu' & & \downarrow \cong & \downarrow I_{Q'} \otimes \tau' & & \downarrow \cong \\ A \otimes_{{}^*\mathcal{C}} {}^*\mathcal{C} & \xrightarrow{\cong} & A & Q' \otimes_{B'} B' & \xrightarrow{\cong} & Q' \end{array}$$

Take  $a, a' \in A$  and  $q, q' \in Q'$ . We compute

$$a \cdot \mu'(q \otimes a') = a \cdot (q \# i(a')) = (q \# i(a'))(xa) = q(xa)a' = \tau'(a \otimes q)a'$$

and this proves that the first diagram commutes. The second diagram commutes if  $(q \# i(a)) \# q' = q \# i(q'(xa))$ . Indeed, for all  $c \in \mathcal{C}$ ,

$$(q \# i(q'(xa)))(c) = q(c)q'(xa) = q'(q(c)xa) = q'(c_{(1)}q(c_{(2)})a) = ((q \# i(a)) \# q')(c).$$

The second statement is obvious: the morphism is given by the inclusion maps  $B' \subset B$ ,  $Q' \subset Q$ , and the identity maps on  $A$  and  ${}^*\mathcal{C}$ .  $\square$

We now give necessary and sufficient conditions for  $\tau'$  to be surjective. It will follow that our two Morita contexts coincide if  $\tau'$  is surjective.

**Theorem 3.3.** *Consider the Morita context  $(B', {}^*\mathcal{C}, A, Q', \tau', \mu')$  of Theorem 3.2. The following statements are equivalent:*

- 1)  $\tau'$  is surjective (and, a fortiori, bijective);
- 2) there exists  $\Lambda \in Q'$  such that  $\Lambda(x) = 1$ ;
- 3) for every right  ${}^*\mathcal{C}$ -module  $M$ , the map  $\omega_M : M \otimes_{{}^*\mathcal{C}} Q' \rightarrow M^{*\mathcal{C}}$ ,  $\omega_M(m \otimes_{{}^*\mathcal{C}} q) = m \cdot q$ , is bijective.

*Proof.* 1)  $\Rightarrow$  2). If  $\tau'$  is surjective, then there exist  $a_j \in A$  and  $q_j \in Q'$  such that  $1 = \tau'(\sum_j a_j \otimes_{{}^*\mathcal{C}} q_j) = \sum_j q_j(xa_j) = \sum_j (i(a_j) \# q_j)(x)$ . Now  $\Lambda = \sum_i (a_i) \# q_i \in Q'$  because  $Q'$  is a left ideal in  ${}^*\mathcal{C}$ .

2)  $\Rightarrow$  3). Consider  $\eta_M : M^{*\mathcal{C}} \rightarrow M \otimes_{{}^*\mathcal{C}} Q'$ ,  $\eta_M(m) = m \otimes_{{}^*\mathcal{C}} \Lambda$ . It is clear that  $\omega_M \circ \eta_M = I_{M^{*\mathcal{C}}}$ . Furthermore, for all  $m \in M$  and  $q \in Q'$ ,

$$\eta_M(\omega_M(m \otimes_{{}^*\mathcal{C}} q)) = (m \cdot q) \otimes_{{}^*\mathcal{C}} \Lambda = m \otimes_{{}^*\mathcal{C}} q \# \Lambda = m \otimes_{{}^*\mathcal{C}} q,$$

since  $q \# \Lambda = q\Lambda(x) = q$ .

3)  $\Rightarrow$  1).  $\omega_A = \tau'$  is bijective.  $\square$

**Corollary 3.4.** *Consider the Morita context  $(B', {}^*\mathcal{C}, A, Q', \tau', \mu')$  of Theorem 3.2. Assume that  $\tau'$  is surjective, and take  $\Lambda \in Q'$  such that  $\Lambda(x) = 1$ . Then we have:*

- 1) *For all  $M \in \mathcal{M}^{\mathcal{C}}$ ,  $M^{*\mathcal{C}} = M^{\text{co}\mathcal{C}}$ . In particular  $B = B'$ .*
- 2)  *$Q = Q'$ .*
- 3) *The two Morita contexts in Theorem 3.3 coincide.*

*Proof.* 1) From Theorem 3.3, we know that there exists  $\Lambda \in Q$  such that  $\Lambda(x) = 1$ . Take  $m \in M^{*\mathcal{C}}$ . Then  $m = m\Lambda(x) = m \cdot \Lambda = m_{[0]}\Lambda(m_{[1]})$  and  $\rho(m) = \rho(m_{[0]}\Lambda(m_{[1]})) = m_{[0]} \otimes_A m_{[1]}\Lambda(m_{[2]}) = m_{[0]} \otimes_A \Lambda(m_{[1]})x = m_{[0]}\Lambda(m_{[1]}) \otimes_A x = m \otimes_A x$ , so it follows that  $m \in M^{\text{co}\mathcal{C}}$ .

2) Look at the commutative diagram

$$\begin{array}{ccc} A \otimes_{{}^*\mathcal{C}} Q' & \xrightarrow{\tau'} & B' \\ \downarrow & & \downarrow \\ A \otimes_{{}^*\mathcal{C}} Q & \xrightarrow{\tau} & B \end{array} \quad \begin{array}{c} \\ \\ \\ \\ = \\ \\ \end{array}$$

From the fact that  $B = B'$  and  $\tau'$  is surjective, we easily deduce that  $\tau$  is surjective. Applying [2, Theorem II.3.4], we find that  $Q \cong \text{Hom}_{{}^*\mathcal{C}}(A, {}^*\mathcal{C}) \cong Q'$ .

3) now follows immediately from 1) and 2).  $\square$

Now let us look at the map  $\mu'$ . If we assume that  $\mathcal{C}$  is finitely generated and projective as a left  $A$ -module. As we already noticed, this implies that  $\mathcal{M}^{\mathcal{C}} \cong \mathcal{M}_{{}^*\mathcal{C}}$ , the two Morita contexts coincide, and we can apply Proposition 2.4. Let us state the result, for completeness sake. From [2, II.3.4], recall that we have ring morphisms

$$\begin{aligned} \pi : {}^*\mathcal{C} &\rightarrow {}_B\text{End}(A)^{\text{op}}, \quad \pi(f)(a) = f(xa); \\ \pi' : {}^*\mathcal{C} &\rightarrow \text{End}_B(Q), \quad \pi'(f)(q) = f\#q. \end{aligned}$$

In fact  $\pi = {}^*\text{can}$ , cf. Proposition 1.1. We also have a  $({}^*\mathcal{C}, B)$ -bimodule map

$$\kappa : Q \rightarrow {}_B\text{Hom}(A, B), \quad \kappa(q)(a) = q(xa)$$

and a  $(B, {}^*\mathcal{C})$ -bimodule map

$$\kappa' : A \rightarrow \text{Hom}_B(Q, B), \quad \kappa'(a)(q) = q(xa).$$

If  $\mu$  is surjective, then  $\pi$ ,  $\pi'$ ,  $\kappa$  and  $\kappa'$  are isomorphisms, and  $A$  and  $Q$  are finitely generated and projective, respectively as left and a right  $B$ -module. We now state some necessary and sufficient conditions for  $\mu$  to be surjective.

**Theorem 3.5.** *Assume that  $\mathcal{C}$  is finitely generated and projective as a left  $A$ -module, and consider the Morita context  $(B = B', {}^*\mathcal{C}, A, Q = Q', \tau = \tau', \mu = \mu')$  of Theorem 3.2. Then the following assertions are equivalent.*

- 1)  *$\mu : Q \otimes_B A \rightarrow {}^*\mathcal{C}$  is surjective (and, a fortiori, bijective);*
- 2)  *$({}^*\mathcal{C}, x)$  satisfies the Weak Structure Theorem;*

- 3)  $A$  is a right  ${}^*\mathcal{C}$ -generator;
- 4)  $A$  is projective as a left  $B$ -module and  $\pi$  is bijective;
- 5)  $A$  is projective as a left  $B$ -module,  $\pi$  is injective, and  $\kappa$  is surjective;
- 6)  $Q$  is projective as a right  $B$ -module,  $\pi'$  is injective, and  $\kappa'$  is surjective;
- 7)  $A$  is projective as a left  $B$ -module and  $(\mathcal{C}, x)$  is a Galois coring.

*Proof.* The equivalence of 1)-6) follows immediately from Proposition 2.6. 4)  $\Leftrightarrow$  7) follows from Proposition 1.1, using the fact that  $\mathcal{C}$  is finitely generated and projective as left  $A$ -module.  $\square$

#### 4. CLEFT ENTWINING STRUCTURES

In this Section, we look at the particular situation where  $\mathcal{C} = A \otimes C$  arises from an entwining structure  $(A, C, \psi)$ . First observe that  ${}^*\mathcal{C} = {}_A\text{Hom}(A \otimes C, A) \cong \text{Hom}(C, A)$  as a  $k$ -module. The ring structure on  ${}^*\mathcal{C}$  induces a  $k$ -algebra structure on  $\text{Hom}(C, A)$ , and this  $k$ -algebra is denoted  $\#(C, A)$ . The product is given by the formula

$$(16) \quad (f \# g)(c) = f(c_{(2)})_\psi g(c_{(1)}^\psi).$$

We have a natural algebra homomorphism  $i : A \rightarrow \#(C, A)$ ,  $i(a)(c) = \varepsilon_C(c)a$ , and we have, for all  $a \in A$  and  $f : C \rightarrow A$ :

$$(17) \quad (i(a) \# f)(c) = a_\psi f(c^\psi) \text{ and } (f \# i(a))(c) = f(c)a.$$

$\text{Hom}(C, A)$  will denote the  $k$ -algebra with the usual convolution product, that is

$$(18) \quad (f * g)(c) = f(c_{(1)})g(c_{(2)}).$$

The fact that we have two multiplications on  $\text{Hom}(C, A)$ , namely the usual convolution  $*$  and the smash product  $\#$  creates a difference between the general coring theory and the theory of entwined modules. We fix a grouplike element  $x \in G(C)$ . Then  $1 \otimes x \in G(\mathcal{C})$ , and the results of Section 4 can be applied to this situation. We have that  $A \in \mathcal{M}(\psi)_A^C$ , with right  $C$ -coaction  $\rho^r(a) = a_\psi \otimes x^\psi$ . The ring of coinvariants is

$$(19) \quad B' = A^{\text{co}C} = \{b \in A \mid b_\psi \otimes x^\psi = b \otimes x\}$$

and the bimodule  $Q'$  is naturally isomorphic to

$$Q' = \{q \in \#(C, A) \mid q(c_{(2)})_\psi \otimes c_{(1)}^\psi = q(c) \otimes x\}.$$

We have maps

$$\mu' : Q' \otimes_{B'} A \rightarrow \#(C, A), \quad \mu'(q \otimes_{B'} a)(c) = q(c)a;$$

$$\tau' : A \otimes_{\#(C, A)} Q' \rightarrow B', \quad \tau'(a \otimes q) = a_\psi q(x^\psi)$$

and  $(B', \#(C, A), A, Q', \tau', \mu')$  is a Morita context.

For  $M \in \mathcal{M}_{\#(C, A)}$ , the module of invariants is given by

$$M^{\#(C, A)} = \{m \in M \mid mf = mf(x), \text{ for all } f \in \#(C, A)\}.$$

From Theorem 3.3, we obtain immediately:

**Theorem 4.1.** *With notation as above, the following assertions are equivalent:*

- 1)  $\tau'$  is surjective;
- 2) there exists a  $\Lambda \in Q'$  such that  $\Lambda(x) = 1$ ;
- 3) for all  $M \in \mathcal{M}_{\#(C,A)}$ , the map

$$\omega_M : M \otimes_{\#(C,A)} Q' \rightarrow M^{\#(C,A)}, \quad \omega_M(m \otimes q) = m \cdot q$$

is bijective.

Now assume that  $C$  is finitely generated projective as a  $k$ -module, and let  $\{c_j, c_j^*\}$  be a finite dual basis. Then  $\text{Hom}(C, A) \cong A \otimes C^*$ , and the multiplication  $\#$  on  $\text{Hom}(C, A)$  can be translated into a multiplication on  $A \otimes C^*$ . The  $k$ -algebra that we obtain in this way is denoted  $A\#C^*$ , and is a smash product arising from a factorization structure: the corresponding  $\rho : C^* \otimes A \rightarrow A \otimes C^*$  is given by  $\rho(c^* \otimes a) = a_\rho \otimes c_\rho^* = \sum_j \langle c^*, c_j^\psi \rangle a_\psi \otimes c_j^*$ . We have maps

$$\mu : Q \otimes_B A \rightarrow A\#C^*, \quad \mu(q \otimes_B a) = \sum_j \langle q, c_j \rangle a \# c_j^*;$$

$$\pi : A\#C^* \rightarrow \text{End}_B(A), \quad \pi(b \# c^*)(a) = \langle c^*, xa \rangle b;$$

$$\text{can} : A \otimes_B A \rightarrow A \otimes C, \quad \text{can}(a \otimes b) = ab_\psi \otimes x^\psi.$$

For every  $M \in \mathcal{M}(\psi)_A^C$ , we have

$$\zeta_M : M^{\text{co}C} \otimes_B A \rightarrow M, \quad \zeta_M(m \otimes a) = ma.$$

From Theorem 3.5, we immediately obtain the following:

**Theorem 4.2.** *Let  $(A, C, \psi)$  be an entwining structure, and  $x \in C$  grouplike, and assume that  $C$  is finitely generated and projective as a  $k$ -module. Then the following assertions are equivalent:*

- 1)  $\mu$  is surjective (and a fortiori bijective);
- 2)  $\zeta_M$  is bijective, for every  $M \in \mathcal{M}(\psi)_A^C$ ;
- 3)  $A$  is a right  $A\#C^*$ -generator;
- 4)  $A$  is projective as a left  $B$ -module, and  $\pi$  is bijective;
- 5)  $A$  is projective as a left  $B$ -module, and  $\text{can}$  is bijective, i.e.  $A$  is a  $C$ -coalgebra Galois extension in the sense of [4].

**Proposition 4.3.** *Assume that  $\lambda : C \rightarrow A$  is convolution invertible, with convolution inverse  $\lambda^{-1}$ . Then the following assertions are equivalent:*

- 1)  $\lambda \in Q'$ ;
- 2) for all  $c \in C$ , we have

$$(20) \quad \lambda^{-1}(c_{(1)})\lambda(c_{(3)})_\psi \otimes c_{(2)}^\psi = \varepsilon(c)1_A \otimes x;$$

- 3) for all  $c \in C$ , we have

$$(21) \quad \lambda^{-1}(c_{(1)}) \otimes c_{(2)} = \lambda^{-1}(c)_\psi \otimes x^\psi.$$

Notice that condition 3) means that  $\lambda^{-1}$  is right  $C$ -colinear. If such a  $\lambda \in Q'$  exists, then we call  $(A, C, \psi, x)$  cleft.

*Proof.* 1)  $\Rightarrow$  2).

$$\lambda^{-1}(c_{(1)})\lambda(c_{(3)})_{\psi} \otimes c_{(2)}^{\psi} = \lambda^{-1}(c_{(1)})\lambda(c_{(2)}) \otimes x = \varepsilon_C(c)1_A \otimes x$$

2)  $\Rightarrow$  3).

$$\begin{aligned} \lambda^{-1}(c)_{\psi} \otimes x^{\psi} &= \varepsilon(c_{(1)})1_A \lambda^{-1}(c_{(2)})_{\psi} \otimes x^{\psi} \\ &\stackrel{(20)}{=} \lambda^{-1}(c_{(1)})\lambda(c_{(3)})_{\psi'} \lambda^{-1}(c_{(4)})_{\psi} \otimes c_{(2)}^{\psi'\psi} \\ &\stackrel{(4)}{=} \lambda^{-1}(c_{(1)}) \left( \lambda(c_{(3)})\lambda^{-1}(c_{(4)}) \right)_{\psi} \otimes c_{(2)}^{\psi} = \lambda^{-1}(c_{(1)}) \otimes c_{(2)} \end{aligned}$$

3)  $\Rightarrow$  1).

$$\begin{aligned} \lambda(c_{(2)})_{\psi} \otimes c_{(1)}^{\psi} &= \lambda(c_{(1)})\lambda^{-1}(c_{(2)})\lambda(c_{(4)})_{\psi} \otimes c_{(3)}^{\psi} \\ &\stackrel{(21)}{=} \lambda(c_{(1)})\lambda^{-1}(c_{(2)})_{\psi'} \lambda(c_{(3)})_{\psi} \otimes x^{\psi'\psi} \\ &\stackrel{(4)}{=} \lambda(c_{(1)}) \left( \lambda^{-1}(c_{(2)})\lambda(c_{(3)}) \right)_{\psi} \otimes x^{\psi} = \lambda(c) \otimes x \end{aligned}$$

□

**Proposition 4.4.** *Assume that  $(A, C, \psi, x)$  is a cleft entwining structure. Then the map  $\tau'$  in the associated Morita context is surjective.*

*Proof.* Let  $\lambda$  be as in Proposition 4.3. From condition 3) in Proposition 4.3, we deduce that  $\lambda^{-1}(x) \otimes x = \lambda^{-1}(x)_{\psi} \otimes x^{\psi}$ , hence  $\lambda^{-1}(x) \in B'$ , and  $\Lambda = \lambda \# i(\lambda^{-1}(x)) \in Q'$ , since  $Q'$  is a right  $B'$ -module. Now  $\Lambda(x) = \lambda(x)\lambda^{-1}(x) = \varepsilon_C(x) = 1$  and it follows from Theorem 4.1 that  $\tau$  is surjective. □

We say that the entwining structure  $(A, C, \psi, x)$  satisfies the *right normal basis property* if there exists a left  $B'$ -linear and right  $C$ -colinear isomorphism  $B' \otimes C \rightarrow A$ .  $(A, C, \psi, x)$  satisfies the Strong (resp. Weak) Structure Theorem if  $(A \otimes C, 1 \otimes x)$  satisfies the Strong (resp. Weak) Structure Theorem. We can now state our main result.

**Theorem 4.5.** *Let  $(A, C, \psi, x)$  be an entwining structure with a fixed grouplike element. The following assertions are equivalent:*

- 1)  $(A, C, \psi, x)$  is cleft;
- 2)  $(A, C, \psi, x)$  satisfies the Strong Structure Theorem and the right normal basis property ;
- 3)  $(A, C, \psi, x)$  is Galois, and satisfies the right normal basis property;
- 4) the map  ${}^*\text{can} : \#(C, A) \rightarrow \text{End}_B(A)^{\text{op}}$  is bijective and  $(A, C, \psi, x)$  satisfies the right normal basis property.

*Proof.* 1)  $\Rightarrow$  2). We take  $\lambda \in Q'$  as in Proposition 4.3, and  $M \in \mathcal{M}(\psi)_A^C$ . For any  $m \in \overline{M}$ , we have

$$\begin{aligned} \rho(m \cdot \lambda) &= \rho(m_{[0]}\lambda(m_{[1]})) = m_{[0]}\lambda(m_{[2]})_{\psi} \otimes m_{[1]}^{\psi} \\ &= m_{[0]}\lambda(m_{[1]}) \otimes x = m \cdot \lambda \otimes x, \end{aligned}$$

hence  $m \cdot \lambda \in M^{\text{co}C}$ , and we have a well-defined map

$$\gamma_M : M \rightarrow M^{\text{co}C} \otimes_{B'} A, \quad \gamma_M(m) = m_{[0]} \cdot \lambda \otimes_{B'} \lambda^{-1}(m_{[1]}).$$

We compute easily that

$$\zeta_M(\gamma_M(m)) = m_{[0]} \cdot (\lambda \# i(\lambda^{-1}(m_{[1]}))) = m_{[0]} \lambda(m_{[1]}) \lambda^{-1}(m_{[2]}) = m.$$

Recall that

$$(22) \quad \tau(a \otimes \lambda) = a_\psi \lambda(x^\psi) \in B'.$$

Take  $a \in A$  and  $m \in M^{\text{co}C}$ . Then

$$\begin{aligned} \gamma_M(\zeta_M(m \otimes_{B'} a)) &= \gamma_M(ma) = ma_\psi \cdot \lambda \otimes_{B'} \lambda^{-1}(x^\psi) \\ &= ma_{\psi\psi'} \lambda(x^{\psi'}) \otimes_{B'} \lambda^{-1}(x^\psi) \stackrel{(22)}{=} m \otimes_{B'} a_{\psi\psi'} \lambda(x^{\psi'}) \lambda^{-1}(x^\psi) \\ &\stackrel{(5)}{=} m \otimes_{B'} a_\psi \lambda((x^\psi)_{(1)}) \lambda^{-1}((x^\psi)_{(2)}) = m \otimes_{B'} a_\psi \varepsilon(x^\psi) \stackrel{(5)}{=} m \otimes_{B'} a \end{aligned}$$

and this proves that  $(A, C, \psi, x)$  satisfies the Weak Structure Theorem. From Proposition 4.4, we know that the map  $\tau'$  in the Morita context is surjective, and it follows from general Morita Theory that the functor  $\bullet \otimes_B A$  is fully faithful, which then implies that  $(A, C, \psi, x)$  satisfies the Strong Structure Theorem.

Take  $M \in \mathcal{M}(\psi)_A^C$ , and consider the maps

$$k : M \rightarrow M^{\text{co}C} \otimes C, \quad k(m) = m_{[0]} \cdot \lambda \otimes m_{[1]} = m_{[0]} \lambda(m_{[1]}) \otimes m_{[2]}$$

and

$$k^{-1} : M^{\text{co}C} \otimes C \rightarrow M, \quad k^{-1}(m \otimes c) = m \lambda^{-1}(c).$$

It is clear that  $k^{-1}(k(m)) = m$ , for all  $m \in M$ . If  $m \in M^{\text{co}C}$ , then  $\rho(ma) = ma_\psi \otimes x^\psi$ , and we compute

$$\begin{aligned} k(k^{-1}(m \otimes c)) &= m \lambda^{-1}(c)_\psi \cdot \lambda \otimes x^\psi \\ &\stackrel{(21)}{=} m \lambda^{-1}(c_{(1)}) \cdot \lambda \otimes c_{(2)} = m \lambda^{-1}(c_{(1)})_\psi \lambda(x^\psi) \otimes c_{(2)} \\ &\stackrel{(21)}{=} m \lambda^{-1}(c_{(1)}) \lambda(c_{(2)}) \otimes c_{(3)} = m \otimes c. \end{aligned}$$

It is obvious that  $k$  is right  $C$ -colinear. Now  $A \in \mathcal{M}(\psi)_A^C$ , so we find a right  $C$ -colinear isomorphism  $A \cong B \otimes C$ . It is also left  $B$ -linear, since the right  $C$ -coaction on  $A$  is left  $B$ -linear.

2)  $\Rightarrow$  3) follows from part 4) of Proposition 1.1.

3)  $\Rightarrow$  4) follows from part 1) of Proposition 1.1.

4)  $\Rightarrow$  1). From the right normal basis property, we know that there exists a left  $B'$ -linear, right  $C$ -colinear isomorphism  $h : B' \otimes C \rightarrow A$ . We consider the maps  $\lambda : C \rightarrow A$ ,  $\lambda(c) = h(1 \otimes c)$  and  $j = (I_{B'} \otimes \varepsilon_C) \circ h^{-1} : A \rightarrow B'$ . Clearly  $\lambda$  is right  $C$ -colinear and  $j$  is left  $B'$ -linear. Take  $a \in A$ , and write

$h^{-1}(a) = \sum_i b_i \otimes c_i$ . Then

$$\begin{aligned} \sum_i b_i h(1 \otimes c_{i(1)}) \otimes c_{i(2)} &= \sum_i h(b_i \otimes c_{i(1)}) \otimes c_{i(2)} \\ &= (h \otimes I_C) \rho \left( \sum_i b_i \otimes c_i \right) = \rho \left( h \left( \sum_i b_i \otimes c_i \right) \right) = \rho(a) = a_\psi \otimes x^\psi. \end{aligned}$$

Apply  $j \otimes I_C$  to both sides:

$$\begin{aligned} j(a_\psi) \otimes x^\psi &= \sum_i b_i (j \circ h)(1 \otimes c_{i(1)}) \otimes c_{i(1)} \\ &= \sum_i b_i (I_B \otimes \varepsilon_C)(1 \otimes c_{i(1)}) \otimes c_{i(1)} = \sum_i b_i \otimes c_i = h^{-1}(a). \end{aligned}$$

Now let  $q = (*\text{can})^{-1}(j)$ . We are done if we can show that  $\lambda$  is the convolution inverse of  $q$ , by Proposition 4.3. The fact that  $\lambda$  is right  $C$ -colinear means

$$(23) \quad \lambda(c_{(1)}) \otimes c_{(2)} = \lambda(c)_\psi \otimes x^\psi$$

and we compute, for all  $c \in C$ ,

$$\begin{aligned} (\lambda * q)(c) &= \lambda(c_{(1)}) q(c_{(2)}) = \lambda(c)_\psi q(x^\psi) = *\text{can}(q)(\lambda(c)) \\ &= j(\lambda(c)) = ((I_{B'} \otimes \varepsilon_C) \circ h^{-1} \circ h)(1 \otimes c) = \varepsilon_C(c) 1_A \end{aligned}$$

as needed. For all  $a \in A$ , we have

$$\begin{aligned} *\text{can}(q * \lambda)(a) &= a_\psi (q * \lambda)(x^\psi) = a_\psi q((x^\psi)_{(1)}) \lambda((x^\psi)_{(2)}) \\ (5) \quad &= a_\psi q(x^\psi) \lambda(x^\psi) = *\text{can}(q)(a_\psi) \lambda(x^\psi) = j(a_\psi) \lambda(x^\psi) = j(a_\psi) h(1 \otimes x^\psi) \\ &= h(j(a_\psi) \otimes x^\psi) = h(h^{-1}(a)) = a. \end{aligned}$$

This proves that  $*\text{can}(q * \lambda) = I_A = *\text{can}(\eta_A \circ \varepsilon_C)$ , and  $q * \lambda = \eta_A \circ \varepsilon_C$  by the injectivity of  $*\text{can}$ . Thus  $\lambda$  is the convolution inverse of  $q$ , as needed.  $\square$

## 5. FACTORIZATION STRUCTURES AND THE CFM MORITA CONTEXT

Let  $(A, S, \rho)$  be a factorization structure, and consider the smash product  $R = A \#_\rho S$ . We fix an algebra map  $\chi : S \rightarrow k$ . Then the map

$$X : R = A \#_\rho S \rightarrow A, \quad X(a \# s) = \chi(s) a$$

satisfies the conditions of Section 2 (with right replaced by left):  $X$  is left  $A$ -linear,  $X(rX(s)) = X(rs)$ , and  $X(1) = 1$ . We can therefore apply the results of Section 2. In particular, we obtain that  $A$  is a left  $R$ -module:

$$(a \# s) \leftarrow b = X((a \# s)b) = X(ab_\rho \# s_\rho) = \chi(s_\rho) ab_\rho$$

and  $b \in B = A^R$  if and only if  $\chi(s_\rho) b_\rho = \chi(s) b$  for all  $s \in S$ . Also  $\sum_i a_i \# s_i \in Q$  if and only if  $\sum_i a_i R \# t_\rho s_i = \chi(t) \sum_i a_i \# s_i$ , for all  $t \in S$ . We have a Morita context  $(B, A \# S, A, Q, \tau, \mu)$  with

$$\mu : A \otimes_B Q \rightarrow A \# S, \quad \mu \left( a \otimes_B \left( \sum_i a_i \# s_i \right) \right) = \sum_i a a_i \# s_i;$$

$$\tau : Q \otimes_{\rho} A \rightarrow B, \tau\left(\sum_i a_i \# s_i \otimes_{\rho} a\right) = \sum_i a_i a_{\rho} \chi(s_i R).$$

Now we consider the following particular situation:  $S = H$  is a bialgebra,  $A$  is a left  $H$ -module algebra, and  $\rho : H \otimes A \rightarrow A \otimes H$ ,  $\rho(h \otimes a) = h_{(1)} \cdot b \otimes h_{(2)}$ . We also take  $\chi = \varepsilon_H$ . The above formulas take the following form:  $(a \# h) \leftarrow b = a(h \cdot b)$ ;  $b \in B$  if and only if  $h \cdot b = \varepsilon(h)b$ ;  $\sum_i a_i \# h_i \in Q$  if and only if  $\sum_i h_{(1)} \cdot a_i \# h_{(2)} h_i = \varepsilon(h) \sum_i a_i \# h_i$ , for all  $h \in H$ .

In the particular situation where  $H$  is a finite dimensional Hopf algebra over a field  $k$ , there exists another Morita context connecting  $B$  and  $A \# H$ , due to Cohen, Fischman and Montgomery (see [11]). The construction can be generalized to the case where  $H$  is a Frobenius Hopf algebra over a commutative ring  $k$  (see [10]). This Morita context can be described as follows. Take a free generator  $t$  of the space of left integrals in  $H$ , and let  $\lambda$  be the distinguished grouplike element in  $H^*$ . Then we have that  $ht = \varepsilon(h)t$  and  $th = \lambda(h)t$  for all  $h \in H$ ;  $\lambda$  is an algebra map, and  $A$  is a  $(B, A \# H)$ -bimodule, the right  $A \# H$ -action is given by

$$a \leftarrow (b \# h) = \lambda(h_{(2)}) \bar{S}(h_{(1)}) \cdot (ab)$$

and we have a Morita context

$$(B, A \# H, A, A, \tilde{\tau}, \tilde{\mu})$$

with

$$\begin{aligned} \tilde{\tau} : A \otimes_{\rho} A &\rightarrow B, \tilde{\tau}(a \otimes_{\rho} b) = t \cdot (ab); \\ \tilde{\mu} : A \otimes_B A &\rightarrow R, \tilde{\mu}(a \otimes_B b) = a(t_{(1)} \cdot b) \# t_{(2)}. \end{aligned}$$

We refer to [11] for the details. We will now show that this Morita context can be obtained using Proposition 2.2 and Theorem 2.7.

If  $H$  is Frobenius, then there exists a left integral  $\varphi$  in  $H^*$  such that  $\langle \varphi, t \rangle = 1$ .  $\varphi$  is a free generator of the space of left integrals in  $H^*$ , and  $(t_{(2)} \otimes \bar{S}(t_{(1)}), \varphi)$  is a Frobenius system for  $H/k$  (see for example [7, Theorem 31]). This means that

$$(24) \quad ht_{(2)} \otimes \bar{S}(t_{(1)}) = t_{(2)} \otimes \bar{S}(t_{(1)})h \text{ and } \langle \varphi, t_{(2)} \rangle \bar{S}(t_{(1)}) = t_{(2)} \langle \varphi, \bar{S}(t_{(1)}) \rangle = 1$$

for all  $h \in H$ .

**Proposition 5.1.** *Let  $H$  be a Frobenius Hopf algebra, let  $t$  and  $\varphi$  be as above, and take a left  $H$ -module algebra  $A$ . Then  $A \# H/A$  is Frobenius, with Frobenius system*

$$\left( e = (1 \# t_{(2)}) \otimes_A (1 \# \bar{S}(t_{(1)})), \bar{\nu} = I_A \# \varphi \right)$$

*Proof.* For all  $a \in A$  and  $h \in H$ , we have

$$\begin{aligned} \left( (1 \# t_{(2)}) \otimes_A (1 \# \bar{S}(t_{(1)})) \right) (a \# h) &= (1 \# t_{(3)}) \otimes_A (\bar{S}(t_{(2)}) \cdot a \# \bar{S}(t_{(1)})h) \\ &= ((t_{(3)} \bar{S}(t_{(2)})) \cdot a \# t_{(4)}) \otimes_A (1 \# \bar{S}(t_{(1)})h) \\ &= (a \# t_{(2)}) \otimes_A (1 \# \bar{S}(t_{(1)})h) = (a \# h) \left( (1 \# t_{(2)}) \otimes_A (1 \# \bar{S}(t_{(1)})) \right). \end{aligned}$$

It is obvious that  $\bar{\nu}$  is left  $A$ -linear. It is also right  $A$ -linear since

$$\bar{\nu}((1\#h)a) = \bar{\nu}(h_{(1)}a\#h_{(2)}) = \langle \varphi, h_{(2)} \rangle h_{(1)}a = \langle \varphi, h \rangle a.$$

Finally, using (24), we find that  $\bar{\nu}(1\#t_{(2)})(1\#\bar{S}(t_{(1)})) = 1\#\bar{S}(\langle \varphi, t_{(2)} \rangle t_{(1)}) = 1\#\bar{S}(\langle \varphi, t \rangle 1) = 1\#1$  and  $(1\#t_{(2)})\bar{\nu}(1\#\bar{S}(t_{(1)})) = 1\#t_{(2)}\langle \varphi, \bar{S}(t_{(1)}) \rangle = 1\#1$ .  $\square$

**Corollary 5.2.** *As in Proposition 5.1, let  $H$  be a Frobenius Hopf algebra, and  $A$  a left  $H$ -module algebra. Then  $A$  and  $Q$  are isomorphic as  $(A, A\#H)$ -bimodules and the Morita contexts from Proposition 2.2 and [11],[10] are isomorphic.*

*Proof.* The fact that  $A$  and  $Q$  are isomorphic follows immediately from Theorem 2.7 and Proposition 5.1. The connecting isomorphisms are  $\alpha : A \rightarrow Q$ ,  $\alpha(a) = t_{(1)} \cdot a\#t_{(2)}$  and  $\alpha^{-1} = I_A\#\varphi|_Q$ . Let us check that the right  $A\#H$ -action on  $A$  transported from the one on  $Q$  coincides with the  $A\#H$ -action from [11]:

$$\begin{aligned} a \rightarrow (b\#h) &= \bar{\nu}(\alpha(a)(b\#h)) = \bar{\nu}\left(\left((t_{(1)} \cdot a)\#t_{(2)}\right)(b\#h)\right) \\ &= \bar{\nu}\left((t_{(1)} \cdot a)(t_{(2)} \cdot b)\#t_{(3)}h\right) = \langle \varphi, t_{(2)}h \rangle t_{(1)} \cdot (ab) \\ &= \langle \varphi, t_{(2)}h_{(3)} \rangle (t_{(1)}h_{(2)}\bar{S}(h_{(1)})) \cdot (ab) = \langle \varphi, th_{(2)} \rangle \bar{S}(h_{(1)}) \cdot (ab) \\ &= \langle \lambda h_{(2)} \rangle \langle \varphi, t \rangle \bar{S}(h_{(1)}) \cdot (ab) = \langle \lambda h_{(2)} \rangle \bar{S}(h_{(1)}) \cdot (ab) \end{aligned}$$

as needed.  $\square$

## 6. CLEFT FACTORIZATION STRUCTURES

As in the beginning of Section 5, let  $(A, S, \rho)$  be a factorization structure, and  $\chi : S \rightarrow k$  an algebra map. Recall that  $q = \sum_i a_i\#s_i \in Q$  if and only if

$$(25) \quad \sum_i a_{iR}\#t_\rho s_i = \chi(t) \sum_i a_i\#s_i$$

for all  $t \in S$ . Take  $q = \sum_i a_i\#s_i \in Q$ , and assume that  $q$  is invertible in  $A^{\text{op}} \otimes S$ , i.e. there exists  $\bar{q} = \sum_j \bar{a}_j\#\bar{s}_j \in A\#S$  such that

$$(26) \quad \sum_{i,j} a_i\bar{a}_j\#\bar{s}_j s_i = \sum_{i,j} \bar{a}_j a_i\#s_i \bar{s}_j = 1_A\#1_S.$$

**Proposition 6.1.** *Let  $q = \sum_i a_i\#s_i \in A\#S$  be invertible in  $A^{\text{op}} \otimes S$ , with inverse  $\bar{q} = \sum_j \bar{a}_j\#\bar{s}_j$ . Then the following assertions are equivalent:*

- 1)  $q \in Q$ ;
- 2)  $\sum_{i,j} (a_j)_\rho \bar{a}_i\#\bar{s}_i t_\rho s_j = \chi(t) 1_A\#1_S$ , for all  $t \in S$ ;
- 3)  $\sum_j \chi(t_\rho) (\bar{a}_j)_\rho \#\bar{s}_j = \sum_j \bar{a}_j\#\bar{s}_j t$ , for all  $t \in S$ .

*In this situation, we call  $(A, S, \rho, \chi)$  a cleft factorization structure.*

*Proof.* 1)  $\Rightarrow$  2). Using (25) and (26), we find, for all  $t \in S$ :

$$\sum_{i,j} (a_j)_\rho \bar{a}_i \# \bar{s}_i t_\rho s_j = \chi(t) \sum_{i,j} a_j \bar{a}_i \# \bar{s}_i s_j = \chi(t) 1_A \# 1_S.$$

2)  $\Rightarrow$  3). For all  $t \in S$ , we compute

$$\begin{aligned} \sum_j \chi(t_\rho) (\bar{a}_j)_\rho \# \bar{s}_j &= \sum_j \chi(t_\rho) (\bar{a}_j)_\rho 1_A \# 1_S \bar{s}_j \stackrel{2)}{=} \sum_{i,j,k} (\bar{a}_j)_\rho (a_k)_r \bar{a}_i \# \bar{s}_i t_{Rr} s_k \bar{s}_j \\ &\stackrel{(8)}{=} \sum_{i,j,k} (\bar{a}_j a_k)_\rho \bar{a}_i \# \bar{s}_i t_{Rr} s_k \bar{s}_j \stackrel{(26)}{=} \sum_i (1_A)_\rho \bar{a}_i \# \bar{s}_i t_\rho 1_S \stackrel{(7)}{=} \sum_i \bar{a}_i \# \bar{s}_i t. \end{aligned}$$

3)  $\Rightarrow$  1).

$$\begin{aligned} \sum_i (a_i)_\rho \# t_\rho s_i &= \sum_{i,j,k} (a_i)_\rho \bar{a}_j a_k \# s_k \bar{s}_j t_\rho s_i \stackrel{3)}{=} \sum_{i,j,k} \chi(t_{Rr}) (a_i)_\rho (\bar{a}_j)_r a_k \# s_k \bar{s}_j s_i \\ &\stackrel{(8)}{=} \sum_{i,j,k} \chi(t_R) (a_i \bar{a}_j)_\rho a_k \# s_k \bar{s}_j s_i \stackrel{(26)}{=} \sum_k a_k \# s_k. \end{aligned}$$

□

**Proposition 6.2.** *Assume that  $(A, S, R, \chi)$  is cleft. Then we have an equivalence of categories*

$$F : {}_B\mathcal{M} \rightarrow {}_\rho\mathcal{M}, F(N) = A \otimes_B N ; G : {}_\rho\mathcal{M} \rightarrow {}_B\mathcal{M}, G(M) = {}^R M.$$

Consequently the map  $\text{can} : A \otimes_B A \rightarrow \text{Hom}(S, A)$ ,  $\text{can}(a \otimes a')(s) = a'_\rho \chi(s_\rho) a$  is bijective.

*Proof.* We first prove that the functor  $F$  is fully faithful. This follows from Proposition 2.5 after we show that the map  $\tau$  from the Morita context from Proposition 2.2 is surjective. It suffices to show that there exists  $\Lambda \in Q$  with  $(I_A \otimes \chi)(\Lambda) = 1$  (Proposition 2.4).

Take  $q \in Q$  as in Proposition 6.1. Then  $\sum_j \chi(t_\rho) \chi(\bar{s}_j) (\bar{a}_j)_\rho = \sum_j \chi(t) \chi(\bar{s}_j) \bar{a}_j$ , which means that  $\sum_j \chi(\bar{s}_j) \bar{a}_j \in B$ .  $Q$  is a left  $B$ -module, so  $\Lambda = \sum_{i,j} \chi(\bar{s}_j) \bar{a}_j a_i \# s_i \in Q$ , and it follows from (26) that

$$(I_A \otimes \chi) \left( \sum_{i,j} \chi(\bar{s}_j) \bar{a}_j a_i \# s_i \right) = \sum_{i,j} \chi(\bar{s}_j) \chi(s_i) \bar{a}_j a_i = 1.$$

Now we show that  $G$  is fully faithful, or, equivalently, the counit of the adjunction  $(F, G)$  is an isomorphism. Recall that, for  $M \in {}_\rho\mathcal{M}$ ,  $\zeta_M : A \otimes_B {}^R M \rightarrow M$  is given by  $\zeta_M(a \otimes m) = am$ . Take  $q \in Q$  as in Proposition 6.1, and  $m \in M$ . Then  $q \cdot m \in {}^R M$  since

$$(1 \# t) q \cdot m = \sum_i (1 \# t) (q_i \# s_i) \cdot m = \left( \sum_i (q_i)_\rho \# t_\rho s_i \right) \cdot m \stackrel{(25)}{=} \chi(t) q \cdot m.$$

Now let  $\gamma_M : M \rightarrow A \otimes_B {}^R M$  be defined by  $\gamma_M(m) = \sum_j \bar{a}_j \otimes_B q\bar{s}_j m$ . For all  $m \in M$ , we have, by (26), that  $\zeta_M(\gamma_M(m)) = \sum_j \bar{a}_j q\bar{s}_j m = m$ . Finally, for all  $b \in A$  and  $m \in {}^R M$ :

$$\begin{aligned} \gamma_M(\zeta_M(b \otimes_B m)) &= \gamma_M(bm) = \sum_j \bar{a}_j \otimes_B q\bar{s}_j bm \\ &= \sum_j \bar{a}_j \otimes_B X(q\bar{s}_j b)m = \sum_j \bar{a}_j X(q\bar{s}_j b) \otimes_B m \\ &= \sum_{i,j} \bar{a}_j a_i \chi(s_i \bar{s}_j) b \otimes_B m = b \otimes_B m, \end{aligned}$$

where we used the fact that  $X(q\bar{s}_j b) \in \text{Im}(\tau) \subset B$ .  $\square$

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