GRADED STRUCTURE AND HOPF STRUCTURES IN PARABOSONIC ALGEBRA. AN ALTERNATIVE APPROACH TO BOSONISATION

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Abstract

Parabosonic algebra in infinite degrees of freedom is presented as a generalization of the bosonic algebra, from the viewpoints of both physics and mathematics. The notion of super-Hopf algebra is shortly discussed and the super-Hopf algebraic structure of the parabosonic algebra is established (without appealing to its Lie superalgebraic structure). Two possible variants of the parabosonic algebra are presented and their (ordinary) Hopf algebraic structure is estabished: The first is produced by "bosonising" the original super-Hopf algebra, while the second is constructed via a slightly different path.

INTRODUCTION

Parabosonic algebras have a long history both in theoretical and mathematical physics. Although, formally introduced in the fifties by Green [9], in the context of second quantization, their history traces back to the fundamental conceptual problems of quantum mechanics; in particular to Wigner's approach to first quantization [25]. We begin by outlining this story.
In classical physics, all information describing the dynamics of a given physical system is "encoded" in it's Hamiltonian $H(p_i, q_i)$, $i = 1, \dots, n$ which is a function of the real variables p_i, q_i. These are usually called "canonical variables". Having determined the Hamiltonian of the system, the dynamics is extracted through the well-known Hamilton equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{1}$$

The passage from the classical description to the quantum description, within the framework of the first quantization, consists of the following procedure: The functional dependence of the Hamiltonian on the canonical variables is -roughly- retained but the canonical variables are no more real variables. Instead they become elements of a unital associative non-commutative algebra, described in terms of the generators p_i, q_i, I, $i = 1, \dots, n$ and relations:

$$[q_i, p_j] = i\hbar\delta_{ij} I \qquad [q_i, q_j] = [p_i, p_j] = 0 \tag{2}$$

I is of course the unity of the algebra and $[x, y]$ stands for $xy - yx$. The states of the system are no more described as functions $p_i(t), q_i(t)$ which are solutions of (1) but rather as vectors of a

Hilbert space, where the elements of the above mentioned algebra act. The dynamics is now determined by the Heisenberg equations of motion:

$$i\hbar \frac{dq_i}{dt} = [q_i, H] \qquad i\hbar \frac{dp_i}{dt} = [p_i, H] \tag{3}$$

We of course describe what is known as the Heisenberg picture of quantum mechanics. Relations (2) are known in the physical literature as the Heisenberg algebra, or the Heisenberg-Weyl algebra or more commonly as the Canonical Commutation Relations often abbreviated as CCR. Their central importance for the quantization procedure formerly described, lies in the fact that if one accepts [1] the algebraic relations (2) together with the quantum dynamical equations (3) then it is an easy matter (see [4]) to extract the classical Hamiltonian equations of motion (1) while on the other hand the acceptance of the classical equations (1) together with (2) reproduces the quantum dynamics exactly as described by (3). In this way the CCR emerge as a fundamental link between the classical and the quantum description of the dynamics.
For technical reasons it is common to use -instead of the variables p_i, q_i- the linear combinations:

$$b_j^+ = \frac{1}{\sqrt{2}}(q_j - ip_j) \qquad b_j^- = \frac{1}{\sqrt{2}}(q_j + ip_j) \tag{4}$$

for $j = 1, \dots, n$ in terms of which (2) become (we have set $\hbar = 1$):

$$[b_i^-, b_j^+] = \delta_{ij} I \qquad [b_i^-, b_j^-] = [b_i^+, b_j^+] = 0 \tag{5}$$

for $i, j = 1, \dots, n$. These latter relations are usually called the bosonic algebra (of n bosons), and in they case of the infinite degrees of freedom $i, j = 1, 2, \dots$ they become the starting point of the free field theory (i.e.: second quantisation).
The above mentioned approach to the first quantisation is actually the path followed by the founders of quantum mechanics such as Dirac, Born, Heisenberg, Schröedinger and others. Although it is not our aim to provide systematic references on this fascinating story, many of the original papers which paved the way can be found in [24].
In 1950 E.P. Wigner in a two page publication [25], noticed that what the above approach implies is that the CCR (2) are sufficient conditions -but not necessary- for the equivalence between the classical Hamiltonian equations (1) and the Heisenberg quantum dynamical equations (3). In a kind of reversing the problem, Wigner posed the question of looking for necessary conditions for the simultaneous fulfillment of (1) and (3). He stated an infinite set of solutions for the above mentioned problem (although not claiming to have found the general solution). It is worth noting that CCR were included as one special case among Wigner's infinite solutions.
A few years latter in 1953, Green in his celebrated paper [9] introduced the parabosonic algebra (in possibly infinite degrees of freedom), by means of generators and relations:

$$\begin{gathered} \left[B_m^-, \{B_k^+, B_l^-\}\right] = 2\delta_{km} B_l^- \\ \left[B_m^-, \{B_k^-, B_l^-\}\right] = 0 \\ \left[B_m^+, \{B_k^-, B_l^-\}\right] = -2\delta_{lm} B_k^- - 2\delta_{km} B_l^- \end{gathered} \tag{6}$$

[1] of course we do not consider arbitrary Hamiltonians but functions of the form $H = \sum_{i=1}^{n} p_i^2 + V(q_1, \dots, q_n)$ which however are general enough for simple physical systems

$k,l,m = 1,2,\ldots$ and $\{x,y\}$ stands for $xy+yx$. Green was primarily interested in field theoretic implications of the above mentioned algebra, in the sense that he considered it as an alternative starting point for the second quantisation problem, generalizing (5). However, despite his original motivation he was the first to realize -see also [19]- that Wigner's infinite solutions were nothing else but inequivalent irreducible representations of the parabosonic algebra (6). (See also the discussion in [20]).

1 PRELIMINARIES

In what follows, all vector spaces and algebras and all tensor products will be considered over the field of complex numbers.

1.1 BOSONS AND PARABOSONS AS SUPERALGEBRAS

The parabosonic algebra, was originally defined in terms of generators and relations by Green [9] and Greenberg-Messiah [8]. We begin with restating their definition:
Let us consider the vector space V_X freely generated by the elements: X_i^+, X_j^-, $i,j = 1,2,....$ Let $T(V_X)$ denote the tensor algebra of V_X. $T(V_X)$ is -up to isomorphism- the free algebra generated by the elements of the basis. In $T(V_X)$ we consider the two-sided ideals I_{P_B}, I_B, generated by the following elements:

$$\left[\{X_i^{\xi}, X_j^{\eta}\}, X_k^{\varepsilon}\right] - (\varepsilon-\eta)\delta_{jk}X_i^{\xi} - (\varepsilon-\xi)\delta_{ik}X_j^{\eta} \tag{7}$$

and:

$$\left[X_i^-, X_j^+\right] - \delta_{ij}I_X \quad , \quad \left[X_i^-, X_j^-\right] \quad , \quad \left[X_i^+, X_j^+\right] \tag{8}$$

respectively, for all values of $\xi,\eta,\varepsilon = \pm 1$ and $i,j = 1,2,\ldots$. I_X is the unity of the tensor algebra. We now have the following:

Definition 1.1. The parabosonic algebra in P_B is the quotient algebra of the tensor algebra $T(V_X)$ of V_X with the ideal I_{P_B}:

$$P_B = T(V_X)/I_{P_B}$$

The bosonic algebra B is the quotient algebra of the tensor algebra $T(V_X)$ with the ideal I_B:

$$B = T(V_X)/I_B$$

We denote by $\pi_{P_B} : T(V_X) \to P_B$ and $\pi_B : T(V_X) \to B$ respectively, the canonical projections. The elements X_i^+, X_j^-, I_X, where $i,j = 1,2,...$ and I_X is the unity of the tensor algebra, are the generators of the tensor algebra $T(V_X)$. The elements $\pi_{P_B}(X_i^+), \pi_{P_B}(X_j^-), \pi_{P_B}(I_X)$, $i,j = 1,...$ are a set of generators of the parabosonic algebra P_B, and they will be denoted by B_i^+, B_j^-, I for $i,j = 1,2,...$ respectively, from now on. $\pi_{P_B}(I_X) = I$ is the unity of the parabosonic algebra. On the other hand elements $\pi_B(X_i^+), \pi_B(X_j^-), \pi_B(I_X)$, $i,j = 1,2,...$ are a set of generators of the bosonic algebra B, and they will be denoted by b_i^+, b_j^-, I for $i,j = 1,2,...$ respectively, from now on. $\pi_B(I_X) = I$ is the unity of the bosonic algebra.
Based on the above definitions we prove now the following proposition which clarifies the relationship between bosonic and parabosonic algebras:

Proposition 1.2. *The parabosonic algebra P_B and the bosonic algebra B are both $\mathbb{Z}_2$-graded algebras with their generators $B_i^{\pm}$ and $b_i^{\pm}$ respectively, $i,j = 1,2,...$, being odd elements. The bosonic algebra B is a quotient algebra of the parabosonic algebra P_B. The "replacement" map $\phi : P_B \to B$ defined by: $\phi(B_i^{\pm}) = b_i^{\pm}$ is a $\mathbb{Z}_2$-graded algebra epimorphism (i.e.: an even algebra epimorphism).*

Proof. It is obvious that the tensor algebra $T(V_X)$ is a $\mathbb{Z}_2$-graded algebra with the monomials being homogeneous elements. If x is an arbitrary monomial of the tensor algebra, then $deg(x) = 0$, namely x is an even element, if it constitutes of an even number of factors (an even number of generators of $T(V_X)$) and $deg(x) = 1$, namely x is an odd element, if it constitutes of an odd number of factors (an odd number of generators of $T(V_X)$). The generators X_i^+, X_j^- , $i,j = 1,...,n$ are odd elements in the above mentioned gradation. In view of the above description we can easily conclude that the $\mathbb{Z}_2$-gradation of the tensor algebra is immediately "transfered" to the algebras P_B and B: Both ideals I_{P_B} and I_B are homogeneous ideals of the tensor algebra, since they are generated by homogeneous elements of $T(V_X)$. Consequently, the projection homomorphisms π_{P_B} and π_B are homogeneous algebra maps of degree zero, or we can equivalently say that they are even algebra homomorphisms. We can straightforwardly check that the bosons satisfy the paraboson relations, i.e:

$$\begin{aligned}\pi_B\big(\big[\{X_i^{\xi}, X_j^{\eta}\}, X_k^{\varepsilon}\big] - (\varepsilon-\eta)\delta_{jk}X_i^{\xi} - (\varepsilon-\xi)\delta_{ik}X_j^{\eta}\big) \\ = \big[\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}\big] - (\varepsilon-\eta)\delta_{jk}b_i^{\xi} - (\varepsilon-\xi)\delta_{ik}b_j^{\eta} = 0\end{aligned}$$

which simply means that: $ker(\pi_{P_B}) \subseteq ker(\pi_B)$ or equivalently: $I_{P_B} \subseteq I_B$. By the correspodence theorem for rings, we get that the set $I_B/I_{P_B} = \pi_{P_B}(I_B)$ is an homogeneous ideal of the algebra P_B, and applying the third isomorphism theorem for rings we get:

$$P_B\Big/(I_B/I_{P_B}) = (T(V_X)/I_{P_B})\Big/(I_B/I_{P_B}) \cong T(V_X)\Big/I_B = B \tag{9}$$

Thus we have shown that the bosonic algebra B is a quotient algebra of the parabosonic algebra P_B. The fact that $I_{P_B} \subseteq I_B$ implies that π_B is uniquely extended to an even algebra homomorphism $\phi : P_B \to B$, where ϕ is determined by it's values on the generators $B_i^{\pm}$ of P_B, i.e.: $\phi(B_i^{\pm}) = b_i^{\pm}$. Recalling now that: $ker\phi = I_B/I_{P_B} = \pi_{P_B}(I_B)$ and using (9), we get that: $P_B/ker\phi \cong B$ which completes the proof that ϕ is an epimorphism of $\mathbb{Z}_2$-graded algebras (or: an even epimorphism). □

Note that $ker\phi$ is exactly the ideal of P_B generated by the elements of the form: $\big[B_i^-, B_j^+\big] - \delta_{ij}I$, $\big[B_i^-, B_j^-\big]$, $\big[B_i^+, B_j^+\big]$ for all values of $i,j = 1,2,...$, and I is the unity of the P_B algebra.
The rise of the theory of quasitriangular Hopf algebras from the mid-80's [3] and thereafter and especially the study and abstraction of their representations (see: [13, 14], [18] and references therein), has provided us with a novel understanding [2] of the notion and the properties of $\mathbb{G}$-graded algebras, where $\mathbb{G}$ is a finite abelian group:
Restricting ourselves to the simplest case where $\mathbb{G} = \mathbb{Z}_2$, we recall that an algebra A being a $\mathbb{Z}_2$-graded algebra (in the physics literature the term superalgebra is also of widespread use) is equivalent to saying that A is a $\mathbb{C}\mathbb{Z}_2$-module algebra, via the $\mathbb{Z}_2$-action: $g \triangleright a = (-1)^{|a|}a$ (for a homogeneous in A). What we actually mean is that A, apart from being an algebra is also a $\mathbb{C}\mathbb{Z}_2$-module and at the same time it's structure maps (i.e.: the multiplication and the unity map

[2] it is worth noting, that some of these ideas already appear in [23]

which embeds the field into the center of the algebra) are $\mathbb{C}\mathbb{Z}_2$-module maps which is nothing else but homogeneous linear maps of degree 0 (i.e.: even linear maps). Note, that under the above action, any element of A decomposes uniquely as: $a = \frac{a+(g \triangleright a)}{2} + \frac{a-(g \triangleright a)}{2}$. We can further summarize the above description saying that A is an algebra in the braided monoidal category of $\mathbb{C}\mathbb{Z}_2$-modules ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$. In this case the braiding is induced by the non-trivial quasitriangular structure of the $\mathbb{C}\mathbb{Z}_2$ Hopf algebra i.e. by the non-trivial R-matrix:

$$R_g = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) \tag{10}$$

In the above relation $1, g$ are the elements of the $\mathbb{Z}_2$ group (written multiplicatively).
We digress here for a moment, to recall that (see [13, 14] or [18]) if (H, R_H) is a quasitriangular Hopf algebra, then the category of modules ${}_H\mathcal{M}$ is a braided monoidal category, where the braiding is given by a natural family of isomorphisms $\Psi_{V,W} : V \otimes W \cong W \otimes V$, given explicitly by:

$$\Psi_{V,W}(v \otimes w) = \sum (R_H^{(2)} \triangleright w) \otimes (R_H^{(1)} \triangleright v) \tag{11}$$

for any $V, W \in obj({}_H\mathcal{M})$. By v, w we denote any elements of V, W respectively.
Combining eq. (10) and (11) we immediately get the braiding in the ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$ category:

$$\Psi_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v \tag{12}$$

In the above relation $|.|$ denotes the degree of an homogeneous element of either V or W (i.e.: $|x| = 0$ if x is an even element and $|x| = 1$ if x is an odd element). This is obviously a symmetric braiding, since $\Psi_{V,W} \circ \Psi_{W,V} = Id$, so we actually have a symmetric monoidal category ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$, rather than a truly braided one.
The really important thing about the existence of the braiding (12) is that it provides us with an alternative way of forming tensor products of $\mathbb{Z}_2$-graded algebras: If A is a superalgebra with multiplication $m : A \otimes A \to A$, then the super vector space $A \otimes A$ (with the obvious $\mathbb{Z}_2$-gradation) equipped with the associative multiplication

$$(m \otimes m)(Id \otimes \Psi_{A,A} \otimes Id) : A \otimes A \otimes A \otimes A \longrightarrow A \otimes A \tag{13}$$

given by: $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ (b, c homogeneous in A), readily becomes a superalgebra (or equivalently an algebra in the braided monoidal category of $\mathbb{C}\mathbb{Z}_2$-modules ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$) which we will denote: $A \underline{\otimes} A$ and call the braided tensor product algebra from now on.

1.2 PARABOSONS AS SUPER-HOPF ALGEBRAS

The notion of $\mathbb{G}$-graded Hopf algebra, for $\mathbb{G}$ a finite abelian group, is not a new one neither in physics nor in mathematics. The idea appears already in the work of Milnor and Moore [16], where we actually have $\mathbb{Z}$-graded Hopf algebras. On the other hand, universal enveloping algebras of Lie superalgebras are widely used in physics and they are examples of $\mathbb{Z}_2$-graded Hopf algebras (see for example [12], [22]). These structures are strongly resemblant of Hopf algebras but they are not Hopf algebras at least in the ordinary sense.
Restricting again to the simplest case where $\mathbb{G} = \mathbb{Z}_2$ we briefly recall this idea: An algebra A being a $\mathbb{Z}_2$-graded Hopf algebra (or super-Hopf algebra) means first of all that A is a $\mathbb{Z}_2$-graded associative algebra (or: superalgebra). We now consider the braided tensor product

algebra $A\underline{\otimes}A$. Then A is equipped with a coproduct $\underline{\Delta}: A \rightarrow A\underline{\otimes}A$, which is an superalgebra homomorphism from A to the braided tensor product algebra $A\underline{\otimes}A$:

$$\underline{\Delta}(ab) = \sum (-1)^{|a_2||b_1|} a_1 b_1 \otimes a_2 b_2 = \underline{\Delta}(a) \cdot \underline{\Delta}(b)$$

for any a, b in A, with $\underline{\Delta}(a) = \sum a_1 \otimes a_2$, $\underline{\Delta}(b) = \sum b_1 \otimes b_2$, and a_2, b_1 homogeneous.
Similarly, A is equipped with an antipode $\underline{S}: A \rightarrow A$ which is not an algebra anti-homomorphism (as in ordinary Hopf algebras) but a superalgebra anti-homomorphism (or: "twisted" anti-homomorphism or: braided anti-homomorphism) in the following sense (for any homogeneous $a, b \in A$):

$$\underline{S}(ab) = (-1)^{|a||b|} \underline{S}(b) \underline{S}(a)$$

The rest of the axioms which complete the super-Hopf algebraic structure (i.e.: coassociativity, counity property, and compatibility with the antipode) have the same formal description as in ordinary Hopf algebras.
Once again, the abstraction of the representation theory of quasitriangular Hopf algebras provides us with a language in which the above description becomes much more compact: We simply say that A is a Hopf algebra in the braided monoidal category of $\mathbb{C}\mathbb{Z}_2$-modules ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$ or: a braided group where the braiding is given in equation (12). What we actually mean is that A is simultaneously an algebra, a coalgebra and a $\mathbb{C}\mathbb{Z}_2$-module, while all the structure maps of A (multiplication, comultiplication, unity, counity and the antipode) are also $\mathbb{C}\mathbb{Z}_2$-module maps and at the same time the comultiplication $\underline{\Delta}: A \rightarrow A\underline{\otimes}A$ and the counit are algebra morphisms in the category ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$ (see also [13, 14] or [18] for a more detailed description).
We proceed now to the proof of the following proposition which establishes the super-Hopf algebraic structure of the parabosonic algebra P_B:

Proposition 1.3. *The parabosonic algebra equipped with the even linear maps* $\underline{\Delta}: P_B \rightarrow P_B\underline{\otimes}P_B$, $\underline{S}: P_B \rightarrow P_B$, $\underline{\varepsilon}: P_B \rightarrow \mathbb{C}$ *, determined by their values on the generators:*

$$\underline{\Delta}(B_i^{\pm}) = 1 \otimes B_i^{\pm} + B_i^{\pm} \otimes 1 \qquad \underline{\varepsilon}(B_i^{\pm}) = 0 \qquad \underline{S}(B_i^{\pm}) = -B_i^{\pm} \tag{14}$$

becomes a super-Hopf algebra.

Proof. Recall that by definition $P_B = T(V_X)/I_{P_B}$. Consider the linear map: $\underline{\Delta}: V_X \rightarrow P_B\underline{\otimes}P_B$ determined by it's values on the basis elements specified by: $\underline{\Delta}(X_i^{\pm}) = I \otimes B_i^{\pm} + B_i^{\pm} \otimes I$. By the universality of the tensor algebra this map is uniquely extended to a superalgebra homomorphism: $\underline{\Delta}: T(V_X) \rightarrow P_B\underline{\otimes}P_B$. Now we compute:

$$\underline{\Delta}(\left[\{X_i^{\xi}, X_j^{\eta}\}, X_k^{\varepsilon}\right] - (\varepsilon - \eta)\delta_{jk} X_i^{\xi} - (\varepsilon - \xi)\delta_{ik} X_j^{\eta}) = 0$$

This means that $I_{P_B} \subseteq ker\underline{\Delta}$, which in turn implies that $\underline{\Delta}$ is uniquely extended as a superalgebra homomorphism: $\underline{\Delta}: P_B \rightarrow P_B\underline{\otimes}P_B$, with values on the generators determined by (14). Proceeding the same way we construct the maps $\underline{\varepsilon}$, $\underline{S}$, as determined in (14).
Note here that in the case of the antipode $\underline{S}$ we need the notion of the $\mathbb{Z}_2$-graded opposite algebra (or: opposite superalgera) P_B^{op}, which is a superalgebra defined as follows: P_B^{op} has the same underlying super vector space as P_B, but the multiplication is now defined as: $a \cdot b = (-1)^{|a||b|} ba$, for all $a, b \in P_B$. (In the right hand side, the product is of course the product of P_B). We start by defining a linear map $\underline{S}: V_X \rightarrow P_B^{op}$ by: $\underline{S}(X_i^{\pm}) = -B_i^{\pm}$ which is (uniquely) extended to a superalgebra homomorphism: $\underline{S}: T(V_X) \rightarrow P_B^{op}$. The fact that $I_{P_B} \subseteq ker\underline{S}$ implies that $\underline{S}$ is

uniquely extended to a superalgebra homomorphism $\underline{S} : P_B \to P_B^{op}$, thus to a superalgebra anti-homomorphism: $\underline{S} : P_B \to P_B$ with values on the generators determined by (14).
Now it is sufficient to verify the rest of the super-Hopf algebra axioms (coassociativity, counity and the compatibility condition for the antipode) on the generators of P_B. This can be done with straigthforward computations. $\square$

Let us note here, that the above proposition generalizes a result which -in the case of finite degrees of freedom- is a direct consequence of the work in [7]. In that work the parabosonic algebra in $2n$ generators (n-paraboson algebra) $P_B^{(n)}$ is shown to be isomorphic to the universal enveloping algebra of the orthosymplectic Lie superalgebra: $P_B^{(n)} \cong U(B(0,n))$. See also the discussion in [10, 11].

2 MAIN RESULTS

A general scheme for "transforming" a Hopf algebra A in the braided category ${}_H\mathcal{M}$ (H: some quasitriangular Hopf algebra) into an ordinary one, namely the smash product Hopf algebra: $A \star H$, such that the two algebras have equivalent module categories, has been developed during '90 's. The original reference is [13] (see also [14, 15]). The technique is called bosonisation, the term coming from physics. This technique uses ideas developed in [21], [17]. It is also presented and applied in [5], [6], [1]. We review the main points of the above method:
In general, A being a Hopf algebra in a category, means that A apart from being an algebra and a coalgebra, is also an object of the category and at the same time it's structure maps are morphisms in the category. In particular, if H is some quasitriangular Hopf algebra, A being a Hopf algebra in the braided monoidal category ${}_H\mathcal{M}$, means that the H-module A is an algebra in ${}_H\mathcal{M}$ (or: H-module algebra) and a coalgebra in ${}_H\mathcal{M}$ (or: H-module coalgebra) and at the same time Δ_A and ε_A are algebra morphisms in the category ${}_H\mathcal{M}$. (For more details on the above definitions one may consult for example [18]).
Since A is an H-module algebra we can form the cross product algebra $A \rtimes H$ (also called: smash product algebra) which as a k-vector space is $A \otimes H$ (i.e. we write: $a \rtimes h \equiv a \otimes h$ for every $a \in A$, $h \in H$), with multiplication given by:

$$(b \otimes h)(c \otimes g) = \sum b(h_1 \triangleright c) \otimes h_2 g \tag{15}$$

$\forall\, b, c \in A$ and $h, g \in H$, and the usual tensor product unit.
On the other hand A is a (left) H-module coalgebra with H: quasitriangular through the R-matrix: $R_H = \sum R_H^{(1)} \otimes R_H^{(2)}$. Quasitriangularity "switches" the (left) action of H on A into a (left) coaction $\rho : A \to H \otimes A$ through:

$$\rho(a) = \sum R_H^{(2)} \otimes (R_H^{(1)} \triangleright a) \tag{16}$$

and A endowed with this coaction becomes (see [14, 15]) a (left) H-comodule coalgebra or equivalently a coalgebra in ${}^H\mathcal{M}$ (meaning that Δ_A and ε_A are (left) H-comodule morphisms, see [18]).
We recall here (see: [14, 15]) that when H is a Hopf algebra and A is a (left) H-comodule coalgebra with the (left) H-coaction given by: $\rho(a) = \sum a^{(1)} \otimes a^{(0)}$, one may form the cross coproduct coalgebra $A \rtimes H$, which as a k-vector space is $A \otimes H$ (i.e. we write: $a \rtimes h \equiv a \otimes h$ for every $a \in A$, $h \in H$), with comultiplication given by:

$$\Delta(a \otimes h) = \sum a_1 \otimes a_2^{(1)} h_1 \otimes a_2^{(0)} \otimes h_2 \tag{17}$$

and counit: $\varepsilon(a \otimes h) = \varepsilon_A(a)\varepsilon_H(h)$. (In the above: $\Delta_A(a) = \sum a_1 \otimes a_2$ and we use in the elements of A upper indices included in parenthesis to denote the components of the coaction according to the Sweedler notation, with the convention that $a^{(i)} \in H$ for $i \neq 0$).
Now we proceed by applying the above described construction of the cross coproduct coalgebra $A \rtimes H$, with the special form of the (left) coaction given by eq. (16). Replacing thus eq. (16) into eq. (17) we get for the special case of the quasitriangular Hopf algebra H the cross coproduct comultiplication:

$$\Delta(a \otimes h) = \sum a_1 \otimes R_H^{(2)} h_1 \otimes (R_H^{(1)} \triangleright a_2) \otimes h_2 \tag{18}$$

Finally we can show that the cross product algebra (with multiplication given by (15)) and the cross coproduct coalgebra (with comultiplication given by (18)) fit together and form a bialgebra (see: [14, 15, 17, 21]). This bialgebra, furnished with an antipode:

$$S(a \otimes h) = (S_H(h_2))u(R^{(1)} \triangleright S_A(a)) \otimes S(R^{(2)} h_1) \tag{19}$$

where $u = \sum S_H(R^{(2)})R^{(1)}$, and S_A the (braided) antipode of A, becomes (see [14]) an ordinary Hopf algebra. This is the smash product Hopf algebra denoted $A \star H$.
Apart from the above described construction, it is worth mentioning two more important points proved in [13]: First, it is shown that if H is triangular and A is quasitriangular in the category ${}_H\mathcal{M}$, then $A \star H$ is (ordinarily) quasitriangular. Second, it is shown that the category of the braided modules of A (A-modules in ${}_H\mathcal{M}$) is equivalent to the category of the (ordinary) modules of $A \star H$.

2.1 AN EXAMPLE OF BOSONISATION

In the special case that A is some super-Hopf algebra, then: $H = \mathbb{CZ}_2$, equipped with it's non-trivial quasitriangular structure, formerly mentioned. In this case, the technique simplifies and the ordinary Hopf algebra produced is the smash product Hopf algebra $A \star \mathbb{CZ}_2$. The grading in A is induced by the $\mathbb{CZ}_2$-action on A:

$$g \triangleright a = (-1)^{|a|} a \tag{20}$$

for a homogeneous in A. Utilizing the non-trivial R-matrix R_g and using eq. (10) and eq. (16) we can readily deduce the form of the induced $\mathbb{CZ}_2$-coaction on A:

$$\rho(a) = \begin{cases} 1 \otimes a & , \quad a : \text{even} \\ g \otimes a & , \quad a : \text{odd} \end{cases} \tag{21}$$

Let us note here that instead of invoking the non-trivial quasitriangular structure R_g we could alternatively extract the (left) coaction (21) utilizing the self-duality of the $\mathbb{CZ}_2$ Hopf algebra: For any abelian group $\mathbb{G}$ a (left) action of $\mathbb{CG}$ coincides with a (right) action of $\mathbb{CG}$. On the other hand, for any finite group, a (right) action of $\mathbb{CG}$ is the same thing as a (left) coaction of the dual Hopf algebra $(\mathbb{CG})^*$. Since $\mathbb{CZ}_2$ is both finite and abelian and hence self-dual in the sense that: $\mathbb{CZ}_2 \cong (\mathbb{CZ}_2)^*$ as Hopf algebras, it is immediate to see that the (left) action (20) and the (left) coaction (21) are virtually the same thing.
The above mentioned action and coaction enable us to form the cross product algebra and the cross coproduct coalgebra according to the preceding discussion which finally form the smash product Hopf algebra $A \star \mathbb{CZ}_2$. The grading of A, is "absorbed" in $A \star \mathbb{CZ}_2$, and becomes an inner automorphism:

$$gag = (-1)^{|a|} a$$

where we have identified: $a\star 1\equiv a$ and $1\star g\equiv g$ in $A\star\mathbb{C}\mathbb{Z}_2$ and a homogeneous element in A. This inner automorphism is exactly the adjoint action of g on $A\star\mathbb{C}\mathbb{Z}_2$ (as an ordinary Hopf algebra). The following proposition is proved -as an example of the bosonisation technique- in [14]:

Proposition 2.1. *Corresponding to every super-Hopf algebra A there is an ordinary Hopf algebra $A\star\mathbb{C}\mathbb{Z}_2$, its bosonisation, consisting of A extended by adjoining an element g with relations, coproduct, counit and antipode:*

$$
\begin{array}{cccc}
g^2=1 & ga=(-1)^{|a|}ag & \Delta(g)=g\otimes g & \Delta(a)=\sum a_1 g^{|a_2|}\otimes a_2 \\
S(g)=g \quad S(a)=g^{-|a|}\underline{S}(a) & & \varepsilon(g)=1 & \varepsilon(a)=\underline{\varepsilon}(a)
\end{array}
\tag{22}
$$

where $\underline{S}$ and $\underline{\varepsilon}$ denote the original maps of the super-Hopf algebra A.
In the case that A is super-quasitriangular via the R-matrix $\underline{R}=\sum\underline{R}^{(1)}\otimes\underline{R}^{(2)}$, then the bosonised Hopf algebra $A\star\mathbb{C}\mathbb{Z}_2$ is quasitriangular (in the ordinary sense) via the R-matrix: $R=R_g\sum\underline{R}^{(1)}g^{|\underline{R}^{(2)}|}\otimes\underline{R}^{(2)}$. Moreover, the representations of the bosonised Hopf algebra $A\star\mathbb{C}\mathbb{Z}_2$ are precisely the super-representations of the original superalgebra A.

The application of the above proposition in the case of the parabosonic algebra P_B is straightforward: we immediately get it's bosonised form $P_{B(g)}$ which by definition is: $P_{B(g)}\equiv P_B\star\mathbb{C}\mathbb{Z}_2$ Utilizing equations (14) which describe the super-Hopf algebraic structure of the parabosonic algebra P_B, and replacing them into equations (22) which describe the ordinary Hopf algebra structure of the bosonised superalgebra, we immediately get the explicit form of the (ordinary) Hopf algebra structure of $P_{B(g)}\equiv P_B\star\mathbb{C}\mathbb{Z}_2$ which reads:

$$
\begin{array}{cccc}
\Delta(b_i^{\pm})=b_i^{\pm}\otimes 1+g\otimes b_i^{\pm} & \Delta(g)=g\otimes g & \varepsilon(b_i^{\pm})=0 & \varepsilon(g)=1 \\
S(b_i^{\pm})=b_i^{\pm}g=-gb_i^{\pm} & S(g)=g & g^2=1 & \{g,b_i^{\pm}\}=0
\end{array}
\tag{23}
$$

where we have again identified $b_i^{\pm}\star 1\equiv b_i^{\pm}$ and $1\star g\equiv g$ in $P_B\star\mathbb{C}\mathbb{Z}_2$. Notice that from now and till the end of this paper, we denote by $b_i^{\pm}$ the generators of the parabosonic algebra P_B. Finally, we can easily check that since $\mathbb{C}\mathbb{Z}_2$ is triangular (via R_g) and P_B is super-quasitriangular (trivially since it is super-cocommutative) it is an immediate consequence of the above proposition that $P_{B(g)}$ is quasitriangular (in the ordinary sense) via the R-matrix R_g.

2.2 AN ALTERNATIVE APPROACH

Let us describe now a slightly different construction (see also: [2, 10, 11]), which achieves the same object: the determination of an ordinary Hopf structure for the parabosonic algebra P_B.

Proposition 2.2. *Corresponding to the super-Hopf algebra P_B there is an ordinary Hopf algebra $P_{B(K^{\pm})}$, consisting of P_B extended by adjoining two elements K^+, K^- with relations, coproduct, counit and antipode:*

$$
\begin{array}{cc}
\Delta(b_i^{\pm})=b_i^{\pm}\otimes 1+K^{\pm}\otimes b_i^{\pm} & \Delta(K^{\pm})=K^{\pm}\otimes K^{\pm} \\
\varepsilon(b_i^{\pm})=0 & \varepsilon(K^{\pm})=1 \\
S(b_i^{\pm})=b_i^{\pm}K^{\mp} & S(K^{\pm})=K^{\mp} \\
K^+K^-=K^-K^+=1 & \{K^+,b_i^{\pm}\}=0=\{K^-,b_i^{\pm}\}
\end{array}
\tag{24}
$$

Proof. Consider the vector space $\mathbb{C}\langle b_i^+, b_j^-, K^\pm\rangle$ freely generated by the elements b_i^+, b_j^-, K^+, K^-. Denote $T(b_i^+, b_j^-, K^\pm)$ its tensor algebra. Let I_{BK} be the ideal of the tensor algebra generated by all the elements of the form $[\{b_i^\xi, b_j^\eta\}, b_k^\varepsilon] - (\varepsilon-\eta)\delta_{jk}b_i^\xi - (\varepsilon-\xi)\delta_{ik}b_j^\eta$, for all $\xi,\eta,\varepsilon=\pm 1$ and $i,j=1,2,\cdots 0$, together with all elements of the form $K^+K^- - 1$, $K^-K^+ - 1$, $\{K^+, b_i^\pm\}$, $\{K^-, b_i^\pm\}$, for all $i=1,2,\cdots$. We define:

$$P_{B(K^\pm)} = T(b_i^+, b_j^-, K^\pm)/I_{BK}$$

Consider the linear map $\Delta : \mathbb{C}\langle b_i^+, b_j^-, K^\pm\rangle \to P_{B(K^\pm)} \otimes P_{B(K^\pm)}$ determined by it's values on the basis elements, specified in equation (24). By the universality property of the tensor algebra, this map extends to an algebra homomorphism: $\Delta : T(b_i^+, b_j^-, K^\pm) \to P_{B(K^\pm)} \otimes P_{B(K^\pm)}$. We emphasize that the usual tensor product algebra $P_{B(K^\pm)} \otimes P_{B(K^\pm)}$ is now considered, with multiplication $(a\otimes b)(c\otimes d) = ac\otimes bd$ for any $a,b,c,d \in P_{B(K^\pm)}$. Now we can trivially verify that

$$\Delta(\{K^\pm, b_i^\pm\}) = \Delta(K^+K^- - 1) = \Delta(K^-K^+ - 1) = 0 \tag{25}$$

We also compute:

$$\Delta(\left[\{b_i^\xi, b_j^\eta\}, b_k^\varepsilon\right] - (\varepsilon-\eta)\delta_{jk}b_i^\xi - (\varepsilon-\xi)\delta_{ik}b_j^\eta) = 0 \tag{26}$$

Relations (25), and (26), mean that $I_{BK} \subseteq ker\Delta$ which in turn implies that Δ is uniquely extended as an algebra homomorphism from $P_{B(K^\pm)}$ to the usual tensor product algebra $P_{B(K^\pm)} \otimes P_{B(K^\pm)}$, with the values on the generators determined by (24).
Following the same procedure we construct an algebra homomorphism $\varepsilon : P_{B(K^\pm)} \to \mathbb{C}$ and an algebra antihomomorphism $S : P_{B(K^\pm)} \to P_{B(K^\pm)}$ which are completely determined by their values on the generators of $P_{B(K^\pm)}$ (i.e.: the basis elements of $\mathbb{C}\langle b_i^+, b_j^-, K^\pm\rangle$). Note that in the case of the antipode we start by defining a linear map S from $\mathbb{C}\langle b_i^+, b_j^-, K^\pm\rangle$ to the opposite algebra $P_{B(K^\pm)}^{op}$, with values determined by equation (24) and following the above described procedure we end up with an algebra anti-homomorphism: $S : P_{B(K^\pm)} \to P_{B(K^\pm)}$.
Now it is sufficient to verify the rest of the Hopf algebra axioms (i.e.: coassociativity of Δ, counity property for ε, and the compatibility condition which ensures us that S is an antipode) on the generators of $P_{B(K^\pm)}$. This can be done with straightforward computations (see [2]). □

Let us notice here, that the initiation for the above mentioned construction lies in the case of the finite degrees of freedom: If we consider the parabosonic algebra in $2n$ generators (n-paraboson algebra) and denote it $P_B^{(n)}$, it is possible to construct explicit realizations of the elements K^+ and K^- in terms of formal power series, such that the relations specified in (24) hold. The construction is briefly (see also [2]) as follows: We define

$$\mathcal{N} = \sum_{i=1}^{n} N_{ii} = \frac{1}{2}\sum_{i=1}^{n}\{b_i^+, b_i^-\}$$

We inductively prove:

$$[\mathcal{N}^m, b_i^+] = b_i^+((\mathcal{N}+1)^m - \mathcal{N}^m) \tag{27}$$

We now introduce the following elements:

$$K^+ = \exp(i\pi\mathcal{N}) \qquad K^- = \exp(-i\pi\mathcal{N})$$

Utilizing the above power series expressions and equation (27) we get

$$\{K^+, b_i^{\pm}\} = 0 \quad \{K^-, b_i^{\pm}\} = 0 \tag{28}$$

A direct application of the Baker-Campbell-Hausdorff formula leads also to:

$$K^+K^- = K^-K^+ = 1 \tag{29}$$

Finally let us make a few comments on the above mentioned constructions.
From the point of view of the structure, an obvious question arises: While $P_{B(g)}$ is a quasitriangular Hopf algebra through the R-matrix: R_g given in eq. (10), there is yet no suitable R-matrix for the Hopf algebra $P_{B(K^{\pm})}$. Thus the question of the quasitriangular structure of $P_{B(K^{\pm})}$ is open. On the other hand, regarding representations, we have already noted that the super representations of P_B ($\mathbb{Z}_2$-graded modules of P_B or equivalently: P_B-modules in ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$) are in "$1-1$" correspodence with the (ordinary) representations of $P_{B(g)}$. Although we do not have such a strong result for the representations of $P_{B(K^{\pm})}$, the preceding construction in the case of finite degrees of freedom enables us to uniquely extend the Fock-like representations of $P_B^{(n)}$ to representations of $P_{B(K^{\pm})}^{(n)}$. Since the Fock-like representations of P_B are unique up to unitary equivalence (see the proof in [8] or [19]), this is a point which deserves to be discussed analytically in a forthcoming work.

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