HOPF CATEGORIES

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Abstract. We introduce Hopf categories enriched over braided monoidal categories. The notion is linked to several recently developed notions in Hopf algebra theory, such as Hopf group (co)algebras, weak Hopf algebras and duoidal categories. We generalize the fundamental theorem for Hopf modules and some of its applications to Hopf categories.

Introduction

The starting point of this paper is enriched category theory. Given a (strict) monoidal category $\mathcal{V}$, we can consider the notion of $\mathcal{V}$-category. For example, if $\mathcal{V}$ is the category of sets, then a $\mathcal{V}$-category is an ordinary category. If $\mathcal{V}$ is the category of vector spaces, then a $\mathcal{V}$-category is a linear category. A $\mathcal{V}$-category with one object is an algebra (or monoid) in $\mathcal{V}$.
Now consider a braided monoidal category. The category $\underline{\mathcal{C}}(\mathcal{V})$ of coalgebras in $\mathcal{V}$ is a monoidal category, so we can consider $\underline{\mathcal{C}}(\mathcal{V})$-categories. A Hopf $\mathcal{V}$-category is a $\underline{\mathcal{C}}(\mathcal{V})$-category with an antipode. These definitions are designed in such a way that $\underline{\mathcal{C}}(\mathcal{V})$-categories, resp. Hopf $\mathcal{V}$-categories, with one object correspond to bialgebras, resp. Hopf algebras in $\mathcal{V}$. In the world of sets, the notion is not of great interest, since $\underline{\mathcal{C}}(\underline{\text{Sets}}) = \underline{\text{Sets}}$: it is well-known that every set has a unique structure of a coalgebra in $\underline{\text{Sets}}$. Hopf categories are groupoids, that is, categories in which every morphism is invertible. In fact, $\underline{\mathcal{C}}(\mathcal{V})$-categories only come to life when we pass to the k-linear world!
Hopf categories are related to several recent generalizations of Hopf algebras and monoidal categories. For example, Hopf group algebras and Hopf group coalgebras give rise to examples of Hopf categories, respectively over the category of vector spaces and its dual category, see Section 5. In Section 7 we will show that k-linear Hopf categories with a set of objects are Hopf monoids in the sense of [7] (in particular bimonoids in the sense of [1, 5]) in a suitable duoidal category. This also indicates the relation with other generalized Hopf-like structures, such as Hopf monads [10].

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Hopf categories with a finite number of objects can be used to construct examples of weak Hopf algebras, see Section 6. As we have mentioned above, groupoids are Hopf categories over sets. Applying the linearization functor, we obtain a Hopf category over the category of vector spaces, Putting this into packed form, we obtain a weak Hopf algebra, which turns out to be the groupoid algebra, the basic example of a weak Hopf algebra.
This brings us to duality. The second author made attempts to construct a satisfactory duality theory for group algebras, based on the philosophy developed in [12]. For Hopf categories, duality works. The dual of a (finite) Hopf $\mathcal{M}_k$-category (also termed a k-linear Hopf category) is a Hopf $\mathcal{M}_k^{\rm op}$-category, see Theorems 4.5 and 4.6. We also have a categorical version of the well-known property that C-comodules correspond to C^*-modules, in the case where C is a finitely generated projective coalgebra, see Proposition 4.4.
It also turns out that some well-known results about Hopf algebras can be generalized to Hopf categories. We mention a few first results. We have a categorical version of the important fact that the representation category of a bialgebra carries a monoidal structure, see Section 3. The fundamental theorem extends to Hopf categories, see Section 9.
It is well-known that Morita contexts can be viewed as k-linear categories with two objects. This is the starting point of Section 8, where the relationship between Hopf categories, H-Galois objects and Morita theory is investigated. It is possible to develop descent and Galois theory for Hopf categories, this is the topic of a forthcoming paper. Hopf categories are also related to partial actions of groups and Hopf algebras (see [2, 14, 15, 17]), this will be investigated in [4].

1. Preliminary results on enriched category theory

Let $(\mathcal{V},\otimes,k)$ be a monoidal category. We will assume that $\mathcal{V}$ is strict. Our results extend easily to arbitrary monoidal categories, in view of the classical result that every monoidal category is equivalent to a strict one, see for example [16]. For a class X, we construct a new monoidal category $\mathcal{V}(X)$. An object is a family of objects M in $\mathcal{V}$ indexed by $X\times X$:

$$M=(M_{x,y})_{x,y\in X}.$$

A morphism $\varphi:\ M\to N$ consists of a family of morphisms $\varphi_{x,y}:\ M_{x,y}\to N_{x,y}$ in $\mathcal{V}$, indexed by $X\times X$. The tensor product $M\bullet N$ is defined by the formula

$$(M\bullet N)_{x,y}=M_{x,y}\otimes N_{x,y},$$

and the unit object is J, with $J_{x,y}=k$, for all $x,y\in X$. To make our notation more transparent, we will write $J_{x,y}=ke_{x,y}$, where $e_{x,y}$ can be viewed as an elementary matrix.
We have a functor $(-)^{\rm op}:\ \mathcal{V}(X)\to\mathcal{V}(X)$. The opposite $V^{\rm op}$ of an object $V\in\mathcal{V}(X)$ is given by $V^{\rm op}_{y,x}=V_{x,y}$, for all $x,y\in X$, and the opposite $\varphi^{\rm op}$ of a morphism φ is given by $\varphi^{\rm op}_{y,x}=\varphi_{x,y}$.

From [9, Sec. 6.2], we recall the notion of a $\mathcal{V}$-category. A $\mathcal{V}$-category A consists of a class $|A| = X$, and an object $A \in \mathcal{V}(X)$ together with two classes of morphisms in $\mathcal{V}$, namely,

(1) the multiplication morphisms $m = m_{x,y,z} : A_{x,y} \otimes A_{y,z} \to A_{x,z}$, defined for each $x, y, z \in X$;
(2) unit morphisms $\eta_x : J_{x,x} = ke_{x,x} \to A_{x,x}$, defined for each $x \in X$,

such that the following associativity and unit conditions are satisfied:

$$m_{x,y,t} \circ (A_{x,y} \otimes m_{y,z,t}) = m_{x,z,t} \circ (m_{x,y,z} \otimes A_{z,t}) = m^2_{x,y,z,t}; \tag{1}$$

$$m_{x,x,y} \circ (\eta_x \otimes A_{x,y}) = A_{x,y} = m_{x,y,y} \circ (A_{x,y} \otimes \eta_y). \tag{2}$$

Observe that J is a $\mathcal{V}$-category; the multiplication maps $ke_{x,y} \otimes ke_{y,z} \to ke_{x,z}$ and the unit maps $ke_{x,x} \to ke_{x,x}$ are all the identity maps.
If $(\mathcal{V}, \otimes, k) = (\underline{\text{Sets}}, \times, \{*\})$, then a $\mathcal{V}$-category is an ordinary category. Indeed, for a $\underline{\text{Sets}}$-category A with underlying class X, set $\mathrm{Hom}_A(x, y) = A_{y,x}$. For $a \in \mathrm{Hom}_A(x, y) = A_{y,x}$ and $b \in \mathrm{Hom}_A(y, z) = A_{z,y}$, we define the composition $b \circ a = m_{z,y,x}(b, a)$. The unit morphism in $\mathrm{Hom}_A(x, x) = A_{x,x}$ is $\eta_x(*)$.
If $(\mathcal{V}, \otimes, k) = (\mathcal{M}_k, \otimes, k)$, the category of modules over a commutative ring k, then a $\mathcal{V}$-category is also called a k-linear category.
If $(\mathcal{V}, \otimes, k, c)$ is a braided monoidal category, then the tensor product $A \bullet B$ in $\mathcal{V}(X)$ of two $\mathcal{V}$-categories A and B is again a $\mathcal{V}$-category: the multiplication morphisms are the compositions

$$\begin{aligned} m^{A\bullet B}_{x,y,z} &= (m_{x,y,z} \otimes m_{x,y,z}) \circ (A_{x,y} \otimes c_{B_{x,y},A_{y,z}} \otimes B_{y,z}) : \\ & A_{x,y} \otimes B_{x,y} \otimes A_{y,z} \otimes B_{y,z} \to A_{x,y} \otimes A_{y,z} \otimes B_{x,y} \otimes B_{y,z} \to A_{x,z} \otimes B_{x,z}. \end{aligned}$$

$\mathcal{V}$-categories can be organized into a 2-category ${}_{\mathcal{V}}\underline{\underline{\text{Cat}}}$.
Let A and B be $\mathcal{V}$-categories, with underlying classes $|A| = X$ and $|B| = Y$. A $\mathcal{V}$-functor $f : A \to B$ consists of the following data: for each $x \in X$, we have $f(x) \in Y$, and we have morphisms

$$f_{x,y} : A_{x,y} \to B_{f(x),f(y)}$$

in $\mathcal{V}$ such that the following diagrams commute, for all $x, y, z \in X$:

$$\begin{array}{ccc} A_{x,y} \otimes A_{y,z} & \xrightarrow{m_{x,y,z}} & A_{x,z} \\ \downarrow{\scriptstyle f_{x,y}\otimes f_{y,z}} & & \downarrow{\scriptstyle f_{x,z}} \\ B_{f(x),f(y)} \otimes B_{f(y),f(z)} & \xrightarrow{m_{f(x),f(y),f(z)}} & B_{f(x),f(z)} \end{array} \qquad \begin{array}{ccc} ke_{x,x} & \xrightarrow{\eta_x} & A_{x,x} \\ & \searrow{\scriptstyle \eta_{f(x)}} & \downarrow{\scriptstyle f_{x,x}} \\ & & B_{f(x),f(x)} \end{array} \tag{3}$$

Let $f, g : A \to B$ be $\mathcal{V}$-functors. A $\mathcal{V}$-natural transformation $\alpha : f \Rightarrow g$ consists of a class of morphisms $\alpha_x : k \to B_{g(x),f(x)}$ in $\mathcal{V}$ such that the diagrams

$$\begin{array}{ccc}
A_{x,y} & \xrightarrow{g_{x,y}\otimes\alpha_y} & B_{g(x),g(y)}\otimes B_{g(y),f(y)} \\
\Big\downarrow{\scriptstyle \alpha_x\otimes f_{x,y}} & & \Big\downarrow{\scriptstyle m_{g(x),g(y),f(y)}} \\
B_{g(x),f(x)}\otimes B_{f(x),f(y)} & \xrightarrow{m_{g(x),f(x),f(y)}} & B_{g(x),f(y)}
\end{array}$$

commute, for all $x, y \in X$. We have a 2-category ${}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$ with $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformation as 0-cells, 1-cells and 2-cells. Let us describe the composition of 1-cells and 2-cells. Given 1-cells $f,\ f' :\ A \to B$ and $g,\ g' :\ B \to C$, $g \circ f :\ A \to C$ is given by the formulas

$$(g \circ f)_{x,y} = g_{f(x),f(y)} \circ f_{x,y} :\ A_{x,y} \to C_{(g\circ f)(x),(g\circ f)(y)}.$$

Now consider 2-cells $\alpha :\ f \Rightarrow f'$ and $\beta :\ g \Rightarrow g'$. $\alpha * \beta :\ g \circ f \Rightarrow g' \circ f'$ is defined as follows:

$$\begin{aligned}
(\alpha * \beta)_x &= m_{g'(f'(x)),g'(f(x)),g(f(x))} \circ ((g'_{f'(x),f(x)} \circ \alpha_x) \otimes \beta_{f(x)}) \\
&= m_{g'(f'(x)),g(f'(x)),g(f(x))} \circ (\beta_{f'(x)} \otimes (g_{f'(x),f(x)} \circ \alpha_x))
\end{aligned}$$

Now let $f, g, h :\ A \to B$ be 1-cells, and let $\alpha :\ f \Rightarrow g$, $\beta :\ g \Rightarrow h$ be 2-cells. We define the vertical decomposition $\beta \circ \alpha :\ f \Rightarrow h$ by the rule

$$(\beta \circ \alpha)_x = m_{h(x),g(x),f(x)} \circ (\beta_x \otimes \alpha_x).$$

Now fix a class X. A $\mathcal{V}$-category with underlying class X is called a $\mathcal{V}$-X-category. A $\mathcal{V}$-functor $f :\ A \to B$ between two $\mathcal{V}$-X-categories A and B is called a $\mathcal{V}$-X-functor if $f(x) = x$ for all $x \in X$, that is, f is the identity on objects. ${}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}(X)$ is the 2-subcategory of ${}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$ with $\mathcal{V}$-X-categories as 0-cells, $\mathcal{V}$-X-functors as 1-cells and $\mathcal{V}$-natural transformations as 2-cells.
If X is a singleton, then the 0-cells and 1-cells of ${}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}(X)$ are $\mathcal{V}$-algebras and $\mathcal{V}$-algebra morphisms. A 2-cell $\alpha :\ f \Rightarrow g$ between two algebra morphisms $f, g :\ A \to B$ is a morphism $\alpha :\ k \to B$ such that $m \circ (g \otimes \alpha) = m \circ (\alpha \otimes f)$. Consider the particular situation where $\mathcal{V} = \mathcal{M}_k$. Then morphisms α_x : $k \to B_{x,x}$ correspond to elements $\alpha_x \in B_{x,x}$, and a 2-cell $\alpha :\ f \Rightarrow g$ between two k-linear X-functors consists of elements $\alpha_x \in B_{x,x}$ such that

$$g_{x,y}(a)\alpha_y = \alpha_x f_{x,y}(a), \tag{4}$$

for all $a \in A_{x,y}$ and $x, y \in X$.
Let $(\mathcal{V}, \otimes, k)$ and $(\mathcal{W}, \square, l)$ be two strict monoidal categories. Recall that a monoidal functor $\mathcal{V} \to \mathcal{W}$ is a triple $(F, \varphi_0, \varphi_2)$, where $F :\ \mathcal{V} \to \mathcal{W}$ is a functor, $\varphi_0 :\ l \to F(k)$ is a morphism in $\mathcal{W}$, and $\varphi_2 :\ F\square F \Rightarrow F \circ \otimes$ is a natural transformation, satisfying certain properties, we refer to [16, XI.4] for detail. A monoidal functor is called strong if φ_0 and φ_2 are isomorphisms.

Proposition 1.1. *A monoidal functor $F :\ \mathcal{V} \to \mathcal{W}$ induces a bifunctor $F :\ {}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}} \to {}_{\mathcal{W}}\underline{\underline{\mathrm{Cat}}}$. If F is a strong monoidal equivalence of categories, then the induced bifunctor is a biequivalence.*

Proof. (sketch). Let A be a $\mathcal{V}$-category, and define $F(A)$ as follows: $F(A)_{x,y} = F(A_{x,y})$. The multiplication and unit maps are given by the formulas

$$\begin{aligned} m'_{x,y,z} &= F(m_{x,y,z}) \circ \varphi_2(A_{x,y}, A_{y,z}) \\ &: \ F(A_{x,y}) \otimes F(A_{y,z}) \to F(A_{x,y} \otimes A_{y,z}) \to F(A_{x,z}); \\ \eta'_x &= F(\eta_x) \circ \varphi_0 : \ l \to F(k) \to F(A_{x,x}). \end{aligned}$$

It is straightforward to show that $F(A)$ is a $\mathcal{W}$-category.
Now let $f : A \to B$ be a $\mathcal{V}$-functor. $F(f) : F(A) \to F(B)$ is given by the data

$$F(f)_{x,y} = F(f_{x,y}) : \ F(A_{x,y}) \to F(B_{f(x),f(y)}).$$

We leave it to the reader to show that $F(f)$ is a $\mathcal{W}$-functor.
Let $f, g : A \to B$ be $\mathcal{V}$-functors, and let $\alpha : f \to g$ be a $\mathcal{V}$-natural transformation. $F(\alpha)$ is defined as follows.

$$F(\alpha)_x = F(\alpha_x) \circ \varphi_0 : \ l \to F(k) \to F(B_{g(x),f(x)}).$$

$F(\alpha)$ is a $\mathcal{W}$-natural transformation, and $F : {}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}} \to {}_{\mathcal{W}}\underline{\underline{\mathrm{Cat}}}$ is a bifunctor. Further details are left to the reader. □

Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a monoidal category, and consider its opposite $\mathcal{V}^{\mathrm{op}} = (\mathcal{V}^{\mathrm{op}}, \otimes^{\mathrm{op}}, k)$. For later use, we provide a brief description of $\mathcal{V}^{\mathrm{op}}$-categories. A $\mathcal{V}^{\mathrm{op}}$-category consists of a class X, $A \in \mathcal{V}(X)$ and a collection of morphisms

$$m_{x,y,z} : \ A_{x,z} \to A_{y,z} \otimes A_{x,y} \ ; \ \eta_x : \ A_{x,x} \to k$$

in $\mathcal{V}$. A $\mathcal{V}^{\mathrm{op}}$-functor $f : A \to B$ consists of $f : X \to Y$ together with morphisms $f_{x,y} : B_{f(x),f(y)} \to A_{x,y}$ in $\mathcal{V}$. A $\mathcal{V}^{\mathrm{op}}$-natural transformation $\alpha : f \Rightarrow g$ consists of a collection of morphisms $\alpha_x : B_{g(x),f(x)} \to k$ in $\mathcal{V}$. We leave it to the reader to formulate all the necessary axioms that have to be satisfied.

2. Hopf categories

Let $\mathcal{V}$ be a strict braided monoidal category, and consider $\underline{\mathcal{C}}(\mathcal{V})$, the category of coalgebras (or comonoids) and coalgebra morphisms in $\mathcal{V}$. $\underline{\mathcal{C}}(\mathcal{V})$ is again a monoidal category: the tensor product of two coalgebras, resp. two coalgebra morphisms is again a coalgebra (resp. a coalgebra morphism), and the unit object k of $\mathcal{V}$ is a coalgebra.
Now we can consider $\underline{\mathcal{C}}(\mathcal{V})$-categories, that is, categories enriched in $\underline{\mathcal{C}}(\mathcal{V})$. According to the definitions in Section 1, a $\underline{\mathcal{C}}(\mathcal{V})$-category A consists of a class $|A| = X$, and coalgebras $A_{x,y}$, for all $x, y \in X$, together with coalgebra morphisms $m_{x,y,z} : A_{x,y} \otimes A_{y,z} \to A_{x,z}$ and $\eta_x : J_{x,x} = ke_{x,x} \to A_{x,x}$ satisfying (1-2).
The definition of a $\underline{\mathcal{C}}(\mathcal{V})$-category can be restated. Before we do this, we first make the elementary observation that a coalgebra in $\mathcal{V}(X)$ is an object $C \in \mathcal{V}(X)$, together with families of morphisms $\Delta_{x,y} : C_{x,y} \to C_{x,y} \otimes C_{x,y}$ and $\varepsilon_{x,y} : C_{x,y} \to J_{x,y} = ke_{x,y}$ such that $(C_{x,y}, \Delta_{x,y}, \varepsilon_{x,y})$ is a coalgebra in $\mathcal{V}$, for all $x, y \in X$. A coalgebra morphism between two coalgebras C and

D in $\mathcal{V}(X)$ is a morphism $f: C \to D$ in $\mathcal{V}(X)$ such that $f_{x,y}$ is a coalgebra map, for all $x, y \in X$.

Proposition 2.1. *Let X be a class and let $\mathcal{V}$ be a strict braided monoidal category. A $\underline{\mathcal{C}}(\mathcal{V})$-category with underlying class X is an object in $\mathcal{V}(X)$ which has the structure of $\mathcal{V}$-category and of a coalgebra in $\mathcal{V}(X)$ such that the morphisms $\Delta_{x,y}$ and $\varepsilon_{x,y}$ define $\mathcal{V}$-X-functors $\Delta: A \to A \bullet A$ and $\varepsilon: A \to J$.*

Proof. Assume that A is a $\mathcal{V}$-category and a coalgebra in $\mathcal{V}(X)$, and consider the following diagrams in $\mathcal{V}$.

(5)
$$\begin{array}{ccc}
A_{x,y} \otimes A_{y,z} & \xrightarrow{\;m_{x,y,z}\;} & A_{x,z} \\
{\scriptstyle \Delta_{x,y}\otimes\Delta_{y,z}}\downarrow & & \downarrow{\scriptstyle \Delta_{x,z}} \\
A_{x,y} \otimes A_{x,y} \otimes A_{y,z} \otimes A_{y,z} & \xrightarrow{\;m^{A\bullet A}_{x,y,z}\;} & A_{x,z} \otimes A_{x,z}
\end{array},$$

(6)
$$\begin{array}{ccc}
ke_{x,x} & \xrightarrow{\;\eta_x\;} & A_{x,x} \\
 & {\scriptstyle \eta_x\otimes\eta_x}\searrow & \downarrow{\scriptstyle \Delta_{x,x}} \\
 & & A_{x,x} \otimes A_{x,x}
\end{array},$$

(7)
$$\begin{array}{ccc}
A_{x,y} \otimes A_{y,z} & \xrightarrow{\;m_{x,y,z}\;} & A_{x,z} \\
{\scriptstyle \varepsilon_{x,y}\otimes\varepsilon_{y,z}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{x,z}} \\
ke_{x,y} \otimes ke_{y,z} & \xrightarrow{\;=\;} & ke_{x,z}
\end{array},$$

and

(8)
$$\begin{array}{ccc}
ke_{x,x} & \xrightarrow{\;\eta_x\;} & A_{x,x} \\
 & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \varepsilon_{x,x}} \\
 & & ke_{x,x}
\end{array}.$$

Δ is a $\mathcal{V}$-X-functor if and only if the diagrams (5) and (6) commute, for all $x, y, z \in X$. ε is a $\mathcal{V}$-X-functor if and only if the diagrams (7) and (8) commute, for all $x, y, z \in X$. $m_{x,y,z}$ is a coalgebra map if and only if (5) and (7) commute, and η_x is a coalgebra map if and only if (6) and (8) commute. □

Observe that $\underline{\mathcal{C}}(\mathcal{V})$-categories with one object correspond to bialgebras in $\mathcal{V}$. It follows from the results in Section 1 that $\underline{\mathcal{C}}(\mathcal{V})$-categories can be organized into a 2-category ${}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\text{Cat}}}$. In particular, a $\underline{\mathcal{C}}(\mathcal{V})$-functor between two $\underline{\mathcal{C}}(\mathcal{V})$-categories A and B is a $\mathcal{V}$-functor $f: A \to B$ such that every $f_{x,y}: A_{x,y} \to B_{x,y}$ is a morphism of coalgebras. For a fixed class

X, $\underline{\mathcal{C}}(\mathcal{V})$-categories with underlying class X can be organized into a 2-category ${}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\mathrm{Cat}}}(X)$. A $\underline{\mathcal{C}}(\mathcal{V})$-natural transformation between two $\underline{\mathcal{C}}(\mathcal{V})$-functors $f,\ g:\ A \to B$ consists of grouplike elements $\alpha_x \in B_{x,x}$ satisfying (4).

Let A be a $\mathcal{V}$-category, and consider its opposite A^{op} in $\mathcal{V}(X)$. A^{op} is also a $\mathcal{V}$-category, with multiplication morphisms

$$m^{\mathrm{op}}_{x,y,z} = m_{z,y,x} \circ c_{A_{y,x},A_{x,y}} :\ A^{\mathrm{op}}_{x,y} \otimes A^{\mathrm{op}}_{y,z} = A_{y,x} \otimes A_{z,y} \to A^{\mathrm{op}}_{x,z} = A_{z,x}$$

and unit morphisms $\eta^{\mathrm{op}}_x = \eta_x$. Observe that we need the inverse braiding here, compare to [26, 1.3].

Let C be a coalgebra in $\mathcal{V}(X)$. The coopposite coalgebra C^{cop} is equal to C as an object of $\mathcal{V}(X)$, with comultiplication morphisms

$$\Delta^{\mathrm{cop}}_{x,y} = c^{-1}_{C_{x,y},C_{x,y}} \circ \Delta_{x,y} :\ C_{x,y} \to C_{x,y} \otimes C_{x,y},$$

and counit morphisms $\varepsilon^{\mathrm{cop}}_{x,y} = \varepsilon_{x,y}$.

Proposition 2.2. *Let $\mathcal{V}$ be a strict braided monoidal category, and let A be a $\underline{\mathcal{C}}(\mathcal{V})$-category. Then A^{opcop} is also a $\underline{\mathcal{C}}(\mathcal{V})$-category.*

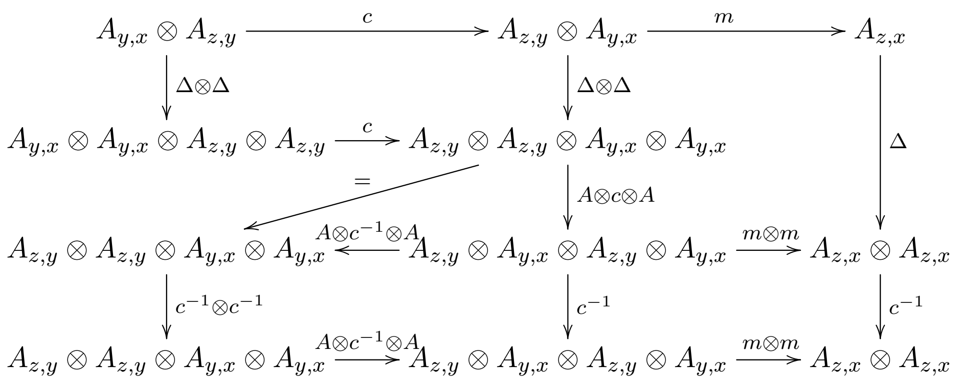
Proof. We have to show that the diagrams (5-8) applied to A^{opcop} commute. (5) takes the following form:

$$\begin{array}{ccc} A_{y,x} \otimes A_{z,y} & \xrightarrow{\ m^{\mathrm{op}}_{x,y,z}\ } & A_{z,x} \\ {\scriptstyle \Delta^{\mathrm{cop}}_{y,x} \otimes \Delta^{\mathrm{cop}}_{z,y}}\downarrow & & \downarrow{\scriptstyle \Delta^{\mathrm{cop}}_{z,x}} \\ A_{y,x} \otimes A_{y,x} \otimes A_{z,y} \otimes A_{z,y} & \xrightarrow{\ m^{\mathrm{op}}_{A\bullet A,x,y,z}\ } & A_{z,x} \otimes A_{z,x} \end{array} \tag{9}$$

From the axioms for a braiding c, we have the following formula, for all $A, B, C, D \in \mathcal{V}$:

$$c_{A\otimes B, C\otimes D} = (C \otimes c_{A,D} \otimes B) \circ (c_{A,C} \otimes c_{B,D}) \circ (A \otimes c_{B,C} \otimes D). \tag{10}$$

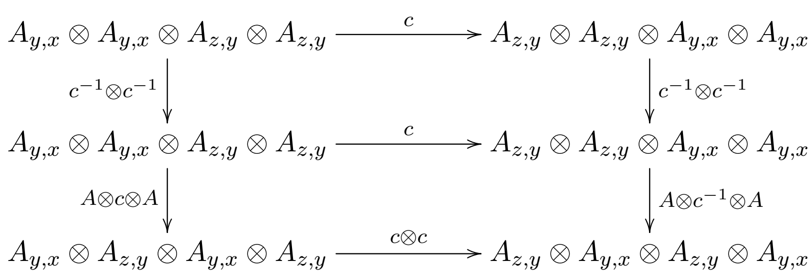
The triangle, the squares and the pentangle in the next diagram all commute: the top square commutes because c is natural; the pentangle is just (5); the bottom right square commutes because c^{-1} is natural; commutativity of the bottom left square follows from (10). We deleted the indices in the morphisms in the diagram; they are pretty obvious.



From the commutativity of the whole diagram, it follows that

$$\begin{aligned}\Delta^{\rm cop}_{z,x} \circ m^{\rm op}_{x,y,z} &= (m_{z,y,x} \otimes m_{z,y,x}) \circ (A_{z,y} \otimes c^{-1}_{A_{y,x},A_{x,y}} \otimes A_{y,x})\\ &\circ \quad (c^{-1}_{A_{z,y},A_{z,y}} \otimes c^{-1}_{A_{y,x},A_{y,x}}) \circ c_{A_{y,x}\otimes A_{y,x},A_{z,y}\otimes A_{z,y}} \circ (\Delta_{y,x} \otimes \Delta_{y,x}).\end{aligned}$$

The square at the top of the next diagram commutes because c is natural; commutativity of the bottom triangle follows from (10).



It follows that (9) commutes. The commutativity of the three other diagrams is obvious. □

Proposition 2.2 generalizes the fact that the opposite-cooposite of a bialgebra is again a bialgebra: take X a singleton. We refer to Sweedler [25] for the case where $\mathcal{V}$ is the category of vector spaces, and to [26, 1.6] for the case where $\mathcal{V}$ is an arbitrary braided monoidal category.

Definition 2.3. A Hopf $\mathcal{V}$-category is a $\underline{\mathcal{C}}(\mathcal{V})$-category A together with a morphism $S: A \to A^{\rm op}$ in $\mathcal{V}(X)$ such that

$$m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} = \eta_x \circ \varepsilon_{x,y} : A_{x,y} \to A_{x,x}; \tag{11}$$
$$m_{y,x,y} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} = \eta_y \circ \varepsilon_{x,y} : A_{x,y} \to A_{y,y}, \tag{12}$$

for all $x, y \in X$.

Observe that a Hopf $\mathcal{V}$-category with one object is a Hopf algebra in $\mathcal{V}$. If $\mathcal{V} = \mathcal{M}_k$, then a Hopf $\mathcal{V}$-category is also termed a k-linear Hopf category.

Example 2.4. Sets.
Let $\mathcal{V} = (\underline{\text{Sets}}, \times, \{*\})$. We have seen above that a $\mathcal{V}$-category is an ordinary category. It is well-known that every set G is in a unique way a coalgebra in $\underline{\text{Sets}}$: the comultiplication is the diagonal map $G \to G \times G$, sending g to (g, g). The counit is the unique map $G \to \{*\}$. This means that the categories $\underline{\text{Sets}}$ and $\mathcal{C}(\underline{\text{Sets}})$ are identical, and therefore the same is true for the 2-categories $\underline{\underline{\text{Cat}}} = {}_{\underline{\text{Sets}}}\underline{\underline{\text{Cat}}}$ and ${}_{\mathcal{C}(\underline{\text{Sets}})}\underline{\underline{\text{Cat}}}$.
Now let us investigate Hopf categories. Assume that G is a Hopf category. For all $x, y \in X = |G|$, we have a map $S_{x,y} : G_{x,y} \to G_{y,x}$, satisfying (11-12). Take $a \in G_{x,y}$, this means that $a : y \to x$ is a morphism in G. It is easily checked that (11) implies that $aS_{x,y}(a) = 1_x$ and that (12) implies that $S_{x,y}(a)a = 1_y$. This shows that every morphism of G is invertible, hence G is a groupoid. Conversely, it is easy to show that a groupoid is a Hopf category.

Proposition 2.5. *Let $\mathcal{V} = (\underline{\text{Sets}}, \times, \{*\})$. Then a Hopf $\mathcal{V}$-category is the same thing as a groupoid.*

Lemma 2.6. *Let A be a Hopf $\mathcal{V}$-category. Then the following statements hold, for all $x, y, z \in X$:*

$$S_{x,z} \circ m_{x,y,z} = m_{z,y,x} \circ (S_{y,z} \otimes S_{x,y}) \circ c_{A_{x,y},A_{y,z}}; \tag{13}$$

$$\Delta_{y,x} \circ S_{x,y} = c_{A_{y,x},A_{y,x}} \circ (S_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y}. \tag{14}$$

Proof. In order to make our computations more transparant, we introduce some notation. $A_{x,y} \otimes A_{y,z}$ is a coalgebra, with comultiplication

$$\Delta_{x,y,z} = (A_{x,y} \otimes c_{A_{x,y},A_{y,z}} \otimes A_{y,z}) \circ (\Delta_{x,y} \otimes \Delta_{y,z})$$

and counit $\varepsilon_{x,y,z} = \varepsilon_{x,y} \otimes \varepsilon_{y,z}$. (5) can be restated as

$$\Delta_{x,z} \circ m_{x,y,z} = (m_{x,y,z} \otimes m_{x,y,z}) \circ \Delta_{x,y,z}. \tag{15}$$

The coassociativity of $\Delta_{x,y,z}$ is expressed by the formula

$$\Delta^2_{x,y,z} = (\Delta_{x,y,z} \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z} = (A_{x,y} \otimes A_{y,z} \otimes \Delta_{x,y,z}) \circ \Delta_{x,y,z}. \tag{16}$$

Now consider the morphisms $f, g, h : A_{x,y} \otimes A_{y,z} \to Z_{z,x}$ given by the formulas

$$\begin{aligned} f &= m_{z,y,x} \circ (S_{y,z} \otimes S_{x,y}) \circ c_{A_{x,y},A_{y,z}}; \\ g &= S_{x,z} \circ m_{x,y,z}; \\ h &= m^3_{z,x,y,z,x} \circ (f \otimes A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta^2_{x,y,z}. \end{aligned}$$

We compute that

$$\begin{aligned} & m^2_{x,y,z,x} \circ (A_{x,y} \otimes A_{y,z} \otimes g) \circ \Delta_{x,y,z} \\ = \;& m_{x,z,x} \circ (A_{x,z} \otimes S_{x,z}) \circ (m_{x,y,z} \otimes m_{x,y,z}) \circ \Delta_{x,y,z} \\ \overset{(15)}{=} \;& m_{x,z,x} \circ (A_{x,z} \otimes S_{x,z}) \circ \Delta_{x,z} \circ m_{x,y,z} \\ \overset{(11)}{=} \;& \eta_x \circ \varepsilon_{x,z} \circ m_{x,y,z} \overset{(7)}{=} \eta_x \circ \varepsilon_{x,y,z}, \end{aligned}$$

and
$$h = m_{z,x,x} \circ (f \otimes \eta_x) \circ (A_{x,y} \otimes A_{y,z} \circ \varepsilon_{x,y,z}) \circ \Delta_{x,y,z} = f.$$
On the other hand, we have that
$$\begin{aligned}
&m^2_{z,x,y,z} \circ (f \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z}\\
&= m^3_{z,y,x,y,z} \circ (S_{y,z} \otimes S_{x,y} \otimes A_{x,y} \otimes A_{y,z}) \circ (c_{A_{x,y},A_{y,z}} \otimes A_{x,y} \otimes A_{y,z})\\
&\quad \circ (A_{x,y} \otimes c_{A_{x,y},A_{y,z}} \otimes A_{y,z}) \circ (\Delta_{x,y} \otimes \Delta_{y,z})\\
&= m^3_{z,y,x,y,z} \circ (S_{y,z} \otimes S_{x,y} \otimes A_{x,y} \otimes A_{y,z})\\
&\quad \circ (c_{A_{x,y}\otimes A_{x,y},A_{y,z}} \otimes A_{y,z}) \circ (\Delta_{x,y} \otimes \Delta_{y,z})\\
&\overset{(*)}{=} m^2_{z,y,y,z} \circ (c_{A_{y,y},A_{z,y}} \otimes A_{y,z}) \circ (m_{y,x,y} \otimes A_{z,y} \otimes A_{z,y})\\
&\quad \circ (S_{x,y} \otimes A_{x,y} \otimes S_{y,z} \otimes A_{y,z}) \circ (\Delta_{x,y} \otimes \Delta_{y,z})\\
&\overset{(12)}{=} m^2_{z,y,y,z} \circ (c_{A_{y,y},A_{z,y}} \otimes A_{y,z}) \circ (\eta_y \otimes A_{z,y} \otimes A_{z,y})\\
&\quad \circ (S_{y,z} \otimes A_{y,z}) \circ (\varepsilon_{x,y} \otimes \Delta_{y,z})\\
&= m^2_{z,y,y,z} \circ (A_{z,y} \otimes \eta_y \otimes A_{z,y}) \circ (S_{y,z} \otimes A_{y,z}) \circ (\varepsilon_{x,y} \otimes \Delta_{y,z})\\
&= m_{z,y,z} \circ (S_{y,z} \otimes A_{y,z}) \circ \Delta_{y,z} \circ (\varepsilon_{x,y} \otimes A_{y,z})\\
&\overset{(12)}{=} \eta_z \circ \varepsilon_{y,z} \circ (\varepsilon_{x,y} \otimes A_{y,z}) = \eta_z \circ \varepsilon_{x,y,z}.
\end{aligned}$$
At $(*)$, we used the naturality of the braiding c, resulting in the commutativity of the diagram
$$\begin{array}{ccc}
(A_{x,y} \otimes A_{x,y}) \otimes A_{y,z} & \xrightarrow{c_{A_{x,y}\otimes A_{x,y},A_{y,z}}} & A_{y,z} \otimes (A_{x,y} \otimes A_{x,y})\\
\downarrow{\scriptstyle (S_{x,y}\otimes A_{x,y})\otimes S_{y,z}} & & \downarrow{\scriptstyle S_{y,z}\otimes(S_{x,y}\otimes A_{x,y})}\\
(A_{y,x} \otimes A_{x,y}) \otimes A_{z,y} & \xrightarrow{c_{A_{y,x}\otimes A_{x,y},A_{z,y}}} & A_{z,y} \otimes (A_{y,x} \otimes A_{x,y})\\
\downarrow{\scriptstyle m_{y,x,y}\otimes A_{z,y}} & & \downarrow{\scriptstyle A_{z,y}\otimes m_{y,x,y}}\\
A_{y,y} \otimes A_{z,y} & \xrightarrow{c_{A_{y,y},A_{z,y}}} & A_{z,y} \otimes A_{y,y}
\end{array}$$
Finally,
$$\begin{aligned}
f = h &= m_{z,z,x} \circ ((\eta_z \circ \varepsilon_{x,y,z}) \otimes g) \circ \Delta_{x,y,z}\\
&= m_{z,z,x} \circ (\eta_z \otimes A_{z,x}) \circ g \circ (\varepsilon_{x,y,z}) \otimes A_{x,y} \otimes A_{y,z}) \circ \Delta_{x,y,z} = g.
\end{aligned}$$
This proves formula (13). (14) is proved using similar techniques. Now we consider $f, g, h:\ A_{x,y} \to A_{y,x} \otimes A_{y,x}$ given by the formulas
$$\begin{aligned}
f &= c_{A_{y,x},A_{y,x}} \circ (S_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y};\\
g &= \Delta_{y,x} \circ S_{x,y};\\
h &= m^2_{A\bullet A,y,x,y,x} \circ (g \otimes A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta^3_{x,y},
\end{aligned}$$
In the subsequent computations, the coassociativity of $m^{A\bullet A}$ will be used frequently. We first compute that
$$m^{A\bullet A}_{y,x,y} \circ (g \otimes A_{x,y} \otimes A_{x,y}) \circ \Delta^2_{x,y}$$

$$
\begin{aligned}
&= m^{A\bullet A}_{y,x,y} \circ (\Delta_{y,x} \otimes \Delta_{x,y}) \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
&\overset{(5)}{=} \Delta_{y,y} \circ m_{y,x,y} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
&\overset{(12)}{=} \Delta_{y,y} \circ \eta_\circ \varepsilon_{x,y} = \eta^{A\bullet A}_y \circ \varepsilon_{x,y}.
\end{aligned}
$$

It follows that

$$h = m^{A\bullet A}_{y,x,y} \circ (\eta^{A\bullet A}_y \otimes A_{y,x} \otimes A_{y,x}) \circ (\varepsilon_{x,y} \otimes f) \circ \Delta_{x,y} = f.$$

Now

$$
\begin{aligned}
& m^{A\bullet A}_{x,y,x} \circ (A_{x,y} \otimes A_{x,y} \otimes f) \circ \Delta^2_{x,y} \\
&= (m_{x,y,x} \otimes m_{x,y,x}) \circ (A_{x,y} \otimes c_{A_{x,y},A_{y,x}} \otimes A_{y,x}) \\
&\quad \circ (A_{x,y} \otimes A_{x,y} \otimes c_{A_{y,x},A_{y,x}}) \circ (A_{x,y} \otimes A_{x,y} \otimes S_{x,y} \otimes S_{x,y}) \circ \Delta^3_{x,y} \\
&= (m_{x,y,x} \otimes m_{x,y,x}) \circ (A_{x,y} \otimes c_{A_{x,y}\otimes A_{y,x},A_{y,x}}) \\
&\quad \circ (A_{x,y} \otimes A_{x,y} \otimes S_{x,y} \otimes S_{x,y}) \circ \Delta^3_{x,y} \\
&\overset{(x)}{=} (m_{x,y,x} \otimes m_{x,y,x}) \circ (A_{x,y} \otimes S_{x,y} \otimes A_{x,y} \otimes S_{x,y}) \\
&\quad \circ (A_{x,y} \otimes A_{x,y} \otimes \Delta_{x,y}) \circ (A_{x,y} \otimes c_{A_{x,y},A_{x,y}}) \circ \Delta^2_{x,y} \\
&\overset{(12)}{=} (m_{x,y,x} \otimes \eta_x) \circ (A_{x,y} \otimes S_{x,y}) \circ (A_{x,y} \otimes A_{x,y} \otimes \varepsilon_{x,y}) \\
&\quad \circ (A_{x,y} \otimes c_{A_{x,y},A_{x,y}}) \circ \Delta^2_{x,y} \\
&= (A_{x,x} \otimes \eta_x) \circ m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ (A_{x,y} \otimes \varepsilon_{x,y} \otimes A_{x,y}) \\
&\quad \circ (A_{x,y} \otimes \Delta_{x,y}) \circ \Delta_{x,y} \\
&= (A_{x,x} \otimes \eta_x) \circ m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \circ \Delta_{x,y} \\
&\overset{(12)}{=} (A_{x,x} \otimes \eta_x) \circ \eta_x \circ \varepsilon_{x,y} = (\eta_x \otimes \eta_x) \circ \varepsilon_{x,y}.
\end{aligned}
$$

At (x), we used the naturality of c, resulting in the commutative diagram

$$
\begin{array}{ccc}
A_{x,y} \otimes A_{x,y} & \xrightarrow{c_{A_{x,y},A_{x,y}}} & A_{x,y} \otimes A_{x,y} \\
\downarrow{\scriptstyle \Delta_{x,y}\otimes A} & & \downarrow{\scriptstyle A\otimes\Delta_{x,y}} \\
A_{x,y} \otimes A_{x,y} \otimes A_{x,y} & & A_{x,y} \otimes A_{x,y} \otimes A_{x,y} \\
\downarrow{\scriptstyle A_{x,y}\otimes S_{x,y}\otimes S_{x,y}} & & \downarrow{\scriptstyle S_{x,y}\otimes A_{x,y}\otimes S_{x,y}} \\
A_{x,y} \otimes A_{y,x} \otimes A_{y,x} & \xrightarrow{c_{A_{x,y}\otimes A_{y,x},A_{y,x}}} & A_{y,x} \otimes A_{x,y} \otimes A_{y,x}
\end{array}
$$

Finally

$$
\begin{aligned}
f = h &= m^{A\bullet A}_{y,x,y} \circ (g \otimes ((\eta_x \otimes \eta_x) \circ \varepsilon_{x,y}) \circ \Delta_{x,y} \\
&= m^{A\bullet A}_{y,x,y} \circ (A_{y,x} \otimes A_{y,x} \otimes \eta_{A\bullet A,x}) \circ g \circ (A_{x,y} \otimes \varepsilon_{x,y}) \circ \Delta_{x,y} = g
\end{aligned}
$$

□

Theorem 2.7. *Let A be a Hopf $\mathcal{V}$-category. The antipode $S: A \to A^{\mathrm{opcop}}$ is a $\underline{\mathcal{C}}(\mathcal{V})$-$X$-functor.*

Proof. First of all, we need to verify that every $S_{x,y}$ is a morphism in $\mathcal{C}(\mathcal{V})$, that is, $S_{x,y}:\ A_{x,y}\to A_{y,x}^{\rm cop}$ is a morphism of coalgebras. To this end, we need the commutativity of the next two diagrams

$$\begin{array}{ccc} A_{x,y} & \xrightarrow{\Delta_{x,y}} & A_{x,y}\otimes A_{x,y} \\ {\scriptstyle S_{x,y}}\downarrow & & \downarrow{\scriptstyle S_{x,y}\otimes S_{x,y}} \\ A_{y,x} & \xrightarrow{\Delta_{y,x}^{\rm cop}} & A_{y,x}\otimes A_{y,x} \end{array} \qquad\qquad \begin{array}{ccc} A_{x,y} & \xrightarrow{\varepsilon_{x,y}} & k \\ {\scriptstyle S_{x,y}}\downarrow & \nearrow{\scriptstyle \varepsilon_{y,x}} & \\ A_{y,x} & & \end{array}$$

The commutativity of the first diagram follows immediately from (14). For the second one, we proceed as follows:

$$\begin{aligned} \varepsilon_{x,y} &\overset{(8)}{=} \varepsilon_{x,x}\circ\eta_x\circ\varepsilon_{x,y}\overset{(11)}{=}\varepsilon_{x,x}\circ m_{x,y,x}\circ(A_{x,y}\otimes S_{x,y})\circ\Delta_{x,y}\\ &\overset{(7)}{=} (\varepsilon_{x,y}\otimes\varepsilon_{y,x})\circ(A_{x,y}\otimes S_{x,y})\circ\Delta_{x,y}=\varepsilon_{y,x}\circ S_{x,y}\circ(\varepsilon_{x,y}\otimes_{x,y})\Delta_{x,y}\\ &= \varepsilon_{y,x}\circ S_{x,y}. \end{aligned}$$

Now we show that S is a $\underline{\mathcal{C}}(\mathcal{V})$-functor. The diagrams (3) take the following form

$$\begin{array}{ccc} A_{x,y}\otimes A_{y,z} & \xrightarrow{m_{x,y,z}} & A_{x,z} \\ {\scriptstyle S_{x,y}\otimes S_{y,z}}\downarrow & & \downarrow{\scriptstyle S_{x,z}} \\ A_{y,x}\otimes A_{z,y} & \xrightarrow{m_{x,y,z}^{\rm op}} & A_{z,x} \end{array} \qquad\qquad \begin{array}{ccc} k & \xrightarrow{\eta_x} & A_{x,x} \\ & {\scriptstyle \eta_x}\searrow & \downarrow{\scriptstyle S_{x,x}} \\ & & A_{x,x} \end{array}$$

The commutativity of the first diagram follows from (13), after making the observation that $m_{x,y,z}^{\rm op}=m_{z,y,x}\circ c_{A_{y,x},A_{z,y}}$, and taking into account the formula

$$(S_{y,z}\otimes S_{x,y})\circ c_{A_{x,y},A_{y,z}}=c_{A_{y,x},A_{z,y}}\circ(S_{x,y}\otimes S_{y,z}),$$

resulting from the naturality of c. The commutativity of the second diagram goes as follows:

$$\begin{aligned} \eta_x &= (\varepsilon_{x,x}\otimes A_{x,x})\circ\Delta_x\circ\eta_x\overset{(6)}{=}(\varepsilon_{x,x}\otimes A_{x,x})\circ(\eta_x\otimes\eta_x)\\ &= (\varepsilon_{x,x}\circ\eta_x)\otimes\eta_x=\eta_x\circ\varepsilon_{x,x}\circ\eta_x\\ &\overset{(11)}{=} m_{x,x,x}\circ(A_{x,x}\otimes S_{x,x})\circ\Delta_{x,x}\circ\eta_x\\ &\overset{(6)}{=} m_{x,x,x}\circ(\eta_x\otimes A_{x,x})\circ S_{x,x}\circ\eta_x=S_{x,x}\circ\eta_x. \end{aligned}$$

□

Proposition 2.8. *Let A be a Hopf $\mathcal{V}$-category. For $x,y\in X$, consider the following statements:*

$$\eta_y\circ\varepsilon_{y,x} = m_{y,x,y}\circ(A_{y,x}\otimes S_{y,x})\circ\Delta_{y,x}^{\rm cop}\ ; \tag{17}$$

$$\eta_x\circ\varepsilon_{y,x} = m_{x,y,x}\circ(S_{y,x}\otimes A_{y,x})\circ\Delta_{y,x}^{\rm cop}\ ; \tag{18}$$

$$S_{y,x}\circ S_{x,y} = A_{x,y}\ ; \tag{19}$$

$$\eta_x\circ\varepsilon_{x,y} = m_{x,y,x}^{\rm op}\circ(S_{x,y}\otimes A_{x,y})\circ\Delta_{x,y}\ ; \tag{20}$$

$$\eta_y\circ\varepsilon_{x,y} = m_{y,x,y}^{\rm op}\circ(A_{x,y}\otimes S_{x,y})\circ\Delta_{x,y}. \tag{21}$$

The following implications hold:

$$\begin{array}{ccccc} (17) & \Longrightarrow & (19) & \Longrightarrow & (20)\\ & \nearrow & & \searrow & \\ (18) & & & & (21) \end{array}$$

Proof. $\underline{(17) \Rightarrow (19)}$. This goes in two steps. First we compute that

$$\begin{aligned} &m_{y,x,y} \circ (S_{x,y} \otimes (S_{y,x} \circ S_{x,y})) \circ \Delta_{x,y} \\ &\quad = \quad m_{y,x,y} \circ (A_{y,x} \otimes S_{y,x}) \circ c^{-1}_{A_{y,x},A_{y,x}} \circ c_{A_{y,x},A_{y,x}} \circ (S_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\ &\quad \overset{(14)}{=} \quad m_{y,x,y} \circ (A_{y,x} \otimes S_{y,x}) \circ \Delta^{\rm cop}_{y,x} \circ S_{x,y} \overset{(17)}{=} \eta_y \circ \varepsilon_{y,x} \circ S_{x,y} = \eta_y \circ \varepsilon_{x,y}. \end{aligned}$$

Then we compute that

$$m^2_{x,y,x,y} \circ (A_{x,y} \otimes S_{x,y} \otimes (S_{y,x} \circ S_{x,y})) \circ \Delta^2_{x,y}$$

is equal to

$$m_{x,y,y} \circ (A_{x,y} \otimes \eta_y) \circ (A_{x,y} \otimes \varepsilon_{x,y}) \circ \Delta_{x,y} = A_{x,y}$$

and, using (11), to

$$m_{x,x,y}(\eta_x \otimes A_{x,y}) \circ S_{y,x} \circ S_{x,y} \circ (\varepsilon_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} = S_{y,x} \circ S_{x,y}.$$

$\underline{(19) \Rightarrow (20)}$.

$$\begin{aligned} \eta_x \circ \varepsilon_{x,y} &= S_{x,x} \circ \eta_x \circ \varepsilon_{x,y} \overset{(11)}{=} S_{x,x} \circ m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\ &\overset{(13)}{=} \quad m_{x,y,x} \circ (S_{y,x} \otimes S_{x,y}) \circ c_{A_{x,y},A_{y,x}} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y} \\ &= \quad m_{x,y,x} \circ c_{A_{y,x},A_{x,y}} \circ (S_{x,y} \otimes (S_{y,x} \circ S_{x,y})) \circ \Delta_{x,y} \\ &\overset{(19)}{=} \quad m^{\rm op}_{x,y,x} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y}. \end{aligned}$$

The proof of the remaining two implications is similar. □

Corollary 2.9. *Suppose that $\mathcal{V}$ is a symmetric monoidal category. For a Hopf $\mathcal{V}$-category, the following assertions are equivalent:*

1. *(17) holds, for all $x, y \in X$;*
2. *(18) holds, for all $x, y \in X$;*
3. *$S_{y,x} \circ S_{x,y} = A_{x,y}$, for all $x, y \in X$.*

Proof. Using the naturality of c and the fact that c is a symmetry, we obtain that

$$\begin{aligned} &m^{\rm op}_{x,y,x} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\ &\qquad = \quad m_{x,y,x} \circ c_{A_{y,x},A_{x,y}} \circ (S_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\ &\qquad = \quad m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ c_{A_{x,y},A_{y,x}} \circ \Delta_{x,y} \\ &\qquad = \quad m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ c^{-1}_{A_{x,y},A_{y,x}} \circ \Delta_{x,y} \\ &\qquad = \quad m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta^{\rm cop}_{x,y}. \end{aligned}$$

This tells us that (20) considered for $(x, y) \in X \times X$ is equivalent to (17) considered for $(y, x) \in X \times X$. The statement now follows easily. □

Let A and B be Hopf $\mathcal{V}$-categories. A $\underline{\mathcal{C}}(\mathcal{V})$-functor $f:\ A\to B$ is called a Hopf $\mathcal{V}$-functor if

$$S^B_{f(x),f(y)}\circ f_{x,y}=f_{y,x}\circ S^A_{x,y}, \tag{22}$$

for all $x,y\in X$.

Proposition 2.10. *Let A and B be Hopf $\mathcal{V}$-categories. If $f:\ A\to B$ is a $\underline{\mathcal{C}}(\mathcal{V})$-functor, then it is also a Hopf $\mathcal{V}$-functor.*

Proof. Consider the morphisms $k,g,h:\ A_{x,y}\to F_{f(y),f(x)}$ defined by the formulas

$$k=S_{f(x),f(h)}\circ f_{x,y}\ ;\ g=f_{y,x}\circ S_{x,y}\ ;\ h=m^2_{f(y),f(x),f(y),f(x)}\circ(k\otimes f_{x,y}\otimes g)\circ\Delta^2_{x,y}.$$

We have that

$$\begin{aligned} &m_{f(x),f(y),f(x)}\circ(f_{x,y}\otimes g)\circ\Delta_{x,y}\\ &\quad= m_{f(x),f(y),f(x)}\circ(f_{x,y}\otimes f_{x,y})\circ(A_{x,y}\otimes S_{x,y})\circ\Delta_{x,y}\\ &\quad= f_{x,x}\circ m_{x,y,x}\circ(A_{x,y}\otimes S_{x,y})\circ\Delta_{x,y}\\ &\quad\overset{(11)}{=} f_{x,x}\circ\eta_x\circ\varepsilon_{x,y}=\eta_{f(x)}\circ\varepsilon_{x,y}, \end{aligned}$$

hence

$$h=m_{f(y),f(y),f(x)}\circ(B_{f(y),f(x)}\otimes\eta_{f(x)})\circ k\circ(A_{x,y}\otimes\varepsilon_{x,y})\circ\Delta_{x,y}=k.$$

We also have that

$$\begin{aligned} &m_{f(y),f(x),f(y)}\circ(k\otimes f_{x,y})\circ\Delta_{x,y}\\ &\quad= m_{f(y),f(x),f(y)}\circ(S_{f(x),f(y)}\otimes B_{f(x),f(y)})\circ(f_{x,y}\otimes f_{x,y})\circ\Delta_{x,y}\\ &\quad= m_{f(y),f(x),f(y)}\circ(S_{f(x),f(y)}\otimes B_{f(x),f(y)})\circ\Delta_{f(x),f(y)}\circ f_{x,y}\\ &\quad\overset{(12)}{=} \eta_{f(y)}\circ\varepsilon_{f(x),f(y)}\circ f_{x,y}=\eta_{f(y)}\circ\varepsilon_{x,y}, \end{aligned}$$

so that

$$k=h=m_{f(y),f(y),f(x)}\circ(\eta_{f(y)}\otimes B_{f(y),f(x)})\circ g\circ(\varepsilon_{x,y}\otimes A_{x,y})\circ\Delta_{x,y}=g.$$

□

We introduce ${}_{\mathcal{V}}\underline{\underline{\text{HopfCat}}}$ as the full 2-subcategory of ${}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\text{Cat}}}$, with Hopf $\mathcal{V}$-categories as 0-cells. For two Hopf $\mathcal{V}$-categories A and B, the category of morphisms $A\to B$ in ${}_{\mathcal{V}}\underline{\underline{\text{HopfCat}}}$ coincides with the category of morphisms $A\to B$ in ${}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\text{Cat}}}$. Thus 1-cells are Hopf $\mathcal{V}$-functors (in view of Proposition 2.10) and 2-cells are $\underline{\mathcal{C}}(\mathcal{V})$-natural transformations.

Proposition 2.11. *Let $F:\ \mathcal{V}\to\mathcal{W}$ be a strong monoidal functor. F induces bifunctors $F:\ {}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\text{Cat}}}\to{}_{\underline{\mathcal{C}}(\mathcal{W})}\underline{\underline{\text{Cat}}}$ and ${}_{\mathcal{V}}\underline{\underline{\text{HopfCat}}}\to{}_{\mathcal{W}}\underline{\underline{\text{HopfCat}}}$.*

Proof. F induces a strong monoidal functor $F:\ \underline{\mathcal{C}}(\mathcal{V})\to\underline{\mathcal{C}}(\mathcal{W})$. For a $\mathcal{V}$-coalgebra C, $F(C)$ is a $\mathcal{W}$-coalgebra. The comultiplication is $\varphi_2^{-1}\circ F(\Delta):\ F(C)\to F(C)\otimes F(C)\to F(C\otimes C)$, and the counit is $\varphi_0^{-1}\circ F(\varepsilon):\ F(C)\to F(k)\to l$.
Now apply Proposition 1.1 to $F:\ \underline{\mathcal{C}}(\mathcal{V})\to\underline{\mathcal{C}}(\mathcal{W})$. We obtain a bifunctor

$F: {}_{\mathcal{C}(\mathcal{V})}\underline{\underline{\mathrm{Cat}}} \to {}_{\mathcal{C}(\mathcal{W})}\underline{\underline{\mathrm{Cat}}}$. For a $\mathcal{C}(\mathcal{V})$-category A, we have that $F(A)_{x,y} = F(A_{x,y})$, with multiplication maps

$$F(m_{x,y,z}) \circ \varphi_2 : \; F(A_{x,y}) \otimes F(A_{y,z}) \to F(A_{x,y} \otimes A_{y,z}) \to F(A_{x,y})$$

and unit maps $F(\eta_x) \circ \varphi_0 : \; l \to F(k) \to F(A)$.
Now let A be a Hopf $\mathcal{V}$-category. We claim that the maps $F(S_{x,y}) : \; F(A_{x,y}) \to F(A_{y,x})$ define an antipode on $F(A)$. Let us show that (11) is satisfied. Using the fact that φ_2 is natural, we obtain that

$$\begin{aligned} & F(m_{x,y,x}) \circ \varphi_2 \circ (F(A_{x,y}) \otimes F(S_{x,y})) \circ \varphi_2^{-1} \circ F(\Delta_{x,y}) \\ = \; & F(m_{x,y,x}) \circ F(A_{x,y} \otimes S_{x,y}) \circ \varphi_2 \circ \varphi_2^{-1} \circ F(\Delta_{x,y}) \\ = \; & F(m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y}) \\ \overset{(11)}{=} \; & F(\eta_x \circ \varepsilon_{x,y}) = F(\eta_x) \circ \varphi_0 \circ \varphi_0^{-1} \circ F(\varepsilon_{x,y}), \end{aligned}$$

as needed. The proof of (12) is similar. □

Example 2.12. Consider the linearization functor $L : \; \underline{\mathrm{Sets}} \to \mathcal{M}_k$. It is well-known that L is strong monoidal, so, by Proposition 2.11, it sends Hopf categories (which are groupoids, see Proposition 5.2) to k-linear Hopf categories. More precisely, consider a groupoid G, and let $G_{x,y}$ be the set of maps from y to x. Then $L(G) = A$ is defined as follows:

$$A_{x,y} = kG_{x,y}.$$

The multiplication is the obvious one: the multiplication on G is extended linearly. $kG_{x,y}$ has the structure of grouplike coalgebra: $\Delta_{x,y}(g) = g \otimes g$ and $\varepsilon_{x,y}(g) = 1$ for $g \in G_{x,y}$. The antipode is given by the formula $S_{x,y}(g) = g^{-1} \in G_{y,x}$.

3. The representation category

Definition 3.1. Let A be a $\mathcal{V}$-category. A left A-module is an object M in $\mathcal{V}(X)$ together with a family of morphisms

$$\psi = \psi_{x,y,z} : \; A_{x,y} \otimes M_{y,z} \to M_{x,z}$$

in $\mathcal{V}$ such that the following associativity and unit conditions hold:

$$\psi_{x,y,u} \circ (A_{x,y} \otimes \psi_{y,z,u}) = \psi_{x,z,u} \circ (m_{x,y,z} \otimes M_{z,u}); \tag{23}$$

$$\psi_{x,x,y} \circ (\eta_x \otimes M_{x,y}) = M_{x,y}. \tag{24}$$

Let M and N be left A-modules. A morphism $\varphi : \; M \to N$ in $\mathcal{V}(X)$ is called left A-linear if

$$\varphi_{x,z} \circ \psi_{x,y,z} = \psi_{x,y,z} \circ (A_{x,y} \otimes \varphi_{y,z}) : \; A_{x,y} \otimes M_{y,z} \to N_{x,z}, \tag{25}$$

for all $x, y, z \in X$.

${}_A\mathcal{V}(X)$ will denote the category of left A-modules and left A-linear morphisms. Right A-modules and (A,B)-bimodules are defined in a similar way, and they form categories $\mathcal{V}(X)_A$ and ${}_A\mathcal{V}(X)_B$.

Proposition 3.2. *Let A be a $\mathcal{V}$-bicategory. Then there is a monoidal structure on ${}_A\mathcal{V}(X)$ such that the forgetful functor ${}_A\mathcal{V}(X)\to\mathcal{V}(X)$ is monoidal.*

Proof. Let M and N be left A-modules. We have a left A-action on $M\otimes N$ as follows:

$$\begin{aligned}
&(\psi_{x,y,z}\otimes\psi_{x,y,z})\circ(A_{x,y}\otimes c_{A_{x,y},M_{y,z}}\otimes N_{y,z})\circ(\Delta_{x,y}\otimes M_{y,z}\otimes N_{y,z}):\\
&\quad A_{x,y}\otimes M_{y,z}\otimes N_{y,z} \to A_{x,y}\otimes A_{x,y}\otimes M_{y,z}\otimes N_{y,z}\\
&\quad\to A_{x,y}\otimes M_{y,z}\otimes A_{x,y}\otimes N_{y,z}\\
&\quad\to M_{x,z}\otimes N_{x,z}=(M\otimes N)_{x,z}.
\end{aligned}$$

J is a left H-module with structure morphisms

$$\varepsilon_{x,y}\otimes ke_{y,z}:\ A_{x,y}\otimes ke_{y,z}\to ke_{x,y}\otimes ke_{y,z}=ke_{x,z}.$$

Verification of all the other details is left to the reader. □

4. Duality

4.1. **Dual $\mathcal{V}$-categories.** The notion of $\mathcal{V}$-category can be dualized. A dual $\mathcal{V}$-category C consists of a class $|C|=X$ and $C\in\mathcal{V}(X)$ together with two classes of morphisms in $\mathcal{V}$, namely

$$\Delta_{x,y,z}:\ C_{x,z}\to C_{x,y}\otimes C_{y,z} \text{ and } \varepsilon_x:\ C_{x,x}\to k,$$

satisfying the following coassociativity and counit conditions

$$\begin{aligned}
&(\Delta_{x,y,z}\otimes C_{z,u})\circ\Delta_{x,z,u}=(C_{x,y}\otimes\Delta_{y,z,u})\circ\Delta_{x,y,u};\\
&(\varepsilon_x\otimes C_{x,y})\circ\Delta_{x,x,y}=(C_{x,y}\otimes\varepsilon_y)\circ\Delta_{x,y,y}.
\end{aligned}$$

Dual $\mathcal{V}$-categories can be organized into a 2-category ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$. A 1-cell $f:\ C\to D$ between two dual $\mathcal{V}$-categories C and D is a dual $\mathcal{V}$-functor, and consists of the following data. For each $x\in X=|C|$, we have $f(x)\in Y=|D|$, and for each $x,y\in X$, the morphisms $f_{x,y}:\ D_{f(x),f(y)}\to C_{x,y}$ such that

$$\begin{aligned}
&(f_{x,y}\otimes f_{y,z})\circ\Delta_{f(x),f(y),f(z)}=\Delta_{x,y,z}\circ f_{x,z};\\
&\varepsilon_{f(x)}=\varepsilon_x\circ f_{x,x}.
\end{aligned}$$

Let $f,g:\ C\to D$ be dual $\mathcal{V}$-functors. A dual $\mathcal{V}$-natural transformation $\alpha:\ f\Rightarrow g$ consists of morphisms $\alpha_x:\ D_{f(x),g(x)}\to k$ in $\mathcal{V}$ such that

$$(f_{x,y}\otimes\alpha_y)\circ\Delta_{f(x),f(y),g(y)}=(\alpha_x\otimes g_{x,y})\circ\Delta_{f(x),g(x),g(y)},$$

for all $x,y\in X$. Dual $\mathcal{V}$-natural transformations are the 2-cells in ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$. The composition of 1-cells goes as follows. Let $f:\ C\to D$ and $g:\ D\to E$ be dual $\mathcal{V}$-functors. $g\circ f$ is defined by the formulas

$$(g\circ f)_{x,y}=f_{x,y}\circ g_{f(x),f(y)}:\ E_{(g\circ f)(x),(g\circ f)(y)}\to C_{x,y}.$$

Now let $f':\ C\to D$ and $g':\ D\to E$ be two more dual $\mathcal{V}$-functors, and let $\alpha:\ f\Rightarrow f'$ and $\beta:\ g\Rightarrow g'$ be dual $\mathcal{V}$-natural transformations. $\alpha*\beta:\ g\circ f\Rightarrow g'\circ f'$ is defined by the formulas

$$(\alpha*\beta)_x = \big(\beta_{f(x)}\otimes(\alpha_x\circ g'_{f(x),f'(x)})\big)\circ\Delta_{(g\circ f)(x),(g'\circ f)(x),(g'\circ f')(x)}$$

$$= \ \big((\alpha_x \circ g_{f(x),f'(x)}) \otimes \beta_{f'(x)}\big) \circ \Delta_{(g\circ f)(x),(g\circ f')(x),(g'\circ f')(x)}$$

Now let $f, g, h : C \to D$ be dual $\mathcal{V}$-functors, and let $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$ be dual $\mathcal{V}$-natural transformations. The vertical composition $\beta \circ \alpha : f \Rightarrow h$ is the following:

$$(\beta \circ \alpha)_x = (\alpha_x \otimes \beta_x) \circ \Delta_{f(x),g(x),h(x)} : B_{f(x),h(x)} \to k.$$

Let $\mathcal{V}^{\mathrm{op}} = (\mathcal{V}^{\mathrm{op}}, \otimes^{\mathrm{op}}, k)$ be the opposite of the monoidal category $\mathcal{V}$. Recall that $\mathrm{Hom}_{\mathcal{V}^{\mathrm{op}}}(M, N) = \mathrm{Hom}_{\mathcal{V}}(N, M)$, and that the opposite tensor product $\otimes$ is given by $M \otimes^{\mathrm{op}} N = N \otimes M$ and $f \otimes^{\mathrm{op}} g = g \otimes f$.

Proposition 4.1. *Let $\mathcal{V}$ be a strict monoidal category. Then the 2-categories ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$ and ${}_{\mathcal{V}^{\mathrm{op}}}\underline{\underline{\mathrm{Cat}}}$ are 2-isomorphic.*

Proof. (Sketch) We will define a 2-functor $F : {}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}} \to {}_{\mathcal{V}^{\mathrm{op}}}\underline{\underline{\mathrm{Cat}}}$. Take a dual $\mathcal{V}$-category C, with underlying class X, and consider $A = C^{\mathrm{op}}$ in $\mathcal{V}(X)$. We have $\mathcal{V}$-morphisms

$$\Delta_{x,y,z} : C_{x,z} = A_{z,x} \to C_{x,y} \otimes C_{y,z} = A_{z,y} \otimes^{\mathrm{op}} A_{y,x},$$

and $\mathcal{V}^{\mathrm{op}}$-morphisms

$$m_{z,y,x} = \Delta_{x,y,z} : A_{z,y} \otimes^{\mathrm{op}} A_{y,x} \to A_{z,x}.$$

Also $\eta_x = \varepsilon_x : k \to A_{x,x} = C_{x,x}$ is a $\mathcal{V}^{\mathrm{op}}$-morphism, and straightforward computations show that this makes A a $\mathcal{V}^{\mathrm{op}}$-category. We define $F(C) = A$.
Let $f : C \to D$ be a dual $\mathcal{V}$-functor, and let $F(D) = B$. For all $x, y \in X$, we have $\mathcal{V}$-morphisms

$$f_{x,y} : D_{f(x),f(y)} = B_{f(y),f(x)} \to C_{x,y} = A_{y,x}.$$

For all $x, y \in X$, let $g(x) = f(x)$ and $g_{y,x} = f_{x,y}$. Then $g_{y,x} : A_{y,x} \to B_{f(y),f(x)}$ is a $\mathcal{V}^{\mathrm{op}}$-morphism, and standard arguments tell us that $g : A \to B$ is a $\mathcal{V}^{\mathrm{op}}$-functor, and we define $F(f) = g$.
Finally let $f, f' : C \to D$ be dual $\mathcal{V}$-functors and let $\alpha : f \Rightarrow f'$ be a dual $\mathcal{V}$-natural transformation. For every $x \in X$, we have a $\mathcal{V}$-morphism $\alpha_x : B_{f'(x),f(x)} = D_{f(x),f'(x)} \to k$, and therefore a $\mathcal{V}^{\mathrm{op}}$-morphism $\alpha_x : k \to B_{f'(x),f(x)} = B_{g'(x),g(x)}$. We leave it to the reader to show that this defines a $\mathcal{V}^{\mathrm{op}}$-natural transformation $\alpha : g = F(f) \Rightarrow g' = F(f')$. We define $F(\alpha) = \alpha$. Standard computations show that F is a 2-functor. The inverse of F is defined in a similar way. □

A dual $\mathcal{V}$-category with underlying class X is called a dual $\mathcal{V}$-X-category. A dual $\mathcal{V}$-functor f between two dual $\mathcal{V}$-X-categories is called a dual $\mathcal{V}$-X-functor if $f(x) = x$, for all $x \in X$. ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}(X)$ is the subcategory of ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$, consisting of dual $\mathcal{V}$-X-categories, dual $\mathcal{V}$-X-functors and dual $\mathcal{V}$-natural transformations. As an immediate corollary of Proposition 4.1, we have the following result.

Corollary 4.2. *Let X be a class, and let $\mathcal{V}$ be a strict monoidal category. Then the 2-categories ${}_{\mathcal{V}^{\mathrm{op}}}\underline{\underline{\mathrm{Cat}}}(X)$ and ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}(X)$ are 2-isomorphic.*

If X is a singleton, then the objects in ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}(X)$ are $\mathcal{V}$-coalgebras. Deleting the non-unit 2-cells in ${}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}(X)$, we obtain $\underline{\mathcal{C}}(\mathcal{V})^{\mathrm{op}}$, the opposite of the category of coalgebras.

4.2. **Modules versus comodules.** We now consider $\mathcal{V} = (\mathcal{M}_k^{\mathrm{f}}, \otimes, k)$, the category of finitely generated projective modules over a commutative ring k, and its opposite $\mathcal{V}^{\mathrm{op}} = (\mathcal{M}_k^{\mathrm{fop}}, \otimes^{\mathrm{op}}, k)$. It is well-known that the functor $(-)^* : \mathcal{M}_k^{\mathrm{f}} \to \mathcal{M}_k^{\mathrm{fop}}$ taking a module M to its dual $M^* = \mathrm{Hom}(M, k)$ is an equivalence of categories. Moreover, we have a strong monoidal functor

$$((-)^*, \varphi_0, \varphi_2) : (\mathcal{M}_k^{\mathrm{f}}, \otimes, k) \to (\mathcal{M}_k^{\mathrm{fop}}, \otimes^{\mathrm{op}}, k).$$

Let $\varphi_0 : k \to (k)^* = k$ be the identity map. We now construct a natural isomorphism

$$\varphi_2 : \otimes^{\mathrm{op}} \circ ((-)^*, (-)^*) \Rightarrow (-)^* \circ \otimes.$$

For two finitely generated projective k-modules M and N, we need an isomorphism

$$\varphi_2(M, N) : M^* \otimes^{\mathrm{op}} N^* \to (M \otimes N)^*$$

in $\mathcal{M}_k^{\mathrm{fop}}$, or, equivalently, an isomorphism

$$\varphi_2(M, N) : (M \otimes N)^* \to N^* \otimes M^*$$

in $\mathcal{M}_k^{\mathrm{f}}$. It is well-known that the map

$$\iota : N^* \otimes M^* \to (M \otimes N)^*, \ \langle \iota(n^* \otimes m^*), m \otimes n\rangle = \langle n^*, n\rangle\langle m^*, m\rangle$$

is invertible, with inverse given by the formula

$$\iota^{-1}(\mu) = \sum_{i,j} \langle \mu, m_i \otimes n_j\rangle n_j^* \otimes m_i^*,$$

where $\sum_i m_i \otimes m_i^*$ and $\sum_j n_j \otimes n_j^*$ are the finite dual bases of M and N. We now define $\varphi_2(M, N)$ as the inverse of ι. As $((-)^*, \varphi_0, \varphi_2)$ is strong monoidal, it follows from Proposition 1.1 that we have a biequivalence between ${}_{\mathcal{M}_k^{\mathrm{f}}}\underline{\underline{\mathrm{Cat}}}$ and ${}_{\mathcal{M}_k^{\mathrm{fop}}}\underline{\underline{\mathrm{Cat}}}$. Applying Proposition 4.1, we find that ${}_{\mathcal{M}_k^{\mathrm{fop}}}\underline{\underline{\mathrm{Cat}}}$ is 2-isomorphic to ${}^{\mathcal{M}_k^{\mathrm{f}}}\underline{\underline{\mathrm{Cat}}}$. Combining these two biequivalences, we obtain the following result.

Theorem 4.3. *Let k be a commutative ring. $(-)^*$ induces a biequivalence*

$${}_{\mathcal{M}_k^{\mathrm{f}}}\underline{\underline{\mathrm{Cat}}} \to {}^{\mathcal{M}_k^{\mathrm{f}}}\underline{\underline{\mathrm{Cat}}}.$$

Let us describe this biequivalence at the level of 0-cells. Suppose that A is a k-linear category, with all underlying $A_{x,y}$ finitely generated and projective. First we have to apply the duality functor $(-)^*$, sending A to A^*, with $(A^*)_{x,y} = A^*_{x,y}$. In order to compute the multiplication and unit maps, we have to apply the construction sketched in the proof of Proposition 1.1. The multiplication is the following composition in $\mathcal{M}_k^{\mathrm{fop}}$:

$$m^*_{x,y,z} \circ \varphi_2(A_{x,y}, A_{y,z}) : A^*_{y,z} \otimes A^*_{x,y} \to (A_{x,y} \otimes A_{y,z})^* \to A^*_{x,z}.$$

The unit map is $\eta_x^*:\ k \to A_{x,x}^*$ in $\mathcal{M}_k^{\rm fop}$. To A^*, we apply the construction performed in the proof of Proposition 4.1, which sends A^* to C, with $C_{x,y} = A_{y,x}^*$. The comultiplication maps are the following maps in $\mathcal{M}_k^{\rm f}$:

$$\Delta_{z,y,x} = \varphi_2(A_{x,y}, A_{y,z}) \circ m_{x,y,z}^*:\ A_{x,z}^* = C_{z,x} \to A_{y,z}^* \otimes A_{x,y}^* = C_{z,y} \otimes C_{y,x}.$$

The counit maps are $\varepsilon_x = \eta_x^*:\ C_{x,x} = A_{x,x}^* \to k$.
Let us also give a brief description of the inverse construction. Let (C, Δ, ε) be a dual $\mathcal{M}_k^{\rm f}$-category. We will use the following Sweedler-Heyneman type notation: for $c \in C_{x,z}$, $\Delta_{x,y,z}(c) = c_{(1,y)} \otimes c_{(2,y)} \in C_{x,y} \otimes C_{y,z}$. Let $A \in \mathcal{M}_k^{\rm fop}(X)$ be defined as $A_{x,y} = C_{y,x}^*$. The multiplication map $m_{x,y,z}:\ A_{x,y} \otimes A_{y,z} \to A_{x,z} = C_{z,x}^*$ is defined by the formula

$$\langle ab, c\rangle = \langle a, c_{(2,y)}\rangle\langle b, c_{(1,y)}\rangle.$$

for $a \in A_{x,y}$, $b \in A_{y,z}$, $c \in C_{z,x}$. The unit elements are $\varepsilon_x \in C_{x,x}^* = A_{x,x}$.

Let C be a dual k-linear category. A right C-comodule M is an object $M \in \mathcal{V}(X)$ together with a family of maps

$$\rho_{x,y,z}:\ M_{x,z} \to M_{x,y} \otimes C_{y,z}$$

such that the coassociativity and counit conditions (26-27) are satisfied. For $m \in M_{x,z}$, we will write

$$\rho_{x,y,z}(m) = m_{[0,y]} \otimes m_{[1,y]}.$$

For all $m \in M_{x,z}$, we need that

$$m_{[0,y][0,u]} \otimes m_{[0,y][1,u]} \otimes m_{[1,y]} = m_{[0,u]} \otimes m_{[1,u](1,y)} \otimes m_{[1,u](2,y)}, \tag{26}$$

in $M_{x,u} \otimes C_{u,y} \otimes C_{y,z}$, and

$$m_{[0,z]}\varepsilon_z(m_{[1,z]}) = m. \tag{27}$$

Proposition 4.4. *Let k be a commutative ring, and let C be a dual k-linear category,with underlying class X, and with all $C_{x,y}$ finitely generated and projective. Let A be the corresponding k-linear category. Then the categories $\mathcal{M}_k^{\rm fop}(X)^C$ and $\mathcal{M}_k^{\rm f}(X)_A$ are isomorphic.*

Proof. Let M be a right C-comodule. We have the structure maps

$$\rho_{x,y,z}:\ M_{x,z} \to M_{x,y} \otimes C_{y,z}$$

Now we claim that M is also a right A-module, with structure maps

$$\psi_{x,z,y}:\ M_{x,z} \otimes A_{z,y} \to M_{x,y},\ \psi_{x,z,y}(m \otimes a) = ma = \langle a, m_{[1,y]}\rangle m_{[0,y]}.$$

Let us first show that this right A-action is associative. Take $m \in M_{x,z}$, $a \in A_{z,y}$ and $b \in A_{y,u}$. Then

$$\begin{aligned}(ma)b &= \langle a, m_{[1,y]}\rangle\langle b, m_{[0,y][1,u]}\rangle m_{[0,y][0,u]}\\ &\overset{(26)}{=} \langle a, m_{[1,u](2,y)}\rangle\langle b, m_{[1,u](1,y)}\rangle m_{[0,u]}\\ &= \langle ab, m_{[1,u]}\rangle m_{[0,u]} = m(ab).\end{aligned}$$

Now we prove the unit property. The unit element of $A_{x,x}$ is ε_x, and for all $m\in M_{x,x}$, we have that $m\varepsilon_x=\langle\varepsilon_x,m_{[1,x]}\rangle m_{[0,x]}=m$.
Conversely, let M be a right A-module. As before, let $\sum_i a_i^{y,z}\otimes c_i^{y,z}\in A_{z,y}\otimes C_{y,z}$ be the finite dual basis of $C_{y,z}$. We define a right C-coaction on M, via the structure maps

$$\rho_{x,y,z}:\ M_{x,z}\to M_{x,y}\otimes C_{y,z},\ \ \rho_{x,y,z}(m)=\sum_i ma_i^{y,z}\otimes c_i^{y,z}.$$

It is straightforward to show that this makes M into a right C-comodule.
These two constructions are inverses. First we start with a right C-coaction on M. The above construction then provides a right A-action on M, and the a new right C-coaction $\tilde\rho$, which coincides with the original ρ. Indeed, for all $m\in M_{x,z}$, we have that

$$\begin{aligned}\tilde\rho_{x,y,z}(m) &= \sum_i ma_i^{y,z}\otimes c_i^{y,z}=\sum_i\langle a_i^{y,z},m_{[1,y]}\rangle m_{[0,y]}\otimes c_i^{y,z}\\ &= m_{[0,y]}\otimes m_{[1,y]}=\rho_{x,y,z}(m).\end{aligned}$$

Now start from a right A-action on M. Applying the two constructions from above, we arrive first at a right C-coaction on M, and then a new right A-action that coincides with the original one: for $m\in M_{x,z}$ and $a\in A_{z,y}$, we have that

$$m\cdot a=\langle a,m_{[1,y]}\rangle m_{[0,y]}=\sum_i\langle a,c_i^{y,z}\rangle ma_i^{y,z}=ma.$$

□

4.3. **Duality between Hopf categories and dual Hopf categories.** $(-)^*$ induces an equivalence of categories $(-)^*:\ \underline{\mathcal{C}}(\mathcal{M}_k^{\rm f})\to\underline{\mathcal{C}}(\mathcal{M}_k^{\rm fop})$. Observing that the categories $\underline{\mathcal{C}}(\mathcal{M}_k^{\rm fop})$ and $\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})^{\rm op}$ are isomorphic, we obtain an equivalence of categories

$$(-)^*:\ \underline{\mathcal{C}}(\mathcal{M}_k^{\rm f})\to\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})^{\rm op}.$$

Let us compute the algebra structure on the dual C^* of a coalgebra C. The coalgebra structure in $\mathcal{M}_k^{\rm fop}$ is the composition

$$\varphi_2(C,C)^{-1}\circ\Delta^*:\ C^*\to(C\otimes C)^*\to C^*\otimes C^*,$$

in $\mathcal{M}_k^{\rm fop}$ which is the composition

$$m=\Delta^*\circ\iota:\ C^*\otimes C^*\to(C\otimes C)^*\to C^*.$$

It easily computed that m is the opposite of the convolution product, that is $m(c^*\otimes d^*)=c^*d^*$, with $\langle c^*d^*,c\rangle=\langle c^*,c_{(2)}\rangle\langle d^*,c_{(1)}\rangle$. Now we claim that we have a strong monoidal equivalence

$$((-)^*,\varphi_0,\varphi_2):\ (\underline{\mathcal{C}}(\mathcal{M}_k^{\rm f}),\otimes,k)\to(\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})^{\rm op},\otimes^{\rm op},k).$$

φ_0 is again the identity on k, and

$$\varphi_2(C,D):\ D^*\otimes C^*\to(C\otimes D)^*$$

in $\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})^{\rm op}$ is the inverse of the map ι defined above. It follows from Proposition 1.1 that $(-)^*$ induces a biequivalence

$$(-)^*: \ {}_{\underline{\mathcal{C}}(\mathcal{M}_k^{\rm f})}\underline{\underline{\rm Cat}} \to {}_{\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})^{\rm op}}\underline{\underline{\rm Cat}}.$$

We now from Proposition 4.1 that ${}_{\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})^{\rm op}}\underline{\underline{\rm Cat}}$ is 2-isomorphic to ${}^{\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})}\underline{\underline{\rm Cat}}$. Hence we have the following result.

Theorem 4.5. *Let k be a commutative ring. We have a biequivalence*

$${}_{\underline{\mathcal{C}}(\mathcal{M}_k^{\rm f})}\underline{\underline{\rm Cat}} \to {}^{\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})}\underline{\underline{\rm Cat}}.$$

For a ${}_{\underline{\mathcal{C}}(\mathcal{M}_k^{\rm f})}\underline{\underline{\rm Cat}}$-category A, we provide the corresponding dual ${}^{\underline{\mathcal{A}}(\mathcal{M}_k^{\rm f})}\underline{\underline{\rm Cat}}$-category C. First we have to apply the duality functor $(-)^*$, sending A to A^*, with $(A^*)_{x,y} = A^*_{x,y}$. Then we apply the construction performed in the proof of Proposition 4.1, which sends A^* to C, with $C_{x,y} = A^*_{y,x}$. From Theorem 4.3, we already know the dual k-linear category structure on C. Each $C_{x,y} = A^*_{y,x}$ is a k-coalgebra, with opposite convolution as multiplication, and $1_{x,y} = \varepsilon_{y,x}$ as unit element.

Let us also give a brief description of the inverse construction. Let (C,Δ,ε) be a dual $\mathcal{M}_k^{\rm f}$-category. The k-linear category structure on A has already been given in the comments following Theorem 4.3. Each $A_{x,y} = C^*_{y,x}$ is a k-coalgebra with comultiplication

$$\Delta(a) = \sum_{i,j} \langle a, c_i c_j\rangle a_j^* \otimes a_i^*,$$

where $\sum_i c_i \otimes a_i \in C_{y,x}\otimes A_{x,y}$ is the dual basis of $C_{y,x}$.

Let C be a dual $\mathcal{V}$-category. C is called a dual Hopf $\mathcal{V}$-category if there exist morphisms $S_{x,y}: \ C_{y,x}\to C_{x,y}$ in $\mathcal{V}$ such that

$$m_{x,y}\circ (C_{x,y}\otimes S_{x,y})\circ \Delta_{x,y,x} \ = \ \eta_{x,y}\circ\varepsilon_x; \tag{28}$$

$$m_{y,x}\circ (S_{y,x}\otimes C_{y,x})\circ \Delta_{x,y,x} \ = \ \eta_{y,x}\circ\varepsilon_x. \tag{29}$$

Theorem 4.6. *Let k be a commutative ring. In the biequivalence from Theorem 4.5, Hopf $\mathcal{M}_k^{\rm f}$-categories correspond to dual Hopf $\mathcal{M}_k^{\rm f}$-categories.*

Proof. Assume that C is a dual Hopf $\mathcal{M}_k^{\rm f}$-category with antipode S, and let A be the corresponding Hopf $\mathcal{M}_k^{\rm f}$-category. We claim that T defined by

$$T_{x,y} = S^*_{y,x}: \ A_{x,y}\to A_{y,x}$$

is an antipode for A. We have to show that (51) holds. The first formula in (51) reduces to

$$a_{(1)}T_{x,y}(a_{(2)}) = \langle a, 1_{y,x}\rangle\varepsilon_x,$$

in $A_{x,x} = C^*_{x,x}$, for all $a\in A_{x,y}$. For all $c\in C_{x,x}$, we have that

$$\begin{aligned}\langle a_{(1)}T_{x,y}(a_{(2)}), c\rangle &= \langle a_{(1)}, c_{(2,y)}\rangle\langle T_{x,y}(a_{(2)}), c_{(1,y)}\rangle\\ &= \ \langle a_{(1)}, c_{(2,y)}\rangle\langle a_{(2)}, S_{y,x}(c_{(1,y)})\rangle\end{aligned}$$

$$= \quad \langle a, S_{y,x}(c_{(1,y)})c_{(2,y)}\rangle \overset{(29)}{=} \langle a, 1_{y,x}\rangle\langle\varepsilon_x, c\rangle.$$

The second formula in (51) is proved in a similar way. □

5. Hopf categories and Hopf group (co)algebras

Let $(\mathcal{V}, \otimes, k)$ be a monoidal category. A group graded $\mathcal{V}$-algebra consists of a group G together with a family of objects $A = \{A_\sigma \mid \sigma \in G\}$ in $\mathcal{V}$ and morphisms

$$m_{\sigma,\tau} : A_\sigma \otimes A_\tau \to A_{\sigma\tau} \; ; \; \eta : k \to A_e$$

in $\mathcal{V}$ such that the following associativity and unit properties hold, for all $\sigma, \tau, \rho \in G$:

$$\begin{aligned} m_{\sigma\tau,\rho} \circ (m_{\sigma,\tau} \otimes A_\rho) &= m_{\sigma,\tau\rho} \circ (A_\sigma \otimes m_{\tau,\rho}); \\ m_{e,\sigma} \circ (\eta \otimes A_\sigma) &= m_{\sigma,e} \circ (A_\sigma \otimes \eta) = A_\sigma. \end{aligned}$$

Consider the case where $\mathcal{V}$ is the category of modules over a commutative ring k, and let $A = \{A_\sigma \mid \sigma \in G\}$ be a graded algebra. Then $A = \oplus_{\sigma\in G} A_\sigma$ is a G-graded algebra in the usual sense (see [22] for the general theory of graded algebras), and is called a graded algebra in packed form. Graded algebras can be organized into a 2-category ${}_{\mathcal{V}}\underline{\underline{\text{gr}}}$.

A 1-cell $f : (G, A) \to (H, B)$ consists of a a group morphism $f : G \to H$ together with a family of morphisms $f_\sigma : A_\sigma \to B_{f(\sigma)}$ in $\mathcal{V}$ such that $f_{\sigma\tau} \circ m_{\sigma,\tau} = m_{f(\sigma),f(\tau)} \circ (f_\sigma \otimes f_\tau)$ and $f_e \circ \eta = \eta$.

Let $f, g : (G, A) \to (H, B)$ be 1-cells; a 2-cell $\alpha : f \Rightarrow g$ consists of a family of morphisms $\alpha_\sigma : k \to B_{g(\sigma)^{-1}f(\sigma)}$ such that the following diagrams commute:

$$\begin{array}{ccc} A_{\sigma^{-1}\tau} & \xrightarrow{g_{\sigma^{-1}\tau}\otimes\alpha_\tau} & B_{g(\sigma)^{-1}g(\tau)} \otimes B_{g(\tau)^{-1}f(\tau)} \\ \Big\downarrow{\scriptstyle \alpha_\sigma\otimes f_{\sigma^{-1}\tau}} & & \Big\downarrow{\scriptstyle m_{g(\sigma)^{-1}g(\tau),g(\tau)^{-1}f(\tau)}} \\ B_{g(\sigma)^{-1}f(\sigma)} \otimes B_{f(\sigma)^{-1}f(\tau)} & \xrightarrow{m_{g(\sigma)^{-1}f(\sigma),f(\sigma)^{-1}f(\tau)}} & B_{g(\sigma)^{-1}f(\tau)} \end{array}$$

We have the dual notion of graded coalgebra. A group graded coalgebra in $\mathcal{V}$ consists of a group G together with a family of objects $C = \{C_\sigma \mid \sigma \in C\}$ in $\mathcal{V}$ and morphisms

$$\Delta_{\sigma,\tau} : C_{\sigma\tau} \to C_\sigma \otimes C_\tau \; ; \; \varepsilon : C_e \to k$$

such that

$$\begin{aligned} (\Delta_{\sigma,\tau} \otimes C_\rho) \circ \Delta_{\sigma\tau,\rho} &= (C_\sigma \otimes \Delta_{\tau,\rho}) \circ \Delta_{\sigma,\tau\rho} \\ (\varepsilon \otimes C_\rho) \circ \Delta_{e,\sigma} &= (C_\sigma \otimes \varepsilon) \circ \Delta_{\sigma,e} = C_\sigma. \end{aligned}$$

Let $\mathcal{V} = \mathcal{M}_k$, and suppose that G is a finite group. If C is a G-graded coalgebra, then $\oplus_{\sigma\in G} C_\sigma$ is a G-graded coalgebra in the sense of [21].

Graded coalgebras can be organized into a 2-category ${}^{\mathcal{V}}\underline{\underline{\text{gr}}}$.

A 1-cell $f : (G, C) \to (H, D)$ is a morphism of graded coalgebras. This consists of a a group morphism $f : G \to H$ together with a family of

morphisms $f_\sigma : D_{f(\sigma)} \to C_\sigma$ such that $(f_\sigma \otimes f_\tau) \circ \Delta_{f(\sigma),f(\tau)} = \Delta_{\sigma,\tau} \circ f_{\sigma\tau}$ and $\varepsilon \circ f_e = \varepsilon$.
Now let $f, g : C \to D$ be 1-cells. A 2-cell $\alpha : f \to g$ consists of a family of morphisms $\alpha_\sigma : D_{f(\sigma)^{-1}g(\sigma)} \to k$ such that

$$(f_{\sigma^{-1}\tau} \otimes \alpha_\tau) \circ \Delta_{f(\sigma)^{-1}f(\tau),f(\tau)^{-1}g(\tau)} = (\alpha_\sigma \otimes g_{\sigma^{-1}\tau}) \circ \Delta_{f(\sigma)^{-1}g(\sigma),g(\sigma)^{-1}g(\tau)}.$$

Proposition 5.1. *Let $\mathcal{V}$ be a strict monoidal category. Then the 2-categories ${}^{\mathcal{V}}\underline{\underline{\mathrm{gr}}}$ and ${}_{\mathcal{V}^{\mathrm{op}}}\underline{\underline{\mathrm{gr}}}$ are 2-isomorphic.*

Proof. The proof is similar to the proof of Proposition 4.1. We will describe the 2-functor $F : {}^{\mathcal{V}}\underline{\underline{\mathrm{gr}}}$ and ${}_{\mathcal{V}^{\mathrm{op}}}\underline{\underline{\mathrm{gr}}}$. Let (G, C) be a graded coalgebra, and let $F(G, C) = (G, A)$, with $A_\sigma = C_{\sigma^{-1}}$. The multiplication map $m_{\sigma,\tau} : A_\sigma \otimes^{\mathrm{op}} A_\tau \to A_{\sigma\tau}$ in $\mathcal{V}^{\mathrm{op}}$ is given by $\Delta_{\tau^{-1},\sigma^{-1}} C_{\tau^{-1}\sigma^{-1}} \to C_{\tau^{-1}} \otimes C_{\sigma^{-1}}$ in $\mathcal{V}$.
Let $f : (G, C) \to (H, D)$ be a morphism of graded coalgebras. We define $F(f) = g : F(G, C) = (G, A) \to F(H, D) = (H, B)$ as follows: $g(\sigma) = \sigma$, for all $\sigma \in G$, and $g_\sigma : A_\sigma \to B_{f(\sigma)}$ in $\mathcal{V}^{\mathrm{op}}$ is the map $f_{\sigma^{-1}} : D_{f(\sigma)^{-1}} = B_{f(\sigma)} \to C_{\sigma^{-1}} = A_\sigma$ in $\mathcal{V}$.
Let $f, f' : (G, C) \to (H, D)$ be morphisms of graded coalgebras, and let $\alpha : f \Rightarrow f'$ be a 2-cell in ${}^{\mathcal{V}}\underline{\underline{\mathrm{gr}}}$. We have morphisms $\alpha_\sigma : D_{f(\sigma)^{-1}f'(\sigma)} \to k$ in $\mathcal{V}$, which are also morphisms $\alpha_\sigma : k \to B_{f'(\sigma)^{-1}f(\sigma)}$ in $\mathcal{V}^{\mathrm{op}}$, defining a 2-cell $F(f) \Rightarrow F(f')$ in ${}_{\mathcal{V}^{\mathrm{op}}}\underline{\underline{\mathrm{gr}}}$. □

Proposition 5.2. *Let $\mathcal{V}$ be a strict monoidal category. We have 2-functors $K : {}_{\mathcal{V}}\underline{\underline{\mathrm{gr}}} \to {}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$ and $H : {}^{\mathcal{V}}\underline{\underline{\mathrm{gr}}} \to {}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$.*

Proof. Let A be a G-graded algebra. We define a $\mathcal{V}$-category $K(A) = K(G, A)$ as follows. The underlying class is G, and $K(A)_{\sigma,\tau} = A_{\sigma^{-1}\tau}$. The multiplication maps are

$$\begin{aligned} m_{\sigma,\rho,\tau} &= m_{\sigma^{-1}\rho,\rho^{-1}\tau} \\ &: \ K(A)_{\sigma,\rho} = A_{\sigma^{-1}\rho} \otimes K(A)_{\rho,\tau} = A_{\rho^{-1}\tau} \to K(A)_{\sigma,\tau} = A_{\sigma^{-1}\tau}, \end{aligned}$$

and the unit maps are $\eta_x = \eta : k \to A_e = A_{\sigma,\sigma}$.
Let $f : (G, A) \to (H, B)$ be a morphism of graded algebras. $K(f) = g : K(G, A) \to K(H, B)$ is then defined as follows. $g(\sigma) = f(\sigma)$, for all $\sigma \in G$, and $g_{\sigma,\tau} = f_{\sigma^{-1}\tau} : K(A)_{\sigma,\tau} = A_{\sigma^{-1}\tau} \to K(B)_{f(\sigma),f(\tau)} = B_{f(\sigma)^{-1}f(\tau)}$.
Now let $\alpha : f \Rightarrow f'$ be a 2-cell in ${}_{\mathcal{V}}\underline{\underline{\mathrm{gr}}}$. We have morphisms $\alpha_\sigma : k \to B_{g(\sigma)^{-1}f(\sigma)} = K(B)_{g(\sigma),f(\sigma)}$, and these also define a 2-cell $g \Rightarrow g'$ in ${}_{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$.
The 2-functor $H : {}^{\mathcal{V}}\underline{\underline{\mathrm{gr}}} \to {}^{\mathcal{V}}\underline{\underline{\mathrm{Cat}}}$ is constructed in a similar way. Let us just mention that, for a G-graded coalgebra C, $H(C)_{\sigma,\tau} = C_{\sigma^{-1}\tau}$. □

Let $\mathcal{V}$ be a braided (strict) monoidal category. We can consider graded coalgebras in $\underline{\mathcal{A}}(\mathcal{V})$ and graded algebras in $\underline{\mathcal{C}}(\mathcal{V})$. A graded coalgebra in $\underline{\mathcal{A}}(\mathcal{V})$ is a graded coalgebra C in $\mathcal{V}$, such that every C_σ is an algebra in $\mathcal{V}$, and the comultiplication and counit morphisms $\Delta_{\sigma,\tau}$ and ε are algebra maps. Graded coalgebras in $\underline{\mathcal{A}}(\mathcal{V})$ are known in the literature as semi-Hopf

group coalgebras. They appeared in [27] (see also [28]), and a systematic algebraic study was initiated in [30].
In a similar way, a graded algebra in $\underline{\mathcal{C}}(\mathcal{V})$ is a graded algebra A in $\mathcal{V}$ such that every A_σ is a coalgebra in $\mathcal{V}$, and the multiplication and counit morphisms $m_{\sigma,\tau}$ and η are coalgebra morphisms. In the literature, this is also called a semi-Hopf group algebra.
This provides us with a new categorical interpretation of semi-Hopf group algebras and coalgebras. We also obtain that semi-Hopf group algebras (resp. coalgebras) can be organized into a 2-category ${}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\mathrm{gr}}}$ (resp. ${}^{\underline{\mathcal{A}}(\mathcal{V})}\underline{\underline{\mathrm{gr}}}$). Note that a different interpretation, where group algebras and coalgebras appear as bialgebras in a suitable symmetric monoidal category was given by the second author and De Lombaerde in [12].
Recall that a semi-Hopf group coalgebra C is called a Hopf group coalgebra if there exist morphisms $S_\sigma :\ C_{\sigma^{-1}} \to C_\sigma$ such that

$$m_\sigma \circ (C_\sigma \otimes S_\sigma) \circ \Delta_{\sigma,\sigma^{-1}} = m_\sigma \circ (S_\sigma \otimes C_\sigma) \circ \Delta_{\sigma^{-1},\sigma} = \eta_\sigma \circ \varepsilon.$$

A semi-Hopf group algebra A is called a Hopf group algebra if there exist morphisms $S_\sigma :\ A_\sigma \to A_{\sigma^{-1}}$ such that

$$m_{\sigma,\sigma^{-1}} \circ (A_\sigma \otimes S_\sigma) \circ \Delta_\sigma = m_{\sigma^{-1},\sigma} \circ (S_\sigma \otimes A_\sigma) \circ \Delta_\sigma = \eta \circ \varepsilon_\sigma.$$

Proposition 5.3. *Let $\mathcal{V}$ be a braided strict monoidal category. We have 2-functors K : ${}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\mathrm{gr}}} \to {}_{\underline{\mathcal{C}}(\mathcal{V})}\underline{\underline{\mathrm{Cat}}}$ and K : ${}^{\underline{\mathcal{A}}(\mathcal{V})}\underline{\underline{\mathrm{gr}}} \to {}^{\underline{\mathcal{A}}(\mathcal{V})}\underline{\underline{\mathrm{Cat}}}$. The first functor sends Hopf group algebras to Hopf $\mathcal{V}$-categories, and the second one sends Hopf group coalgebras to dual Hopf $\mathcal{V}$-categories.*

Proof. The first statement is an immediate corollary of Proposition 5.2. The proof of the second statement is straightforward. Let A be a Hopf group algebra. $K(S)_{\sigma,\tau} = S_{\sigma^{-1}\tau} :\ K(A)_{\sigma,\tau} = A_{\sigma^{-1}\tau} \to K(A)_{\tau,\sigma} = A_{\tau^{-1}\sigma}$ makes $K(A)$ into a Hopf $\mathcal{V}$-category. □

6. Hopf categories and weak Hopf algebras

Let A be a k-linear Hopf category, with $|A| = X$ a finite set, and consider

$$A = \oplus_{x,y\in X} A_{x,y}.$$

We define a multiplication on A in the usual way: for $h \in A_{x,y}$ and $k \in A_{z,u}$, the product of hk is the image of $h\otimes k$ under the map $m_{x,y,u} : A_{x,y}\otimes A_{y,u} \to A_{x,u}$ if $y = z$, and $hk = 0$ if $y \neq z$. This multiplication is extended linearly to the whole of A. Then A is a k-algebra with unit $1 = \sum_{x\in X} 1_x$, where 1_x is the identity morphism $x \to x$.
Now we define $\Delta :\ A \to A \otimes A$, $\varepsilon :\ A \to k$, $S :\ A \to A$ in such a way that their restrictions to $A_{x,y}$ are respectively $\Delta_{x,y}$, $\varepsilon_{x,y}$ and $S_{x,y}$.

Proposition 6.1. *Let A be a k-linear Hopf category, with $|A| = X$ a finite set. Then $A = \oplus_{x,y\in X} A_{x,y}$ is a weak Hopf algebra.*

Proof. We refer to [8] for the definition of a weak Hopf algebra. We compute that

$$\Delta(1) = 1_{(1)} \otimes 1_{(2)} = \sum_{x\in X} 1_x \otimes 1_x,$$

and

$$1_{(1)}\otimes 1_{(2)}1_{(1')}\otimes 1_{(2')} = \sum_{x,y\in X} 1_x\otimes 1_x 1_y\otimes 1_y = \sum_{x\in X} 1_x\otimes 1_x\otimes 1_x = (\Delta\otimes A)(\Delta(1)),$$

as needed. In a similar way, we show that

$$1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')} = (\Delta \otimes A)(\Delta(1)).$$

Let us now show that

$$\varepsilon(hkl) = \varepsilon(hk_{(1)})\varepsilon(k_{(2)}l).$$

It suffices to show this for $h \in A_{x,y}$, $k \in A_{y',z'}$, $l \in A_{z,u}$. If $y \neq y'$ or $z \neq z'$, then both sides of the equation are 0. Assume that $y = y'$ and $z = z'$. From (7), it follows that $\varepsilon(hk_{(1)})\varepsilon(k_{(2)}l) = \varepsilon(h)\varepsilon(k_{(1)})\varepsilon(k_{(2)}l) = \varepsilon(h)\varepsilon(\varepsilon(k_{(1)})k_{(2)}l) = \varepsilon(h)\varepsilon(kl) = \varepsilon(hkl)$. Similar arguments show that

$$\varepsilon(hkl) = \varepsilon(hk_{(2)})\varepsilon(k_{(1)}l).$$

This proves that A is a weak bialgebra. For $h \in A_{x,y}$, we compute that

$$\varepsilon_t(h) = \sum_{z\in X} \langle\varepsilon, 1_z h\rangle 1_z = \langle\varepsilon, 1_x h\rangle 1_x = \langle\varepsilon_{x,y}, h\rangle 1_x.$$

In a similar way, we show that $\varepsilon_s(h) = \langle\varepsilon, h1_y\rangle 1_y = \langle\varepsilon_{x,y}, h\rangle 1_y$. Now

$$h_{(1)}S_{x,y}(h_{(2)}) \overset{(11)}{=} \eta_x(\varepsilon_{x,y}(h)) = \varepsilon_t(h);$$
$$S_{x,y}(h_{(1)})h_{(2)} \overset{(12)}{=} \eta_y(\varepsilon_{x,y}(h)) = \varepsilon_s(h),$$

and, finally,

$$S_{x,y}(h_{(1)})h_{(2)}S_{x,y}(h_{(3)}) = \varepsilon_{x,y}(h_{(1)})1_y S_{x,y}(h_{(2)}) = S_{x,y}(h).$$

□

Remark 6.2. Let G be a groupoid. Using Example 2.12, we obtain a k-linear Hopf category. Then applying Proposition 6.1, we find a weak Hopf algebra, which is precisely the groupoid algebra kG.

Now let C be a dual k-linear Hopf category. Then every $C_{x,y}$ is an algebra, and we have k-linear maps $\Delta_{x,y,z} : C_{x,z} \to C_{x,y} \otimes C_{y,z}$, $\varepsilon_x : C_{x,x} \to k$ and $S_{x,y} : C_{y,x} \to C_{x,y}$ such that the following axioms are satisfied, for all $h, k \in C_{x,z}$ and $l, m \in C_{x,x}$:

$$\Delta_{x,u,y}(h_{(1,y)}) \otimes h_{(2,y)} = h_{(1,y)} \otimes \Delta_{y,u,z}(h_{(2,y)}) \tag{30}$$
$$\varepsilon_x(h_{(1,x)})h_{(2,x)} = h_{(1,z)}\varepsilon_z(h_{(2,z)}) = h; \tag{31}$$
$$\Delta_{x,y,z}(hk) = h_{(1,y)}k_{(1,y)} \otimes h_{(2,y)}k_{(2,y)}; \tag{32}$$
$$\varepsilon_x(lm) = \varepsilon_x(l)\varepsilon_x(m); \tag{33}$$
$$\Delta_{x,y,z}(1_{x,z}) = 1_{x,y} \otimes 1_{y,z}; \tag{34}$$

$$\varepsilon_x(1_{x,x}) = 1; \tag{35}$$

$$l_{(1,y)}S_{x,y}(l_{(2,y)}) = \varepsilon_x(l)1_{x,y}; \tag{36}$$

$$S_{y,x}(l_{(1,y)})l_{(2,y)} = \varepsilon_x(l)1_{y,x}. \tag{37}$$

Cere $1_{x,y}$ is the unit element of $C_{x,y}$, and we used the Sweedler-Heyneman notation

$$\Delta_{x,y,z}(h) = h_{(1,y)} \otimes h_{(2,y)}.$$

Proposition 6.3. *Let C be a dual k-linear Hopf category, and assume that $|C| = X$ is finite. Then $C = \oplus_{x,y\in X} C_{x,y}$ is a weak Hopf algebra.*

Proof. Being the direct product of a finite number of k-algebras, C is itself a k-algebra, with unit $1 = \sum_{x,z\in X} 1_{x,z}$. We define a comultiplication on C as follows:

$$\Delta(h) = \sum_{y\in X} \Delta_{x,y,z}(h),$$

for $h \in C_{x,z}$. It follows immediately from (30) that Δ is coassociative. The counit is defined by ($h \in C_{x,y}$):

$$\varepsilon(h) = \begin{cases} \varepsilon_x(h) & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

We verify the left counit condition:

$$((\varepsilon \otimes C) \circ \Delta)(h) = \sum_{y\in X} \varepsilon(h_{(1,y)})h_{(2,y)} = \varepsilon_x(h_{(1,x)})h_{(2,x)} \overset{(31)}{=} h.$$

The right counit condition can be verified in a similar way, and we conclude that C is a coalgebra. It follows from (32) and (33) that Δ and ε preserve the multiplication. it follows from (34) that

$$\Delta(1) = 1_{(1)} \otimes 1_{(2)} = \sum_{x,y,z\in X} 1_{x,y} \otimes 1_{y,z}.$$

We now find easily that

$$\begin{aligned} 1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} &= \sum_{x,y,z,u,v,w\in X} 1_{x,y} \otimes 1_{y,z}1_{u,v} \otimes 1_{v,w} \\ &= \sum_{x,y,z,w\in X} 1_{x,y} \otimes 1_{y,z} \otimes 1_{z,w} = 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)}. \end{aligned}$$

In a similar way, we find that $1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)}$. Now take $h, k, l \in C_{x,x}$.

$$\begin{aligned} \varepsilon(hk_{(1)})\varepsilon(k_{(2)}l) &= \sum_{y\in X} \varepsilon(hk_{(1,y)})\varepsilon(k_{(2,y)}l) = \varepsilon_x(hk_{(1,x)})\varepsilon_x(k_{(2,x)}l) \\ &\overset{(33)}{=} \varepsilon_x(h)\varepsilon_x(k_{(1,x)})\varepsilon_x(k_{(2,x)}l) = \varepsilon_x(h)\varepsilon_x(\varepsilon_x(k_{(1,x)})k_{(2,x)}l) \\ &\overset{(31)}{=} \varepsilon_x(h)\varepsilon_x(kl) \overset{(33)}{=} \varepsilon_x(hkl) = \varepsilon_x(hkl). \end{aligned}$$

We conclude that

$$\varepsilon(hk_{(1)})\varepsilon(k_{(2)}l) = \varepsilon(hkl), \tag{38}$$

if $h, k, l \in C_{x,x}$. If $h, k, l \in C_{x,y}$ with $y \neq x$, then both sides of (38) are zero. So we can conclude that (38) holds for all $h, k, l \in C$. In a similar way, we can show that

$$\varepsilon(hk_{(2)})\varepsilon(k_{(1)}l) = \varepsilon\Delta_{(g\circ f)(x),(g'\circ f)(x),(g'\circ f')(x)}(hkl),$$

for all $h, k, l \in C$. This shows that C is a weak bialgebra.
Recall from [8] that the maps $\varepsilon_s,\ \varepsilon_t:\ C \to C$ are given by the formulas

$$\varepsilon_s(h) = 1_{(1)}\varepsilon(h1_{(2)})\ ;\ \varepsilon_t(h) = \varepsilon(1_{(1)}h)1_{(2)}.$$

These maps can be easily computed: for $h \in C_{x,z}$, we have

$$\varepsilon_t(h) = \sum_{u,v,y\in X} \varepsilon(1_{u,v}h)1_{v,y} = \sum_{y\in X} \varepsilon(h)1_{z,y} = \begin{cases} \sum_{y\in X} \varepsilon_x(h)1_{x,y} & \text{if } x = z \\ 0 & \text{if } x \neq z \end{cases}$$

In a similar way, we find that

$$\varepsilon_s(h) = \begin{cases} \sum_{y\in X} \varepsilon_x(h)1_{y,x} & \text{if } x = z \\ 0 & \text{if } x \neq z \end{cases}$$

Now we define $S:\ C \to C$ as follows: the restriction of S to $C_{x,y}$ is $S_{y,x}$, and then we extend linearly. Then we have, for $h \in C_{x,z}$:

$$(S * C)(h) = \sum_{y\in X} S_{y,x}(h_{(1,y)})h_{(2,y)}.$$

If $x \neq z$, then we find easily that $(S * C)(h) = 0 = \varepsilon_s(h)$. If $x = z$, then we find

$$(S * C)(h) \overset{(37)}{=} \sum_{y\in X} \varepsilon_x(h)1_{y,x} = \varepsilon_s(h).$$

This shows that $S * C = \varepsilon_s$. In a similar way, we have that $C * S = \varepsilon_t$. Finally we have that

$$(S * C * S)(h) = \sum_{y,u\in X} S_{u,x}(h_{(1,y)(1,u)})h_{(1,y)(2,u)}S_{z,y}(h_{(2,y)}).$$

The terms on the right hand side are products of an element of $C_{u,x}$, an element of $C_{u,y}$ and an element of $C_{z,y}$. These products are zero if $x \neq y$ of $z \neq u$. Hence we find

$$\begin{aligned}(S * C * S)(h) &= S_{z,x}(h_{(1,x)(1,z)})h_{(1,x)(2,z)}S_{z,x}(h_{(2,x)}) \\ &\overset{(37)}{=} \varepsilon_x(h_{(1,x)})1_{z,x}S_{z,x}(h_{(2,x)}) \overset{(31)}{=} S_{x,z}(h) = S(h).\end{aligned}$$

This proves that C satisfies all the axioms of a weak Hopf algebra, see [8]. $\square$

Remark 6.4. If A be a k-linear Hopf category, with $|A| = X$ an infinite set, then $A = \oplus_{x,y\in X} A_{x,y}$ is an algebra without unit, but with (idempotent) local units. We believe that if A is a Hopf category and using similar constructions as above, the associated algebra A can be endowed with the structure of a *weak multiplier Hopf algebra* (see [29] and [6]), but we haven't worked out the details of this construction.

7. Hopf categories and duoidal categories

Let X be a set. We have seen in Section 1 that $(\mathcal{M}_k(X), \bullet, J)$ is a monoidal category. We will define a second monoidal structure on $\mathcal{M}_k(X)$, in such a way that $\mathcal{M}_k(X)$ becomes a duoidal category (also called 2-monoidal category) in the sense of [1]. We will follow the notation of [5], and we call $\bullet$ the black tensor product on $\mathcal{M}_k(X)$. The second tensor product is called the white tensor product and is defined as follows. For $M, N \in \mathcal{M}_k(X)$, let

$$(M \odot N)_{x,z} = \oplus_{y\in X} M_{x,y} \otimes N_{y,z}.$$

The unit object for the white tensor product is I, defined by

$$I_{x,y} = \begin{cases} ke_{x,x} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

We will simply write

$$I_{x,y} = k\delta_{x,y},$$

where the Kronecker symbol $\delta_{x,y}$ stands formally for the element of the identity matrix in the (x, y)-position. Let

$$\tau :\ I \to J$$

be the natural inclusion. We compute that

$$(I \bullet I)_{x,y} = k\delta_{x,y} \otimes k\delta_{x,y} = k\delta_{x,y} = I_{x,y},$$

hence $I \bullet I = I$, and we let

$$\delta :\ I \to I \bullet I$$

be the identity map. Now we compute that

$$(J \odot J)_{x,y} = \oplus_{z\in X} ke_{x,z} \otimes ke_{z,y} = \oplus_{z\in X} kze_{x,y} = kXe_{x,y}.$$

We now define $\varpi :\ J \odot J \to J$. For all $x, y \in X$,

$$\varpi_{x,y} :\ \oplus_{z\in X} kze_{x,y} \to ke_{x,y},\ \varpi_{x,y}(\sum_{z\in X} \alpha_z ze_{x,y}) = \sum_{z\in X} \alpha_z e_{x,y}.$$

For $M, N, P, Q \in \mathcal{V}(X)$ we have that

$$((M \bullet N) \odot (P \bullet Q))_{x,y} = \bigoplus_{z\in X} M_{x,z} \otimes N_{x,z} \otimes P_{z,y} \otimes Q_{z,y};$$

$$((M \odot P) \bullet (N \odot Q))_{x,y} = \bigoplus_{u,v\in X} M_{x,u} \otimes P_{u,y} \otimes N_{x,v} \otimes Q_{v,y},$$

and we define

$$\zeta_{M,N,P,Q} :\ (M \bullet N) \odot (P \bullet Q) \to (M \odot P) \bullet (N \odot Q)$$

as follows: for $x, y \in X$, $\zeta_{M,N,P,Q,x,y}$ is the map switching the second and third tensor factor, followed by the natural inclusion.

Theorem 7.1. *Let X be a set. $(\mathcal{M}_k(X), \odot, I, \bullet, J, \delta, \varpi, \tau, \zeta)$ is a duoidal category.*

Proof. We have to show that the axioms in [5, Def. 1.1] are satisfied.
1) (J, ϖ, τ) is a monoid in $(\mathcal{M}_k(X), \odot, I)$.
Associativity: first compute that

$$(J \odot J \odot J)_{x,y} = k(X \times X)e_{x,y} = \oplus_{u,v \in X} k(u, v)e_{x,y},$$

and

$$\begin{aligned}(\varpi(J \odot \varpi))(\sum_{u,v} \alpha_{(u,v)}(u, v)e_{x,y}) &= \varpi(\sum_{u,v} \alpha_{(u,v)} u e_{x,y}) \\ = \ \sum_{u,v} \alpha_{(u,v)} e_{x,y} &= (\varpi(\varpi \odot J))(\sum_{u,v} \alpha_{(u,v)}(u, v)e_{x,y}).\end{aligned}$$

Left unit property: we have to show that the diagram

$$\begin{array}{ccc} (J \odot I)_{x,y} & \xrightarrow{(J\odot\tau)_{x,y}} & (J \odot J)_{x,y} \\ & \searrow^{=} & \downarrow{\scriptstyle \varpi_{x,y}} \\ & & J_{x,y} \end{array}$$

commutes, for all $x, y \in X$. Observe that $(J \odot I)_{x,y} = \oplus_{z \in X} ke_{x,z} \otimes k\delta_{z,y} = ke_{x,y} = J_{x,y}$ and $(J \odot J)_{x,y} = kXe_{x,y}$. Now

$$\varpi_{x,y}\big((J \odot \tau)_{x,y}(\alpha e_{x,y}\big) = \varpi_{x,y}(\alpha y e_{x,y}) = \alpha e_{x,y},$$

for all $\alpha \in k$. The right unit property can be shown in a similar way.
2) (I, δ, τ) is a comonoid in $(\mathcal{M}_k(X), \bullet, J)$.
The coassociativity of δ is clear, since δ is the identity map. For the left counit property: oberve that the diagram

$$\begin{array}{ccc} I_{x,y} = k\delta_{x,y} & & \\ {\scriptstyle \delta_{x,y}}\downarrow & \searrow^{=} & \\ I_{x,y} = k\delta_{x,y} & \xrightarrow{(J\bullet\tau)_{x,y}} & (J \bullet I)_{x,y} = k\delta_{x,y} \end{array}$$

commutes: the three maps in the diagram are the identity map.
3) Verification of the associativity and unitality axioms [5, 1.6-7] is obvious and is left to the reader. □

Recall the following definition from [1, Def. 6.25] (see also [5, Def. 1.2]).

Definition 7.2. Let $(\mathcal{M}, \odot, I, \bullet, J, \delta, \varpi, \tau, \zeta)$ be a duoidal category. A bimonoid is an object A, together with an algebra structure (μ, η) in $(\mathcal{M}, \odot, I)$ and a coalgebra structure (Δ, ε) in $(\mathcal{M}, \bullet, J)$ subject to the compatibility conditions

$$\Delta \circ \mu = (\mu \bullet \mu) \circ \zeta \circ (\Delta \odot \Delta); \tag{39}$$

$$\varpi \circ (\varepsilon \odot \varepsilon) = \varepsilon \circ \mu; \tag{40}$$

$$(\eta \bullet \eta) \circ \delta = \Delta \circ \eta; \tag{41}$$

$$\varepsilon \circ \eta = \tau. \tag{42}$$

Theorem 7.3. *Let X be a set, and let $A \in \mathcal{M}_k(X)$. We have a bijective correspondence between bimonoid structures on A over the duoidal category $(\mathcal{M}_k(X), \odot, I, \bullet, J, \delta, \varpi, \tau, \zeta)$ from Theorem 7.1 and $\underline{\mathcal{C}}(\mathcal{M}_k)$-category structures on A.*

Proof. First let A be a bimonoid. A has an algebra structure (μ, η) on $(\mathcal{M}_k(X), \odot, I)$. Consider the (x, y)-component of the multiplication map $\mu : A \odot A \to A$, namely

$$\mu_{x,y} : \oplus_{u \in X} A_{x,u} \otimes A_{u,y} \to A_{x,y},$$

and let $\mu_{x,z,y}$ be the composition

$$\mu_{x,y} \circ i_z : A_{x,z} \otimes A_{z,y} \to \oplus_{u \in X} A_{x,u} \otimes A_{u,y} \to A_{x,y},$$

where i_z is the natural inclusion. Also consider the (x, x)-component of the unit map $\eta : I \to A$, namely $\eta_x = \eta_{x,x} : k \to A_{x,x}$. Now it is easy to see that (1-2) are satisfied, so that A becomes a k-linear category.
A has a coalgebra structure (Δ, ε) on $(\mathcal{M}_k(X), \bullet, J)$. Consider the (x, y)-component of the comultiplication $\Delta : A \to A \bullet A$ and of the counit $\varepsilon : A \to J$. This gives k-linear maps $\Delta_{x,y} : A_{x,y} \to A_{x,y} \otimes A_{x,y}$ and $\varepsilon_{x,y} : A_{x,y} \to k$ making $A_{x,y}$ into a k-coalgebra.
Now we write the (x, y)-component of (39) and (40) as commutative diagrams. This gives us

$$\begin{array}{ccccc}
\oplus_z A_{x,z} \otimes A_{z,y} & \xrightarrow{\mu_{x,y}} & A_{x,y} & \xrightarrow{\Delta_{x,y}} & A_{x,y} \otimes A_{x,y} \\
\downarrow{\scriptstyle \oplus_z \Delta_{x,z} \otimes \Delta_{z,y}} & & & & \uparrow{\scriptstyle \mu_{x,y} \otimes \mu_{x,y}} \\
\oplus_z A_{x,z} \otimes A_{x,z} \otimes A_{z,y} \otimes A_{z,y} & & \xrightarrow{\zeta_{x,y}} & & \oplus_{u,v} A_{x,u} \otimes A_{u,y} \otimes A_{x,v} \otimes A_{v,y}
\end{array}$$

and

$$\begin{array}{ccc}
\oplus_z A_{x,z} \otimes A_{z,y} & \xrightarrow{\oplus_z \varepsilon_{x,z} \otimes \varepsilon_{z,y}} & \oplus_z k e_{x,z} \otimes e_{z,y} = \oplus_z k z e_{x,y} \\
\downarrow{\scriptstyle \mu_{x,y}} & & \downarrow{\scriptstyle \varpi} \\
A_{x,y} & \xrightarrow{\varepsilon_{x,y}} & k e_{x,y}
\end{array}$$

Evaluating the two diagrams at $a \otimes b \in A_{x,z} \otimes A_{z,y}$, we find that

$$\Delta_{x,y}(ab) = a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} \text{ and } \varepsilon_{x,y}(ab) = \varepsilon_{x,z}(a) \varepsilon_{y,z}(b).$$

Now we write the (x,x)-component of (41) and (42) as commutative diagrams. This gives

$$\begin{array}{ccc} k & \xrightarrow{\eta_x} & A_{x,x} \\ {\scriptstyle \delta_{x,x}}\downarrow & & \downarrow{\scriptstyle \Delta_{x,x}} \\ k\otimes k & \xrightarrow{\eta_x\otimes\eta_x} & A_{x,x}\otimes A_{x,x} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} k & \xrightarrow{\eta_x} & A_{x,x} \\ & {\scriptstyle \tau_{x,x}}\searrow & \downarrow{\scriptstyle \varepsilon_{x,x}} \\ & & k \end{array}$$

Evaluating these diagrams at 1, we find that $\Delta_{x,x}(1_x) = 1_x \otimes 1_x$ and $\varepsilon_{x,x}(1_x) = 1$, and we conclude that A is a $\underline{\mathcal{C}}(\mathcal{M}_k)$-category.
Conversely, let A be a $\underline{\mathcal{C}}(\mathcal{M}_k)$-category. Define $\mu :\ A \odot A \to A$, $\eta :\ I \to A$, $\Delta :\ A \to A \bullet A$ and $\varepsilon :\ A \to J$ as follows. $\mu_{x,y} = \sum_u \mu_{x,u,y} :\ \oplus_{u\in X} A_{x,u} \otimes A_{u,y} \to A_{x,y}$; $\eta_{x,y} = 0$ if $x \neq y$ and $\eta_{x,x} = \eta_x$; the components of Δ and ε are just $\Delta_{x,y}$ and $\varepsilon_{x,y}$. Straightforward computations show that this turns A into a bimonoid. It is clear that these two operations are inverses. This completes the proof. □

Linearization and the duoidal category of spans. We have seen in Theorem 7.1 that we can associate a duoidal category $\mathcal{M}_k(X)$ to a set X. In [1, 5], two other classes of duoidal categories are investigated, namely the category $\mathrm{span}(X)$ consisting of spans, and the category ${}_R\mathcal{M}_R$ of bimodules over a commutative k-algebra R. We will now discuss how these three classes of examples are related. To this end, we need to give alternative descriptions of $\mathcal{M}_k(X)$ and $\mathrm{span}(X)$.
As we have seen in Example 2.4, every set X carries a unique comonoid structure in $\underline{\mathrm{Sets}}$. A right X-coaction on a set V consists of a map $\rho :\ V \to V \times X$ of the form $\rho(v) = (v, s(v))$, where $s :\ V \to X$ is a function. So right X-coactions on V correspond bijectively to X^V. In a similar way, giving a two-sided coaction of X on V amounts to giving two functions $s, t :\ V \to X$, which means precisely that (V, t, s) is a span, see [5, Sec. 4.2]. Morphisms of spans correspond to bicomodule maps, and we conclude that the categories ${}^X\underline{\mathrm{Sets}}^X$ and $\mathrm{span}(X)$ are isomorphic. The white product of two spans V and W is

$$V \odot W = \{(v,w) \in V \times W \mid s(v) = t(w)\}$$

is precisely the cocarthesian product $V \times^X W$. Now observe that the category ${}^X\underline{\mathrm{Sets}}^X$ is isomorphic to $\underline{\mathrm{Sets}}^{X\times X}$. The black product is

$$V \bullet W = \{(v,w) \in V \times W \mid s(v) = s(w),\ t(v) = t(w)\}$$

and this is the cocarthesian product $V \times^{X\times X} W$. The white unit object is X, and the black unit object is $X \times X$.
A similar description applies to $\mathcal{M}_k(X)$. kX is a coalgebra, and we have isomorphisms of categories

$$\mathcal{M}_k(X) \cong {}^{kX}\mathcal{M}_k^{kX} \cong \mathcal{M}_k^{k(X\times X)}.$$

An object $(M_{x,y})_{x,y\in X}$ corresponds to $M = \oplus_{x,y\in X} M_{x,y}$, with left and right kX-coaction given by the formulas

$$\lambda(m) = x \otimes m \ ; \ \rho(m) = m \otimes y,$$

for $m \in M_{x,y}$, extended linearly. The black tensor product in $\mathcal{M}_k(X)$ is precisely the cotensor product over $k(X \times X)$, and the white one is the cotensor product over kX.
The linearization functor $L : \underline{\text{Sets}} \to \mathcal{M}_k$ is strongly monoidal, sends X to the grouplike coalgebra kX and a set V with a two-sided X-coaction to the kX-bicomodule kV. We find the following result.

Proposition 7.4. *The linearization functor induces a functor* $L : \text{span}(X) \to \mathcal{M}_k(X)$ *preserving the black and white tensor products.*

This construction can be generalized, replacing kX by a cocommutative coalgebra C. We have to assume that the cotensor product is associative, which can be done by requiring that k is a field, or else that k is a commutative ring and that C is finitely generated and projective over k. Then the category ${}^C\mathcal{M}_k^C \cong \mathcal{M}_k^{C\otimes C}$ of C-bicomodules is duoidal, with the cotensor product over C and $C \otimes C$ as the white and black tensor product. This brings us back to the second example of duoidal category studied in [1, 5]. For a commutative k-algebra A, the category ${}_A\mathcal{M}_A \cong \mathcal{M}_{A\otimes A}$ is a duoidal category, with the tensor products over A and $A \otimes A$ as the black and white tensor product. This is precisely the dual construction.

Generalized Hopf monoids in monoidal bicategories. Now we focus attention to the recent work by Böhm and Lack [7] on generalized Hopf monoids in monoidal bicategories.
It is well-known that the category of endomorphisms of an object of a bicategory is a monoidal category. It was observed in [24] that, in a similar way, duoidal categories arise as the category of endomorphisms in a monoidal bicategory of a pseudomonoid whose multiplication 1-cell and unit 1-cell have a right adjoint (such an object is known as a *map-monoidale*). In this case, the second monoidal structure is obtained using a *convolution product*. Consider the monoidal bicategory of free k-coalgebras, bicomodules and bicomodule maps, with the cotensor product as horizontal composition, the opposite composition as vertical composition and the k-tensor product as monoidal product. kX is a map-monoidale in this monoidal bicategory. Hence the category $\mathcal{M}_k(X) \cong {}^{kX}\mathcal{M}_k^{kX}$ of kX-bicomodules is the category of endomorphisms over a map-monoidale, so it can be endowed with a duoidal structure. This duoidal structure coincides with the one described above, the black monoidal product being the convolution product. It also follows from [24] that A is a bimonoid over the duoidal endohom category $\mathcal{M}_k(X)$ if and only if it is a monoidal comonad on kX in the monoidal bicategory described above, hence it induces a monoidal comonad on $\mathcal{M}_k(X)$.
Furthermore, Böhm and Lack provide equivalent conditions for the bimonoid A in the duoidal endohom category to have an antipode (i.e. to be a Hopf

monoid), in terms of a fundamental theorem of Hopf modules (see also our Section 9) and in terms of the associated monoidal comonad to be a Hopf (co)monad. In particular, this leads us to the following result.

Theorem 7.5. *Let X be a set, and let $A \in \mathcal{M}_k(X)$. We have a bijective correspondence between Hopf monoid structures on A (in the sense of [7]) over the duoidal category $(\mathcal{M}_k(X), \odot, I, \bullet, J, \delta, \varpi, \tau, \zeta)$ from Theorem 7.1 and Hopf $\mathcal{M}_k$-category structures on A. In particular, if A is Hopf $\mathcal{M}_k$-category, then this induces a Hopf monad on $\mathcal{M}_k(X)$.*

Proof. From the discussion above, we already know that the structure of an $\underline{\mathcal{C}}(\mathcal{M}_k)$-category on A corresponds to the structure of a bimonoid in the duoidal category $\mathcal{M}_k(X)$. Hence it only remains to compare the antipode axioms for Hopf categories (11) and (12) with the antipode axioms of [7, Theorem 7.2]. We leave out the details, but remark that the monoidal bicategory of bicomodules over free coalgebras has duals. Given a kX-kY bicomodule $M = \oplus_{(x,y)\in X\times Y} M_{x,y}$, then $M^- = M^{op} = \oplus_{(y,x)\in Y\times X} M_{y,x}$ is a kY-kX bicomodule. Furthermore, the 2-cell φ in [7] should in our setting be interpreted as the inclusion map $A_{x,y} \otimes A_{y,x} \to \oplus_{y\in X} A_{x,y} \otimes A_{y,x}$. □

8. Hopf categories and Morita contexts

Let k be a commutative ring, and $\mathcal{V} = \mathcal{M}_k$, the category of k-modules.

Definition 8.1. A Morita context consists of the following data:

(1) a class X;
(2) $A_{x,x}$ is a k-algebra, for all $x \in X$;
(3) $A_{x,y}$ is an $(A_{x,x}, A_{y,y})$-bimodule, for all $x, y \in X$;
(4) $\overline{m}_{x,y,z} : A_{x,y} \otimes_{A_{y,y}} A_{y,z} \to A_{x,z}$ is an $(A_{x,x}, A_{z,z})$-bimodule map,

satsifying the following conditions:

(1) $\overline{m}_{x,x,y} : A_{x,x} \otimes_{A_{x,x}} A_{x,y} \to A_{x,y}$ and $\overline{m}_{x,y,y} : A_{x,y} \otimes_{A_{y,y}} A_{y,y} \to A_{x,y}$ are the canonical isomorphisms;
(2) the associativity condition (43) is satisfied, for all $x, y, z, u \in X$

$$\overline{m}_{x,y,u} \circ (A_{x,y} \otimes_{A_{y,y}} \overline{m}_{y,z,u}) = \overline{m}_{x,z,u} \circ (\overline{m}_{x,y,z} \otimes_{A_{z,z}} A_{z,u}). \tag{43}$$

For $a \in A_{x,y}$ and $b \in A_{y,z}$, we will write $\overline{m}_{x,y,z}(a \otimes_{A_{y,y}} n) = ab$.

Morita contexts can be organized into a 2-category ${}_k\underline{\underline{\mathrm{Mor}}}$. Before we describe the 1-cells, we recall the following result. Let $f : A \to B$ be a morphism of k-algebras, and consider $M, N \in \mathcal{M}_A$, $M', N' \in \mathcal{M}_B$, and k-linear maps $g : M \to M'$ and $h : N \to N'$ such that $g(ma) = g(m)f(a)$ and $h(an) = f(a)h(n)$, for all $a \in A$, $m \in M$ and $n \in N$. Then we have a well-defined map

$$g \otimes_f h : M \otimes_A N \to M' \otimes_B N',\ (g \otimes_f h)(m \otimes_A n) = g(m) \otimes_B h(n).$$

A 1-cell $f : A \to B$ in ${}_k\underline{\underline{\mathrm{Mor}}}$ consists of $f : X \to Y$, and maps $f_{x,y} : A_{x,y} \to B_{f(x),f(y)}$ such that

- every $f_{x,x}$ is an algebra map;
- $f_{x,y}(a'aa'') = f_{x,x}(a')f_{x,y}(a)f_{y,y}(a'')$, for all $a'inA_{x,x}$, $a \in A_{x,y}$ and $a'' \in A_{y,y}$;
- $f_{x,y} \circ \overline{m}_{x,y,z} = \overline{m}_{f(x),f(y),f(z)} \circ (f_{x,y} \otimes_{f_{y,y}} f_{y,z})$.

For two given 1-cells $f, g : A \to B$, a 2-cell $\alpha : f \Rightarrow g$ consists of a family of elements $\alpha_x \in B_{g(x),f(x)}$ indexed by x such that

$$\overline{m}_{g(x),g(y),f(y)}(g_{x,y}(a) \otimes_{B_{g(y),g(y)}} \alpha_y) = \overline{m}_{g(x),f(x),f(y)}(\alpha_x \otimes_{B_{g(x),g(x)}} f_{x,y}(a)),$$

for all $x, y \in X$ and $a \in A_{x,y}$.

Let A be a Morita context, and take $x \neq y \in X$. Take $p, r \in A_{x,y}$ and $q \in A_{y,x}$. It follows from (43) that

$$\overline{m}_{x,y,x}(p \otimes_{A_{y,y}} q)r = p\overline{m}_{y,x,y}(q \otimes_{A_{x,x}} r).$$

It follows that $(A_{x,x}, A_{y,y}, A_{x,y}, A_{y,x}, \overline{m}_{x,y,x}, \overline{m}_{y,x,y})$ is a Morita context. In particular, Morita contexts with a pair as underlying class are Morita contexts in the classical sense.

Theorem 8.2. *The 2-categories ${}_{\mathcal{M}_k}\underline{\underline{\mathrm{Cat}}}$ and ${}_k\underline{\underline{\mathrm{Mor}}}$ are isomorphic.*

Proof. (sketch) Let A be a k-linear category, with underlying class X. It is clear that $A_{x,x}$ is a k-algebra, and that $A_{x,y}$ is an $(A_{x,x}, A_{y,y})$-bimodule, for all $x, y \in X$. Take $a \in A_{x,y}$, $b \in A_{y,y}$ and $c \in A_{y,z}$. From (1), it follows that $m_{x,y,z}(ab \otimes c) = m_{x,y,z}(a \otimes bc)$, so we have a well-defined map

$$\overline{m}_{x,y,z} : A_{x,y} \otimes_{A_{y,y}} A_{y,z} \to A_{x,z},\ \overline{m}_{x,y,z}(a \otimes_{A_{y,y}} c) = m_{x,y,z}(a \otimes c).$$

From (2), it follows that $\overline{m}_{y,y,z}(1_y \otimes_{A_{y,y}} c) = m_{y,y,z}(1_y \otimes c) = c$, so that $\overline{m}_{y,y,z}$ is the canonical isomorphism $A_{y,y} \otimes_{A_{y,y}} A_{y,z} \cong A_{y,z}$. It is easy to verify that the associativity axiom (43) is satisfied, and it follows that A is a Morita X-context.
Conversely, let A be a Morita context with underlying class X. Define $m_{x,y,z}$ as the composition of $\overline{m}_{x,y,z}$ and the canonical surjection $A_{x,y} \otimes A_{y,z} \to A_{x,y} \otimes_{A_{y,y}} A_{y,z}$. It is a straightforward verification to check that A is k-linear category.
It is clear that these two constructions are inverses, and this defines 2-functors between our two 2-categories at the level of 0-cells. We leave it to the reader that we have a one-to-one correspondence between 1-cells and 2-cells in ${}_{\mathcal{M}_k}\underline{\underline{\mathrm{Cat}}}$ and ${}_k\underline{\underline{\mathrm{Mor}}}$. □

Theorem 8.3. *Let A be a k-linear category with underlying class X, and consider the corresponding Morita context. The following statements are equivalent.*

(1) $m_{x,y,z}$ *is surjective, for all* $x, y, z \in X$
(2) $m_{x,y,x}$ *is surjective, for all* $x, y \in X$;
(3) $\overline{m}_{x,y,x}$ *is bijective, for all* $x, y \in X$;
(4) $\overline{m}_{x,y,z}$ *is bijective, for all* $x, y, z \in X$.

A is called strict if these four equivalent conditions are satisfied.

Proof. The implications 4) $\Rightarrow$ 1) $\Rightarrow$ 2) are obvious.
2) $\Rightarrow$ 3). If $m_{x,y,x}$ is surjective, then $\overline{m}_{x,y,x}$ is also surjective. We have seen that $(A_{x,x}, A_{y,y}, A_{x,y}, A_{y,x}, \overline{m}_{x,y,x}, \overline{m}_{y,x,y})$ is a Morita context, hence surjectivity of $\overline{m}_{x,y,x}$ implies injectivity, by a classical property of Morita contexts, see [3].
3) $\Rightarrow$ 4). For all $x, y \in X$, we have that $\overline{m}_{x,x,y}$ and $\overline{m}_{x,y,y}$ are bijective (by definition), and $\overline{m}_{x,y,x}$ is bijective by assumption. It follows from (43) that

$$\overline{m}_{x,y,z} \circ (A_{x,y} \otimes_{A_{y,y}} \overline{m}_{y,x,z}) = \overline{m}_{x,x,z} \circ (\overline{m}_{x,y,x} \otimes_{A_{x,x}} A_{x,z}).$$

The right hand side is invertible, and therefore $\overline{m}_{x,y,z} \circ (A_{x,y} \otimes_{A_{y,y}} \overline{m}_{y,x,z})$ is also invertible. This implies that $\overline{m}_{x,y,z}$ has a right inverse, and that $A_{x,y} \otimes_{A_{y,y}} \overline{m}_{y,x,z}$ has a left inverse. Having a right inverse, $\overline{m}_{x,y,z}$ is surjective, for all $x, y, z \in X$.
It also follows that $A_{y,x} \otimes_{A_{x,x}} A_{x,y} \otimes_{A_{y,y}} \overline{m}_{y,x,z}$ and $\overline{m}_{y,x,y} \otimes_{A_{y,y}} \overline{m}_{y,x,z}$ have a left inverse, because $\overline{m}_{y,x,y}$ is bijective. Let f be the left inverse of $\overline{m}_{y,x,y} \otimes_{A_{y,y}} \overline{m}_{y,x,z}$, and take $\alpha \in \mathrm{Ker}\overline{m}_{y,x,z}$. $\overline{m}_{y,x,y}$ is surjective, hence there exists $\beta \in A_{y,x} \otimes_{A_{x,x}} A_{x,y}$ such that $\overline{m}_{y,x,y}(\beta) = 1_y$. Now

$$\beta \otimes_{A_{y,y}} \alpha = \big(f \circ (\overline{m}_{y,x,y} \otimes_{A_{y,y}} \overline{m}_{y,x,z})\big)(\beta \otimes_{A_{y,y}} \alpha) = 0,$$

and

$$0 = \overline{m}_{y,x,y}(\beta) \otimes_{A_{y,y}} \alpha = 1_y \otimes_{A_{y,y}} \alpha$$

in $A_{y,y} \otimes_{A_{y,y}} A_{y,x} \otimes_{A_{x,x}} A_{x,z} \cong A_{y,x} \otimes_{A_{x,x}} A_{x,z}$, and, finally, $\alpha = 0$. We conclude that $\overline{m}_{y,x,z}$ is injective. □

Example 8.4. The category A of k-progenerators, is a strict k-linear category. For two finitely generated projective k-modules P and Q, we have that $A_{P,Q} = \mathrm{Hom}(Q, P)$, and $m_{P,Q,P} : A_{P,Q} \otimes A_{Q,P} \to A_{P,P}$ is given by composition: $m_{P,Q,P}(f \otimes g) = f \circ g$. We have to show that $m_{P,Q,P}$ is surjective.
Q is a generator of ${}_k\mathcal{M}$, so there exist $q_i \in Q$ and $q_i^* \in Q^*$ such that $\sum_i \langle q_i^*, q_i\rangle = 1$.
P is finitely generated projective, so there exist $p_j \in P$ and $p_j^* \in P^*$ such that $p = \sum_j \langle p_j^*, p\rangle p_j$, for all $p \in P$. Now consider

$$f_{ij} : Q \to P \quad ; \quad f_{ij}(q) = \langle q_i^*, q\rangle p_j;$$
$$g_{ij} : P \to Q \quad ; \quad g_{ij}(p) = \langle p_j^*, p\rangle q_i.$$

Now

$$m_{P,Q,P}(\sum_{i,j} f_{ij} \otimes g_{ij})(p) = \sum_{i,j} \langle p_j^*, p\rangle \langle q_i^*, q_i\rangle p_j = p,$$

hence $m_{P,Q,P}(\sum_{i,j} f_{ij} \otimes g_{ij}) = P$ and $m_{P,Q,P}$ is surjective.

Example 8.5. Let A be a G-graded k-algebra, and consider the corresponding k-linear category $K(A)$ (see Proposition 5.2). $K(A)$ is strict if and only if the multiplication maps $A_{g^{-1}h} \otimes A_{h^{-1}g} \to A_e$ are surjective, for all $g, h \in G$. This is equivalent to surjectivity of $A_{g^{-1}} \otimes A_g \to A_e$, for all $g \in G$. This is one of the equivalent definitions of a strongly graded k-algebra, see for example [22]. We conclude that $K(A)$ is strict if and only if A is a strongly graded k-algebra.

Now assume that A is a $\underline{\mathcal{C}}(\mathcal{M}_k)$-category. It follows from the axioms that every $A_{x,x}$ is a bialgebra and that every $A_{x,y}$ is an $(A_{x,x}, A_{y,y})$-bimodule coalgebra. In this case the induction functors $A_{x,y} \otimes - : {}_{A_{y,y}}\mathcal{M} \to {}_{A_{x,x}}\mathcal{M}$ are comonoidal.

Example 8.6. Let H be Hopf algebra with bijective antipode S, and let A be a faithfully flat right H-Galois object. In [23], a new Hopf algebra L is constructed in such a way that A is a faitfhully flat left L-Galois object, and even an (L, H)-bigalois object. $A^{\rm op}$ is an (H, L)-bigalois object (see [23, Remark 4.4]). The left H-coaction on $A^{\rm op}$ is the following:

$$\lambda(a) = S^{-1}(a_{[1]}) \otimes a_{[0]}.$$

We now have a dual $\underline{\mathcal{A}}(\mathcal{M}_k)$-category A with underlying class $\{x, y\}$ defined as follows:

$$A_{x,x} = H;\ A_{y,y} = L;\ A_{x,y} = A;\ A_{y,x} = A^{\rm op}.$$

A is even a dual Hopf category; the antipode maps are the following: $S_H : H \to H$, $S_L :\ L \to L$ and the identity $A_{x,y} = A \to A_{y,x} = A^{\rm op}$.
Now let H be finitely generated and projective; then A and L are also finitely generated and projective, and the dual category of A is an example of a k-linear Hopf category.

9. Hopf modules and the fundamental theorem

Let $\mathcal{V}$ be a strict monoidal category with equalizers, and let A be a $\underline{\mathcal{C}}(\mathcal{V})$-category, with underlying class $|A| = X$. Assume that $M \in \mathcal{V}(X)$, with the following additional structure:

- $M \in \mathcal{V}_A$ in the sense of Definition 3.1, with structure morphisms $\psi_{x,y,z} :\ M_{x,y} \otimes A_{y,z} \to M_{x,z}$;
- $M \in \mathcal{V}^A$, that is, M is a right comodule over A considered as a coalgebra in $\mathcal{V}(X)$; this means that every $M_{x,y}$ is a right $A_{x,y}$-comodule, with coaction $\rho_{x,y} :\ M_{x,y} \to M_{x,y} \otimes A_{x,y}$.

Recall that $A \bullet A$ is also a $\mathcal{V}$-category. $M \bullet A \in \mathcal{V}_{A\bullet A}$, with structure maps

$$\psi^{M\bullet A}_{x,y,z} = (\psi_{x,y,z} \otimes m_{x,y,z}) \circ (M_{x,y} \otimes c_{A_{x,y},A_{y,z}} \otimes A_{y,z}).$$

M is called a Hopf module if the compatibility relation

$$\rho_{x,z} \circ \psi_{x,y,z} = \psi^{M\bullet A}_{x,y,z} \circ (\rho_{x,y} \otimes A_{y,z}) \tag{44}$$

holds for all $x, y, z \in X$. A morphism between Hopf modules is a morphism in $\mathcal{V}$ that is a morphism in $\mathcal{V}_A$ and $\mathcal{V}^A$. The category of Hopf modules is denoted $\mathcal{V}(X)^A_A$.
We introduce the category $\mathcal{D}(X)$ ($\mathcal{D}$ stands for "diagonal"). Its objects are families of objects in $\mathcal{V}$ indexed by X, and a morphism $N \to N'$ consists of a family of morphisms $N_x \to N'_x$ in $\mathcal{V}$.

Proposition 9.1. *We have a pair of adjoint functors (F, G) between $\mathcal{D}(X)$ and $\mathcal{V}(X)^A_A$.*

Proof. We define a functor $F: \mathcal{D}(X) \to \mathcal{V}(X)^A_A$ as follows. For $N \in \mathcal{D}(X)$, let $F(N) \in \mathcal{V}(X)^A_A$ be given by the data

$$F(N)_{x,y} = N_x \otimes A_{x,y};\ \psi_{x,y,z} = N_x \otimes m_{x,y,z};\ \rho_{x,y} = N_x \otimes \Delta_{x,y}.$$

For $f: N \to N'$ in $\mathcal{D}(X)$, let $F(f)_{x,y} = f_x \otimes A_{x,y}$. Verification of further details is straightforward.
Now we define $G: \mathcal{V}(X)^A_A \to \mathcal{D}(X)$. Let $M \in \mathcal{V}(X)^A_A$. $M_{x,x}$ is a right $A_{x,x}$-module, for every $x \in X$, and we define $G(M) = M^{\mathrm{co}A}$ as follows:

$$G(M)_x = M_x^{\mathrm{co}A} = M_{x,x}^{\mathrm{co}A_{x,x}},$$

the equalizer of the parallel morphisms $\rho_{x,x},\ M_{x,x}\otimes\eta_x: M_{x,x} \to M_{x,x}\otimes A_{x,x}$.
For $g: M \to M'$ in $\mathcal{V}(X)^A_A$, $G(g) = g^{\mathrm{co}A}$ is defined as follows: $G(g)_x = g_x^{\mathrm{co}A}$ is the unique morphism in $\mathcal{V}$ making the diagram

$$\begin{array}{ccccc}
M_x^{\mathrm{co}A} & \xrightarrow{i_x} & M_{xx} & \overset{M_{x,x}\otimes\eta_X}{\underset{\rho_{x,x}}{\rightrightarrows}} & M_{x,x}\otimes A_{x,x} \\
\Big\downarrow{\scriptstyle \exists! g_x^{\mathrm{co}A}} & & \Big\downarrow{\scriptstyle f_{x,x}} & & \Big\downarrow{\scriptstyle f_{x,x}\otimes A_{x.x}} \\
M'^{\mathrm{co}A}_x & \xrightarrow{i_x} & M'_{x,x} & \overset{M'_{x,x}\otimes\eta_X}{\underset{\rho'_{x,x}}{\rightrightarrows}} & M'_{x,x}\otimes A_{x,x}
\end{array}$$

commutative. The existence and uniqueness of $g_x^{\mathrm{co}A}$ is guaranteed by the universal property of equalizers.
Next we describe the unit and the counit of the adjunction. For $N \in \mathcal{D}(X)$, the unit $\eta_N: N \otimes GF(N)$ has X component $\eta_x^N: N_x \to GF(N)_x = (N_x \otimes A_{x,x})^{\mathrm{co}A_{x,x}}$, the unique morphism in $\mathcal{V}$ such that

$$i \circ \eta_x^N = N_x \otimes \eta_x: N_x \to (N_x \otimes A_{x,x})^{\mathrm{co}A_{x,x}} \to N_x \otimes A_{x,x}. \tag{45}$$

For $M \in \mathcal{V}(X)^A_A$, the (x, y)-component of $\varepsilon^M: FG(M) \to M$ is

$$\varepsilon^M = \psi_{x,x,y} \circ (i \otimes A_{x,y}): FG(M)_{x,y} = M_x^{\mathrm{co}A} \otimes A_{x,y} \to M_{x,y}.$$

In order to show that (F, G) is an adjoint pair, we have verify that

$$F(N) = \varepsilon^{F(N)} \circ F(\eta^N) \text{ and } G(M) = G(\varepsilon_M) \circ \eta^{G(M)},$$

for all $N \in \mathcal{D}(X)$ and $M \in \mathcal{V}(X)^A_A$. Now

$$\begin{aligned}
\varepsilon_{x,y}^{F(N)} \circ F(\eta^N)_{x,y} &= (N_x \otimes m_{x.x.y}) \circ (i \otimes A_{x,y}) \circ (\eta_x^N \otimes A_{x,y}) \\
&= (N_x \otimes m_{x,x,y}) \circ (N_x \otimes \eta_x \otimes A_{x,y}) = N_x \otimes A_{x,y} = F(N)_{x,y},
\end{aligned}$$

proving the first formula. For the second formula, we consider the diagram

$$
\begin{array}{ccc}
M_x^{\mathrm{co}A} & & \\
\Big\downarrow \eta_x^{G(M)} & \searrow^{M_x^{\mathrm{co}A}\otimes\eta_x} & \\
(M_x^{\mathrm{co}A}\otimes A_{x,x})^{\mathrm{co}A_{x,x}} & \xrightarrow{\ i\ } & M_x^{\mathrm{co}A}\otimes A_{x,x} \\
\Big\downarrow (i\otimes A_{x,x})^{\mathrm{co}A_{x,x}} & & \Big\downarrow i\otimes A_{x,x} \\
(M_{x,x}\otimes A_{x,x})^{\mathrm{co}A_{x,x}} & \xrightarrow{\ i\ } & M_{x,x}\otimes A_{x,x} \\
\Big\downarrow \psi_{x,x,x}^{\mathrm{co}A_{x,x}} & & \Big\downarrow \psi_{x,x,x} \\
M_x^{\mathrm{co}A} & \xrightarrow{\ i\ } & M_x
\end{array}
$$

The commutativity of the triangle follows from the definition of $\eta_x^{G(M)}$; the commutativity of the two squares follows from the definition of G at the level of morphisms. Now

$$\psi_{x,x,x}\circ(i\otimes A_{x,x})\circ(M_x^{\mathrm{co}A}\otimes\eta_x)=\psi_{x,x,x}\circ(M_{x,x}\otimes\eta_x)\circ i=i,$$

and it follows from the uniqueness in the universal property of equalizers that the vertical composition in the diagram is the identity on $M_x^{\mathrm{co}A}=G(M)_x$; this vertical composition is the x-component of the right hand side in the second formula. □

Let A be a $\underline{\mathcal{C}}(\mathcal{V})$-category, with underlying class $|A|=X$. For all $x,y,z\in X$, we consider the canonical map

$$\mathrm{can}^z_{x,y}=(m_{z,x,y}\otimes A_{x,})\circ(A_{z,x}\otimes\Delta_{x,y}):\ A_{z,x}\otimes A_{x,y}\to A_{z,y}\otimes A_{x,y}.$$

With respect to the observations made at the end of Section 7, the following theorem should be compared to [7, Theorem 7.14].

Theorem 9.2. (Fundamental Theorem for Hopf Modules) *Let $\mathcal{V}$ be a strict braided monoidal category with equalizers. For a $\underline{\mathcal{C}}(\mathcal{V})$-category A with underlying class X, the following assertions are equivalent.*

(1) *A is a Hopf $\mathcal{V}$-category;*
(2) *the pair of adjoint functors (F,G) from Proposition 9.1 is a pair of inverse equivalences between the categories $\mathcal{D}(X)$ and $\mathcal{V}(X)^A_A$;*
(3) *the functor G from Proposition 9.1 is fully faithful;*
(4) *$\mathrm{can}^z_{x,y}$ is an isomorphism, for all $x,y,z\in X$;*
(5) *$\mathrm{can}^x_{x,y}$ has a left inverse $f_{x,y}$ and $\mathrm{can}^y_{x,y}$ is an isomorphism, with inverse $g_{x,y}$, for all $x,y\in X$.*

Proof. (1) ⇒ (2). **Part 1.** ε^M has an inverse α^M, for all $M\in\mathcal{V}(X)^A_A$. We first show that the morphism

$$\gamma_{x,y}=\psi_{x,y,x}\circ(M_{x,y}\otimes S_{x,y})\circ\rho_{x,y}:\ M_{x,y}\to M_{x,x}$$

satisfies the equality

$$\rho_{x,x} \circ \gamma_{x,y} = (M_{x,x} \otimes \eta_x) \circ \gamma_{x,x}. \tag{46}$$

$$\begin{aligned}
\rho_{x,x} \circ \gamma_{x,y} &= \rho_{x,x} \circ \psi_{x,y,x} \circ (M_{x,y} \otimes S_{x,y}) \circ \rho_{x,y} \\
&\overset{(44)}{=} (\psi_{x,y,x} \otimes m_{x,y,x}) \circ (M_{x,y} \otimes c_{A_{x,y},A_{y,x}} \otimes A_{y,x}) \circ (\rho_{x,y} \otimes \Delta_{y,x}) \\
&\qquad \circ (M_{x,y} \otimes S_{x,y}) \circ \rho_{x,y} \\
&\overset{(14)}{=} (\psi_{x,y,x} \otimes m_{x,y,x}) \circ (M_{x,y} \otimes c_{A_{x,y},A_{y,x}} \otimes A_{y,x}) \\
&\qquad \circ (M_{x,y} \otimes A_{x,y} \otimes c_{A_{y,x},A_{y,x}}) \circ (\rho_{x,y} \otimes S_{x,y} \otimes S_{x,y}) \\
&\qquad \circ (M_{x,y} \otimes \Delta_{x,y}) \circ \rho_{x,y} \\
&= (\psi_{x,y,x} \otimes A_{x,x}) \circ (M_{x,y} \otimes A_{y,x} \otimes m_{x,y,x}) \circ (M_{x,y} \otimes c_{A_{x,y}\otimes A_{y,x},A_{y,x}}) \\
&\qquad \circ (M_{x,y} \otimes A_{x,y} \otimes S_{x,y} \otimes S_{x,y}) \circ \rho^3_{x,y} \\
&\overset{(*)}{=} (\psi_{x,y,x} \otimes A_{x,x}) \circ (M_{x,y} \otimes c_{A_{x,x},A_{y,x}}) \circ (M_{x,y} \otimes m_{x,y,x} \circ A_{y,x}) \\
&\qquad \circ (M_{x,y} \otimes A_{x,y} \otimes S_{x,y} \otimes S_{x,y}) \circ (M_{x,y} \otimes \Delta_{x,y} \otimes A_{x,y}) \circ \rho^2_{x,y} \\
&\overset{(11)}{=} (\psi_{x,y,x} \otimes A_{x,x}) \circ (M_{x,y} \otimes c_{A_{x,x},A_{y,x}}) \circ (M_{x,y} \otimes \eta_x \otimes A_{y,x}) \\
&\qquad \circ (M_{x,y} \otimes \varepsilon_{x,y} \otimes S_{x,y}) \circ \rho^2_{x,y} \\
&= (\psi_{x,y,x} \otimes A_{x,x}) \circ (M_{x,y} \otimes A_{y,x} \otimes \eta_x) \circ (M_{x,y} \otimes S_{x,y}) \circ \rho_{x,y} \\
&= (M_{x,y} \otimes \eta_x) \circ \psi_{x,y,x} \circ (M_{x,y} \otimes S_{x,y}) \circ \rho_{x,y} = (M_{x,x} \otimes \eta_x) \circ \gamma_{x,x}.
\end{aligned}$$

At $(*)$ we used the naturality of c resulting in the commutative diagram

$$\begin{array}{ccc}
A_{x,y} \otimes A_{y,x} \otimes A_{y,x} & \xrightarrow{c_{A_{x,y}\otimes A_{y,x},A_{y,x}}} & A_{y,x} \otimes A_{x,y} \otimes A_{y,x} \\
\Big\downarrow {\scriptstyle m_{x,y,x}\otimes A_{y,x}} & & \Big\downarrow {\scriptstyle A_{y,x}\otimes m_{x,y,x}} \\
A_{x,x} \otimes A_{y,x} & \xrightarrow{c_{A_{x,x},A_{y,x}}} & A_{y,x} \otimes A_{x,x}
\end{array}$$

From (46) and the universal property of equalizers, it follows that there is a unique morphism $\tilde{\gamma}_{x,y} :\ M_{x,y} \to M_x^{\mathrm{co}A}$ such that $i \circ \tilde{\gamma}_{x,y} = \gamma_{x,y}$.
Now we are ready to define $\alpha^M :\ M \to FG(M)$. The (x,y)-component is

$$\alpha^M_{x,y} = (\tilde{\gamma}_{x,y} \otimes A_{x,y}) \circ \rho_{x,y} :\ M_{x,y} \to M_x^{\mathrm{co}A} \otimes A_{x,y}.$$

$$\begin{aligned}
\varepsilon^M_{x,y} \circ \alpha^M_{x,y} &= \psi_{x,x,y} \circ (i \otimes A_{x,y}) \circ (\tilde{\gamma}_{x,y} \otimes A_{x,y}) \circ \rho_{x,y} \\
&= \psi^2_{x,y,x,y} \circ (M_{x,y} \otimes S_{x,y} \otimes A_{x,y}) \circ \rho^2_{x,y} \\
&\overset{(12)}{=} \psi_{x,y,y} \circ (M_{x,y} \otimes \eta_y) \circ (M_{x,y} \otimes \varepsilon_{x,y}) \circ \rho_{x,y} = M_{x,y}.
\end{aligned}$$

The proof of the fact that α^M is also a left inverse of ε^M is more involved. We first compute

$$\rho_{x,y} \circ \psi_{x,x,y} \circ (i \otimes A_{x,y}) :\ M_x^{\mathrm{co}A} \otimes A_{x,y} \to M_{x,y} \otimes A_{x,y}.$$

$$\begin{aligned}
&\rho_{x,y} \circ \psi_{x,x,y} \circ (i \otimes A_{x,y}) \\
&\overset{(44)}{=} (\psi_{x,x,y} \otimes M_{x,x,y}) \circ (M_{x,x} \otimes c_{A_{x,x},A_{x,y}} \otimes A_{x,y}) \\
&\qquad \circ (\rho_{x,x} \otimes \Delta_{x,y}) \circ (i \otimes A_{x,y})
\end{aligned}$$

$$\begin{aligned}
&= (\psi_{x,x,y}\otimes M_{x,x,y})\circ(M_{x,x}\otimes c_{A_{x,x},A_{x,y}}\otimes A_{x,y})\\
&\quad\circ(M_{x,x}\otimes\eta_x\otimes\Delta_{x,y})\circ(i\otimes A_{x,y})\\
&= (\psi_{x,x,y}\otimes M_{x,x,y})\circ(M_{x,x}\otimes A_{x,y}\otimes\eta_x\otimes A_{x,y})\circ(i\otimes\Delta_{x,y})\\
&= (\psi_{x,x,y}\otimes A_{x,y})\circ(i\otimes\Delta_{x,y}). \qquad (47)
\end{aligned}$$

Our next step is to compute

$$\begin{aligned}
&i\circ\tilde{\gamma}_{x,y}\circ\psi_{x,x,y}\circ(i\otimes A_{x,y})\\
&\quad= \psi_{x,y,x}\circ(M_{x,y}\otimes S_{x,y})\circ\rho_{x,y}\circ\psi_{x,x,y}\circ(i\otimes A_{x,y})\\
&\quad\overset{(47)}{=} \psi_{x,y,x}\circ(M_{x,y}\otimes S_{x,y})\circ(\psi_{x,x,y}\otimes A_{x,y})\circ(i\otimes\Delta_{x,y})\\
&\quad= \psi^2_{x,x,y,x}\circ(M_{x.x}\otimes A_{x,y}\otimes S_{x,y})\circ(M_{x.x}\otimes\Delta_{x,y})\circ(i\otimes A_{x,y})\\
&\quad\overset{(11)}{=} \psi_{x,x,x}\circ(M_{x,x}\otimes\eta_x)\circ(M_{x,x}\otimes\varepsilon_{x,y})\circ(i\otimes A_{x,y})\\
&\quad= i\otimes\varepsilon_{x,y}=i\circ(M_x^{\mathrm{co}A}\otimes\varepsilon_{x,y}).
\end{aligned}$$

The universal property of equalizers tells us that there is a unique $f:\ M_x^{\mathrm{co}A}\otimes A_{x,y}\to M_x^{\mathrm{co}A}$ such that $i\circ f=i\otimes\varepsilon_{x,y}$. This implies that

$$\tilde{\gamma}_{x,y}\circ\psi_{x,x,y}\circ(i\otimes A_{x,y})=M_x^{\mathrm{co}A}\otimes\varepsilon_{x,y}. \qquad (48)$$

Finally

$$\begin{aligned}
\alpha^M_{x,y}\circ\varepsilon^M_{x,y} &= (\tilde{\gamma}_{x,y}\otimes A_{x,y})\circ\rho_{x,y}\circ\psi_{x,x,y}\circ(i\otimes A_{x,y})\\
&\overset{(47)}{=} (\tilde{\gamma}_{x,y}\otimes A_{x,y})\circ(\psi_{x,x,y}\otimes A_{x,y})\circ(i\otimes\Delta_{x,y})\\
&= (\tilde{\gamma}_{x,y}\otimes A_{x,y})\circ(\psi_{x,x,y}\otimes A_{x,y})\circ(i\otimes A_{x,y}\otimes A_{x,y}))\circ(M_x^{\mathrm{co}A}\otimes\Delta_{x,y})\\
&\overset{(48)}{=} (M_x^{\mathrm{co}A}\otimes\varepsilon_{x,y}\otimes A_{x,y})\circ(M_x^{\mathrm{co}A}\otimes\Delta_{x,y})=M_x^{\mathrm{co}A}.
\end{aligned}$$

Part 2. η^N has an inverse β^N, for all $N\in\mathcal{D}(X)$.
The x-component of β^N is

$$\beta^N_x=(N_x\otimes\varepsilon_{x,x})\circ i:\ (N_x\otimes A_{x,x})^{\mathrm{co}A_{x,x}}\to N_x$$

It is easy to see that

$$\beta^N_x\circ\eta^N_x=(N_x\otimes\varepsilon_{x,x})\circ i\circ\eta^N_x\overset{(9)}{=}(N_x\otimes\varepsilon_{x,x})\circ(N_x\otimes\eta_x)=N_x.$$

The universal property of the equalizer entails that there is only one endomorphism f of $(N_x\otimes A_{x,x})^{\mathrm{co}A_{x,x}}$ such that $i\circ f=i$, namely the identity. Now

$$\begin{aligned}
i\circ\eta^N_x\circ\beta^N_x&\overset{(9)}{=}(N_x\otimes\eta_x)\circ(N_x\otimes\varepsilon_{x,x})\circ i\\
&= (N_x\otimes\varepsilon_{x,x}\otimes A_{x,x})\circ(N_x\otimes A_{x,x}\otimes\eta_x)\circ i\\
&= (N_x\otimes\varepsilon_{x,x}\otimes A_{x,x})\circ(N_x\otimes\Delta_{x,x})\circ i=i,
\end{aligned}$$

so it follows that $\eta^N_x\circ\beta^N_x=(N_x\otimes A_{x,x})^{\mathrm{co}A_{x,x}}$.

(2) $\Rightarrow$ (3) is obvious.

<u>(3) ⇒ (4)</u>. For every $z\in X$, consider the object $M^z\in \mathcal{V}(X)$ given by $M^z_{x,y}=A_{z,y}\otimes A_{x,y}$. The structure morphisms $\rho^z_{x,y}=A_{z,y}\otimes\Delta_{x,y}:\ M^z_{x,y}\otimes A_{x,y}$ and

$$\begin{aligned}\psi^z_{x,y,u}&=(m_{z,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\circ(M^z_{x,y}\otimes\Delta_{y,u})\\ &:\ M^z_{x,y}\otimes A_{y,u}\to M^z_{x,u}\end{aligned}$$

make M^z into an object of $\mathcal{V}(X)^A_A$. Let us verify that the compatibility relation (44) holds. We compute both sides of the equation, and see that they are equal.

$$\begin{aligned}
\rho^z_{x,u}\circ\psi^z_{x,y,u}&=(A_{z,u}\otimes\Delta_{x,u})\circ(m_{z,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes\Delta_{y,u})\\
&=(m_{z,y,u}\otimes m_{x,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes A_{y,u}\otimes A_{x,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{y,u}\otimes\Delta_{x,y}\otimes\Delta_{y,u})\circ(A_{z,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes\Delta_{y,u})\\
&=(m_{z,y,u}\otimes m_{x,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes A_{y,u}\otimes A_{x,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes c_{A_{x,y}\otimes A_{x,y},A_{y,u}}\otimes A_{y,u}\otimes A_{y,u})\circ(A_{z,y}\otimes\Delta_{x,y}\otimes A_{y,u}\otimes\Delta_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes\Delta_{y,u})\\
&=(m_{z,y,u}\otimes m_{x,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes c_{A_{x,y},A_{y,u}}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u}\otimes A_{y,u})\circ(A_{z,y}\otimes\Delta_{x,y}\otimes\Delta^2_{y,u});
\end{aligned}$$

$$\begin{aligned}
&(\psi^z_{x,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes A_{x,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\circ(\rho^z_{x,y}\otimes\Delta_{y,u})\\
&=(m_{z,y,u}\otimes m_{x,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u}\otimes A_{x,y}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes\Delta_{y,u}\otimes A_{x,y}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\circ(A_{z,y}\otimes\Delta_{x,y}\otimes\Delta_{y,u})\\
&=(m_{z,y,u}\otimes m_{x,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u}\otimes A_{x,y}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes c_{A_{x,y},A_{y,u}\otimes A_{y,u}}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes A_{x,y}\otimes\Delta_{y,u}\otimes A_{y,u})\circ(A_{z,y}\otimes\Delta_{x,y}\otimes\Delta_{y,u})\\
&=(m_{z,y,u}\otimes m_{x,y,u}\otimes m_{x,y,u})\circ(A_{z,y}\otimes c_{A_{x,y},A_{y,u}}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u})\\
&\quad\circ(A_{z,y}\otimes A_{x,y}\otimes c_{A_{x,y},A_{y,u}}\otimes A_{y,u}\otimes A_{y,u})\circ(A_{z,y}\otimes\Delta_{x,y}\otimes\Delta^2_{y,u}).
\end{aligned}$$

Consider the morphism $f=A_{z,x}\otimes\eta_{x,x}:\ A_{z,x}\to A_{z,x}\otimes A_{x,x}=M^z_{x,x}$. Since

$$\begin{aligned}\rho^z_{x,x}\circ f&=(A_{z,x}\otimes\Delta_{x,x})\circ(A_{z,x}\otimes\eta_{x,x})=(A_{z,x}\otimes\eta_{x,x}\otimes\eta_x)\\ &=(A_{z,x}\otimes A_{x,x}\otimes\eta_x)\circ(A_{z,x}\otimes\eta_x)=(M^z_{x,x}\otimes\eta_x)\circ f,\end{aligned}$$

there exists a unique $\tilde f:\ A_{z,x}\to M_x^{z\mathrm{co}A}$ such that $i\circ\tilde f=f$. $\tilde f$ is invertible, with inverse $g=(A_{z,x}\otimes\varepsilon_{x,x})\circ i$. Indeed,

$$g\circ\tilde f=(A_{z,x}\otimes\varepsilon_{x,x})\circ f=(A_{z,x}\otimes\varepsilon_{x,x})\circ(A_{z,x}\otimes\eta_{x,x})=A_{z,x}.$$

We also have that

$$\begin{aligned}i\circ\tilde f\circ g&=f\circ g=(A_{z,x}\otimes\eta_{x,x})\circ(A_{z,x}\otimes\varepsilon_{x,x})\circ i\\ &=(A_{z,x}\otimes\varepsilon_{x,x}\otimes A_{x,x})\circ(A_{z,x}\otimes A_{x,x}\otimes\eta_x)\circ i\end{aligned}$$

$$= \quad (A_{z,x}\otimes\varepsilon_{x,x}\otimes A_{x,x})\circ(A_{z,x}\otimes\Delta_{x,x})\circ i=i,$$

and it follows from the uniqueness in the universal property of equalizers that $\tilde{f}\circ g=M_x^{z\mathrm{co}A}$. We know by assumption that

$$\varepsilon^z_{x,y}=\psi^z_{x,x,y}\circ(i\otimes A_{x,y}):\ M_x^{z\mathrm{co}A}\to A_{x,y}\to M^z_{x,y}$$

is an isomorphism. It follows that

$$\begin{aligned}
\varepsilon^z_{x,y}\circ(\tilde{f}\otimes A_{x,y}) &= (m_{z,x,y}\otimes m_{x,x,y})\circ(A_{z,x}\otimes c_{A_{x,x},A_{x,y}}\otimes A_{x,y})\\
&\quad\circ(A_{z,x}\otimes A_{x,x}\otimes\Delta_{x,y})\circ(i\otimes A_{x,y})\circ(\tilde{f}\otimes A_{x,y})\\
&= (m_{z,x,y}\otimes m_{x,x,y})\circ(A_{z,x}\otimes c_{A_{x,x},A_{x,y}}\otimes A_{x,y})\\
&\quad\circ(A_{z,x}\otimes A_{x,x}\otimes\Delta_{x,y})\circ(A_{x,x}\otimes\eta_{x,x}\otimes A_{x,y})\\
&= (m_{z,x,y}\otimes m_{x,x,y})\circ(A_{z,x}\otimes c_{A_{x,x},A_{x,y}}\otimes A_{x,y})\\
&\quad\circ(A_{x,x}\otimes\eta_{x,x}\otimes A_{x,y}\otimes A_{x,y})\circ(A_{z,x}\otimes\Delta_{x,y})\\
&= (m_{z,x,y}\otimes m_{x,x,y})\circ(A_{z,x}\otimes A_{x,y}\otimes\eta_x\otimes A_{x,y})\circ(A_{z,x}\otimes\Delta_{x,y})\\
&= (m_{z,x,y}\otimes A_{x,y})\circ(A_{z,x}\otimes\Delta_{x,y})=\mathrm{can}^z_{x,y}
\end{aligned}$$

is an isomorphism.

<u>(4) ⇒ (5)</u> is obvious.

<u>(5) ⇒ (1)</u>.
We define the antipode as follows:

$$S_{x,y}=(A_{y,x}\otimes\varepsilon_{x,y})\circ g_{x,y}\circ(\eta_y\otimes A_{x,y}).$$

We have to show that the equations (11-12) are satisfied. To this end, we first need some auxiliary formulas. Composing the equality

$$\begin{aligned}
&(m_{x,y,y}\otimes A_{x,y})\circ(A_{x,y}\otimes\mathrm{can}^y_{x,y})\\
&\quad= (m_{x,y,y}\otimes A_{x,y})\circ(A_{x,y}\otimes m_{y,x,y}\otimes A_{x,y})\circ(A_{x,y}\otimes A_{y,x}\circ\Delta_{x,y})\\
&\quad= (m_{x,x,y}\otimes A_{x,y})\circ(m_{x,y,x}\otimes A_{x,y}\otimes A_{x,y})\circ(A_{x,y}\otimes A_{y,x}\circ\Delta_{x,y})\\
&\quad= (m_{x,x,y}\otimes A_{x,y})\circ(A_{x,x}\otimes\Delta_{x,y})\circ(m_{x,y,x}\otimes A_{x,y})\\
&\quad= \mathrm{can}^x_{x,y}\circ(m_{x,y,x}\otimes A_{x,y})
\end{aligned}$$

to the left with $f_{x,y}$ and to the right with $A_{x,y}\otimes g_{x,y}$, we find that

$$f_{x,y}\circ(m_{x,y,y}\otimes A_{x,y})=(m_{x,y,x}\otimes A_{x,y})\circ(A_{x,y}\otimes g_{x,y}). \tag{49}$$

Composing the equality

$$\begin{aligned}
&(\mathrm{can}^y_{x,y}\otimes A_{x,y})\circ(A_{y,x}\otimes\Delta_{x,y})\\
&\quad= (m_{y,x,y}\otimes A_{x,y}\otimes A_{x,y})\circ(A_{y,x}\otimes\Delta_{x,y}\otimes A_{x,y})\circ(A_{y,x}\otimes\Delta_{x,y})\\
&\quad= (m_{y,x,y}\otimes A_{x,y}\otimes A_{x,y})\circ(A_{y,x}\otimes A_{x,y}\otimes\Delta_{x,y})\circ(A_{y,x}\otimes\Delta_{x,y})\\
&\quad= (A_{y,y}\otimes\Delta_{x,y})\circ(m_{y,x,y}\otimes A_{x,y})\circ(A_{y,x}\otimes\Delta_{x,y})\\
&\quad= (A_{y,y}\otimes\Delta_{x,y})\circ\mathrm{can}^y_{x,y}
\end{aligned}$$

to the left and to the right with $g_{x,y}$, we find that

$$(A_{y,x} \otimes \Delta_{x,y}) \circ g_{x,y} = (g_{x,y} \otimes A_{x,y}) \circ (A_{y,y} \otimes \Delta_{x,y}). \tag{50}$$

$$\begin{aligned}
\eta_x \circ \varepsilon_{x,y} = (A_{x,x} \otimes \varepsilon_{x,y}) \circ (\eta_x \otimes A_{x,y}) & \\
= \quad & (A_{xx,} \otimes \varepsilon_{x,y}) \circ f_{x,y} \circ \mathrm{can}^x_{x,y} \circ (\eta_x \otimes A_{x,y}) \\
= \quad & (A_{xx,} \otimes \varepsilon_{x,y}) \circ f_{x,y} \circ (m_{x,x,y} \otimes A_{x,y}) \circ (A_{x,x} \otimes \Delta_{x,y}) \circ (\eta_x \otimes A_{x,y}) \\
= \quad & (A_{xx,} \otimes \varepsilon_{x,y}) \circ f_{x,y} \circ (m_{x,x,y} \otimes A_{x,y}) \circ (\eta_x \otimes A_{x,y} \otimes A_{x,y}) \circ \Delta_{x,y} \\
= \quad & (A_{xx,} \otimes \varepsilon_{x,y}) \circ f_{x,y} \circ (m_{x,x,y} \otimes A_{x,y}) \circ (A_{x,y} \otimes \eta_y \otimes A_{x,y}) \circ \Delta_{x,y} \\
\overset{(49)}{=} \quad & (A_{xx,} \otimes \varepsilon_{x,y}) \circ (m_{x,y,x} \otimes A_{x,y}) \circ (A_{x,y} \otimes g_{x,y}) \\
& \circ (A_{x,y} \otimes \eta_y \otimes A_{x,y}) \circ \Delta_{x,y} \\
= \quad & m_{x,y,x} \circ (A_{x,y} \otimes S_{x,y}) \circ \Delta_{x,y},
\end{aligned}$$

and this shows that (11) holds.

$$\begin{aligned}
\eta_y \circ \varepsilon_{x,y} = (A_{y,y} \otimes \varepsilon_{x,y}) \circ (\eta_y \otimes A_{x,y}) & \\
= \quad & (A_{y,y} \otimes \varepsilon_{x,y}) \circ \mathrm{can}^y_{x,y} \circ g_{x,y} \circ (\eta_y \otimes A_{x,y}) \\
= \quad & (A_{y,y} \otimes \varepsilon_{x,y}) \circ (m_{y,x,y} \otimes A_{x,y}) \circ (A_{y,x} \otimes \Delta_{x,y}) \circ g_{x,y} \circ (\eta_y \otimes A_{x,y}) \\
= \quad & m_{y,x,y} \circ (A_{y,x} \otimes A_{x,y} \otimes \varepsilon_{x,y}) \circ (A_{y,x} \otimes \Delta_{x,y}) \circ g_{x,y} \circ (\eta_y \otimes A_{x,y}) \\
= \quad & m_{y,x,y} \circ (A_{y,x} \otimes \varepsilon_{x,y} \otimes A_{x,y}) \circ (A_{y,x} \otimes \Delta_{x,y}) \circ g_{x,y} \circ (\eta_y \otimes A_{x,y}) \\
\overset{(50)}{=} \quad & m_{y,x,y} \circ (A_{y,x} \otimes \varepsilon_{x,y} \otimes A_{x,y}) \circ (g_{x,y} \otimes A_{x,y}) \\
& \circ (A_{y,y} \otimes \Delta_{x,y}) \circ (\eta_y \otimes A_{x,y}) \\
= \quad & m_{y,x,y} \circ (A_{y,x} \otimes \varepsilon_{x,y} \otimes A_{x,y}) \circ (g_{x,y} \otimes A_{x,y}) \circ (\eta_y \otimes A_{x,y} \otimes A_{x,y}) \circ \Delta_y \\
= \quad & m_{y,x,y} \circ \ (S_{x,y} \otimes A_{x,y}) \circ \Delta_y,
\end{aligned}$$

and this shows that (12) holds. □

Remarks 9.3. 1) The implication $\underline{(1) \Rightarrow (4)}$ can easily be proved directly: it is easily verified that

$$(\mathrm{can}^z_{x,y})^{-1} = (m_{z,y,x} \circ A_{x,y}) \circ (A_{z,y} \otimes S_{x,y} \otimes A_{x,y}) \circ (A_{z,y} \otimes \Delta_{x,y}).$$

2) It follows from the Theorem that a Hopf module over a Hopf category is isomorphic to a free Hopf module, that is a Hopf module in the image of the functor G. This result is known in the literature as the Fundamental Theorem for Hopf modules. Its original form (in the case where $\mathcal{V}$ is de category of vector spaces and X is a singleton) it is due to Larson and Sweedler [18], see also [25, Theorem 1.1]. For the case where $\mathcal{V}$ is an arbitrary braided monoidal category with equalizers and X is a singleton, see [26, Theorem 3.4] and [19, Theorem 1.4].

Let us now proceed to some applications of the Fundamental Theorem. We restrict attention to the case where $\mathcal{V}$ is the category $\mathcal{M}^{\mathrm{f}}_k$ of finitely generated projective modules over a commutative ring k (or finite dimensional vector spaces over a field k). Our applications generalize applications of the

classical Fundamental Theorem as they can be found in [25, Chapter 4].
For $\mathcal{V} = \mathcal{M}_k^{\rm f}$, the axioms (11-12) take the following form

$$h_{(1)}S_{x,y}(h_{(2)}) = \varepsilon_{x,y}(h)1_x \; ; \; S_{x,y}(h_{(1)})h_{(2)} = \varepsilon_{x,y}(h)1_y, \tag{51}$$

for all $x, y \in X$ and $h \in A_{x,y}$. The formula (13-14) can be written as

$$S_{x,z}(hl) = S_{y,z}(l)S_{x,y}(h); \tag{52}$$

$$\Delta_{y,x}(S_{x,y}(h)) = S_{x,y}(h_{(2)}) \otimes S_{x,y}(h_{(1)}), \tag{53}$$

for all $x, y, z \in X$, $h \in A_{x,y}$ and $l \in A_{y,z}$. The compatibility relation for Hopf modules amounts to

$$\rho_{x,z}(ma) = m_{[0]}a_{(1)} \otimes m_{[1]}a_{(2)}, \tag{54}$$

for all $m \in M_{x,y}$ and $a \in A_{y,z}$.

Proposition 9.4. *Let A be a Hopf category in $\mathcal{M}_k^{\rm f}(X)$. Then A^* is a Hopf module, with structure maps $\rho_{x,y} : A^*_{x,y} \to A^*_{x,y} \otimes A_{x,y}$ and $\psi_{x,y,z} : A^*_{x,y} \otimes A_{y,z} \to A^*_{x,z}$ defined as follows:*

1. *For $a^* \in A^*_{x,y}$, $\rho_{x,y}(a^*) = \sum_i a^*a_i^* \otimes a_i$, where $\sum_i a_i^* \otimes a_i \in A^*_{x,y} \otimes A_{x,y}$ is the finite dual basis of $A_{x,y}$. The multiplication on $A^*_{x,y}$ is the opposite convolution.*
2. *For $a^* \in A^*_{x,y}$ and $a \in A_{y,z}$, $\psi_{x,y,z}(a^* \otimes a) = a^* \leftharpoonup a \in A^*_{x,z}$ is given by the formula $\langle a^* \leftharpoonup a, b\rangle = \langle a^*, bS_{y,z}(a)\rangle$, for all $b \in A_{y,z}$.*

Proof. The right A-coaction is obtained as follows: $A_{x,y}$ is a k-coalgebra, hence $A^*_{x,y}$ is a k-algebra (with opposite convolution product). It is therefore a right $A^*_{x,y}$-module, and a right $A_{x,y}$-comodule. The coaction that is opbtained in this way is precisely the one that is described in the Proposition.
Now let us show that the structure maps $\psi_{x,y,z}$ define a right A-module structure on A^*.
Associativity. For all $a^* \in A^*_{x,y}$, $a \in A_{y,z}$, $b \in A_{z,u}$ and $c \in A_{x,u}$, we have that

$$\begin{aligned} \langle a^* \leftharpoonup (ab), c\rangle &= \langle a^*, cS_{y,u}(ab)\rangle \overset{(52)}{=} \langle a^*, cS_{z,u}(b)S_{y,z}(a)\rangle \\ &= \langle a^* \leftharpoonup a, cS_{z,u}(b)\rangle = \langle (a^* \leftharpoonup a) \leftharpoonup b, c\rangle. \end{aligned}$$

Unit property. For all $a^* \in A^*_{x,y}$ and $a \in A_{x,y}$, we have that

$$\langle a^* \leftharpoonup 1_y, a\rangle = \langle a^*, aS_{y,y}(1_i)\rangle \overset{(52)}{=} \langle a^*, a\rangle.$$

Now we verify the Hopf compatibility condition (54). We have to show that

$$\rho_{x,z}(a^* \leftharpoonup a) = \sum_i (a^*a_i^*) \leftharpoonup a_{(1)} \otimes a_i a_{(2)},$$

for all $a^* \in A^*_{x,y}$ and $a \in A_{y,z}$. Now

$$\rho_{x,z}(a^* \leftharpoonup a) = \sum_j (a^* \leftharpoonup a)b_j^* \otimes b_j,$$

where $\sum_j b_j^* \otimes b_j \in A_{x,z}^* \otimes A_{x,z}$ is the dual basis of $A_{x,z}$, so it suffices to show that

$$\sum_j \langle (a^* \leftharpoonup a) b_j^*, c\rangle b_j = \sum_i \langle (a^* a_i^*) \leftharpoonup a_{(1)}, c\rangle a_i a_{(2)},$$

for all $c \in A_{x,z}$. This can be done as follows:

$$\begin{aligned}\sum_i \langle (a^* a_i^*) \leftharpoonup a_{(1)}, c\rangle a_i a_{(2)} &\overset{(53)}{=} \sum_i \langle a^*, c_{(2)} S_{y,z}(a_{(1)})\rangle \langle a_i^*, c_{(1)} S_{y,z}(a_{(2)})\rangle a_i a_{(3)} \\ &= \langle a^*, c_{(2)} S_{y,z}(a_{(1)})\rangle c_{(1)} S_{y,z}(a_{(2)}) a_{(3)} \\ &\overset{(51)}{=} \langle a^*, c_{(2)} S_{y,z}(a_{(1)})\rangle c_{(1)} \varepsilon_{y,z}(a_{(2)}) 1_z = \langle a^*, c_{(2)} S_{y,z}(a)\rangle c_{(1)} \\ &= \sum_j \langle a^* \leftharpoonup a, c_{(2)}\rangle \langle b_j^*, c_{(1)}\rangle b_j = \sum_j \langle (a^* \leftharpoonup a) b_j^*, c\rangle b_j.\end{aligned}$$

□

We compute $A^{*\mathrm{co}A}$. Recall that $A_{x,x}$ is a Hopf algebra, for every $x \in X$, and that

$$A_x^{*\mathrm{co}A} = (A_{x,x}^*)^{\mathrm{co}A_{x,x}} = \int_{A_{x,x}^*}^l = \{\varphi \in A_{x,x}^* \mid \varphi a^* = \langle a^*, 1_x\rangle \varphi, \text{ for all } a^* \in A_{x,x}^*\},$$

the space of left integrals on $A_{x,x}$. From Theorem 9.2 and Proposition 9.4, we obtain the following result.

Corollary 9.5. *Let A be a Hopf category in $\mathcal{M}_k^{\mathrm{f}}(X)$. For all $x, y \in X$, we have an isomorphism*

$$\alpha_{x,y} = \varepsilon_{x,y}^{A^*} : \int_{A_{x,x}^*}^l \otimes A_{x,y} \to A_{x,y}^*, \ \varepsilon_{x,y}^{A^*}(\varphi \otimes a) = \varphi \leftharpoonup a.$$

Proposition 9.6. *Let A be a Hopf category in $\mathcal{M}_k^{\mathrm{f}}(X)$. The antipode maps $S_{x,y}:\ A_{x,y} \to A_{y,x}$ are bijective, for all $x, y \in X$.*

Proof. It is well-known (and it also follows from Corollary 9.5) that $J = \int_{A_{x,x}^*}^l$ is finitely generated projective of rank one as a k-module. Therefore the evaluation map

$$\mathrm{ev}:\ J^* \otimes J \to k,\ \mathrm{ev}(p \otimes \varphi) = p(\varphi)$$

is an isomorphism of k-modules. The isomorphism

$$\tilde{\alpha}_{x,y} = (J^* \otimes \alpha) \circ (\mathrm{ev}^{-1} \otimes A_{x,y}) :\ A_{x,y} \to J^* \otimes A_{x,y}$$

can be described explicitly as follows:

$$\tilde{\alpha}_{x,y}(a) = \sum_l p_l \otimes \varphi_l \leftharpoonup a,$$

where $\mathrm{ev}^{-1}(1) = \sum_l p_l \otimes \varphi_l$.
Now assume that $S_{x,y}(a) = 0$, for some $a \in A_{x,y}$. For all $\varphi \in A_{x,x}^*$ and $b \in A_{x,y}$, we have that

$$\langle \varphi \leftharpoonup a, b\rangle = \langle \varphi, b S_{x,y}(a)\rangle = 0,$$

so it follows that $\tilde{\alpha}_{x,y}(a) = 0$, and $a = 0$, since $\tilde{\alpha}_{x,y}$ is a bijection. This proves that $S_{x,y}$ is injective.
Now assume that k is a field. The maps

$$\alpha = S_{x,y} \circ S_{y,x} \text{ and } \beta = S_{y,x} \circ S_{x,y}$$

are injective endomorphisms of the finite dimensional vector spaces $A_{y,x}$ and $A_{x,y}$. From the dimension formulas, it follows that they are automorphisms. We then have that

$$\begin{aligned} A_{y,x} &= \alpha \circ \alpha^{-1} = S_{x,y} \circ S_{y,x} \circ \alpha^{-1}; \\ A_{y,x} &= \beta^{-1} \circ \beta = \beta^{-1} \circ S_{y,x} \circ S_{x,y}. \end{aligned}$$

This tells us that $S_{x,y}$ has a left inverse and a right inverse; these are necessarily equal, hence $S_{x,y}$ is bijective.
Now consider the general case where k is a commutative ring. The surjectivity of $S_{x,y}$ follows from a local-global argument. Let $Q = \text{Coker}\,(S_{x,y})$. For every prime ideal p of k, we can consider the localized Hopf category A_p, with $A_{p,x,y} = A_{x,y} \otimes k_p$. For every prime ideal p of k, $\text{Coker}\,(S_{p,x,y}) = Q_p$, since localization at a prime ideal is an exact functor. Now the spaces $A_{p,x,y}/pA_{p,x,y}$ define a finite dimensional Hopf category A_p/pA_p over the field k_p/pk_p, and its antipode maps are bijective. It follows from Nakayama's Lemma that the localized maps $S_{p,x,y} : A_{p,x,y} \to A_{p,y,x}$ are all bijective, and then it follows that $S_{x,y}$ is bijective. □

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