

DGLA STRUCTURE ON THE DEFORMATION COMPLEX OF LEIBNIZ PAIRS

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Abstract

In this note we outline the construction of a DGLA structure on the deformation complex of a Leibniz Pair. The Lie bracket is given intrinsically as the commutator of coderivations on a certain coalgebra related to Kajiura and Stasheff's Open Closed Homotopy Algebras.

Introduction

The cohomology and deformations of Leibniz Pairs were studied by Flato, Gerstenhaber and Voronov [5] in 1995. The concept of Leibniz Pairs has appeared in the work of H. Cartan [2] under the name of \mathfrak{g} -algebra. It is also related to the concept of Lie-Rinehart algebras [8, 9]. A new type of homotopy algebra (called OCHA) related to Open-Closed String Field Theory was introduced by Kajiura and Stasheff [10] in 2004. OCHAs can be given a nice geometrical definition involving the Axelrod-Singer compactification of the configuration space of points on the upper closed half plane [7]. A Leibniz pair is a particular example of an OCHA with most of its higher structure maps equal to zero.

The main aim of this note is to outline the construction of a differential graded Lie algebra (DGLA) structure on the deformation complex of Leibniz Pairs using an intuitive argument based on the geometrical definition of OCHA. Our main result is the following (see Section (3.2)).

Theorem. *There is an isomorphism of DG spaces:*

$$\xi : (C^*(L, A), d_{tot}) \longrightarrow (\text{Coder}({}^c\Lambda(\downarrow L)) \times \text{Coder}({}^c\Lambda(\downarrow L) \otimes {}^c\bar{T}(\downarrow A)), \delta = [l + n, -]).$$

A DGLA structure on the deformation complex $C^*(L, A)$ follows immediately. A proof will be outlined in this note. Detailed constructions and precise proofs will appear in a forthcoming paper.

1 Preliminaries

Let us fix a field k of characteristic zero. All vector spaces are assumed to be over k . By a 'graded vector space' we will always mean a ' \mathbb{Z} -graded vector space'. For a graded vector space V , we define a left action of the symmetric group S_n on $V^{\otimes n}$ in the following way: if

$\tau \in S_2$ is a transposition, then the action is given by $x_1 \otimes x_2 \xrightarrow{\tau} (-1)^{|x_1||x_2|} x_2 \otimes x_1$. Since any $\sigma \in S_n$ is a composition of transpositions, the sign of the action of σ on $V^{\otimes n}$ is well defined:

$$x_1 \otimes \cdots \otimes x_n \xrightarrow{\sigma} \varepsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

We will refer to $\varepsilon(\sigma)$ as the Koszul sign of the permutation. Let us define $\chi(\sigma) = (-1)^\sigma \varepsilon(\sigma)$, where $(-1)^\sigma$ is the sign of the permutation. Given two homogeneous maps $f, g : V \rightarrow W$, we will follow the Koszul sign convention for the tensor product: $(f \otimes g)(v_1 \otimes v_2) = (-1)^{|g||v_1|} (f(v_1) \otimes g(v_2))$.

We will use the notation of Lada-Markl [12] for the suspension and desuspension operators. Let $\uparrow V$ (resp. $\downarrow V$) denote the suspension (resp. desuspension) of the graded vector space V defined by: $(\uparrow V)^p = V^{p-1}$ (resp. $(\downarrow V)^p = V^{p+1}$). We thus have the natural maps $\uparrow : V \rightarrow \uparrow V$ of degree 1, and $\downarrow : V \rightarrow \downarrow V$ of degree -1 . Let $\uparrow^{\otimes n}$ denote $\otimes^n \uparrow : \otimes^n V \rightarrow \otimes^n \uparrow V$ ($\downarrow^{\otimes n}$ is defined analogously). The operators $\uparrow^{\otimes n}$ and $\downarrow^{\otimes n}$ transform symmetric operations into anti-symmetric ones. In fact, let E (resp. A) denote the symmetric (resp. anti-symmetric) action of the group of permutations S_n on $V^{\otimes n}$:

$$\begin{aligned} E(\sigma)(x_1 \otimes \cdots \otimes x_n) &= \varepsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \\ (\text{resp. } A(\sigma)(x_1 \otimes \cdots \otimes x_n) &= \chi(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}). \end{aligned} \quad (1)$$

The convenience of Lada-Markl notation becomes apparent in the following properties which are crucial in some computations: $\uparrow^{\otimes n} E(\sigma) \downarrow^{\otimes n} = (-1)^{n(n-1)/2} A(\sigma)$, for any $\sigma \in S_n$. In particular, $\uparrow^{\otimes n} \circ \downarrow^{\otimes n} = (-1)^{n(n-1)/2} \cdot \text{Id}$. The sign $(-1)^{n(n-1)/2}$ is a consequence of the Koszul sign convention (see also [3]).

2 OCHAs and Leibniz Pairs

A Leibniz pair (L, A) consists of a Lie algebra L , an associative algebra A and a morphism of Lie algebras: $\mu : L \rightarrow \text{Der}A$, i.e., a Lie algebra action by derivations of L on A . So, a Leibniz pair is a structure (L, A, l, m, μ) , where l is the Lie bracket on L , m is the associative product on A and μ is the Lie action. We will see below that such a structure is a particular case of an OCHA structure.

Let us first recall the definition of SH Lie [13] algebras in a grading and signs convention compatible with its compactified configuration space description (see [14, 11]).

Definition 2.1 (Strong Homotopy Lie algebra). A strong homotopy Lie algebra (or L_∞ -algebra) is a \mathbb{Z} -graded vector space V endowed with a collection of graded symmetric n -ary brackets $l_n : V^{\otimes n} \rightarrow V$, of degree $3 - 2n$ such that $l_1^2 = 0$ and for $n \geq 2$:

$$\partial l_n(v_1, \dots, v_n) = \sum_{\substack{\sigma \in \Sigma_{k+l=n} \\ k \geq 2, l \geq 1}} \varepsilon(\sigma) l_{1+l}(l_k(v_{\sigma(1)}, \dots, v_{\sigma(k)}), v_{\sigma(k+1)}, \dots, v_{\sigma(n)}) = 0 \quad (2)$$

where σ runs over all (k, l) -unshuffles, i.e., permutations $\sigma \in S_n$ such that $\sigma(i) < \sigma(j)$ for $1 \leq i < j \leq k$ and for $k+1 \leq i < j \leq k+l$.

Remark 2.2. The operator ∂ in the above definition denotes the induced differential on the endomorphism complex, i.e.:

$$\partial l_n = l_1 l_n + l_n(l_1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes l_1).$$

Definition 2.3 (Open-Closed Homotopy Algebra – OCHA). An OCHA consists of a 4-tuple $(L, A, \mathfrak{l}, \mathfrak{n})$ where L and A are \mathbb{Z} -graded vector spaces, $\mathfrak{l} = \{l_n : L^{\otimes n} \rightarrow L\}_{n \geq 1}$ and $\mathfrak{n} = \{n_{p,q} : L^{\otimes p} \otimes A^{\otimes q} \rightarrow A\}_{p+q \geq 1}$ are two families of multilinear maps where l_n has degree $3 - 2n$ and $n_{p,q}$ has degree $2 - 2p - q$, such that (L, \mathfrak{l}) is an L_∞ -algebra and the two families satisfy the following compatibility condition:

$$\begin{aligned} \partial n_{n,m}(v_1, \dots, v_n, a_1, \dots, a_m) = & \\ & \sum_{\sigma \in \Sigma_{p+r=n}, p \geq 2} (-1)^{\varepsilon(\sigma)} n_{1+r,m}(l_p(v_{\sigma(1)}, \dots, v_{\sigma(p)}), v_{\sigma(p+1)}, \dots, v_{\sigma(n)}, a_1, \dots, a_m) + \\ + \sum_{\substack{\sigma \in \Sigma_{p+r=n}, i+j=m-s \\ (r,s) \neq (0,1), (n,m)}} (-1)^{\mu_{p,i}(\sigma)} n_{p,i+1+j}(v_{\sigma(1)}, \dots, v_{\sigma(p)}, a_1, \dots, a_i, n_{r,s}(v_{\sigma(p+1)}, \dots, v_{\sigma(n)}, a_{i+1}, \dots, a_{i+s}), a_{i+s+1}, \dots, a_m). \end{aligned}$$

$$\begin{aligned} \text{where } \mu_{p,i}(\sigma) = s + i + si + ms + \varepsilon(\sigma) + s(v_{\sigma(1)} + \dots + v_{\sigma(p)} + a_1 + \dots + a_i) + \\ + (a_1 + \dots + a_i)(v_{\sigma(i+1)} + \dots + v_{\sigma(n)}). \end{aligned}$$

Remark 2.4. The operator ∂ in the above definition denotes the induced differential on the endomorphism complex, i.e.:

$$\partial n_{n,m} = n_{0,1} n_{n,m} - (-1)^m n_{n,m}(d_{L^n} \otimes \mathbf{1}_A^{\otimes m} + \mathbf{1}_L^{\otimes n} \otimes d_{A^m})$$

where $d_{L^n} = l_1 \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes l_1$ and $d_{A^m} = n_{0,1} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes n_{0,1}$.

It is convenient to have a shorthand expression for the OCHA relations:

$$\begin{aligned} \partial n_{n,m} = \sum_{\sigma \in \Sigma_{p+r=n}, p \geq 2} n_{1+r,m}(l_p \otimes \mathbf{1}_L^{\otimes r} \otimes \mathbf{1}_A^{\otimes m})(E(\sigma) \otimes \mathbf{1}_A^{\otimes m}) + \\ + \sum_{\substack{\sigma \in \Sigma_{p+r=n}, i+j=m-s \\ (r,s) \neq (0,1), (n,m)}} (-1)^{s+i+si+ms} n_{p,i+1+j}(\mathbf{1}_L^{\otimes p} \otimes \mathbf{1}_A^{\otimes i} \otimes n_{r,s} \otimes \mathbf{1}_A^{\otimes j})(E(\sigma) \otimes \mathbf{1}_A^{\otimes m}) \quad (3) \end{aligned}$$

where $E(\sigma)$ is defined by (1). The complicated sign of the definition is absorbed in the above expression if we assume the following standard convention: given two maps $h_1, h_2 : V \otimes W \rightarrow U$, the tensor product $h_1 \otimes h_2$ defined on $V^{\otimes 2} \otimes W^{\otimes 2}$ is given by: $(h_1 \otimes h_2)((v_1 \otimes v_2) \otimes (w_1 \otimes w_2)) = (-1)^{|v_2||w_1|} h_1(v_1, w_1) \otimes h_2(v_2, w_2)$.

Example 2.5. Here is a list of the first few OCHA relations:

$$\partial n_{0,1} = 2(n_{0,1})^2 = 0 \quad (4)$$

$$\partial n_{1,1} = n_{0,2}(n_{1,0} \otimes \mathbb{1}_A) - n_{0,2}(\mathbb{1}_A \otimes n_{1,0}) \quad (5)$$

$$\partial n_{2,0} = n_{1,0} l_2 + n_{1,1}(\mathbb{1}_L \otimes n_{1,0}) + n_{1,1}(\mathbb{1}_L \otimes n_{1,0})E(\tau_{1,2}) \quad (6)$$

$$\begin{aligned} \partial n_{1,2} = n_{1,1}(\mathbf{1}_L \otimes n_{0,2}) - n_{0,2}(n_{1,1} \otimes \mathbf{1}_A) - n_{0,2}(\mathbf{1}_A \otimes n_{1,1}) + \\ + n_{0,3}(n_{1,0} \otimes \mathbf{1}_A \otimes \mathbf{1}_A) - n_{0,3}(\mathbf{1}_A \otimes n_{1,0} \otimes \mathbf{1}_A) + n_{0,3}(\mathbf{1}_A \otimes \mathbf{1}_A \otimes n_{1,0}) \quad (7) \end{aligned}$$

$$\begin{aligned} \partial n_{2,1} = n_{1,1}(l_2 \otimes \mathbf{1}_A) + n_{1,1}(\mathbf{1}_L \otimes n_{1,1}) + n_{1,1}(\mathbf{1}_L \otimes n_{1,1})(E(1\ 2) \otimes \mathbf{1}_A) + \\ + n_{0,2}(n_{2,0} \otimes \mathbf{1}_A) - n_{0,2}(\mathbf{1}_A \otimes n_{2,0}) + n_{1,2}(\mathbf{1}_L \otimes n_{1,0} \otimes \mathbf{1}_A) - n_{1,2}(\mathbf{1}_L \otimes \mathbf{1}_A \otimes n_{1,0}) \\ + n_{1,2}(\mathbf{1}_L \otimes n_{1,0} \otimes \mathbf{1}_A)(E(1\ 2) \otimes \mathbf{1}_A) - n_{1,2}(\mathbf{1}_L \otimes \mathbf{1}_A \otimes n_{1,0})(E(1\ 2) \otimes \mathbf{1}_A) \quad (8) \end{aligned}$$

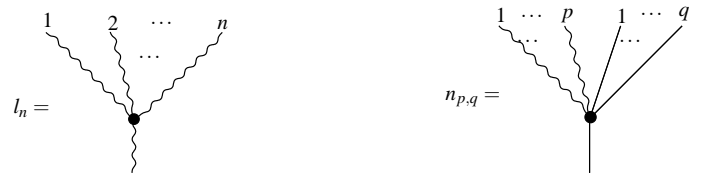
Relation (4) simply says that $n_{0,1}$ is a differential operator. On the other hand, (5) means that $n_{1,0}$ takes L into the homotopy center of A where $n_{1,1}$ is the homotopy operator. If we consider an OCHA structure where the maps $n_{1,0}$ and $n_{2,0}$ are set equal to zero, then relations (7) and (8) together say that $n_{1,1} : L \otimes A \rightarrow A$ is a Lie algebra action by derivations up to homotopy. Each OCHA defining relation correspond to a manifold with corners whose boundary is labeled by trees. Interesting figures illustrating relations (5),(6),(7) and (8) above can be found in [7]. From the above relations, one can see that a Leibniz pair is a particular case of an OCHA structure. Given its importance in this note, we state this fact more precisely in the following Proposition.

Proposition 2.6. *Let A and L be vector spaces, a Leibniz pair structure on (L,A) is equivalent to an OCHA structure $(L,A, \mathfrak{l} = \{l_n\}, \mathfrak{n} = \{n_{p,q}\})$ where only $l_2, n_{1,1}$ and $n_{0,2}$ are different from zero.*

2.1 The geometrical description of OCHA

The geometrical meaning of the OCHA relations can be described in the language of operads. Here we will show very briefly how those relations are obtained from the Axelrod-Singer [1] compactification of configuration space of points in the closed disc.

The OCHA operad OC_∞ is the DG 2-colored operad generated by the following corollae:



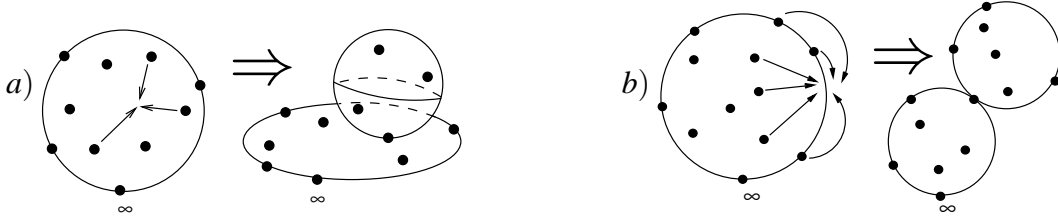
for $n \geq 2$ and $2p + q \geq 2$. The above trees are partially planar trees, where the wiggly edges are spatial and the straight edges are planar. We endow OC_∞ with a differential operator d defined on corollae l_n and $n_{p,q}$ as shown below and extended to all trees by the Leibniz rule for operadic composition:

$$d l_n = \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{\substack{\text{unshuffles } \sigma: \\ \{1,2,\dots,n\} = I_1 \cup I_2 \\ \#I_1 = k, \#I_2 = l-1}} \left(\text{Diagram of } l_n \text{ with } I_1 \text{ and } I_2 \text{ subsets} \right) \quad (9)$$

observing that an unshuffle σ is equivalent to a partition $\{1, 2, \dots, n\} = I_1 \cup I_2$ into two ordered subsets

$$d n_{n,m} = \sum_{\substack{k+l=n+1 \\ k,l \geq 2}} \sum_{\substack{\text{unshuffles } \sigma: \\ \{1,2,\dots,n\} = I_1 \cup I_2 \\ \#I_1 = k, \#I_2 = l-1}} \left(\text{Diagram 1} + \sum_{0 \leq i \leq m} (-1)^{s+i+si+ms} \text{Diagram 2} \right). \quad (10)$$

The Axelrod-Singer compactification of configuration space of points in the closed disc is a manifold with corners denoted $\overline{C(p, q)}$. Points in $\overline{C(p, q)}$ can be intuitively described through “bubbling offs” on the disc. Those bubbling offs are pictured on the next figures.



The boundary strata of $\overline{C(p, q)}$ are labeled by the trees in \mathcal{OC}_∞ . The topological boundary of the manifold induces on trees precisely the coboundary operator d defined above. Indeed, the signs in formulas (9) and (10) result from comparing the induced orientation on the boundary with the product orientation induced by the operadic composition. For details about the geometrical description of OCHA, see [7].

3 Main results

Given two graded vector spaces L and A , Kajiwara and Stasheff [10] have presented an OCHA structure $(L, A, \mathfrak{l}, \mathfrak{n})$ as a coderivation in $\text{Coder}({}^c\Lambda(L) \otimes {}^cT(A))$ having the specific form $\mathfrak{l} + \mathfrak{n}$ and satisfying $(\mathfrak{l} + \mathfrak{n})^2 = 0$. Here ${}^c\Lambda(L)$ and ${}^cT(A)$ denote the (graded) symmetric and tensor coalgebras cogenerated by L and A , respectively. In [6], following suggestions by Markl, we have shown that *any* coderivation on the coalgebra ${}^c\Lambda(L) \otimes {}^cT(A)$ has the form $\mathfrak{l} + \mathfrak{n}$. We can thus state our first result.

Theorem 3.1. *Given two DG-spaces L and A , an OCHA structure on the pair (L, A) is equivalent to a degree one coderivation $D \in \text{Coder}^1({}^c\Lambda(L) \otimes {}^cT(A))$ such that $D^2 = 0$.*

The above theorem along with the geometrical description of OCHA mentioned in the previous section, led us to find a DGLA structure on the Cohomology of Leibniz.

3.1 Cohomology of Leibniz pairs

We begin by recalling the definition of the cochain complex $C^*(A, L)$ given in [5]. Consider a Leibniz pair (A, L) with Lie algebra action $\mu : L \otimes A \rightarrow A$, $\mu(X, a) = [X, a]$. Let $C^p(A, A) = \text{Hom}_k(A^{\otimes p}, A)$ denote the p th Hochschild cochain group of A with coefficients in itself. For any $p \geq 1$, there is a Lie algebra action of L on $C^p(A, A)$:

$$[X, f] := [\mu(X, -), f]_G \quad \forall X \in L, \forall f \in C^p(A, A)$$

where $[-, -]_G$ denotes the Gerstenhaber bracket. More explicitly, we have $[X, f] \in C^p(A, A)$ given by:

$$[X, f](a_1, \dots, a_p) = [X, f(a_1, \dots, a_p)] - \sum_{i=1}^p f(a_1, \dots, [X, a_i], \dots, a_p).$$

Since the Hochschild differential $\delta_H : C^p(A, A) \rightarrow C^{p+1}(A, A)$ is also defined by the Gerstenhaber bracket $\delta_H = [m, -]_G$, we have:

$$\begin{aligned} \delta_H([X, f]) &= [m, [\mu(X, -), f]_G]_G = [[m, \mu(X, -)]_G, f]_G + [\mu(X, -), [m, f]_G]_G = \\ &= [\mu(X, -), [m, f]_G]_G = [X, \delta_H f] \end{aligned}$$

where $[m, [\mu(X, -), f]_G]_G = \delta_H(\mu(X, -)) = 0$ because $\mu(X, -)$ is a derivation of A . So, the following proposition is proved.

Proposition 3.2. *The Hochschild differential $\delta_H : C^p(A, A) \rightarrow C^{p+1}(A, A)$ is a morphism of L -modules.*

Now define:

$$C^{p,q}(A, L) = \text{Hom}_k(A^{\otimes p} \otimes \Lambda^q L, A) = \text{Hom}_k(\Lambda^q L, C^p(A, A)) \quad \text{for } p > 0 \text{ and } q \geq 0$$

$$C^{0,q}(A, L) = \text{Hom}_k(\Lambda^q L, L) \quad \text{for } q \geq 0.$$

Let $\delta_{CE} : \text{Hom}_k(\Lambda^q L, C^p(A, A)) \rightarrow \text{Hom}_k(\Lambda^q L, C^{p+1}(A, A))$ be the Chevalley-Eilenberg differential defined by the L -module structure of $C^p(A, A)$ and, for $p \geq 1$ and let $\delta_H : \text{Hom}_k(A^{\otimes p} \otimes \Lambda^q L, A) \rightarrow \text{Hom}_k(A^{\otimes p+1} \otimes \Lambda^q L, A)$ be the Hochschild differential. We thus have a double complex:

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \delta_H \uparrow & & \delta_H \uparrow & & \delta_H \uparrow & \\
\text{Hom}(A^{\otimes 2}, A) & \xrightarrow{\delta_{CE}} & \text{Hom}(A^{\otimes 2} \otimes L, A) & \xrightarrow{\delta_{CE}} & \text{Hom}(A^{\otimes 2} \otimes \Lambda^2 L, A) & \xrightarrow{\delta_{CE}} & \dots \\
& \delta_H \uparrow & & \delta_H \uparrow & & \delta_H \uparrow & \\
\text{Hom}(A, A) & \xrightarrow{\delta_{CE}} & \text{Hom}(A \otimes L, A) & \xrightarrow{\delta_{CE}} & \text{Hom}(A \otimes \Lambda^2 L, A) & \xrightarrow{\delta_{CE}} & \dots \\
& \delta_v \uparrow & & \delta_v \uparrow & & \delta_v \uparrow & \\
\text{Hom}(k, L) & \xrightarrow{\delta_{CE}} & \text{Hom}(L, L) & \xrightarrow{\delta_{CE}} & \text{Hom}(\Lambda^2 L, L) & \xrightarrow{\delta_{CE}} & \dots
\end{array}$$

where $\delta_v : \text{Hom}(\Lambda^p L, L) \rightarrow \text{Hom}(A \otimes \Lambda^p L, A) = \text{Hom}(\Lambda^p L, C^1(A, A))$ is the map induced by the L -module structure of A . From Proposition 3.2, one can see that all the vertical and horizontal coboundaries in the above diagram commute.

Definition 3.3 (Flato, Gerstenhaber, Voronov [5]). The *Deformation Complex* $C^*(L, A)$ of a Leibniz pair (L, A) is the total complex associated to the above double complex.

3.2 DGLA structure on $C^*(L, A)$

Let us now outline the construction of a DGLA structure on the deformation complex of Leibniz pairs. Recall the tensor (resp. symmetric) coalgebra ${}^c T(A)$ (resp. ${}^c \Lambda(L)$) is coaugmented, so let us denote its coaugmentation ideal by ${}^c \bar{T}(A)$ (resp. ${}^c \bar{\Lambda}(L)$).

We first define a natural Lie algebra action of $\text{Coder}({}^c \Lambda(L))$ on $\text{Coder}({}^c \bar{T}(A) \otimes {}^c \Lambda(L))$. Given $\varphi \in \text{Coder}({}^c \Lambda(L))$. The tensor product of a coderivation φ on some coalgebra with the identity map of other coalgebra always results in a coderivation on the product of the two coalgebras, so $\text{Id}_{{}^c \bar{T}(A)} \otimes \varphi$ is a coderivation of ${}^c \bar{T}(A) \otimes {}^c \Lambda(L)$. For any $\varphi \in \text{Coder}({}^c \Lambda(L))$ and $\psi \in \text{Coder}({}^c \bar{T}(A) \otimes {}^c \Lambda(L))$, we define the Lie action of φ on ψ by:

$$[\varphi, \psi] := (\text{Id}_{{}^c \bar{T}(A)} \otimes \varphi)\psi - (-1)^{|\varphi||\psi|}\psi(\text{Id}_{{}^c \bar{T}(A)} \otimes \varphi). \quad (11)$$

Let $\text{Coder}({}^c \Lambda(L)) \ltimes \text{Coder}({}^c \bar{T}(A) \otimes {}^c \Lambda(L))$ denote the *semidirect product* induced by that action. We can now state our main results.

Theorem 3.4. *A Leibniz pair structure on (L, A) is equivalent to a degree one quadratic element $l + n \in \text{Coder}_2^1({}^c\Lambda(\downarrow L)) \times \text{Coder}_2^1({}^c\Lambda(\downarrow L) \otimes {}^c\bar{T}(\downarrow A))$ such that $[l + n, l + n] = 0$.*

Sketch of Proof. From Theorem 3.1, every coderivation in $\text{Coder}({}^c\Lambda(\downarrow L) \otimes {}^c\bar{T}(\downarrow A))$ has the form $l + n$ where l is obtained by the lifting as a coderivation of multilinear maps of the form $l_n : \Lambda^n L \rightarrow L$ for $n \geq 1$, while n is obtained by lifting maps of the form $n_{p,q+1} : \Lambda^p L \otimes A^{\otimes q+1} \rightarrow A$. Notice that n has no components of the form $n_{p,0}$ because the coalgebra ${}^c\Lambda(\downarrow L) \otimes {}^c\bar{T}(\downarrow A)$ has no components of the form $\Lambda^p L \otimes k \simeq \Lambda^p L$. Restricting attention to degree one quadratic coderivations, we see that l is the lifting of l_2 , while n is the lifting of $n_{1,1} + n_{0,2}$. The result follows from Proposition 2.6 and the fact that $\text{Coder}({}^c\Lambda(L)) \times \text{Coder}({}^c\bar{T}(A) \otimes {}^c\Lambda(L))$ is isomorphic to the Lie subalgebra of $\text{Coder}({}^cT(A) \otimes {}^c\Lambda(L))$ of those coderivations obtained by lifting maps of the form l_n and $n_{p,q+1}$. \square

The geometrical intuition behind the above Theorem came from observing the OCHA relations in Example 2.5. If we set $n_{p,0} = 0$ for $p \geq 1$ and $n_{p,q} = 0$ for $p + q > 2$, then relations (7) and (8) define precisely a Lie algebra action by derivations. The semi-direct product $\text{Coder}({}^c\Lambda(L)) \times \text{Coder}({}^c\bar{T}(A) \otimes {}^c\Lambda(L))$ is well suited to take precisely those OCHA coderivations where $n_{p,0} = 0$ for all $p \geq 1$, and by restricting to quadratic coderivation we get $n_{p,q} = 0$ for $p + q > 2$.

If $l + n$ defines a Leibniz pair, then $\delta = [l + n, -]$ is a differential operator and we have a DGLA structure on $\text{Coder}({}^c\Lambda(\downarrow L)) \times \text{Coder}({}^c\Lambda(\downarrow L) \otimes {}^c\bar{T}(\downarrow A))$. The isomorphism ξ in the next Theorem is defined by the lifting as a coderivation.

Theorem 3.5. *There is an isomorphism of DG spaces:*

$$\xi : (C^*(L, A), d_{\text{tot}}) \longrightarrow (\text{Coder}({}^c\Lambda(\downarrow L)) \times \text{Coder}({}^c\Lambda(\downarrow L) \otimes {}^c\bar{T}(\downarrow A)), \delta = [l + n, -]).$$

Corollary 3.6. *The deformation complex $C^*(L, A)$ of a Leibniz pair (L, A) admits a natural structure of differential graded Lie algebra.*

4 Conclusion

Homotopy algebras in general can be studied through operads. This general principle also applies to homotopy algebraic structures defined on pairs, or even n -tuples of spaces, see [4]. In those cases, the operads must be colored. OCHA is a natural example of a homotopy algebra defined by a 2-colored operad. In this note we have presented an application of this new structure to the study of the cohomology of Leibniz pairs by showing that its deformation complex is a semi-direct product of two DG-Lie algebras. The deformation complex of algebras over colored operads is a rich subject. Open-Closed Homotopy Algebras, as a natural example, is likely to guide us to new discoveries.

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