QUANTUM SYMMETRIES OF HIGHER COXETER-DYNKIN GRAPHS. A NON-COMMUTATIVE CASE: THE $\mathcal{D}_3$ GRAPH OF $SU(3)$ SYSTEM

D. Hammaoui[1,2] and E. H. Tahri[1]

[1] *Equipe de physique mathématique et plasmas, Laboratoire de Physique Théorique, Physique des Particules et Modélisation (LPTPM), Faculté des Sciences, Oujda, Morocco*
[2] *Centre de CPGE, Taza, Morocco*
e-mail: hammaouidahmane@yahoo.fr, tahrie@sciences.univ-oujda.ac.ma

Abstract

The purpose of this contribution is to show how quantum geometry of higher Coxeter graphs of $SU(N)$ type gives a common algebraic formulation for both RCFT and quantum groupoïds. These apparently two different fields are hardly studied by wide communities of physicists and mathematicians. To carry out this formulation we determine all Nim-reps describing CFT and weak Hopf algebra structures based on combinatorial data of graphs. We will pay attention to a particular case: the $\mathcal{D}_3$ orbifold graph of $SU(3)$ system for which the algebra of quantum symmetries is non-commutative.

Introduction

Higher Coxeter graphs of $SU(N)$ type and their quantum symmetries are well known both in physics and in mathematics. They were introduced by A. Ocneanu as a generalization to higher ranks the usual ADE Coxeter-Dynkin diagrams associated to SU(2) system [27]. In mathematics these graphs are implicated in various fields such as representation theory of quantum groups, (weak) Hopf algebras, classification of semi-simple Lie algebras, operator algebras, subfactors, category theory and bimodules [28, 27, 3, 26, 33, 1, 2, 25, 30]... In physics, many models are based on this kind of graphs like lattice integrable models in statistical mechanics or models of quantum gravity in sting theory and D-branes [31, 36, 17, 32, 37].
- From an algebraic point of view, higher Coxeter graphs of the $SU(N)$ type are related to the classification of affine Lie algebras and describe their representation theory [27, 17].
- In topological field theory, they enable to compute the set of $3j$ and $6j$-symbols in many theories of strings [18].
- In a conformal viewpoint, they are related to the classification of modular invariants of conformal models and describe the different aspects of quantum conformal field theory [4, 19, 23, 13, 11, 34].

Quantum geometry on graphs (introduced by A. Ocneanu) describes quantum symmetries of these graphs together with the corresponding Ocneanu graphs which index the defect lines of the CFT and brings out new structures of weak Hopf algebras called algebras of double triangles [28, 29].

The main task here is to show how a triplet $(G, \mathcal{A}(G), \Gamma(G))$ of graphs can, on one hand describe $2D$-RCFT and determines all types of its Nim-Reps, and on the other hand encode a weak

Hopf C*-algebra structure known as the double triangle algebra. The graph G is a generalized Coxeter-Dynkin graph of $SU(N)$ type associated to a given modular invariant, the graph $\mathcal{A}$ (with same dual Coxeter number as G) is the graph describing the fusion algebra and Γ is the Ocneanu graph describing hidden quantum symmetries of G [6, 13, 14, 9, 22, 20]. A system of several generalized 6-j symbols, called Ocneanu cells represented by 3-simplice which are labeled by vertices and edges of the previous graphs, appears as consistency conditions to ensure the axioms of the underlying Ocneanu quantum groupoïd [28, 15, 7, 8, 33, 21]. The starting point is a given modular invariant partition function associated to a graph G. It includes many pieces of information on spectral and quantum data. The algebra $Oc(G)$ of quantum symmetries and the associated Ocneanu graph $\Gamma(G)$ are deduced from the use of the so-called modular splitting technique needing only the fusion nimreps and the modular invariant [28, 10, 24, 20].

Outline:

- Fusion algebras and the properties of the $\mathcal{A}_k$ graphs.
- The module graphs G and self-fusion.
- The algebra of quantum symmetries and the Ocneanu graph $\Gamma(G)$.
- The double triangle algebra.
- References.

Notations:

For the $\mathcal{A}_k$ graphs: vertices are denoted by $\lambda, \mu, \nu \ldots$, and the $\mathbb{N}_\lambda$ are the fusion matrices.

For the G graph: vertices are denoted by $a, b, c \ldots$, the $\mathbb{G}_a$ are the graph algebra matrices and the $\mathbb{F}_\lambda$ are the annular matrices.

For the $\Gamma(G)$ graph: vertices are denoted denoted $x, y, z \ldots$, the $\mathbb{O}_x$ are the quantum matrices, the $\mathbb{S}_x$ are the dual annular matrices, the $\mathbb{W}_{xy}$ are the toric matrices and the $\mathbb{V}_{\lambda\mu}$ the double annular matrices.

1 Fusion algebras and the properties of the $\mathcal{A}_k$ graphs

Fusion algebras describe fusion of primary fields in CFT and their structure constants are nimreps (non-negative integer matrix representations) denoted $(N_i)_{jk}$ and are related to the OPE of fields:

$$\mathcal{V}_i \star \mathcal{V}_j = \oplus_k N_{ij}^{\ k} \mathcal{V}_k,$$

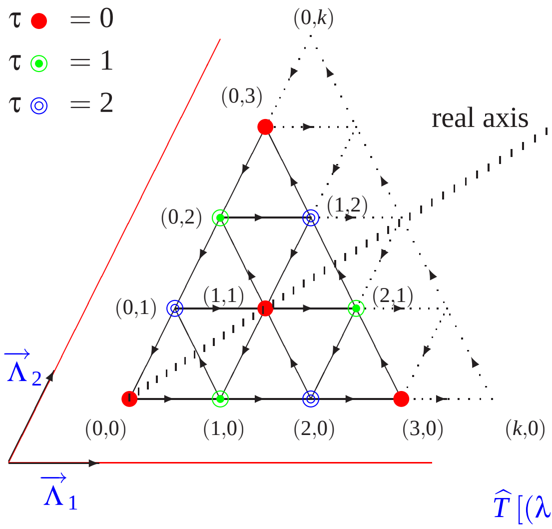
where $\mathcal{V}_i$ are irreducible representations of the chiral algebra, an extension of the Virasoro algebra, of the RCFT.

$\mathcal{A}_k$ graphs are the Weyl alcoves of $SU(N)$ type truncated at some level k. Each graph is characterized by the generalized (dual) Coxeter number $\kappa = N + k$. Vertices λ, μ, $\nu \ldots$ represent irreducible representations (irreps) of quantum sub-groups of $SU(N)_k$ at a root of unity $q = e^{i\pi/\kappa}$. The A_k graphs encode the tensor product of irreps inherited from fusion of fields [16, 17].

1.1 General properties of $\mathcal{A}_k$ graphs of the SU(3) system

In the following, in Figure 1, we display the Weyl alcove of $SU(3)$ at a level k, as well as some symmetry automorphisms and the modular generators [17, 20]:

$$\mathcal{A}_k = \{\lambda = (\lambda_1, \lambda_2) = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2 / \ \lambda_1, \lambda_2 \in \mathbb{N}, \ \ \lambda_1 + \lambda_2 \leq k\}$$



Triality τ: $\tau(\lambda_1,\lambda_2)=\lambda_1-\lambda_2 \mod 3$

$\mathbb{Z}_3$-orbifold z: $z(\lambda_1,\lambda_2)=(k-\lambda_1-\lambda_2,\lambda_1)$

Gannon twist: $\rho(\lambda)=z^{k\tau(\lambda)}(\lambda)$

Conjugation $\star$: $\lambda^\star=(\lambda_2,\lambda_1)$

Modular transformations ($SL(2,\mathbb{Z})$):

$\mathcal{S}:\tau\longrightarrow -1/\tau$ and $\mathcal{T}:\tau\longrightarrow 1+\tau$

characters of irreps of $SU(3)$:

$\chi^k_\lambda(\tau)=Tr\left[\exp\left(2i\pi\left(L_0-c/24\right)\right)\right]$

Modular exponents:

$$\widehat{T}\left[(\lambda_1,\lambda_2)\right]=(\lambda_1+1)^2+(\lambda_1+1)(\lambda_2+1)+(\lambda_2+1)^2-\kappa$$

⋆ **Vertices:** are irreps of $SU(3)$ $(1,0)=\ydiagram{1}$ $(0,1)=\ydiagram{1,1}$ $(1,1)=\ydiagram{2,1}$ $\cdots$

⋆ **Edges:** are oriented

Figure 1: General data on Weyl alcoves of SU(3) type.

Λ_1 and Λ_2 are the fundamental weights of the $SU(3)$ Lie group and λ_1, λ_2 are the corresponding Dynkin labels. $(0,0)$ is the unit representation which index the unit vertex of $\mathcal{A}_k$ and is related to the "vacuum state", $(1,0)$ is the fundamental generator (irrep) of $SU(3)_k$ and $(0,1)$ is its conjugate. The number of vertices of $\mathcal{A}_k$ is $d_{\mathcal{A}_k}=(k+1)(k+2)/2$.

The nimreps $N_{ij}^{\ k}$ (non-negative integer valued matrix representations), subject to the Verlinde formula [39], give a $d_{\mathcal{A}_k}\times d_{\mathcal{A}_k}$-dimensional matrix representation for the fusion algebra:

$$\mathbb{N}_\lambda\mathbb{N}_\mu \;=\; \Sigma_{\nu\in\mathcal{A}_k}N_{\lambda\mu}^{\nu}\mathbb{N}_\nu \qquad \text{and} \qquad N_{ij}^{\ k} \;=\; \Sigma_{m\in\mathcal{A}_k}\frac{S_{im}S_{jm}S^*_{km}}{S_{0m}}.$$

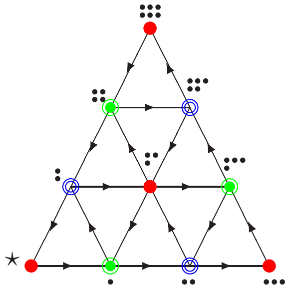
The $\mathbb{N}_\lambda$ satisfy the recurrence relation for coupling of irreducible $SU(3)$ representations: $\mathbb{N}_{(0,0)}=\mathbb{I}_{d_{\mathcal{A}_k}}$, $\mathbb{N}_{(1,0)}=\mathbb{A}(\mathcal{A}_k)$ is the adjacency matrix of the graph $\mathcal{A}_k$, and

$$\begin{aligned}
\mathbb{N}_{(\lambda,\mu)} &= \mathbb{N}_{(1,0)}\,\mathbb{N}_{(\lambda-1,\mu)}-\mathbb{N}_{(\lambda-1,\mu-1)}-\mathbb{N}_{(\lambda-2,\mu+1)} && \text{if } \mu\neq 0\\
\mathbb{N}_{(\lambda,0)} &= \mathbb{N}_{(1,0)}\,\mathbb{N}_{(\lambda-1,0)}-\mathbb{N}_{(\lambda-2,1)}\\
\mathbb{N}_{(0,\lambda)} &= \left(\mathbb{N}_{(\lambda,0)}\right)^{tr}
\end{aligned}$$

Note that the $S-$matrix is given by $S_{\lambda\mu}=\left[(\psi_\lambda)_\mu\right]$, where $[\psi]$ is the vector class matrix of $\mathcal{A}$ deduced from the matrix eigenvectors of the adjacency matrix $\mathbb{A}(\mathcal{A}_k)$ by choosing an appropriate ordering of lines and columns [9]. While the T-matrix is diagonal and given by $T_{\lambda\mu}=\exp(2i\pi\widehat{T}[\lambda])\delta_{\lambda\mu}$.

1.2 First example: The $\mathcal{A}_3$ graph of SU(3) type

In the following, we present, in Figure 2, the $\mathcal{A}_3$ graph, with $\kappa = 6$, and the corresponding adjacency matrix $\mathbb{A}(\mathcal{A}_3) = \mathbb{N}_{(1,0)}$.



$$\mathbb{A}(\mathcal{A}_3) = \begin{pmatrix} . & 1 & . & . & . & . & . & . & . & . \\ . & . & 1 & 1 & . & . & . & . & . & . \\ 1 & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & 1 & . & 1 & . & . & . \\ . & 1 & . & . & . & 1 & . & 1 & . & . \\ . & . & 1 & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & 1 & . & . & . & . & 1 & . \\ . & . & . & . & 1 & . & . & . & . & 1 \\ . & . & . & . & . & 1 & . & . & . & . \end{pmatrix}$$

Figure 2: The $\mathcal{A}_3$ graph and its adjacency matrix.

The biggest eigenvalue of $\mathbb{A}(\mathcal{A}_3)$ is the norm of the graph: $\beta = [3]_q = 2$. The corresponding normalized Perron-Frobenius vector $\overrightarrow{\mu}(\mathcal{A}_3) = \{[1],[3],[3],[3],[2][4],[3],[1],[3],[3],[1]\}$ gives the quantum dimensions of vertices of $\mathcal{A}_3$.

2 The module graphs G and self-fusion

2.1 Definitions

Each modular invariant of affine $SU(N)$ type is associated to a graph G. Such graphs are considered as modules over the fusion algebras $\mathcal{A}(G)$ with same Coxeter number κ [6, 13, 9].

$$\begin{array}{ccc} \mathcal{A}(G) \times Vert(G) & \longrightarrow & Vert(G) \\ \lambda.a & \hookrightarrow & \sum_b \mathbb{F}_{\lambda a}^{\ \ b} b \end{array}$$

The action of A-module on $Vert(G)$, the vector space spanned by vertices of G, is encoded in nim-reps $(F_\lambda)_{ab}$ giving new $d_G \times d_G$-dimensional matrix representation of the fusion algebra, which provides solutions to the Cardy equation in boundary conformal field theory (BCFT) [5],

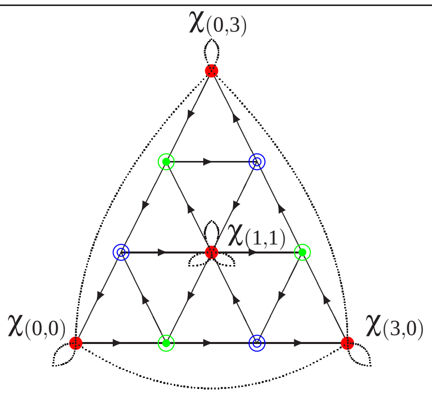
$$\mathbb{F}_\lambda \mathbb{F}_\mu \;=\; \sum_\nu N_{\lambda\mu}^{\ \ \nu} \mathbb{F}_\nu, \qquad \text{and} \qquad F_{\lambda a}^{\ \ b} \;=\; \sum_{m \in \mathcal{E}xp(G)} \frac{S_{\lambda m}}{S_{0m}} \psi_a^m (\psi_b^m)^*.$$

To each irrep λ of $\mathcal{A}(G)$ is associated a matrix $\mathbb{F}_\lambda$ (called annular matrix) such that $\mathbb{F}_{(0,0)} = \mathbb{I}_{d_G}$ and $\mathbb{F}_{(1,0)} = \mathbb{A}(G)$, where $\mathbb{A}(G)$ is the adjacency matrix of the graph G. The vertices $a, b, c\ldots$ of G represent the boundary states of the BCFT.

2.2 Second example: The $\mathcal{D}_3$ orbifold graph of SU(3) system

2.2.1 General data on $\mathcal{D}_3$

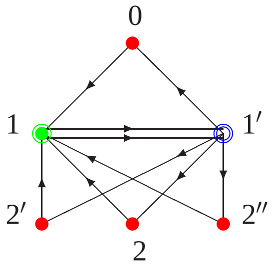
Consider the following modular invariant coded in a modular diagram as displayed in Figure 3. It is associated to the $\mathcal{D}_3$ graph of $SU(3)$ type obtained from the $\mathcal{A}_3$ graph by a $\mathbb{Z}_3$-orbifold operation. In Figure 4 we display the $\mathcal{D}_3$ graph and its adjacency matrix $\mathbb{A}(\mathcal{D}_3)$ associated with the multiplication by the fundamental generator **1** (the unit vertex is **0**).



$$\mathcal{M}(\mathcal{D}_3) = \mathbb{W}_{00} = \begin{pmatrix} 1 & . & . & . & . & . & 1 & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 3 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & 1 & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & 1 & . & . & 1 \end{pmatrix},$$

$$\mathcal{Z}_{\mathcal{D}_3} = |\chi_{(0,0)} + \chi_{(3,0)} + \chi_{(0,3)}|^2 + 3|\chi_{(1,1)}|^2$$

Figure 3: The modular diagram of the $\mathcal{D}_3$ graph.



$$\mathbb{A}(\mathcal{D}_3) = \begin{pmatrix} . & 1 & . & . & . & . \\ . & . & 2 & . & . & . \\ 1 & . & . & 1 & 1 & 1 \\ . & 1 & . & . & . & . \\ . & 1 & . & . & . & . \\ . & 1 & . & . & . & . \end{pmatrix}$$

Figure 4: The $\mathcal{D}_3$ graph and its adjacency matrix.

Once the adjacency matrix $\mathbb{A}(\mathcal{D}_3) = \mathbb{F}_{(1,0)}$ is known, one can easily deduce all other annular matrices $\mathbb{F}_\lambda$. Many interesting information could be extracted from the modular invariant toric matrix $\mathcal{M}$. First of all are the spectral data. Indeed, the diagonal entries $\mathcal{M}_{\lambda\lambda}$ are in bi-univoque correspondence with the Coxeter exponents $(r_1, r_2) \in Exp(G)$ (a subset of $Exp(\mathcal{A}(G)) \equiv \mathcal{A}(G)$) [35]. So, the $\mathcal{D}_3$ graph has six eigenvalues $\gamma^{(r_1,r2)}$ (the last one is triple) associated with the exponents $(0,0)$, $(3,0)$, $(0,3)$ and $(1,1)$ respectively:

$$\gamma^{(r_1,r2)} = \exp\frac{-2i\pi(2(r_1+1)+(r_2+1))}{3\kappa}[1+\exp\frac{2i\pi(r_1+1)}{\kappa}+\exp\frac{2i\pi((r_1+1)+(r_2+1))}{\kappa}]$$

The norm of the $\mathcal{D}_3$ graph is $\beta = \gamma^{(0,0)} = 1+2\cos(2\pi/\kappa) = [3]_q = 2$ and the corresponding normalized Perron-Frobenius vector [6] is $\vec{\mu}(\mathcal{D}_3) = \{1,2,2,1,1,1\}$.

2.2.2 The graph algebra of $\mathcal{D}_3$

The $\mathcal{D}_3$ graph possesses a self-fusion i.e there exists an associative, unital commutative algebra structure in the sense that one can multiply generators of $\mathcal{D}_3$ by each other and this multiplication is compatible with the module multiplication by elements of $\mathcal{A}$ described by the $\mathbb{F}_\lambda$ matrices. Such graph is called a subgroup graph of the quantum group $SU(3)_q$. To each generator $a \in \mathcal{D}_3$ we associate a $\mathbb{N}$-valued matrix $\mathbb{G}_a$ of size $d_G \times d_G$ such that the coefficients $G_{ab}^{\ c}$ satisfy a generalized Verlinde formula

$$\mathbb{G}_{\mathbf{0}} = \mathbb{I}_{d_G}, \qquad \mathbb{G}_{\mathbf{1}} = \mathbb{A}(\mathcal{D}_3), \qquad \mathbb{G}_a\mathbb{G}_b = \sum_{c\in\mathcal{D}_3} G_{ab}^{\ \ c}\mathbb{G}_c \qquad \text{and} \qquad G_{ab}^{\ c} = \sum_{\alpha\in Exp(G)} \frac{\psi_{a\alpha}\psi_{b\alpha}\psi^*_{c\alpha}}{\psi_{0\alpha}}$$

The trace of $\mathcal{M}$ gives the cardinality of the graph: $d_G = Tr(\mathcal{M})$. Explicit values of $\mathbb{G}_a$ for $\mathcal{D}_3$ are deduced from the multiplication by the fundamental generator

$$\mathbb{G}_{\mathbf{0}} = \mathbb{I}_6, \qquad \mathbb{G}_{\mathbf{1}} = (\mathbb{G}_{\mathbf{1'}})^{tr} = \mathbb{A}(\mathcal{D}_3), \qquad \text{and} \qquad \mathbb{G}_{\mathbf{2}} + \mathbb{G}_{\mathbf{2'}} + \mathbb{G}_{\mathbf{2''}} = \mathbb{G}_{\mathbf{1}}\mathbb{G}_{\mathbf{1'}} - \mathbb{G}_{\mathbf{0}}$$

The subset $\mathcal{J} = \{\mathbf{0}, \mathbf{2}, \mathbf{2'}, \mathbf{2''}\}$, called the modular subalgebra, is a subalgebra of the graph algebra $\mathcal{D}_3$ and is determined by T-modular properties of the graph $\mathcal{D}_3$. The compatibility condition between the $\mathcal{A}$-module structure on G and the graph algebra structure of G is written as: $\lambda(ab) = (\lambda a)b$, for all $\lambda \in \mathcal{A}(G)$ and $a, b \in G$.

3 The algebras of quantum symmetries and the Ocneanu graphs

Defect lines of a twisted chiral CFT are encoded by vertices $x, y, z \dots$ of an Ocneanu graph $\Gamma(G)$ associated to a higher Coxeter graph G. Their fusion is described by an algebra structure $Oc(G)$ called the algebra of quantum symmetries. In the $SU(3)$ case [9, 20], the Ocneanu graph has four fundamental chiral generators (twice the number of fundamental representations).

3.1 The Ocneanu algebra of quantum symmetries

A technical method to determine the algebra of quantum symmetries of a graph is the modular splitting formula [28, 10, 24, 20] based only on the knowledge of $\mathcal{M}(G)$ and the fusion matrices $\mathbb{N}_\lambda$ of $\mathcal{A}(G)$:

$$\Sigma_{\lambda''\mu''} \mathbb{N}_{\lambda\lambda'}^{\lambda''} \mathbb{N}_{\mu\mu'}^{\mu''} \mathcal{M}_{\lambda''\mu''} \quad = \quad \Sigma_z (\mathbb{W}_{0z})_{\lambda\mu} (\mathbb{W}_{z0})_{\lambda'\mu'}.$$

which enable to compute all toric matrices $\mathbb{W}_{0z}$ or $\mathbb{W}_{z0}$ with one defect line z. its generalization gives a relation between twisted toric matrices with two defect lines.

$$\Sigma_z (\mathbb{W}_{xz})_{\lambda\mu} (\mathbb{W}_{zy})_{\lambda'\mu'} \quad = \quad \Sigma_{\lambda''\mu''} \mathbb{N}_{\lambda\lambda'}^{\lambda''} \mathbb{N}_{\mu\mu'}^{\mu''} (\mathbb{W}_{xy})_{\lambda''\mu''},$$

The others $\mathbb{W}_{xy}$, which give the generalized twisted partition functions of the theory with two defect lines (say x and y), are given by

$$\Sigma_z (\mathbb{W}_{xz})_{\lambda\mu} \mathbb{W}_{z0} \quad = \quad \mathbb{N}_\lambda \mathbb{W}_{x0} (\mathbb{N}_\mu)^{tr}.$$

Note that very often the algebra $Oc(G)$ can be realized as an appropriate quotient of the tensor product of a graph algebra H on which G acts as an H-module (H could be $\mathcal{A}(G)$ or G itself): $Oc(G) = H \otimes_{\mathcal{J}(H)} H$. This fact can shows that $Oc(G)$ is always block diagonalisable and isomorphic to a direct sum of finite dimensional matrix algebras as $\oplus_{\lambda,\mu} M\left(\mathcal{M}_{\lambda\mu}, \mathbb{C}\right)$. When some $\mathcal{M}_{\lambda\mu}$ are greater than 2, the $Oc(G)$ is a non-commutative algebra due to the presence of this matrix blocks. A faithful anti-representation[1] of the Ocneanu algebra $Oc(G)$ is carried by $\mathbb{N}$-valued matrices called Ocneanu quantum matrices $\mathbb{O}_x$ of size $d_\Gamma \times d_\Gamma$, which attached attached to the generators x of $\Gamma(G)$ and satisfy

$$\mathbb{O}_x \mathbb{O}_y \quad = \quad \Sigma_z O_{yx}^{\ z} \mathbb{O}_z.$$

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[1] In general $Oc(G)$ is a non-commutative algebra, otherwise its structure constants satisfy $O_{xy}^z = O_{yx}^z$ and the set of quantum matrices forms a representation.

3.2 The double fusion algebra

It is convenient to introduce new nimreps for the tensor square of the fusion algebra $\mathcal{A}_k \otimes \mathcal{A}_k$ called the double fusion algebra. To each pair (λ,μ) of vertices of $\mathcal{A}_k$ one associates a matrix $\mathbb{V}_{\lambda\mu}$, called double annular matrix related to the toric matrices by:

$$(\mathbb{V}_{\lambda\mu})_{xy} \quad = \quad (\mathbb{W}_{xy})_{\lambda\mu} \qquad \text{and} \qquad \mathbb{V}_{\lambda\mu}\mathbb{V}_{\lambda'\mu'} \quad = \quad \sum_{\lambda''\mu''} \mathbb{N}_{\lambda\lambda'}^{\lambda''} \mathbb{N}_{\mu\mu'}^{\mu''} \mathbb{V}_{\lambda''\mu''}$$

3.3 $\Gamma(G)$ as an $\mathcal{A}$-$\mathcal{A}$ bimodule

The Ocneanu graph $\Gamma(G)$ is a bimodule on the double fusion algebra $\mathcal{A} \otimes \mathcal{A}$.

$$\begin{array}{lcc} \mathcal{A}(G) \times \Gamma(G) \times \mathcal{A}(G) & \longrightarrow & \Gamma(G) \\ \lambda x \mu & \longmapsto & \sum_y (\mathbb{V}_{\lambda\mu})_{xy} y = \sum_y (\mathbb{W}_{xy})_{\lambda\mu} y \end{array}$$

There are some compatibility conditions between the algebra of quantum symmetries and this bimodule structure given by

$$\mathbb{O}_x \mathbb{V}_{\lambda\mu} \quad = \quad \mathbb{V}_{\lambda\mu}\mathbb{O}_x \quad = \quad \sum_z (\mathbb{V}_{\lambda\mu})_{xz} \mathbb{O}_z.$$

in particular if we set $x=0$, we can deduce the twisted toric matrices $\mathbb{W}_{xy}$ from the data of $\mathbb{O}_z$ and $\mathbb{W}_{0z}$

$$\mathbb{W}_{xy} \quad = \quad \sum_z (\mathbb{O}_z)_{xy} \mathbb{W}_{0z}.$$

Furthermore the knowledge of the matrices $\mathbb{V}_{\lambda\mu}$ allows to give the Ocneanu matrices associated with the fundamental chiral generators by

$$\begin{array}{ll} \mathbb{V}_{(1,0)(0,0)} = O_{1_L} & \qquad \mathbb{V}_{(0,0)(1,0)} = O_{1_R} \\ \mathbb{V}_{(0,1)(0,0)} = O_{1_L^*} & \qquad \mathbb{V}_{(0,0)(0,1)} = O_{1_R^*} \end{array}$$

which are the adjacency matrices for the Ocneanu graph $\Gamma(G)$, its unit vertex is assigned to $\mathbb{V}_{(0,0)(0,0)} = \mathbb{I}_{d_\Gamma}$.The cardinality of $\Gamma(G)$ is also encoded in the modular invariant $\mathcal{M}$ as: $d_\Gamma = Tr(\mathcal{M}\,\mathcal{M}^{tr})$ and the exponents of the spectrum of $\mathbb{A}(\Gamma)$ is given by the non-zero diagonal entries of $\mathcal{M}\,\mathcal{M}^{tr}$.

3.4 G as an $Oc(G)$-module

The graph G acts as a module on the Ocneanu algebra $Oc(G)$. This structure is encoded in a set of nim-reps $(S_x)_{ab}$ called dual annular coefficients:

$$\begin{array}{lcc} Oc(G) \times G & \longrightarrow & G \\ x.a & \longrightarrow & \sum_b S_{xa}^{\ b} b \end{array}$$

The set of matrices $\mathbb{S}_x$ of size $d_G \times d_G$ associated to vertices $x \in \Gamma(G)$ forms a new anti-representation of the Ocneanu algebra:

$$\mathbb{S}_x \mathbb{S}_y \quad = \quad \sum_z O_{yx}^{\ z} \mathbb{S}_z.$$

3.5 Example: The Ocneanu graph $\Gamma(\mathcal{D}_3)$

Applying the above formalism to the $\mathcal{D}_3$ case, we find the expressions of the left and right chiral generators of $\Gamma(\mathcal{D}_3)$ [20]:

$$\mathbb{O}_{1_0} = \begin{pmatrix} \mathbb{G}_1 & . & . \\ . & \mathbb{G}_1 & . \\ . & . & \mathbb{G}_1 \end{pmatrix} \qquad \mathbb{O}_{1_{1'}} = \begin{pmatrix} . & . & \mathbb{G}_{1'} \\ \mathbb{G}_{1'} & . & . \\ . & \mathbb{G}_{1'} & . \end{pmatrix}$$

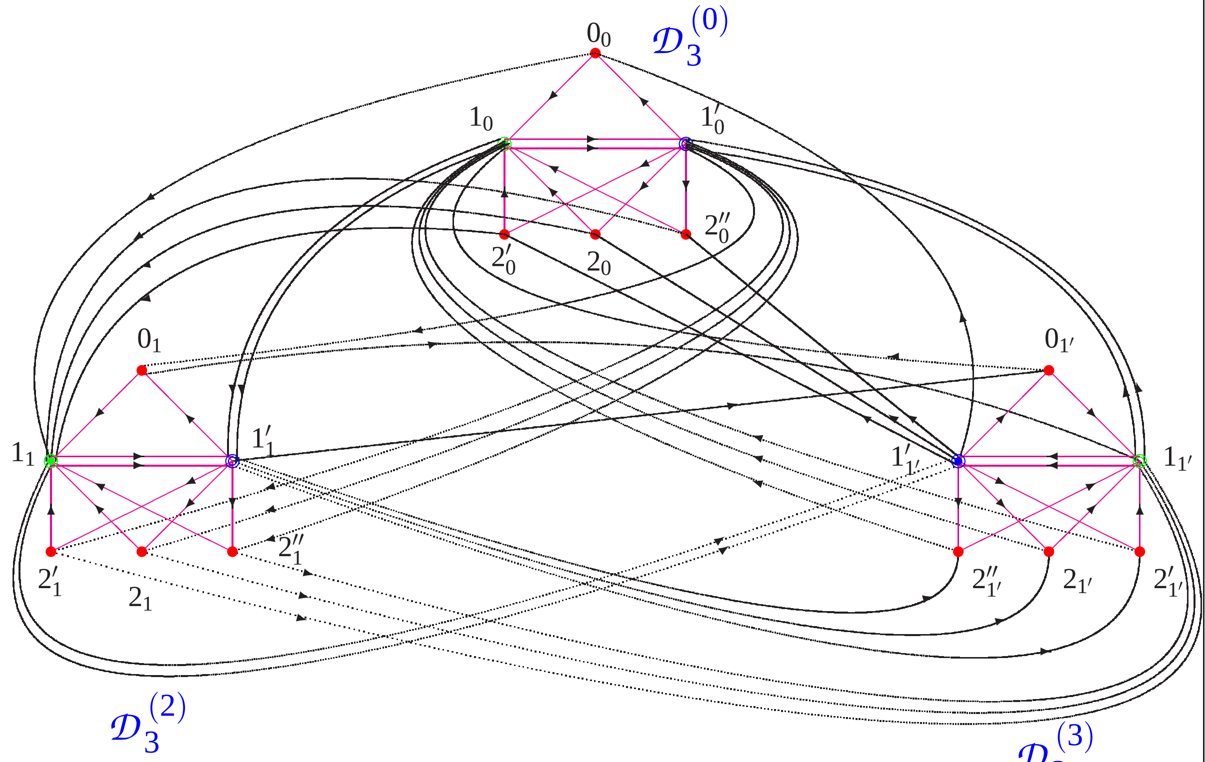


Figure 5: The Ocneanu graph $\Gamma(\mathcal{D}_3)$

The graph $\Gamma(\mathcal{D}_3)$ has $d_\Gamma = 18$ vertices according to the formula $d_\Gamma = Tr\left(\mathcal{M}\,\mathcal{M}^{tr}\right)$. The analysis of the graph shows that we can split it into three subgraphs as

$$\Gamma(\mathcal{D}_3) \sim \mathcal{D}_3^{(0)} \oplus \mathcal{D}_3^{(1)} \oplus \mathcal{D}_3^{(2)}$$

The algebra of quantum symmetries is non-commutative and can be written as a square tensor product of the graph algebra $\mathcal{D}_3$ over the modular subalgebra $\mathcal{J}$ or as a semi-direct product of $\mathcal{D}_3$ by the discrete group $\mathbb{Z}_3$ as it is seen in Figure 5:

$$Oc(\mathcal{D}_3) \;\equiv\; \mathcal{D}_3 \otimes_{\mathcal{J}} \mathcal{D}_3 \;\cong\; \mathcal{D}_3 \times \mathbb{Z}_3.$$

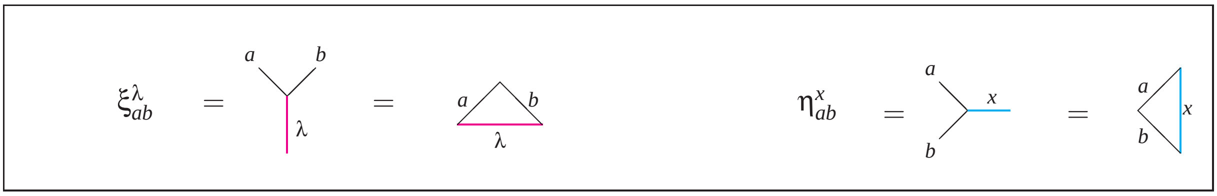
This noncommutativity comes from the fact that the decomposition [2, 27] of $Oc(G)$ contains a matrix bloc $M(3,\mathbb{C})$, coming from the fact that the modular invariant partition function is of the form: $\mathcal{Z}(\mathcal{D}_3) = |00+30+03|^2 + 3\,|11|^2$. So it is written as:

$$Oc(\mathcal{D}_3) \;\cong\; \oplus_{r=1}^{r=9}\mathbb{C}_r \oplus M(3,\mathbb{C})$$

4 The double triangle algebra

4.1 Essential paths on G and the double triangle algebras

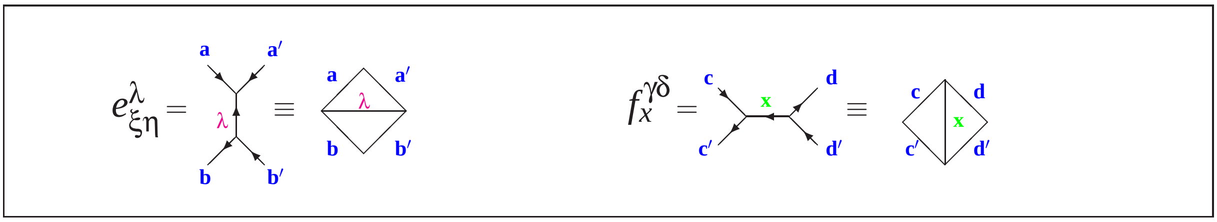
We move from the geometry on the graph G to the geometry describing the paths on G.
Essential horizontal paths [28, 6] (or horizontal triangle) ξ^{λ}_{ab} of type λ going from a to b span a vector space $\mathcal{H}\,paths(G)$ graded by λ: $\mathcal{H}\,paths(G) = \oplus_{\lambda\in\mathcal{A}}\mathcal{H}_{\lambda}$ with cardinality $\sum_{\lambda\in\mathcal{A}} d_{\lambda}$ where $d_{\lambda} = \sum_{a,b\in G}(F_{\lambda})_{ab}$.
Essential vertical paths (or vertical triangles) η^{x}_{ab} of type x going from a to b span a vector space $\mathcal{V}\,paths(G)$ graded by x: $\mathcal{V}\,paths(G) = \oplus_{x\in\Gamma}\mathcal{V}_{x}$. Its dimension is $\sum_{x\in\Gamma} d_{x}$ where $d_{x} = \sum_{a,b\in G}(S_{x})_{ab}$.



Now, if we consider the vector spaces of endomorphisms on essential paths, we obtain two (dual) algebras[2] which are both semi-simple and cosemi-simple:

$$\mathcal{B} = \oplus_{\lambda}\mathcal{B}_{\lambda} = \textstyle\sum_{\lambda}\mathcal{H}_{\lambda}(G)\otimes\widehat{\mathcal{H}}_{\lambda}(G) \qquad \widehat{\mathcal{B}} = \oplus_{x}\widehat{\mathcal{B}}_{x} = \textstyle\sum_{x}\mathcal{V}_{x}(G)\otimes\widehat{\mathcal{V}}_{x}(G).$$

A basis for the algebra $(\mathcal{B},\circ)$ is defined by $\{e_{\xi\eta}\}$ where $e_{\xi\eta}$ stands for $\xi^{\lambda}_{ab}\otimes\widehat{\eta}^{\lambda}_{cd}$ and are represented by matrix units. For the dual algebra $\left(\widehat{\mathcal{B}},\widehat{\circ}\right)$ a basis is defined by $\left\{f^{\gamma\delta}\right\}$ where $f^{\gamma\delta}$ means $\gamma^{x}_{ab}\otimes\widehat{\delta}^{x}_{cd}$.



4.2 A weak Hopf algebra structure

4.2.1 The algebras $(\mathcal{B},\circ)$ and $\left(\widehat{\mathcal{B}},\widehat{\circ}\right)$

The two type of products $\circ$ and $\widehat{\circ}$ on both isomorphic algebras $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are defined respectively by:

$$e^{\lambda}_{\xi\kappa}\circ e^{\lambda'}_{\zeta\eta} = \delta_{\lambda\lambda'}\delta_{\kappa\zeta}e_{\xi\eta} \qquad \text{and} \qquad f^{\alpha\beta}_{x}\,\widehat{\circ}\,f^{\gamma\delta}_{x'} = \delta_{xx'}\delta_{\beta\gamma}f^{\alpha\delta}$$

---

[2] $\dim\mathcal{B} = \sum_{\lambda}(d_{\lambda})^{2} = \sum_{x}(d_{x})^{2} = \dim\widehat{\mathcal{B}}$ (= 1032 for the $\mathcal{D}_{3}$ case).

4.2.2 The coalgebras $(\mathcal{B},\Delta)$ and $\left(\widehat{\mathcal{B}},\widehat{\Delta}\right)$

The existence of a product $\widehat{\circ}$ in $\widehat{\mathcal{B}}$ allows to define a coproduct on $\mathcal{B}$ by: $\left\langle f^{\alpha\beta}\otimes f^{\gamma\delta},\Delta e_{\xi\eta}\right\rangle = \left\langle f^{\alpha\beta}\widehat{\circ} f^{\gamma\delta},e_{\xi\eta}\right\rangle$. In analogous way a coproduct $\widehat{\Delta}$ is defined on $\widehat{\mathcal{B}}$ using the product $\circ$ in its dual $\mathcal{B}$: $\left\langle \widehat{\Delta}\left(f^{\alpha\beta}\right),e_{\xi\eta}\otimes e_{\kappa\xi}\right\rangle = \left\langle f^{\alpha\beta},e_{\xi\eta}\circ e_{\kappa\xi}\right\rangle$.

4.2.3 The units and counits

The unit elements of $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are defined via the corresponding minimal central projectors[3] as $\mathbb{I}=\sum_\lambda \pi_\lambda$ for $\mathcal{B}$ and $\widehat{\mathbb{I}}=\sum_x \omega^x$ for $\widehat{\mathcal{B}}$.

We can show that the axiom $\Delta(\mathbb{I})=\mathbb{I}\otimes\mathbb{I}$ for usual Hopf algebras is "weakened" for $\mathcal{B}$ and $\widehat{\mathcal{B}}$ and one get $\Delta(\mathbb{I})=\mathbb{I}_{(1)}\otimes\mathbb{I}_{(2)}$ and $\widehat{\Delta}\left(\widehat{\mathbb{I}}\right)=\widehat{\mathbb{I}}_{(1)}\otimes\widehat{\mathbb{I}}_{(2)}$, where Swedler notation is used.

The counit ε of $\mathcal{B}$ satisfy $\varepsilon(e\circ e')=\varepsilon\left(e\mathbb{I}_{(1)}\right)\varepsilon\left(\mathbb{I}_{(2)}e'\right)$, for all $e,e'\in\mathcal{B}$, and the same for the counit $\widehat{\varepsilon}$ of $\widehat{\mathcal{B}}$ we have $\widehat{\varepsilon}(f\circ f')=\widehat{\varepsilon}\left(f\widehat{\mathbb{I}}_{(1)}\right)\widehat{\varepsilon}\left(\widehat{\mathbb{I}}_{(2)}f'\right)$, for all $f,f'\in\widehat{\mathcal{B}}$. This means that these two maps are not algebra homomorphisms.

4.2.4 The antipodes

An antipode S on $\mathcal{B}$ can be defined as an algebra anti-homomorphism like a conjugation of elements of $\mathcal{B}$ in the following way: $S\left(e^{\lambda}_{\xi\eta}\right)=ke^{\lambda}_{\overline{\eta}\overline{\xi}}$, where $\overline{\xi}=\overline{\xi}^{\lambda}_{ab}=\xi^{\lambda}_{ba}$ and $k=\sqrt{\frac{\mu(a)\mu(d)}{\mu(b)\mu(c)}}$ is a function of quantum dimensions of the vertices of G. By analogy, an antipode $\widehat{S}$ on $\widehat{\mathcal{B}}$ can be defined as an anti-homomorphism of algebras by $\widehat{S}(f^{\kappa\sigma}_x)=kf^{\overline{\sigma\kappa}}_x$. S and $\widehat{S}$ fulfill all the properties defining the weak Hopf algebras.

4.2.5 Gathering the pieces

To ensure the axioms of weak Hopf algebras [3, 26] for both algebra structures $(\mathcal{B},\circ,\mathbb{I},\Delta,\varepsilon,S)$ and $\left(\widehat{\mathcal{B}},\widehat{\circ},\widehat{\mathbb{I}},\widehat{\Delta},\widehat{\varepsilon},\widehat{S}\right)$, some $\mathbb{C}$-valued 3-simplices ${}^{(i)}\mathrm{C}$, $i=0,1,\ldots 4$, called cells are introduced and are related to quantum standard Racah symbols. There is only two kind of vertices for each tetrahedron: $\bullet\in\mathcal{A}$ and $\circ\in\Gamma$ and only three types of oriented edges, the one going from $\bullet$ to $\circ$ is excluded.

${}^{(1)}C$ and ${}^{(3)}C$ cells are introduced to define the explicit actions of Δ and $\widehat{\Delta}$ respectively and ${}^{(0)}C$ and ${}^{(43)}C$ cells to ensure their associativity. ${}^{(2)}C$ are called Ocneanu cells and are quite special in the sense that the two set of bases $\left\{e_{\xi\eta}\right\}$ and $\left\{f^{\alpha\beta}\right\}$ are not dual: If we try the basis change in $\widehat{\mathcal{B}}$ between $\left\{f^{\alpha\beta}\right\}$ and $\left\{\widehat{e}_{\xi\eta}\right\}$, the dual one of $\left\{e_{\xi\eta}\right\}$, then their pairing is given by: $\left\langle f^{\alpha\beta},\widehat{e}_{\xi\eta}\right\rangle ={}^{(2)}C={}^{(2)}C^{\alpha\beta x}_{\xi\eta\lambda}$.

In TQFT language, the set of cells ${}^{(i)}C$ gives a generalization of quantum $3j$ and $6j$ symbols. In RCFT, there is a natural link between these cells and the coefficients of Moore and Sieberg.

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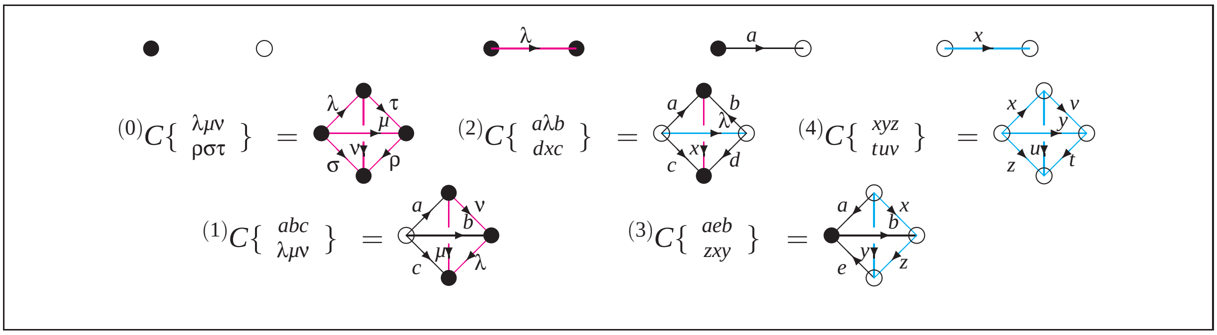
[3] The minimal central projector of the block λ is defined by $\pi_\lambda=\sum_\xi e^{\lambda}_{\xi\xi}$ such that $\pi_\lambda\circ\pi_{\lambda'}=\delta_{\lambda\lambda'}\pi_\lambda$, and projects $\mathcal{B}$ on the block $\mathcal{B}_\lambda$ by $\pi_\lambda(\mathcal{B})=\mathcal{B}_\lambda$.The minimal central projector of the block x is defined by $\omega^x=\sum_\alpha f^{\alpha\alpha}_x$ such that $\omega^x\widehat{\circ}\omega^y=\delta^{xy}\omega^x$, and projects $\widehat{\mathcal{B}}$ on the block $\widehat{\mathcal{B}}_x$ by $\omega^x\left(\widehat{\mathcal{B}}\right)=\widehat{\mathcal{B}}_x$.



The correspondence to Nimreps of the theory is the following:

$$ {}^{(0)}C \leftrightarrows \mathbb{N}_{\lambda\mu}{}^{\nu}, \qquad {}^{(1)}C \leftrightarrows \mathbb{F}_{ab}{}^{\lambda}, \qquad {}^{(3)}C \leftrightarrows \mathbb{S}_{ab}{}^{x}, \qquad {}^{(4)}C \leftrightarrows O_{xy}{}^{z}. $$

In this way, we give the necessary data needed to construct two finite dimensional algebras[4] $(\mathcal{B},\circ,\mathbb{I},\Delta,\varepsilon,S)$ and $\left(\widehat{\mathcal{B}},\widehat{\circ},\widehat{\mathbb{I}},\widehat{\Delta},\widehat{\varepsilon},\widehat{S}\right)$. In RCFT, the representation theory of $\mathcal{B}$ is described by the fusion algebra $\mathcal{A}(G)$ via the coefficients $\mathbb{N}_{\lambda\mu}^{\nu}$ and that one of $\widehat{\mathcal{B}}$ is encoded in the Ocneanu algebra of quantum symmetries via its nimreps $\mathbb{O}_{xy}^{z}$.
In the $\mathcal{D}_3$ case, the computation of dimensions d_λ and d_x for the two types of blocks from the determination of annular and dual annular coefficients allows to check the quadratic sum rule: $\sum_\lambda (d_\lambda)^2 = \sum_x (d_x)^2 = 1032$ which is the common dimension of $\mathcal{B}$ and $\widehat{\mathcal{B}}$. However, the linear sum rule is not fulfilled in this case $\sum_\lambda d_\lambda \neq \sum_x d_x$ and one must introduce suitable symmetry factors. It is worth to mention that more details can be found in [9, 20, 21] for the SU(3) cases and many others explicit examples in [7, 15, 12, 8] for the SU(2) cases. Finally, there are many open problems in this direction like the explicit computation of quantum 6-j symbols for SU(N) models or the determination of quantum symmetries related to systems of general Lie groups other than SU(N) groups.

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[4] In fact these are two isomorphic algebras and they describe the same double triangle algebra (DTA) [28]

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