

DESCENT AND GALOIS THEORY FOR HOPF CATEGORIES

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ABSTRACT. Descent theory for linear categories is developed. Given a linear category as an extension of a diagonal category, we introduce descent data, and the category of descent data is isomorphic to the category of representations of the diagonal category, if some flatness assumptions are satisfied. Then Hopf-Galois descent theory for linear Hopf categories, the Hopf algebra version of a linear category, is developed. This leads to the notion of Hopf-Galois category extension. We have a dual theory, where actions by dual linear Hopf categories on linear categories are considered. Hopf-Galois category extensions over groupoid algebras correspond to strongly graded linear categories.

INTRODUCTION

A k -linear category is a category enriched in the monoidal category of vector spaces \mathcal{M}_k . It is a generalization of a k -algebra in the sense that a k -linear category with one object is simply a k -algebra. Thus we can regard a k -linear category as a multi-object version of a k -algebra. This philosophy was further examined in [3], leading to multi-object versions of bialgebras and Hopf algebras, respectively termed k -linear semi-Hopf categories and k -linear Hopf categories. It turns out that several classical properties of Hopf algebras can be generalized to Hopf categories, see [3] for some examples. One of the results in [3] is the fundamental theorem for Hopf modules, opening the way to Hopf-Galois theory. The main aim of this paper is to develop Hopf-Galois theory for Hopf categories.

Hopf-Galois objects were introduced by Chase and Sweedler [7], and was generalized by Kreimer and Takeuchi [14]. One of the important properties is the Fundamental Theorem, which can be interpreted as Hopf-Galois descent, and can be stated as follows: if A is an H -comodule algebra, with coinvariant subalgebra B , then there is an adjunction between B -modules and relative Hopf modules, which is a pair of inverse equivalences if A is a Hopf-Galois extension of B , which is faithfully flat as a left B -module. From the point of view of descent theory: an A -module can be descended to a B -module if it has the additional structure of a relative Hopf module, in other words, the relative Hopf modules become the Hopf-Galois descent

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data. An elegant formulation of the theory was given by Brzeziński in [5], based on the theory of corings. In this formalism, classical descent data (as introduced in [11] for schemes, in [12] for extensions of commutative rings, and in [8] for extensions of non-commutative rings) as well as Hopf-Galois descent data become comodules over certain corings. A more detailed account of this approach is presented in the survey paper [6], which is at the basis of the methods developed in this paper, with one important drawback, namely the fact that, as far as we could figure it out, the formulation in terms of corings is not working in the setting of Hopf categories. However, the general philosophy survives, and enables us to formulate faithfully flat descent and Hopf-Galois theory for Hopf categories.

The line-up of the paper is as follows. Preliminary results from [3] are given in Section 1. In Section 2, we present faithfully flat descent theory for linear categories. In the classical theory, both the base B and the extension A are algebras, connected by an algebra morphism. In our setting, B and A are linear categories, connected by a so-called extension. But now B is a diagonal category, meaning that $B_{xy} = 0$ if $x \neq y$. Another difference, already mentioned above, is that the descent data cannot be interpreted as comodules over a coring. In Section 3, we generalize the notions of comodule algebra and its coinvariants and relative Hopf module. The strategy to develop Hopf-Galois descent is now the following. There is a functor from descent data to relative Hopf modules, see Proposition 3.4. A Hopf category is called an H -Galois category extension of its coinvariants if a collection of canonical maps is invertible; some equivalent conditions are given in Theorem 3.5 and in this case descent data and relative Hopf modules are isomorphic categories, leading to the desired descent theory if some flatness conditions are satisfied. In the classical case, an alternative description of descent data is possible if A is finitely generated projective as a B -module: it is the category of modules over ${}_B\text{End}(A)$. This is generalized to the categorical situation in Section 5. It turns out that we need clusters of k -linear categories, these are collections of k -linear categories indexed by X , see Section 4. Some duality results are discussed in Sections 6 and 7. In the final Section 8, we focus on Galois category extensions over a Hopf category induced by a groupoid, and link our results to the work of Lundström [15], generalizing an old result of Ulbrich [20] that a Hopf-Galois extension over a group algebra is a strongly graded ring. Another classical result, already observed in [7], is that classical Galois extensions, where a finite group G acts on A , are precisely Hopf-Galois extensions over the dual of the group ring $(kG)^*$. Although we have a duality theory, see Sections 6 and 7, this result cannot be generalized in a satisfactory way at the moment, we refer to the final remark Remark 8.3 for full explanation. This will be the topic of a forthcoming paper.

1. PRELIMINARY RESULTS

1.1. **k -linear categories.** Let \mathcal{V} be a monoidal category. From [2, Sec. 6.2], we recall the notion of \mathcal{V} -category. In particular, in the case where $\mathcal{V} = \mathcal{M}_k$, the category of vector spaces over a field k (or, more generally, the category of modules over a commutative ring k), a \mathcal{V} -category is a k -linear category. In [3], the notions of $\mathcal{C}(\mathcal{V})$ -category and Hopf \mathcal{V} -category are introduced. In this paper, we will work over $\mathcal{V} = \mathcal{M}_k$ and over $\mathcal{V} = \mathcal{M}_k^{\text{op}}$. A Hopf \mathcal{M}_k -category is called a k -linear Hopf category, while a Hopf $\mathcal{M}_k^{\text{op}}$ -category is called a dual k -linear Hopf category. Let us specify the definitions from [3] to this particular situation.

Let A be a k -linear category, and let X be the class of objects in A . For $x, y \in A$, we write A_{xy} for the k -module of morphisms from y to x . For all $x, y, z \in X$, we then have the composition maps $m_{xyz} : A_{xy} \otimes A_{yz} \rightarrow A_{xz}$, $m_{xyz}(a \otimes b) = ab$, for all $a \in A_{xy}$ and $b \in A_{yz}$. The unit element of A_{xx} is denoted by 1_x .

Let A and A' be k -linear categories with the same underlying class of objects X . A k -linear functor $f : A \rightarrow B$ that is the identity on X is called a k -linear X -functor: for all $x, y \in X$, $f_{xy} : A_{xy} \rightarrow B_{xy}$ is a k -linear map preserving multiplication and unit.

For a class X , we introduce the category $\mathcal{M}_k(X)$. An object is a family of objects M in \mathcal{M}_k indexed by $X \times X$:

$$M = (M_{xy})_{x,y \in X}.$$

A morphism $\varphi : M \rightarrow N$ consists of a family of k -linear maps $\varphi_{xy} : M_{xy} \rightarrow N_{xy}$ indexed by $X \times X$.

Let A be a k -linear category. A right A -module is an object M in $\mathcal{M}_k(X)$ together with a family of k -linear maps

$$\psi = \psi_{xyz} : M_{xy} \otimes A_{yz} \rightarrow M_{xz}, \quad \psi_{xyz}(m \otimes a) = ma$$

such that the following associativity and unit conditions hold: $(ma)b = m(ab)$; $m1_y = m$, for all $m \in M_{xy}$, $a \in A_{yz}$ and $b \in A_{zu}$.

Let M and N be right A -modules. A morphism $\varphi : M \rightarrow N$ in $\mathcal{M}_k(X)$ is called right A -linear if $\varphi_{xz}(ma) = \varphi_{xy}(m)a$, for all $m \in M_{xy}$ and $a \in A_{yz}$. The category of right A -modules and right A -linear morphisms is denoted by $\mathcal{M}_k(X)_A$.

We will also need the category $\mathcal{D}_k(X)$. Objects are families of k -modules $N = (N_x)_{x \in X}$ indexed by X , and a morphism $N \rightarrow N'$ consists of a family of k -linear maps $N_x \rightarrow N'_x$. $\mathcal{D}_k(X)$ is a symmetric monoidal category, and an algebra B in $\mathcal{D}_k(X)$ consists of a family of k -algebras $B = (B_x)_{x \in X}$ indexed by X . We can consider B as a k -linear category: $B_{xy} = \{0\}$ if $x \neq y$ and $B_{xx} = B_x$. B is then called a diagonal k -linear category. A diagonal right B -module is an object $N \in \mathcal{D}_k(X)$ such that every N_x is a right B_x -module. $\mathcal{D}_k(X)_B$ is the category of diagonal B -modules and right B -linear morphisms. A morphism g in $\mathcal{D}_k(X)$ between right B -modules is right

B -linear if every g_x is right B_x -linear. The category of left B -modules is defined in a similar way.

1.2. Finitely generated projective modules and dual basis. Let B be a k -algebra, and assume that M is a finitely generated projective left B -module. Then $M^* = {}_B\text{Hom}(M, B)$ is a finitely generated projective right B -module, with action given by the formula $(m^* \cdot b)(m) = m^*(m)b$, for all $m \in M$, $m^* \in M^*$ and $b \in B$. M has a finite dual basis $\sum_i e_i^* \otimes_B e_i \in M^* \otimes_B M$ satisfying the formulas

$$(1) \quad \sum_i e_i^*(m)e_i = m \text{ and } \sum_i e_i^* m^*(e_i) = m^*,$$

for all $m \in M$ and $m^* \in M^*$. For $N \in {}_B\mathcal{M}$ and $P \in \mathcal{M}_B$, we have isomorphisms

$$(2) \quad {}_B\text{Hom}(M, N) \cong M^* \otimes_B N \text{ and } \text{Hom}_B(M^*, P) \cong P \otimes_B M.$$

For later use, we provide the explicit description of $\alpha : P \otimes_B M \rightarrow \text{Hom}_B(M^*, P)$ and its inverse. For $p \in P$, $m \in M$, $m^* \in M^*$ and $f \in \text{Hom}_B(M^*, P)$, we have

$$(3) \quad \alpha(p \otimes_B m)(m^*) = pm^*(m) \text{ and } \alpha^{-1}(f) = \sum_i f(e_i^*) \otimes_B e_i.$$

A left B -progenerator (in the literature also termed as a faithfully projective left B -module) is a finitely generated projective left B -module that is also a generator, that is, $\text{Tr}(M) = \{\sum_i m_i^*(m_i) \mid m_i \in M, m_i^* \in M^*\} = B$. A finitely generated projective module is flat, and a progenerator is faithfully flat.

Let B be a diagonal k -linear category. We can view B as a k -linear category, see Section 1.1. Consider a left B -module M . We introduce the following terminology.

M is called locally flat, resp. locally finite as a left B -module if every M_{xy} is flat, resp. finitely generated projective as a left B_x -module.

A locally flat left B -module M is called locally faithfully flat if every M_{xx} is faithfully flat as a left B_x -module; A locally finite left B -module M is called locally faithfully projective if every M_{xx} is faithfully flat as a left B_x -module.

1.3. Hopf categories. The category $\mathcal{C}(\mathcal{M}_k)$ of k -coalgebras is a monoidal category, so we can consider $\mathcal{C}(\mathcal{M}_k)$ -categories. It is shown in [3] that a $\mathcal{C}(\mathcal{M}_k)$ -category is a k -linear category H with the following additional structure: for all $x, y \in X$, H_{xy} is a k -coalgebra with structure maps Δ_{xy} and ε_{xy} such that the following properties hold, for all $h \in H_{xy}$ and $k \in H_{yz}$:

$$\begin{aligned} \Delta_{xz}(hk) &= h_{(1)}k_{(1)} \otimes h_{(2)}k_{(2)} ; \Delta_{xx}(1_x) = 1_x \otimes 1_x ; \\ \varepsilon_{xz}(hk) &= \varepsilon_{xy}(h)\varepsilon_{yz}(k) ; \varepsilon_{xx}(1_x) = 1. \end{aligned}$$

A $\mathcal{C}(\mathcal{M}_k)$ -category with one object is a bialgebra; an obvious name for $\mathcal{C}(\mathcal{M}_k)$ -categories in general therefore seems to be “ k -linear bicategories”. However, this terminology is badly chosen, because of possible confusion

with the existing notions of 2-categories and bicategories. This is why we introduce the name “ k -linear semi-Hopf categories” for $\underline{\mathcal{C}}(\mathcal{M}_k)$ -categories. A k -linear Hopf category is a $\underline{\mathcal{C}}(\mathcal{M}_k)$ -category together with k -linear maps $S_{xy} : H_{xy} \rightarrow H_{yx}$ such that

$$(4) \quad h_{(1)}S_{xy}(h_{(2)}) = \varepsilon_{xy}(h)1_x ; S_{xy}(h_{(1)})h_{(2)} = \varepsilon_{xy}(h)1_y,$$

for all $x, y \in X$ and $h \in H_{xy}$.

The same construction can be performed in the opposite category of vector spaces, leading to the following notions. A dual k -linear semi-Hopf category K consists of a an object $K \in \mathcal{M}_k(X)$, together with comultiplication and counit maps $\Delta_{xyz} : K_{xz} \rightarrow K_{xy} \otimes K_{yz}$ and $\varepsilon_x : K_{xx} \rightarrow k$, satisfying the obvious coassociativity and counit properties. We adopt the Sweedler notation $\Delta_{xyz}(k) = k_{(1,x,y)} \otimes k_{(2,y,z)}$, for $k \in K_{xz}$. Furthermore, every K_{xy} is a k -algebra, with unit 1_{xy} . The compatibility relations between the two structures are the following, for $h, k \in K_{xz}$ and $l, m \in K_x$:

$$\begin{aligned} \Delta_{xyz}(hk) &= h_{(1,x,y)}k_{(1,x,y)} \otimes h_{(2,y,z)}k_{(2,y,z)} ; \Delta_{xyz}(1_{xz}) = 1_{xy} \otimes 1_{yz} ; \\ \varepsilon_x(lm) &= \varepsilon_x(l)\varepsilon_x(m) ; \varepsilon_x(1_{xx}) = 1. \end{aligned}$$

A dual k -linear Hopf category is a dual k -linear semi-Hopf category K with an antipode T consisting of a family of maps $T_{xy} : K_{yx} \rightarrow K_{xy}$ satisfying the following equations, for all $l \in K_{xx}$ and $x \in X$:

$$l_{(1,x,y)}T_{xy}(l_{(2,y,x)}) = \varepsilon_x(l)1_{xy} \text{ and } T_{yx}(l_{(1,x,y)})l_{(2,y,x)} = \varepsilon_x(l)1_{yx}.$$

From [3, Theorem 5.6], we recall that there is a duality between the categories of locally finite (semi-)Hopf categories and locally finite dual (semi-)Hopf categories.

For later use, we give the explicit description of the dual k -linear Hopf category K corresponding to a locally finite k -linear Hopf category H . As an object of $\mathcal{M}_k(X)$, K is given componentwise as $K_{xy} = H_{yx}^*$. The multiplication on K_{xy} is given by opposite convolution:

$$\langle kl, h \rangle = \langle k, h_{(2)} \rangle \langle l, h_{(1)} \rangle,$$

for all $k, l \in K_{xy}$ and $h \in H_{yx}$. The unit of K_{xy} is $1_{xy} = \varepsilon_{yx}$. The comultiplication maps

$$\Delta_{xyz} : K_{xz} \rightarrow K_{xy} \otimes K_{yz}, \quad \Delta_{xyz}(k) = k_{(1,x,y)} \otimes k_{(2,y,z)}$$

are characterized by the formulas

$$\langle k_{(1,x,y)}, h' \rangle \langle k_{(2,y,z)}, h \rangle = \langle k, hh' \rangle,$$

for all $h \in H_{zy}$ and $h' \in H_{yx}$. The counit maps $\varepsilon_x : K_{xx} \rightarrow k$ are given by $\varepsilon_x(k) = \langle k, 1_x \rangle$. The antipode maps are $T_{yx} = S_{xy}^* : K_{xy} = H_{yx}^* \rightarrow K_{yx} = H_{xy}^*$.

2. DESCENT THEORY FOR k -LINEAR CATEGORIES

Let A and B be k -linear categories, with underlying class X , and assume that B is diagonal. Let $i : B \rightarrow A$ be a k -linear X -functor. Then B consists of a family of k -algebras indexed by X , and for every $x \in X$, we have a k -algebra morphism $i_x : B_x \rightarrow A_{xx}$.

Definition 2.1. A descent datum (M, σ) for the functor i consists of a right A -module M together with a family of k -linear maps $\sigma = (\sigma_{xy})_{x,y \in X}$, where

$$\sigma_{xy} : M_{xy} \rightarrow M_{xx} \otimes_{B_x} A_{xy}.$$

We use the following Sweedler-type notation: $\sigma_{xy}(m) = m_{\langle 0 \rangle} \otimes_{B_x} m_{\langle 1 \rangle}$, for $m \in M_{xy}$. The following conditions have to be satisfied, for all $m \in M_{xy}$ and $a \in A_{yz}$:

$$(5) \quad \sigma_{xz}(ma) = m_{\langle 0 \rangle} \otimes_{B_x} m_{\langle 1 \rangle} a;$$

$$(6) \quad \sigma_{xx}(m_{\langle 0 \rangle}) \otimes_{B_x} m_{\langle 1 \rangle} = m_{\langle 0 \rangle} \otimes_{B_x} 1_x \otimes_{B_x} m_{\langle 1 \rangle};$$

$$(7) \quad m_{\langle 0 \rangle} m_{\langle 1 \rangle} = m.$$

A morphism between two descent data (M, σ) and (M', σ') is a morphism $f : M \rightarrow M'$ in $\mathcal{M}_k(X)_A$ such that

$$(8) \quad f_{xx}(m_{\langle 0 \rangle}) \otimes_{B_x} m_{\langle 1 \rangle} = \sigma'_{xy}(f_{xy}(m)),$$

for all $m \in M_{xy}$. The category of descent data is denoted $\underline{\text{Desc}}_B(A)$.

Proposition 2.2. Let $i : B \rightarrow A$ be a k -linear X -functor. We have an adjoint pair of functors (F, G) between the categories $\mathcal{D}_k(X)_B$ and $\underline{\text{Desc}}_B(A)$.

Proof. For $N \in \mathcal{D}_k(X)_B$, we define $F(N) \in \underline{\text{Desc}}_B(A)$ as follows: $F(N)_{xy} = N_x \otimes_{B_x} A_{xy}$, and $\sigma_{xy} : F(N)_{xy} = N_x \otimes_{B_x} A_{xy} \rightarrow F(N)_{xx} \otimes_{B_x} A_{xy} = N_x \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xy}$ is given by the formula

$$\sigma_{xy}(n \otimes_{B_x} a) = n \otimes_{B_x} 1_x \otimes_{B_x} a.$$

Conditions (5) and (7) are obviously satisfied. We also compute easily that

$$\begin{aligned} \sigma_{xx}(n \otimes_{B_x} 1_x) \otimes_{B_x} a &= n \otimes_{B_x} 1_x \otimes_{B_x} 1_x \otimes_{B_x} a \\ &= (n \otimes_{B_x} a)_{\langle 0 \rangle} \otimes_{B_x} 1_x \otimes_{B_x} (n \otimes_{B_x} a)_{\langle 1 \rangle}, \end{aligned}$$

hence (6) is also satisfied, and $F(N)$ is a descent datum. In particular, $F(B) = A$ is a descent datum, with $\sigma_{xy} : A_{xy} \rightarrow A_{xx} \otimes_{B_x} A_{xy}$, $\sigma_{xy}(a) = 1_x \otimes_{B_x} a$.

At the level of morphisms, F is defined as follows. Take $g : N \rightarrow N'$ in $\mathcal{D}_k(X)_B$. Then $F(g) = f : F(N) \rightarrow F(N')$ has x, y -component

$$f_{xy} = g_{xy} \otimes_{B_x} A_{xy} : N_x \otimes_{B_x} A_{xy} \rightarrow N'_x \otimes_{B_x} A_{xy}.$$

Conversely, for $M \in \underline{\text{Desc}}_B(A)$, let $G(M) \in \mathcal{D}_k(X)$ be given by the formula

$$G(M)_x = \{m \in M_{xx} \mid \sigma_{xx}(m) = m \otimes_{B_x} 1_x\}.$$

We claim that $G(M) \in \mathcal{D}_k(X)_B$, that is, $G(M)_x$ is a right B_x -module, for every $x \in X$. Indeed, for every $m \in G(M)_x$ and $b \in B_x$, we have that $mb \in G(M)_x$ since $\sigma_{xx}(mb) \stackrel{(5)}{=} m \otimes_{B_x} 1_x b = mb \otimes_{B_x} 1_x$. Observe that

$$G(A)_x = \{a \in A_{xx} \mid 1_x \otimes_{B_x} a = a \otimes_{B_x} 1_x\}.$$

$G(A)$ is a diagonal k -linear category: it is easy to show that every $G(A)_x$ is a k -algebra. Also the algebra morphisms $i_x : B_x \rightarrow A_{xx}$ corestrict to $i_x : B_x \rightarrow G(A)_x$. Indeed, for $b \in B_x$, we have that $i_x(b) = b1_x = 1_x b$, and $1_x \otimes_{B_x} b1_x = b1_x \otimes_{B_x} 1_x$, so that $i_x(b) \in G(A)_x$.

Let $f : (M, \sigma) \rightarrow (M', \sigma')$ be a morphism of descent data. For $m \in G(M)_x$, we have that

$$\sigma'_{xx}(f_{xx}(m)) \stackrel{(8)}{=} f_{xx}(m_{\langle 0 \rangle}) \otimes_{B_x} m_{\langle 1 \rangle} = f_{xx}(m) \otimes_{B_x} 1_x,$$

and $f_{xx}(m) \in G(M')_x$. We now define $G(f) : G(M) \rightarrow G(M')$. $G(f)_x$ is the restriction and corestriction of f_{xx} to $G(M)$ and $G(M')$.

Unit of the adjunction. Let N be a diagonal B -module. Then

$$GF(N)_x = \left\{ \sum_i n_i \otimes_{B_x} a_i \in N_x \otimes_{B_x} A_{xx} \mid \sum_i n_i \otimes_{B_x} 1_x \otimes_{B_x} a_i = \sum_i n_i \otimes_{B_x} a_i \otimes_{B_x} 1_x \right\}.$$

$\eta^N : N \rightarrow GF(N)$ is now defined as follows:

$$\eta_x^N : N_x \rightarrow GF(N)_x \subset N_x \otimes_{B_x} A_{xx}, \quad \eta_x^N(n) = n \otimes_{B_x} 1_x.$$

Counit of the adjunction. Let M be a descent datum. $FG(M)_{xy} = G(M)_x \otimes_{B_x} A_{xy}$, and $\varepsilon^M : FG(M) \rightarrow M$ is defined as follows:

$$\varepsilon_{xy}^M : G(M)_x \otimes_{B_x} A_{xy} \rightarrow M_{xy}, \quad \varepsilon_{xy}^M(m \otimes_{B_x} a) = ma.$$

Verification of all the further details is left to the reader. \square

Proposition 2.3. *Let $i : B \rightarrow A$ be a k -linear X -functor. Take $x, y \in X$ and assume that A_{xy} is flat as a left B_x -module. Then the counit morphism ε_{xy}^M from Proposition 2.2 is bijective, for every descent datum (M, σ) .*

Proof. Consider the map

$$i_x^M = M_{xx} \otimes_{B_x} i_x : M_{xx} \rightarrow M_{xx} \otimes_{B_x} A_{xx}, \quad i_x^M(m) = m \otimes_{B_x} 1_x.$$

Then we have an exact sequence

$$0 \rightarrow G(M)_x \xrightarrow{\subset} M_{xx} \xrightarrow[i_x^M]{\sigma_{xx}} M_{xx} \otimes_{B_x} A_{xx}.$$

By assumption, A_{xy} is flat as a left B_x -module, hence the sequence

$$0 \rightarrow G(M)_x \otimes_{B_x} A_{xy} \xrightarrow{\subset} M_{xx} \otimes_{B_x} A_{xy} \xrightarrow[i_x^M \otimes_{B_x} A_{xy}]{\sigma_{xx} \otimes_{B_x} A_{xy}} M_{xx} \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xy}$$

is exact. Take $m \in M_{xy}$. Then $\sigma_{xy}(m) = \tilde{m}_{\langle 0 \rangle} \otimes_{B_x} m_{\langle 1 \rangle} \in M_{xx} \otimes_{B_x} A_{xy}$ and

$$\begin{aligned} (\sigma_{xx} \otimes_{B_x} A_{xy})(\sigma_{xy}(m)) &= \sigma_{xx}(m_{\langle 0 \rangle}) \otimes_{B_x} m_{\langle 1 \rangle} \\ &\stackrel{(6)}{=} m_{\langle 0 \rangle} \otimes_{B_x} 1_x \otimes_{B_x} m_{\langle 1 \rangle} = (i_x^M \otimes_{B_x} A_{xy})(\sigma_{xy}(m)). \end{aligned}$$

It follows that $\sigma_{xy}(m) \in G(M)_x \otimes_{B_x} A_{xy}$, so σ_{xy} corestricts to

$$\sigma_{xy} : M_{xy} \rightarrow G(M)_x \otimes_{B_x} A_{xy}.$$

We now show that this map is the inverse of ε_{xy}^M . For all $m \in M_{xy}$, we have that

$$(\varepsilon_{xy}^M \circ \sigma_{xy})(m) = m_{\langle 0 \rangle} m_{\langle 1 \rangle} \stackrel{(7)}{=} m.$$

For $m \in G(M)_x$ and $a \in A_{xy}$, we easily calculate that

$$(\sigma_{xy} \circ \varepsilon_{xy}^M)(m \otimes_{B_x} a) = \sigma_{xy}(ma) \stackrel{(5)}{=} m_{\langle 0 \rangle} \otimes_{B_x} m_{\langle 1 \rangle} a = m \otimes_{B_x} a.$$

□

Proposition 2.4. *Let $i : B \rightarrow A$ be a k -linear X -functor. Take $x \in X$ and assume that A_{xx} is faithfully flat as a left B_x -module. Then η_x^N is bijective for every $N \in \mathcal{D}_k(X)_B$.*

Proof. Let $f_1, f_2 : N_x \otimes_{B_x} A_{xx} \rightarrow N_x \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xx}$ be defined by the formulas

$$f_1(n \otimes_{B_x} a) = n \otimes_{B_x} 1_x \otimes_{B_x} a ; f_2(n \otimes_{B_x} a) = n \otimes_{B_x} a \otimes_{B_x} 1_x.$$

Since η_x^N is the corestriction of i_x^N to $GF(N)_x$, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_x & \xrightarrow{i_x^N} & N_x \otimes_{B_x} A_{xx} & \xrightarrow[f_2]{f_1} & N_x \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xx} \\ & & \downarrow \eta_x^N & & \downarrow = & & \downarrow = \\ 0 & \longrightarrow & GF(N)_x & \xrightarrow{\subset} & N_x \otimes_{B_x} A_{xx} & \xrightarrow[f_2]{f_1} & N_x \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xx} \end{array}$$

It follows from the definition of G that the bottom row is exact. If we can show that the top row is exact, then it will follow from the five lemma that η_x^N is an isomorphism. Since A_{xx} is faithfully flat as a left B_x -module, it suffices to show that the top row becomes exact after the functor $-\otimes_{B_x} A_{xx}$, is applied to it.

Take $\alpha = \sum_i n_i \otimes_{B_x} a_i \otimes_{B_x} b_i \in N_x \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xx}$, and assume that $(f_1 \otimes_{B_x} A_{xx})(\alpha) = (f_2 \otimes_{B_x} A_{xx})(\alpha)$, that is,

$$\sum_i n_i \otimes_{B_x} 1_x \otimes_{B_x} a_i \otimes_{B_x} b_i = \sum_i n_i \otimes_{B_x} a_i \otimes_{B_x} 1_x \otimes_{B_x} b_i.$$

Multiplying the third and the fourth tensor factor, we obtain that

$$(1_x^N \otimes_{B_x} A_{xx})\left(\sum_i n_i \otimes_{B_x} a_i b_i\right) = \sum_i n_i \otimes_{B_x} 1_x \otimes_{B_x} a_i b_i = \sum_i n_i \otimes_{B_x} a_i \otimes_{B_x} b_i = \alpha,$$

so $\alpha \in \text{Im}(1_x^N \otimes_{B_x} A_{xx})$, which is precisely what we need. □

As an immediate application of Propositions 2.2-2.4, we obtain the following result, which can be viewed as the faithfully flat descent theorem for k -linear categories. We would like to point out that Theorem 2.5 can also be derived

from Beck's Theorem, see [16, Sec. VI.7]; we have preferred to present a direct proof.

Theorem 2.5. *Let $i : B \rightarrow A$ be a k -linear X -functor. Assume that A is locally flat as a left B -module. Then the following assertions are equivalent.*

- (1) $A_{x,x}$ is faithfully flat as a left B_x -module for all $x \in X$;
- (2) the adjoint pair (F, G) from Proposition 2.2 is a pair of inverse equivalences, and the categories $\mathcal{D}_k(X)_B$ and $\underline{\text{Desc}}_B(A)$ are equivalent.

Proof. We only need to prove (2) \Rightarrow (1). Assume that (F, G) is a pair of inverse equivalences. Fix $x \in X$ and let

$$(9) \quad 0 \rightarrow N'_x \rightarrow N_x \rightarrow N''_x \rightarrow 0$$

be a sequence of left B_x -modules such that

$$(10) \quad 0 \rightarrow N'_x \otimes_{B_x} A_{xx} \rightarrow N_x \otimes_{B_x} A_{xx} \rightarrow N''_x \otimes_{B_x} A_{xx} \rightarrow 0$$

is exact. If we apply the functor G to the sequence, and use the fact that every η_x^N is bijective, we find that (10) is exact. \square

For later use, we briefly discuss ${}_B \underline{\text{Desc}}(A)$, the category of left descent data. Let $i : B \rightarrow A$ be as before. A left descent datum is a left A -module M together with a family of linear maps $\tau = (\tau_{xy})_{x,y \in X}$, $\tau_{xy} : M_{xy} \rightarrow A_{xy} \otimes_{B_y} M_{yy}$, $\tau_{xy}(n) = n_{\langle -1 \rangle} \otimes_{B_y} n_{\langle 0 \rangle}$, satisfying the following properties, for all $m \in M_{xy}$ and $a \in A_{zx}$:

$$(11) \quad \tau_{xy}(am) = am_{\langle -1 \rangle} \otimes_{B_y} m_{\langle 0 \rangle};$$

$$(12) \quad m_{\langle -1 \rangle} \otimes_{B_y} \tau_{yy}(m_{\langle 0 \rangle}) = m_{\langle -1 \rangle} \otimes_{B_y} 1_y \otimes_{B_y} m_{\langle 0 \rangle};$$

$$(13) \quad m_{\langle -1 \rangle} m_{\langle 0 \rangle} = m.$$

There is a pair of adjoint functors between ${}_B \mathcal{D}_k(X)$ and ${}_B \underline{\text{Desc}}(A)$, which is a pair of inverse equivalences if every A is locally faithfully flat as a right B -module.

3. RELATIVE HOPF MODULES

3.1. Right relative Hopf modules.

Definition 3.1. Let H be a k -linear semi-Hopf category. A right H -comodule category is a k -linear category A with the following additional structure: every A_{xy} is a right H_{xy} -comodule, with coaction $\rho_{xy} : A_{xy} \rightarrow A_{xy} \otimes H_{xy}$, that is compatible with the multiplication and unit on A in the sense that

$$(14) \quad \rho_{xz}(ab) = a_{[0]} b_{[0]} \otimes a_{[1]} b_{[1]} \text{ and } \rho_{xx}(1_x^A) = 1_x^A \otimes 1_x^H,$$

for all $a \in A_{xy}$ and $b \in A_{yz}$. We used the obvious Sweedler notation for the coaction. A right relative (A, H) -Hopf module M is a right A -module $M \in \mathcal{M}_k(X)_A$ such that every M_{xy} is a right H_{xy} -comodule satisfying the equation

$$(15) \quad \rho_{xz}(ma) = m_{[0]} a_{[0]} \otimes m_{[1]} a_{[1]},$$

for all $m \in M_{xy}$ and $a \in A_{yz}$.

A morphism $f : M \rightarrow M'$ between two relative Hopf modules is a morphism $f : M \rightarrow M'$ that is a morphism in $\mathcal{M}(X)_A$ and in $\mathcal{M}(X)^H$. The category of relative Hopf modules is denoted $\mathcal{M}(X)_A^H$.

Let M be a relative (A, H) -Hopf module. For each $x \in X$, we consider $M_{xx}^{\text{co}H_{xx}}$. Then the N_x are the components of an object $N = M^{\text{co}H} \in \mathcal{D}_k(X)$. We then have that $N_x = M_x^{\text{co}H} = M_{xx}^{\text{co}H_{xx}}$. In particular A is a relative Hopf module, and we can consider $A^{\text{co}H}$. It is easy to see that every $A_x^{\text{co}H}$ is a k -algebra, so that $A^{\text{co}H}$ is an algebra in $\mathcal{D}_k(X)$, and $M \in \mathcal{D}_k(X)_{A^{\text{co}H}}$, for all $M \in \mathcal{M}(X)_A^H$. Assume that B is an algebra in $\mathcal{D}_k(X)$, and that we have an algebra morphism $i : B \rightarrow A^{\text{co}H}$. It follows that every relative Hopf module M is a right B -module, by restriction of scalars.

Proposition 3.2. *For any H -comodule algebra A , we have a pair of adjoint functors (F, G) between the categories $\mathcal{D}_k(X)_B$ and $\mathcal{M}(X)_A^H$.*

Proof. We first define $F : \mathcal{D}_k(X)_B \rightarrow \mathcal{M}_k(X)_A^H$. For $N \in \mathcal{D}_k(X)_B$, we have $M = F(N) \in \mathcal{M}_k(X)_A^H$, with $M_{xy} = N_x \otimes_{B_x} A_{xy}$, with structure maps

$$(n \otimes a)b = n \otimes ab ; \rho_{xy}(n \otimes a) = n \otimes a_{[0]} \otimes a_{[1]},$$

for all $n \in N_x$, $a \in A_{xy}$ and $b \in A_{yz}$. For $g : N \rightarrow N'$ in $\mathcal{D}_k(X)_B$, $F(g)_{xy} = g_x \otimes A_{xy}$.

For $M \in \mathcal{M}_k(X)_A^H$, let $G(M) = M^{\text{co}H}$. Consider $f : M \rightarrow M'$ in $\mathcal{M}(X)_A^H$. It is easy to see that $f_{xx} : M_{xx} \rightarrow M'_{xx}$ restricts and corestricts to a map $M_{xx}^{\text{co}H_{xx}} \rightarrow M'_{xx}{}^{\text{co}H_{xx}}$ which is by definition the x -component of $G(f) = f^{\text{co}H}$. Now we describe the unit and counit of the adjunction. For $N \in \mathcal{D}_k(X)_B$, $\eta^N : N \rightarrow GF(N)$ has components

$$\eta_x^N : N_x \rightarrow (N_x \otimes_{A_x^{\text{co}H}} A_{xx})^{\text{co}H_{xx}}, \eta_x^N(n) = n \otimes 1_x.$$

For $M \in \mathcal{M}(X)_A^H$, $\varepsilon^M : FG(M) \rightarrow M$ has components

$$\varepsilon_{xy}^M : M_x^{\text{co}H} \otimes_{B_x} A_{xy} \rightarrow M_{xy}, \varepsilon_{xy}^M(m \otimes_{B_x} a) = ma.$$

The verification of all further details is left to the reader. \square

We will investigate when (F, G) is a pair of inverse equivalences. We begin with some necessary conditions. For $x, y, z \in X$, we consider the maps

$$\text{can}_{xy}^z : A_{zx} \otimes_{B_x} A_{xy} \rightarrow A_{zy} \otimes_{B_y} A_{xy}, \text{can}_{xy}^z(a \otimes a') = aa'_{[0]} \otimes a'_{[1]}.$$

Proposition 3.3. *We consider the pair of adjoint functors (F, G) from Proposition 3.2.*

- (1) *If F is fully faithful, then $i : B \rightarrow A^{\text{co}H}$ is an isomorphism.*
- (2) *If G is fully faithful, then all the can_{xy}^z are isomorphisms.*

Proof. 1) If F is fully faithful, then η_x^N is an isomorphism, for all $x \in X$ and $N \in \mathcal{D}_k(X)_B$. Take $N = B$. Then we have that

$$i_x = \eta_x^B : B_x \rightarrow (B_x \otimes_{B_x} A_{xx})^{\text{co}H_{xx}} = A_{xx}^{\text{co}H_{xx}} = A_x^{\text{co}H}$$

is an isomorphism.

2) Assume that G is fully faithful. For each $z \in X$, consider $M^z \in \mathcal{M}_k(X)_A^H$ defined as follows: $M_{xy}^z = A_{zy} \otimes H_{xy}$, with structure maps

$$\rho_{xy}(a \otimes h) = a \otimes h_{(1)} \otimes h_{(2)} ; (a \otimes h)a' = aa'_{[0]} \otimes ha'_{[1]},$$

for all $a \in A_{zy}$, $h \in H_{xy}$, $a' \in A_{yu}$. We claim that

$$(16) \quad (M_{xx}^z)^{\text{co}H_{xx}} \cong A_{zx}.$$

It suffices to show that the maps

$$\begin{aligned} f : A_{zx} &\rightarrow (M_{xx}^z)^{\text{co}H_{xx}}, \quad f(a) = a \otimes 1_x; \\ g : (M_{xx}^z)^{\text{co}H_{xx}} &\rightarrow A_{zx}, \quad g\left(\sum_i a_i \otimes h_i\right) = \sum_i a_i \varepsilon_{xy}(h_i); \end{aligned}$$

are inverses. It is obvious that $g \circ f = A_{zx}$. Now take $\sum_i a_i \otimes h_i \in (M_{xx}^z)^{\text{co}H_{xx}}$. Then

$$\sum_i a_i \otimes h_{i(1)} \otimes h_{i(2)} = \sum_i a_i \otimes h_i \otimes 1_x.$$

Applying ε_{xx} to the second tensor factor, we find that

$$\sum_i a_i \otimes h_i = \sum_i a_i \varepsilon_{xx}(h_i) \otimes 1_x = (f \circ g)\left(\sum_i a_i \otimes h_i\right),$$

and this shows that $f \circ g = (M_{xx}^z)^{\text{co}H_{xx}}$. Finally observe that

$$\begin{aligned} \text{can}_{xy}^z &= \varepsilon_{xy}^{M^z} \circ (f \otimes_{B_x} A_{xy}) : \\ A_{zx} \otimes_{B_x} A_{xy} &\rightarrow (M_{xx}^z)^{\text{co}H_{xx}} \otimes_{B_x} A_{xy} \rightarrow M_{xy}^z = A_{zy} \otimes H_{xy} \end{aligned}$$

is an isomorphism. Indeed,

$$(\varepsilon_{xy}^{M^z} \circ (f \otimes_{B_x} A_{xy}))(a \otimes a') = (a \otimes 1_x)a' = aa'_{[0]} \otimes a'_{[1]} = \text{can}_{xy}^z(a \otimes a').$$

□

Proposition 3.4. *Let H be a k -linear semi-Hopf category, let A be a right H -comodule category, and let $B = A^{\text{co}H}$. Then we have a functor*

$$P : \underline{\text{Desc}}_B(A) \rightarrow \mathcal{M}_k(X)_A^H.$$

Proof. Take $(M, \sigma) \in \underline{\text{Desc}}_B(A)$, and consider

$$\rho_{xy} : M_{xy} \rightarrow M_{xy} \otimes H_{xy}, \quad \rho_{xy}(m) = m_{\langle 0 \rangle} m_{\langle 1 \rangle [0]} \otimes m_{\langle 1 \rangle [1]}.$$

We will show that $(M, \rho) \in \mathcal{M}_k(X)_A^H$. Let us first show that ρ_{xy} is coassociative.

$$\begin{aligned} ((\rho_{xy} \otimes A_{xy}) \circ \rho_{xy})(m) &= (m_{\langle 0 \rangle} m_{\langle 1 \rangle [0]})_{\langle 0 \rangle} (m_{\langle 0 \rangle} m_{\langle 1 \rangle [0]})_{\langle 1 \rangle [0]} \\ &\quad \otimes (m_{\langle 0 \rangle} m_{\langle 1 \rangle [0]})_{\langle 1 \rangle [1]} \otimes m_{\langle 1 \rangle [1]} \\ &\stackrel{(5)}{=} m_{\langle 0 \rangle \langle 0 \rangle} (m_{\langle 0 \rangle \langle 1 \rangle} m_{\langle 1 \rangle [0]})_{[0]} \otimes (m_{\langle 0 \rangle \langle 1 \rangle} m_{\langle 1 \rangle [0]})_{[1]} \otimes m_{\langle 1 \rangle [1]} \\ &\stackrel{(6)}{=} m_{\langle 0 \rangle} (1_x m_{\langle 1 \rangle [0]})_{[0]} \otimes (1_x m_{\langle 1 \rangle [0]})_{[1]} \otimes m_{\langle 1 \rangle [1]} \end{aligned}$$

$$= m_{<0>} m_{<1>[0]} \otimes m_{<1>[1]} \otimes m_{<1>[2]} = ((M_{xy} \otimes \Delta_{xy}) \circ \rho_{xy})(m).$$

We proceed with the counit property

$$((M_{xy} \otimes \varepsilon_{xy}) \circ \rho_{xy})(m) = m_{<0>} m_{<1>[0]} \varepsilon_{xy}(m_{<1>[1]}) = m_{<0>} m_{<1>} \stackrel{(6)}{=} m.$$

Finally, the compatibility condition (15) holds. For $m \in M_{xy}$ and $a \in A_{yz}$, we have that

$$\begin{aligned} \rho_{xz}(ma) &\stackrel{(5)}{=} m_{<0>} (m_{<1>} a)_{[0]} \otimes (m_{<1>} a)_{[1]} \\ &= m_{<0>} m_{<1>[0]} a_{[0]} \otimes m_{<1>[1]} a_{[1]} = m_{[0]} a_{[0]} \otimes m_{[1]} a_{[1]}. \end{aligned}$$

We now define $P(M, \sigma) = (M, \rho)$. Let $f : (M, \sigma) \rightarrow (M', \sigma')$ be a morphism in $\underline{\text{Desc}}_B(A)$. We claim that f is also a morphism $(M, \rho) \rightarrow (M', \rho')$ in $\mathcal{M}_k(X)_A^H$. To this end, we need to show that every $f_{xy} : M_{xy} \rightarrow M'_{xy}$ is H_{xy} -colinear. For all $m \in M_{xy}$, we have that

$$\begin{aligned} f_{xy}(m_{[0]}) \otimes m_{[1]} &= f_{xy}(m_{<0>} m_{<1>[0]}) \otimes m_{<1>[1]} \\ &\stackrel{*}{=} f_{xx}(m_{<0>}) m_{<1>[0]} \otimes m_{<1>[1]} \\ &\stackrel{(8)}{=} f_{xy}(m)_{<0>} f_{xy}(m)_{<1>[0]} \otimes f_{xy}(m)_{<1>[1]} = \rho'_{xy}(f_{xy}(m)). \end{aligned}$$

At $*$, we used the fact that f is right A -linear. We now define $P(f) = f$. \square

Theorem 3.5. *Let H be a k -linear semi-Hopf category, let A be a right H -comodule category, and let $B = A^{\text{co}H}$. Then the following assertions are equivalent.*

- (1) can_{xy}^z is bijective, for all $x, y, z \in X$;
- (2) can_{xy}^y is bijective, and can_{xy}^x has a left inverse g_{xy} , for all $x, y \in X$;
- (3) for all $x, y \in X$, there exists $\gamma_{xy} : H_{xy} \rightarrow A_{yx} \otimes_{B_x} A_{xy}$, notation

$$\gamma_{xy}(h) = \sum_i l_i(h) \otimes_{B_x} r_i(h),$$

such that

$$(17) \quad \sum_i l_i(h) r_i(h)_{[0]} \otimes r_i(h)_{[1]} = 1_y \otimes h;$$

$$(18) \quad \sum_i a_{[0]} l_i(a_{[1]}) \otimes_{B_x} r_i(a_{[1]}) = 1_x \otimes_{B_x} a,$$

for all $h \in H_{xy}$ and $a \in A_{xy}$.

If these equivalent conditions are satisfied, then we call A an H -Galois category extension of $B = A^{\text{co}H}$.

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (3). We define γ_{xy} by the formula

$$\gamma_{xy}(h) = (\text{can}_{xy}^y)^{-1}(1_y \otimes h).$$

Then

$$1_y \otimes h = \text{can}_{xy}^y \left(\sum_i l_i(h) \otimes_{B_x} r_i(h) \right) = \sum_i l_i(h) r_i(h)_{[0]} \otimes r_i(h)_{[1]},$$

so that (17) holds. Now we define $f_{xy}^z : A_{zy} \otimes H_{xy} \rightarrow A_{zx} \otimes_{B_x} A_{xy}$ by the formula

$$(19) \quad f_{xy}^z(a \otimes h) = \sum_i al_i(h) \otimes_{B_x} r_i(h).$$

Then

$$(\text{can}_{xy}^z \circ f_{xy}^z)(a \otimes h) = \sum_i al_i(h)r_i(h)_{[0]} \otimes r_i(h)_{[1]} \stackrel{(17)}{=} a1_y \otimes h = a \otimes h.$$

It follows that f_{xy}^z is a right inverse of can_{xy}^z . By assumption, g_{xy} is a left inverse of can_{xy}^x , and it follows easily that

$$g_{xy} = g_{xy} \circ \text{can}_{xy}^x \circ f_{xy}^x = f_{xy}^x.$$

For all $a \in A_{xy}$, we now have that

$$\begin{aligned} 1_x \otimes_{B_x} a &= (f_{xy}^x \circ \text{can}_{xy}^x)(1_x \otimes_{B_x} a) \\ &= f_{xy}^x(a_{[0]} \otimes a_{[1]}) = \sum_i a_{[0]}l_i(a_{[1]}) \otimes_{B_x} r_i(a_{[1]}), \end{aligned}$$

and this proves that (18) holds.

(3) \Rightarrow (1). We define f_{xy}^z using (19). We have shown above that f_{xy}^z is a right inverse of can_{xy}^z . It is also a left inverse since

$$\begin{aligned} (f_{xy}^z \circ \text{can}_{xy}^z)(a \otimes_{B_x} a') &= f_{xy}^z(aa'_{[0]} \otimes a'_{[1]}) \\ &= \sum_i aa'_{[0]}l_i(a'_{[1]}) \otimes_{B_x} r_i(a'_{[1]}) \stackrel{(18)}{=} a1_x \otimes_{B_x} a' = a \otimes_{B_x} a', \end{aligned}$$

for all $a \in A_{zx}$ and $a' \in A_{xy}$. \square

Example 3.6. Let H be a k -linear semi-Hopf category. H is a right H -comodule category, and $H^{\text{co}H} = J$, with $J_x = k$, for all $x \in X$. It follows from [3, Theorem 9.2] that H is an H -Galois category extension of J if and only if H is a Hopf category.

Proposition 3.7. *Assume that the equivalent conditions of Theorem 3.5 are satisfied. The maps γ_{xy} have the following properties, for all $h \in H_{xy}$ and $h' \in H_{yz}$:*

$$(20) \quad \gamma_{xz}(hh') = \sum_{i,j} l_i(h')l_i(h) \otimes_{B_x} r_j(h)r_i(h');$$

$$(21) \quad \gamma_{xy}(h) \in (A_{yx} \otimes_{B_x} A_{xy})^{B_y};$$

$$(22) \quad \gamma_{xy}(h_{(1)}) \otimes h_{(2)} = \sum_i l_i(h) \otimes_{B_x} r_i(h)_{[0]} \otimes r_i(h)_{[1]};$$

$$(23) \quad \sum_i l_i(h)r_i(h) = \varepsilon_{xy}(h)1_y;$$

$$(24) \quad \gamma_{xy}(h_{(2)}) \otimes S_{xy}(h_{(1)}) = \sum_i l_i(h)_{[0]} \otimes_{B_x} r_i(h) \otimes l_i(h)_{[1]}.$$

For (24), we need the additional assumption that H is a Hopf category.

Proof.

$$\begin{aligned}
& \text{can}_{xy}^z \left(\sum_{i,j} l_i(h') l_i(h) \otimes_{B_x} r_j(h) r_i(h') \right) \\
&= \sum_{i,j} l_i(h') l_i(h) r_j(h)_{[0]} r_i(h')_{[0]} \otimes r_j(h)_{[1]} r_i(h')_{[1]} \\
&\stackrel{(17)}{=} \sum_i l_i(h') 1_y r_i(h')_{[0]} \otimes h r_i(h')_{[1]} \stackrel{(17)}{=} 1_z \otimes h h',
\end{aligned}$$

proving (20). (21) follows if we can show that

$$\sum_i l_i(h) \otimes r_i(h) b = \sum_i b l_i(h) \otimes r_i(h),$$

for all $b \in B_y$. Indeed,

$$\begin{aligned}
& \text{can}_{xy}^y \left(\sum_i l_i(h) \otimes r_i(h) b \right) = \sum_i l_i(h) r_i(h)_{[0]} b_{[0]} \otimes_i (h)_{[1]} b_{[1]} \\
&\stackrel{(17)}{=} 1_y b \otimes h = b 1_y \otimes h \stackrel{(17)}{=} \text{can}_{xy}^y \left(b \sum_i l_i(h) \otimes r_i(h) \right).
\end{aligned}$$

(22) is proved in a similar way:

$$\begin{aligned}
& (\text{can}_{xy}^y \otimes H_{xy}) \left(\sum_i l_i(h) \otimes_{B_x} r_i(h)_{[0]} \otimes r_i(h)_{[1]} \right) \\
&= \sum_i l_i(h) r_i(h)_{[0]} \otimes r_i(h)_{[1]} \otimes r_i(h)_{[2]} \\
&= \sum_i l_i(h) r_i(h)_{[0]} \otimes \Delta_{xy}(r_i(h)_{[1]}) \\
&\stackrel{(17)}{=} 1_y \otimes \Delta_{xy}(h) = (\text{can}_{xy}^y \otimes H_{xy}) \left(\gamma_{xy}(h_{(1)}) \otimes h_{(2)} \right).
\end{aligned}$$

(23) follows after we apply $A_{yy} \otimes \varepsilon_{xy}$ to (17). (24) is equivalent to

$$(25) \quad 1 \otimes h_{(2)} \otimes S_{xy}(h_{(1)}) = \sum_i l_i(h)_{[0]} r_i(h)_{[0]} \otimes r_i(h)_{[1]} \otimes l_i(h)_{[1]}.$$

Indeed, applying $\text{can}_{xy}^y \otimes H_{yx}$ to (24), we obtain (25). Applying $\rho_{yy} \otimes ((S_{xy} \otimes H_{xy}) \circ \Delta_{xy})$ to (17), we obtain that

$$\begin{aligned}
& \sum_i l_i(h)_{[0]} r_i(h)_{[0]} \otimes l_i(h)_{[1]} r_i(h)_{[1]} \otimes S_{xy}(r_i(h)_{[2]}) \otimes r_i(h)_{[3]} \\
&= 1_y^A \otimes 1_y^H \otimes S_{xy}(h_{(1)}) \otimes h_{(2)}.
\end{aligned}$$

Now we multiply the second and third tensor factor, and obtain that

$$\sum_i l_i(h)_{[0]} r_i(h)_{[0]} \otimes l_i(h)_{[1]} \otimes r_i(h)_{[1]} = 1_y^A \otimes S_{xy}(h_{(1)}) \otimes h_{(2)}.$$

(25) follows after we switch the second and the third tensor factor. \square

Lemma 3.8 is folklore; we will need it in the proof of Theorem 3.9, and this is why we provide a detailed proof.

Lemma 3.8. *Let A be a k -algebra, and take $M \in \mathcal{M}_A$, $N \in {}_A\mathcal{M}_A$, $P \in {}_A\mathcal{M}$. For $m_i \in M$, $n_i \in N^A$, $p_i \in P$, we have the following implication:*

$$\sum_i n_i \otimes m_i \otimes p_i = 0 \text{ in } N^A \otimes M \otimes_A P \implies \sum_i m_i \otimes_A n_i \otimes p_i = 0 \text{ in } M \otimes_A N \otimes_A P.$$

Proof. From the assumption that $\sum_i n_i \otimes m_i \otimes p_i = 0$, it follows that

$$\sum_i n_i \otimes m_i \otimes p_i = \sum_j x_j \otimes y_j a_j \otimes z_j - x_j \otimes y_j \otimes a_j z_j,$$

for some $x_j \in N^A$, $y_j \in M$, $a_j \in A$ and $z_j \in P$. This implies that

$$\begin{aligned} \sum_i m_i \otimes n_i \otimes p_i &= \sum_j y_j a_j \otimes x_j \otimes z_j - y_j \otimes a_j x_j \otimes z_j \\ &\quad + y_j \otimes x_j a_j \otimes z_j - y_j \otimes x_j \otimes a_j z_j, \end{aligned}$$

where we used the fact that $x_j \in N^A$. This implies that $\sum_i m_i \otimes_A n_i \otimes p_i = 0$ in $M \otimes_A N \otimes_A P$. \square

Theorem 3.9. *Let H be a Hopf category, and let A be an H -Galois category extension of $B = A^{\text{co}H}$. Then the functor $P : \underline{\text{Desc}}_B(A) \rightarrow \mathcal{M}_k(X)_A^H$ from Proposition 3.4 is an isomorphism of categories.*

Proof. We define a functor $Q : \mathcal{M}_k(X)_A^H \rightarrow \underline{\text{Desc}}_B(A)$. For a relative Hopf module (M, ρ) , consider

$$\sigma_{xy} = (\psi_{x,y,x} \otimes_{B_x} A_{xy}) \circ (M_{xy} \otimes \gamma_{xy}) \circ \rho_{xy} : M_{xy} \rightarrow M_{xx} \otimes_{B_x} A_{xy},$$

that is,

$$\sigma_{xy}(m) = m_{\langle 0 \rangle} \otimes_{B_x} m_{\langle 1 \rangle} = m_{[0]} \gamma_{xy}(m_{[1]}) = \sum_i m_{[0]} l_i(m_{[1]}) \otimes_{B_x} r_i(m_{[1]}).$$

We claim that $(M, \sigma) \in \underline{\text{Desc}}_B(A)$. We will first show that (5) holds, that is

$$\sigma_{xz}(ma) = m_{\langle 0 \rangle} \otimes_{B_x} m_{\langle 1 \rangle} a,$$

for all $m \in M_{xy}$ and $a \in A_{yz}$. It follows from (18) that

$$\sum_i a_{[0]} l_i(a_{[1]}) \otimes_{B_y} r_i(a_{[1]}) = 1_y \otimes_{B_y} a.$$

From (21), we know that $\gamma_{xy}(h) \in (A_{yx} \otimes_{B_x} A_{xy})^{B_y}$, for all $h \in H_{xy}$. Therefore we have that

$$\gamma_{xy}(h) \otimes \sum_i a_{[0]} l_i(a_{[1]}) \otimes_{B_y} r_i(a_{[1]}) = \gamma_{xy}(h) \otimes 1_y \otimes_{B_y} a.$$

in $(A_{yx} \otimes_{B_x} A_{xy})^{B_y} \otimes A_{xy} \otimes_{B_y} A_{xy}$. From Lemma 3.8, it follows that

$$\sum_i a_{[0]} l_i(a_{[1]}) \otimes_{B_y} \gamma_{xy}(h) \otimes_{B_y} r_i(a_{[1]}) = 1_y \otimes_{B_y} \gamma_{xy}(h) \otimes_{B_y} a$$

in $A_{xy} \otimes_{B_y} A_{yx} \otimes_{B_x} A_{xy} \otimes_{B_y} A_{yx}$. Multiplying the two first and the two last tensor factors, we find that

$$(26) \quad \sum_{i,j} a_{[0]} l_i(a_{[1]}) l_j(h) \otimes_{B_x} r_j(h) r_i(a_{[1]}) = \sum_j l_j(h) \otimes_{B_x} r_j(h) a.$$

Finally

$$\begin{aligned} \sigma_{xz}(ma) &= \sum_i m_{[0]} a_{[0]} l_i(m_{[1]} a_{[1]}) \otimes_{B_x} r_i(m_{[1]} a_{[1]}) \\ &\stackrel{(20)}{=} \sum_{i,j} m_{[0]} a_{[0]} l_i(a_{[1]}) l_j(m_{[1]}) \otimes_{B_x} r_j(m_{[1]}) r_i(a_{[1]}) \\ &\stackrel{(26)}{=} \sum_j m_{[0]} 1_y l_j(m_{[1]}) \otimes_{B_x} r_j(m_{[1]}) a = \sigma_{xy}(m) a, \end{aligned}$$

and (5) follows. Our next aim is to show that (6) holds. Take $m \in M_{xy}$. It follows from (24) that

$$\begin{aligned} m_{[0]} \otimes m_{[1]} \otimes \gamma_{xy}(m_{[3]}) \otimes S_{xy}(m_{[2]}) \\ = m_{[0]} \otimes m_{[1]} \otimes l_i(m_{[2]})_{[0]} \otimes_{B_x} r_i(m_{[2]}) \otimes l_i(m_{[2]})_{[1]}, \end{aligned}$$

and

$$\begin{aligned} m_{[0]} \otimes m_{[1]} S_{xy}(m_{[2]}) \otimes \gamma_{xy}(m_{[3]}) &= \sum_j m_{[0]} \otimes 1_x^H \otimes l_j(m_{[1]}) \otimes_{B_x} r_j(m_{[1]}) \\ &= m_{[0]} \otimes m_{[1]} l_i(m_{[2]})_{[1]} \otimes l_i(m_{[2]})_{[0]} \otimes_{B_x} r_i(m_{[2]}). \end{aligned}$$

Now we apply $\gamma_{xx} : H_{xx} \rightarrow (A_{xx} \otimes_{B_x} A_{xx})^{B_x}$ (see (21)) to the second tensor factor. Observing that $\gamma_{xx}(1_x^H) = 1_x^A \otimes_{B_x} 1_x^A$, this gives us the following equality in $A_{xy} \otimes (A_{xx} \otimes_{B_x} A_{xx})^{B_x} \otimes_{B_y} A_{yx} \otimes_{B_x} A_{xy}$:

$$\begin{aligned} \sum_j m_{[0]} \otimes 1_x^A \otimes_{B_x} 1_x^A \otimes l_j(m_{[1]}) \otimes_{B_x} r_j(m_{[1]}) \\ = \sum_i m_{[0]} \otimes \gamma_{xx}(m_{[1]} l_i(m_{[2]})_{[1]}) \otimes l_i(m_{[2]})_{[0]} \otimes_{B_x} r_i(m_{[2]}). \end{aligned}$$

Now we apply Lemma 3.8, and obtain the equality

$$\begin{aligned} \sum_j m_{[0]} \otimes l_j(m_{[1]}) \otimes_{B_x} 1_x^A \otimes_{B_x} 1_x^A \otimes_{B_x} r_j(m_{[1]}) \\ = \sum_i m_{[0]} \otimes l_i(m_{[2]})_{[0]} \otimes_{B_x} \gamma_{xx}(m_{[1]} l_i(m_{[2]})_{[1]}) \otimes_{B_x} r_i(m_{[2]}) \end{aligned}$$

in $A_{xy} \otimes A_{yx} \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xy}$. Multiplying the first three tensor factors, we obtain that

$$\begin{aligned} \sum_j m_{[0]} l_j(m_{[1]}) \otimes_{B_x} 1_x^A \otimes_{B_x} r_j(m_{[1]}) \\ = \sum_i m_{[0]} l_i(m_{[2]})_{[0]} \gamma_{xx}(m_{[1]} l_i(m_{[2]})_{[1]}) \otimes_{B_x} r_i(m_{[2]}), \end{aligned}$$

which is precisely (6). (7) follows easily:

$$m_{\langle 0 \rangle} m_{\langle 1 \rangle} = \sum_i m_{[0]} l_i(m_{[1]}) r_i(m_{[1]}) \stackrel{(23)}{=} m_{[0]} \varepsilon_{xy}(m_{[1]}) 1_y = m.$$

We now define $Q(M, \rho) = (M, \sigma)$. If $f : (M, \rho) \rightarrow (M', \rho')$ is a morphism in $\mathcal{M}_k(X)_A^H$, then it is also a morphism $(M, \sigma) \rightarrow (M', \sigma')$ in $\underline{\text{Desc}}_B(A)$. Indeed, for all $m \in M_{xy}$, we have that

$$\begin{aligned} f_{xx}(m_{\langle 0 \rangle}) \otimes_{B_x} m_{\langle 1 \rangle} &= \sum_i f_{xx}(m_{[0]} l_i(m_{[1]})) \otimes_{B_x} r_i(m_{[1]}) \\ &= \sum_i f_{xx}(m_{[0]}) l_i(m_{[1]}) \otimes_{B_x} r_i(m_{[1]}) \\ &= \sum_i (f_{xx}(m))_{[0]} l_i((f_{xx}(m))_{[1]}) \otimes_{B_x} r_i((f_{xx}(m))_{[0]}) \\ &= \sigma'_{xy}(f_{xy}(m)), \end{aligned}$$

so that (8) holds. We now define $Q(f) = f$.

Take $(M, \sigma) \in \underline{\text{Desc}}_B(A)$, and let $(Q \circ P)(M, \sigma) = (M, \sigma')$. Then for all $m \in M_{xy}$, we have that

$$\begin{aligned} \rho_{xy}(m) &= m_{\langle 0 \rangle} m_{\langle 1 \rangle [0]} \otimes m_{\langle 1 \rangle [1]}; \\ \sigma'_{xy}(m) &= \sum_i m_{\langle 0 \rangle} m_{\langle 1 \rangle [0]} l_i(m_{\langle 1 \rangle [1]}) \otimes_{B_x} r_i(m_{\langle 1 \rangle [1]}) \\ &\stackrel{(18)}{=} \sum_i m_{\langle 0 \rangle} 1_x \otimes_{B_x} m_{\langle 1 \rangle} = \sigma_{xy}(m) \end{aligned}$$

Finally take $(M, \rho) \in \mathcal{M}_k(X)_A^H$, and let $(P \circ A)(M, \rho) = (M, \rho')$. Then for all $m \in M_{xy}$

$$\begin{aligned} \sigma_{xy}(m) &= \sum_i m_{[0]} l_i(m_{[1]}) \otimes_{B_x} r_i(m_{[1]}); \\ \rho'_{xy}(m) &= \sum_i m_{[0]} l_i(m_{[1]}) r_i(m_{[1]})_{[0]} \otimes r_i(m_{[1]})_{[1]} \\ &\stackrel{(17)}{=} m_{[0]} 1_y \otimes m_{[1]} = \rho_{xy}(m). \end{aligned}$$

This shows that Q is the inverse of P . \square

3.2. Left relative Hopf modules. As in Section 3.1, let H be a k -linear semi-Hopf category, and let A be a right H -comodule category. We introduce left (A, H) -relative Hopf modules. The results of Section 3.1 have their counterparts for left (A, H) -relative Hopf modules. The proofs are similar, so we restrict to a brief survey of the results. A left (A, H) -relative Hopf module is an object $M \in {}_A \mathcal{M}_k(X)$ such that every M_{xy} is a right H_{xy} -comodule satisfying the compatibility relations

$$\rho_{xz}(am) = a_{[0]} m_{[0]} \otimes a_{[1]} m_{[1]},$$

for all $a \in A_{xy}$ and $m \in M_{yz}$. The category of left (A, H) -relative Hopf modules is denoted as ${}_A\mathcal{M}_k(X)^H$. As in Section 3.1, we assume that B is an algebra in $\mathcal{D}_k(X)$, and that $i : B \rightarrow A^{\text{co}H}$ is an algebra morphism.

Proposition 3.10. *We have a pair of adjoint functors (F', G') between ${}_B\mathcal{D}_k(X)$ and ${}_A\mathcal{M}_k(X)^H$.*

Proof. For $N \in {}_B\mathcal{D}_k(X)$, $F(N)_{xy} = A_{xy} \otimes N_y$, with action and coaction given by the formulas

$$a'(a \otimes n) = a'a \otimes n \text{ and } \rho_{xy}(a \otimes n) = a_{[0]} \otimes n \otimes a_{[1]},$$

for all $a' \in A_{ux}$, $a \in A_{xy}$ and $n \in N_y$. For a relative Hopf module M , $G'(M) = M^{\text{co}H}$. The unit $\eta^N : N \rightarrow G'F'(N)$ and the counit $\varepsilon^M : F'G'(M) \rightarrow M$ are given by the formulas

$$\begin{aligned} \eta_x^N : N_x &\rightarrow (A_{xx} \otimes_{B_x} N_x)^{\text{co}H_{xx}} & ; & \quad \eta_x^N(n) = n \otimes_{B_x} 1_x; \\ \varepsilon_{xy}^M : A_{xy} \otimes_{B_y} M_{xx}^{\text{co}H_{xx}} & & ; & \quad \varepsilon_{xy}^M(a \otimes_{B_y} m) = am. \end{aligned}$$

□

For all $y \in X$, we consider the relative Hopf module M^y , $M_{zx}^y = A_{zy} \otimes H_{zx}$, with action and coaction given by the formulas

$$a'(a \otimes h) = a'_0 a \otimes a'_1 h \text{ and } \rho_{zx}(a \otimes h) = a \otimes h_{(1)} \otimes h_{(2)},$$

for $a' \in A_{uz}$, $a \in A_{zy}$ and $h \in H_{zx}$. We have an isomorphism

$$f : A_{xy} \rightarrow (M_{xx}^y)^{\text{co}H_{xx}}, \quad f(a) = a \otimes 1_x,$$

with inverse given by the formula $f^{-1}(\sum_i a_i \otimes h_i) = \sum_i a_i \varepsilon_{xx}(h_i)$. Now observe that the composition

$$\text{can}_{zx}^{\prime y} = \varepsilon_{zx}^{M^y} \circ (A_{zx} \otimes_{B_x} f) : A_{zx} \otimes_{B_x} A_{xy} \rightarrow M_{zx}^y = A_{zy} \otimes H_{zx},$$

is given by the formula

$$(27) \quad \text{can}_{zx}^{\prime y}(a \otimes_{B_x} a') = a_{[0]} a' \otimes a_{[1]}.$$

Proposition 3.11. *If F' is fully faithful, then $i : B \rightarrow A^{\text{co}H}$ is an isomorphism; If F' is fully faithful, then $\text{can}_{zx}^{\prime y}$ is an isomorphism, for all $x, y, z \in X$.*

Theorem 3.12. *Let H be a k -linear semi-Hopf category, let A be a right H -comodule category, and let $B = A^{\text{co}H}$. Then the following assertions are equivalent.*

- (1) $\text{can}_{zx}^{\prime y}$ is bijective, for all $x, y, z \in X$;
- (2) $\text{can}_{zx}^{\prime z}$ is bijective and $\text{can}_{zx}^{\prime x}$ has a left inverse g'_{zx} , for all $x, z \in X$;
- (3) for all $x, z \in X$, there exists $\gamma'_{zx} : H_{zx} \rightarrow A_{zx} \otimes_{B_x} A_{xz}$, notation

$$\gamma'_{zx}(h) = \sum_i l'_i(h) \otimes_{B_x} r'_i(h),$$

such that

$$(28) \quad \sum_i l'_i(h)_{[0]} r'_i(h) \otimes l'_i(h)_{[1]} = 1_z \otimes h;$$

$$(29) \quad \sum_i l'_i(a_{[1]}) \otimes_{B_x} r'_i(a_{[1]}) a_{[0]} = a \otimes_{B_x} 1_x,$$

for all $h \in H_{zx}$ and $a \in A_{zx}$.

If these equivalent conditions are satisfied, then we call A an H -Galois' category extension of $B = A^{\text{co}H}$.

Proof. (2) \Rightarrow (3). For $h \in H_{zx}$, we define

$$\gamma'_{zx}(h) = (\text{can}'_{zx})^{-1}(1_z \otimes h).$$

(2) \Rightarrow (3). For $a \in A_{zy}$ and $h \in H_{zx}$, we define

$$(\text{can}'_{zx})^{-1}(a \otimes h) = \sum_i l'_i(h) \otimes_{B_x} r'_i(h) a.$$

□

Theorem 3.13. *Let H be a k -linear semi-Hopf category, let A be a right H -comodule category, and let $B = A^{\text{co}H}$. We have a functor $P' : {}_B\text{Desc}(A) \rightarrow {}_A\mathcal{M}_k(X)^H$. If A an H -Galois' category extension of B , then P' is an isomorphism of categories.*

Proof. For a descent datum (M, τ) , we define $P'(M, \tau) = (M, \rho)$, with

$$\rho_{xy}(m) = m_{\langle -1 \rangle [0]} m_{\langle 0 \rangle} \otimes m_{\langle -1 \rangle [1]},$$

for $m \in M_{xy}$. Assume that A an H -Galois' category extension of B . For $(M, \rho) \in {}_A\mathcal{M}_k(X)_A^H$, define $Q'(M, \rho) = (M, \tau)$, with

$$\tau_{zx} : M_{zx} \rightarrow A_{zx} \otimes_{B_x} M_{xx}, \quad \tau_{zx}(m) = \sum_i l'_i(m_{[1]}) \otimes_{B_x} r'_i(m_{[1]}) m_{[0]}.$$

Then Q' is the inverse of P' . □

Theorem 3.14. *Let H be a k -linear Hopf category with bijective antipode, let A be a right H -comodule category, and let $B = A^{\text{co}H}$. Then A is an H -Galois' category extension of B if and only if A an H -Galois category extension of B .*

Proof. The map

$$\phi : A_{zy} \otimes H_{xy} \rightarrow A_{zy} \otimes H_{zx}, \quad \phi(a \otimes h) = a_{[0]} \otimes a_{[1]} S_{xy}(h),$$

is invertible, with inverse given by the formula

$$\phi^{-1}(a \otimes h') = a_{[0]} \otimes S_{xz}^{-1}(h').$$

An easy computation shows that

$$\text{can}'_{xz} = \phi \circ \text{can}_{xy}^z.$$

Consequently can' is invertible if and only if can is invertible. □

4. k -LINEAR CLUSTERS

Definition 4.1. A k -linear cluster \mathcal{A} with underlying class X consists of a class of k -linear categories with underlying class X , indexed by X , that is, for every $x \in X$, we have a k -linear category \mathcal{A}^x .

Let \mathcal{A} be a cluster. A right \mathcal{A} -module is an object $M \in \mathcal{M}_k(X)$, together with morphisms

$$\psi_{xyz} : M_{xy} \otimes \mathcal{A}_{yz}^x \rightarrow M_{xz}, \quad \psi_{xyz}(m \otimes a) = ma,$$

satisfying the appropriate associativity and unit conditions:

$$m(ab) = (ma)b \text{ and } m1_y^x = m,$$

for all $m \in M_{xy}$, $a \in \mathcal{A}_{yz}^x$ and $b \in \mathcal{A}_{zu}^x$. 1_y^x is the unit element of \mathcal{A}_{yy}^x . A morphism $\varphi : M \rightarrow N$ between two right \mathcal{A} -modules M and N is a morphism $\varphi : M \rightarrow N$ in $\mathcal{M}_k(X)$ that is right \mathcal{A} -linear, which means that

$$\varphi_{xz}(ma) = \varphi_{xy}(m)a,$$

for all $m \in M_{xy}$ and $a \in \mathcal{A}_{yz}^x$. The category of right \mathcal{A} -modules will be denoted by $\mathcal{M}_{\mathcal{A}}$.

Example 4.2. Let B be a diagonal k -linear category, and let $M \in \mathcal{M}_k(X)$ be a left B -module, meaning that we have maps $B_x \otimes M_{xy} \rightarrow M_{xy}$ satisfying the appropriate associativity and unit conditions. We have a cluster $\mathcal{A} = {}_B\text{End}(M)$ defined as follows:

$$\mathcal{A}_{yz}^x = {}_B\text{End}(M)_{yz}^x = {}_{B_x}\text{Hom}(M_{xy}, M_{xz})$$

The multiplication maps are given by opposite composition: for $f \in \mathcal{A}_{yz}^x$ and $g \in \mathcal{A}_{zu}^x$, we put $fg = g \circ f$. The unit element of \mathcal{A}_{yy}^x is the identity M_{xy} . ${}_B\text{End}(M)$ is called the left endocluster of M . Note that M is a right ${}_B\text{End}(M)$ -module, via the structure maps

$$M_{xy} \otimes \mathcal{A}_{yz}^x \rightarrow M_{xz}, \quad m \otimes f = f(m).$$

The right endocluster $\mathcal{A}' = \text{End}_B(M)$ of a right B -module M can be defined in a similar way:

$$\mathcal{A}'_{yz}{}^x = \text{End}_B(M)_{yz}{}^x = \text{Hom}_{B_x}(M_{zx}, M_{yx}).$$

Now the multiplication is given by composition.

More examples will be presented in Sections 6 and 7.

5. FAITHFULLY PROJECTIVE DESCENT

Proposition 5.1. *We consider the setting of Section 2: A and B are k -linear categories, with underlying class X , and B is a diagonal algebra. $i : B \rightarrow A$ is a k -linear X -functor. Then A is a left B -module via restriction of scalars, and we can consider $\mathcal{A} = {}_B\text{End}(A)$ as in Example 4.2. We have a functor $H : \underline{\text{Desc}}_B(A) \rightarrow \mathcal{M}_{\mathcal{A}}$.*

Proof. Let $(M, \sigma) \in \underline{\text{Desc}}_B(A)$. We define a right \mathcal{A} -action on M as follows: for $m \in M_{xy}$ and $f \in \mathcal{A}_{yz}^x$, let $mf = m_{\langle 0 \rangle} f(m_{\langle 1 \rangle}) \in M_{xz}$. Let us show that the associativity and unit condition are satisfied. Take $g \in \mathcal{A}_{zu}^x$.

$$\begin{aligned} (mf)g &= (m_{\langle 0 \rangle} f(m_{\langle 1 \rangle}))_{\langle 0 \rangle} g((m_{\langle 0 \rangle} f(m_{\langle 1 \rangle}))_{\langle 1 \rangle}) \\ &\stackrel{(5)}{=} m_{\langle 0 \rangle} g_{\langle 0 \rangle} (m_{\langle 0 \rangle} f(m_{\langle 1 \rangle}))_{\langle 1 \rangle} \\ &\stackrel{(6)}{=} m_{\langle 0 \rangle} (g \circ f)(m_{\langle 1 \rangle}) = m(fg); \\ mA_{xy} &= m_{\langle 0 \rangle} m_{\langle 1 \rangle} \stackrel{(7)}{=} m. \end{aligned}$$

Now we define $H(M, \sigma) = M$. H acts as the identity on morphisms: if $\varphi : (M, \sigma) \rightarrow (M', \sigma')$ is a morphism in $\underline{\text{Desc}}_B(A)$, then φ is right \mathcal{A} -linear. Indeed, for $m \in M_{xy}$ and $f \in \mathcal{A}_{yz}^x$, we have that

$$\begin{aligned} \varphi_{xy}(m)f &= \varphi_{xy}(m)_{\langle 0 \rangle} f(\varphi_{xy}(m)_{\langle 1 \rangle}) \\ &\stackrel{(8)}{=} \varphi_{xx}(m_{\langle 0 \rangle}) f(m_{\langle 1 \rangle}) \\ &= \varphi_{xx}(m_{\langle 0 \rangle}) f(m_{\langle 1 \rangle}) = \varphi_{xz}(mf). \end{aligned}$$

□

Remark 5.2. To any $a \in A_{yz}$, we can associate $r_a \in \mathcal{A}_{yz}^x = {}_{B_x} \text{Hom}(A_{xy}, A_{xz})$, given by right multiplication by a : $r_a(a') = a'a$, for all $a' \in A_{xy}$. For $a' \in A_{xy}$ we have that

$$a'r_a = a'_{\langle 0 \rangle} r_a(a'_{\langle 1 \rangle}) = a'_{\langle 0 \rangle} a'_{\langle 1 \rangle} a = a'a.$$

Theorem 5.3. *Let A and B be as in Proposition 5.1, and assume that A is locally finite as a left B -module. Then the categories $\underline{\text{Desc}}_B(A)$ and \mathcal{M}_A are isomorphic.*

Proof. We will show that the functor H from Proposition 5.1 has an inverse K . Take a right \mathcal{A} -module M . Then M is also a right A -module: for $m \in M_{xy}$ and $a \in A_{yz}$, we just put $ma = mr_a$, see Remark 5.2.

Let $\sum_i e_i^* \otimes_{B_x} e_i$ be a finite dual basis of the left B_x -module A_{xy} . For $a^* \in A_{xy}^*$, we have that $i_x \circ a^* \in \mathcal{A}_{yx}^x$. Now we define $\sigma_{xy} : M_{xy} \rightarrow M_{xx} \otimes_{B_x} A_{xy}$ as follows:

$$\sigma_{xy}(m) = \sum_i m(i_x \circ e_i^*) \otimes_{B_x} e_i.$$

We claim that $(M, \sigma) \in \underline{\text{Desc}}_B(A)$. We need to show that (5-7) hold. Take $m \in M_{xy}$ and $a \in A_{yz}$, and let $\sum_j f_j^* \otimes_{B_x} f_j$ be a finite dual basis for A_{xz} . (5) follows if we can show that

$$\sigma_{xz}(ma) = \sum_j (mr_a)(i_x \circ f_j^*) \otimes_{B_x} f_j$$

equals

$$\sigma_{xz}(m)a = \sum_i m(i_x \circ e_i^*) \otimes_{B_x} e_i a$$

in $M_{xx} \otimes_{B_x} A_{xz} \cong \text{Hom}_{B_x}(A_{xz}^*, M_{xx})$, see (2). To this end, it suffices to show that both terms are equal after we evaluate them at an arbitrary $a^* \in A_{xz}^*$, that is,

$$\sum_j ((mr_a)(i_x \circ f_j^*)) a^*(f_j) = m \left(\sum_j r_{(i_x \circ a^*)(f_j)} \circ (i_x \circ f_j^*) \circ r_a \right)$$

equals

$$\sum_i (m(i_x \circ e_i^*)) a^*(e_i a) = m \left(\sum_i r_{a^*(e_i a)} \circ i_x \circ e_i^* \right).$$

It suffices to show that

$$\sum_j r_{(i_x \circ a^*)(f_j)} \circ (i_x \circ f_j^*) \circ r_a = \sum_i r_{a^*(e_i a)} \circ i_x \circ e_i^*,$$

which can be easily done as follows: for all $a' \in A_{xy}$, we easily find that

$$\left(\sum_j r_{(i_x \circ a^*)(f_j)} \circ (i_x \circ f_j^*) \circ r_a \right) (a') = a^*(a'a) = \left(\sum_i r_{a^*(e_i a)} \circ i_x \circ e_i^* \right) (a').$$

(6) amounts to the equality of

$$\begin{aligned} \sigma_{xx}(m_{\langle 0 \rangle}) \otimes_{B_x} m_{\langle 1 \rangle} &= \sum_i \sigma_{xx}(m(i_x \circ e_i^*)) \otimes_{B_x} e_i \\ &= \sum_{i,j} (m(i_x \circ e_i^*)) (i_x \circ f_j^*) \otimes_{B_x} f_j \otimes_{B_x} e_i \end{aligned}$$

and

$$m_{\langle 0 \rangle} \otimes_{B_x} 1_x \otimes_{B_x} m_{\langle 1 \rangle} = \sum_i m(i_x \circ e_i^*) \otimes_{B_x} 1_x \otimes_{B_x} e_i$$

in $M_{xx} \otimes_{B_x} A_{xx} \otimes_{B_x} A_{xy} \stackrel{(2)}{\cong} \text{Hom}_{B_x}(A_{xx}^*, M_{xx}) \otimes_{B_x} A_{xy}$. To this end, it suffices to show that

$$\sum_j (m(i_x \circ e_i^*)) (i_x \circ f_j^*) \otimes_{B_x} f_j = m(i_x \circ e_i^*) \otimes_{B_x} 1_x$$

in $M_{xx} \otimes_{B_x} A_{xx} \cong \text{Hom}_{B_x}(A_{xx}^*, M_{xx})$, for all i . This is equivalent to proving that

$$\sum_j ((m(i_x \circ e_i^*)) (i_x \circ f_j^*)) a^*(f_j) = (m(i_x \circ e_i^*)) a^*(1_x),$$

for all $a^* \in A_{xx}^*$, or

$$m \left(\sum_i r_{(i_x \circ a^*)(f_j)} \circ i_x \circ f_j^* \circ i_x \circ e_i^* \right) = m \left(r_{(i_x \circ a^*)(1_x)} \circ i_x \circ e_i^* \right),$$

so it suffices to show that

$$\sum_j r_{(i_x \circ a^*)(f_j)} \circ i_x \circ f_j^* \circ i_x \circ e_i^* = r_{(i_x \circ a^*)(1_x)} \circ i_x \circ e_i^*.$$

Keeping in mind that $i_x(b) = b1_{xx}$ for all $b \in B_x$, we compute that

$$\sum_j (r_{(i_x \circ a^*)(f_j)} \circ i_x \circ f_j^* \circ i_x \circ e_i^*)(a) = \sum_j f_j^*(e_i^*(a)1_x) a^*(f_j) 1_x$$

$$= a^*(e_i^*(a)1_x)1_x = e_i^*(a)a^*(1_x)1_x = (r_{(i_x \circ a^*)(1_x)} \circ i_x \circ e_i^*)(a),$$

for all $a \in A_{xy}$. Let us finally show that (7) holds: for all $m \in M_{xy}$, we have that

$$m_{\langle 0 \rangle} m_{\langle 1 \rangle} = \sum_i (m(i_x \circ e_i^*))e_i = m\left(\sum_i r_{e_i} \circ i_x \circ e_i^*\right) = mA_{xy} = m,$$

where we used the fact that $\sum_i r_{e_i} \circ i_x \circ e_i^* = A_{xy}$. Indeed, for all $a \in A_{xy}$, we have that

$$\sum_i (r_{e_i} \circ i_x \circ e_i^*)(a) = \sum_i e_i^*(a)e_i = a.$$

We define $K(M) = (M, \sigma)$ at the level of objects. We leave it to the reader to show that if $\varphi : M \rightarrow M'$ is right \mathcal{A} -linear, then $\varphi : (M, \sigma) \rightarrow (M', \sigma')$ is a morphism of descent data, so that we can define K as the identity at the level of morphisms.

It remains to be shown that $H \circ K$ and $K \circ H$ are the identity functors. Take a descent datum (M, σ) and let $KH(M, \sigma) = (M, \tilde{\sigma})$. We will show that $\tilde{\sigma} = \sigma$. For all $m \in M_{xy}$, we have

$$\begin{aligned} \tilde{\sigma}_{xy}(m) &= \sum_i m((i_x \circ e_i^*) \otimes_{B_x} e_i) = \sum_i m_{\langle 0 \rangle} e_i^*(m_{\langle 1 \rangle}) \otimes_{B_x} e_i \\ &= \sum_i m_{\langle 0 \rangle} \otimes_{B_x} e_i^*(m_{\langle 1 \rangle}) e_i = m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle} = \sigma_{xy}(m). \end{aligned}$$

Finally take a right \mathcal{A} -module M . Then $HK(M) = M$ with a new \mathcal{A} -action \leftarrow . We will show that it coincides with the original one. For all $m \in M_{xy}$ and $f \in \mathcal{A}_{yz}^x = {}_{B_x}\text{Hom}(A_{xy}, A_{xz})$, we have that

$$m \leftarrow f = \sum_i (m(i_x \circ e_i^*))f(e_i) = m\left(\sum_i r_{f(e_i)} \circ i_x \circ e_i^*\right) = mf.$$

Here we used the fact that $\sum_i r_{f(e_i)} \circ i_x \circ e_i^* = f$, which can be seen as follows. For all $a \in A_{xy}$, we have

$$\sum_i (r_{f(e_i)} \circ i_x \circ e_i^*)(a) = \sum_i e_i^*(a)f(e_i) = f(a).$$

□

Theorem 5.4. *Let A and B be as in Proposition 5.1. Then we have a pair of adjoint functors (F_1, G_1) between the categories $\mathcal{D}_k(X)_B$ and \mathcal{M}_A . If A is locally finite as a left B -module, then the counit of this adjunction is an isomorphism. If A is locally faithfully projective as a left B -module, then the unit of the adjunction is also an isomorphism, and (F_1, G_1) is a pair of inverse equivalences.*

Proof. In the case where A is locally finite as a left B -module, the result is an immediate consequence of Propositions 2.3 and 2.4 and Theorem 5.3, taking into account the remarks made in Section 1.2. Let us describe the

functors $F_1 = H \circ F$ and $G_1 = G \circ K$. First take $N \in \mathcal{D}_k(X)$. It is easy to show that $F_1(N)_{xy} = N_x \otimes_{B_x} A_{xy}$, with right \mathcal{A} -action

$$(n \otimes_{B_x} a)f = n \otimes_{B_x} f(a),$$

for $n \in N_x$, $a \in A_{xy}$ and $f \in \mathcal{A}_{yz}^x = {}_{B_x}\text{Hom}(A_{xy}, A_{xz})$.

Now for $M \in \mathcal{M}_{\mathcal{A}}$, $K(M) = (M, \sigma)$ is defined in the proof of Theorem 5.3. Now

$$G_1(M)_x = G(M, \sigma)_x = \{m \in M_{xx} \mid \sigma_{xx}(m) = m \otimes_{B_x} 1_x\}.$$

Now $\sigma_{xx}(m) = \sum_j m(i_x \circ f_j^*) \otimes_{B_x} f_j$, where $\sum_j f_j^* \otimes_{B_x} f_j$ is a dual basis of A_{xx} as a left B_x -module. Now $\sigma_{xx}(m)$ and $m \otimes_{B_x} 1_x$ live in $M_{xx} \otimes_{B_x} A_{xx} \cong \text{Hom}_{B_x}(A_{xx}^*, M_{xx})$, see Section 1.2. For all $a^* \in A_{xx}^*$, we have that

$$\sum_j (m(i_x \circ f_j^*))a^*(f_j) = \sum_j m(r_{i_x \circ a^*}(f_j) \circ i_x \circ f_j^*) \stackrel{(*)}{=} m(i_x \circ a^*).$$

At (*), we used the following: for all $a \in A_{xx}$, we have that

$$r_{i_x \circ a^*}(f_j) \circ i_x \circ f_j^* = \sum_j f_j^*(a)1_x a^*(f_j) = a^*(a)1_x.$$

We conclude that $\sigma_{xx}(m) = m \otimes_{B_x} 1_x$ if and only if $ma^* = ma^*(1_x)$ for all $a^* \in A_{xx}^*$, and

$$G_1(M)_x = \{m \in M_{xx} \mid ma^* = ma^*(1_x) \text{ for all } a^* \in A_{xx}^*\}.$$

Now we drop the assumption that A is locally finite. We can still define the functors F_1 and G_1 : the explicit formulas presented above do not involve the dual basis, and it can easily be established that the same is true for the unit and the counit of the adjunction. \square

Now consider the right endomorphism cluster $\mathcal{A} = \text{End}_B(A)$. We have a functor $H_1 : {}_B\text{Desc}(A) \rightarrow {}_{\mathcal{A}}\mathcal{M}$, which is an isomorphism of categories if A is locally finite as a right B -module. We have an adjunction (F'_1, G'_1) between ${}_B\mathcal{D}_k(X)$ and ${}_{\mathcal{A}}\mathcal{M}$ which is a pair of inverse equivalences if A is locally faithfully projective as a right B -module.

6. DUAL K -GALOIS CATEGORY EXTENSIONS

We begin this Section with a new class of examples of clusters. Let K be a dual k -linear category. A right K -module category is a k -linear category A such that every A_{xy} is a right K_{xy} -module, and the following condition holds, for all $a \in A_{xy}$, $a' \in A_{yz}$ and $k \in K_{xz}$:

$$(30) \quad (aa') \cdot k = (a \cdot k_{(1,x,y)})(a' \cdot k_{(2,y,z)}).$$

We have a cluster $\mathcal{B} = K \# A$, $\mathcal{B}_{yz}^x = K_{xy} \# A_{yz}$. The symbol $\#$ replaces \otimes and indicates that \mathcal{B}^x is a k -linear category. The multiplication and unit are given by the formulas

$$(31) \quad (k \# a)(k' \# a') = kk'_{(1,x,y)} \# (a \cdot k'_{(2,y,z)})a' \text{ and } 1_y^x = 1_{xy} \# 1_y,$$

for $k \in K_{xy}$, $k' \in K_{xz}$, $a \in A_{yz}$, $a' \in A_{zu}$. We call $K \# A$ the smash product cluster. Now we have a diagonal algebra $B = A^K$ defined as follows:

$$B_x = A_{xx}^{K_{xx}} = \{a \in A_{xx} \mid a \cdot k = \varepsilon_x(k)a, \text{ for all } k \in K_{xx}\}.$$

Moreover we have a canonical morphism of clusters

$$(32) \quad \kappa : \mathcal{B} = K \# A \rightarrow \mathcal{A} = {}_B \text{End}(A)$$

from the smash product cluster to the endocluster, given by the formulas

$$\kappa_{yz}^x : K_{xy} \# A_{yz} \rightarrow {}_{B_x} \text{Hom}(A_{xy}, A_{xz}), \quad \kappa_{yz}^x(k \# a)(a') = (a' \cdot k)a.$$

Definition 6.1. We call A a dual (right) K -Galois category extension of $B = A^K$ if κ is an isomorphism of clusters.

Proposition 6.2. *With notation as above, we have an adjoint pair of functors (F_2, G_2) between the categories $\mathcal{D}_k(X)_B$ and \mathcal{M}_B . Moreover $F_1 = R \circ F_2$, where $F_1 : \mathcal{D}_k(X)_B \rightarrow \mathcal{M}_A$ is the functor defined in Theorem 5.4, and $R : \mathcal{M}_B \rightarrow \mathcal{M}_A$ is the restriction of scalars functor via κ . (F_2, G_2) is a pair of inverse equivalences if the following conditions are satisfied*

- (1) A_{xy} is finitely generated and projective as a left B_x -module, for all $x, y \in X$;
- (2) A_{xy} is a left B_{xx} -progenerator, for all $x \in X$;
- (3) A is a dual K -Galois category extension of $B = A^K$.

Proof. For $N \in \mathcal{D}_k(X)_B$, $F_2(N)_{xy} = N_x \otimes_{B_x} A_{xy}$, with right \mathcal{B} -action defined as follows:

$$(n \otimes a)(k \# a') = n \otimes (a \cdot k)a',$$

for all $n \in N_x$, $a \in A_{xy}$, $k \in K_{xy}$ and $a' \in A_{yz}$.

For $M \in \mathcal{M}_B$, $G_2(M)$ is defined as follows:

$$G_2(M)_x = \{m \in M_{xx} \mid m(k \# 1_x) = \varepsilon_x(k)m, \text{ for all } k \in K_x\} = M_{xx}^{K_{xx}}.$$

We have to show that $G_2(M)_x$ is a right B_x -module. First observe that M is a right A -module via restriction of scalars: the maps $M_{xy} \otimes A_{yz} \rightarrow M_{xz}$ are defined as follows:

$$ma = m(1_{xy} \# a).$$

Now for $m \in M_{xx}$ and $b \in B_x \subset A_{xx}$, we define

$$mb = m(1_{xx} \# b).$$

This makes M_{xx} a right B_x -module. $M_{xx}^{K_{xx}}$ is a B_x -submodule. Take $m \in M_{xx}^{K_{xx}}$ and $b \in B_x$. For all $k \in K_{xx}$, we have

$$\begin{aligned} (mb)(k \# 1_x) &= m(1_{xx} \# b)(k \# 1_x) = m(k_{(1,x,x)} \# b \cdot k_{(2,x,x)}) \\ &= m(k_{(1,x,x)} \# \varepsilon_x(k_{(2,x,x)})b) = m(k \# b) \\ &= m(k \# 1_x)(1_{xx} \# b) = \varepsilon_x(k)mb. \end{aligned}$$

We describe the unit and the counit of the adjunction. For $M \in \mathcal{M}_{\mathcal{B}}$, $\varepsilon^M : F_2 G_2(M) \rightarrow M$ has the following components

$$\varepsilon_{xy}^M : M_{xx}^{K_{xx}} \otimes_{B_x} A_{xy} \rightarrow M_{xy}, \quad \varepsilon_{xy}^M(m \otimes_{B_x} a) = ma.$$

For $N \in \mathcal{D}_k(X)_B$, $\eta^N : N \rightarrow G_2 F_2(N)$ has the following components:

$$\eta_x^N : N_x \rightarrow (N_x \otimes_{B_x} A_{xx})^{K_{xx}}, \quad \eta_x^N(n) = n \otimes_{B_x} 1_x.$$

Verification of the details is left to the reader. The second statement amounts to the following assertion. Take $n \in N_x$, $a \in A_{xy}$, $k \in A_{xy}$ and $a' \in A_{yz}$. We need to show that $(n \otimes a)(k \# a')$ in $F_2(N)$ equals $(n \otimes a)\kappa_{yz}^x(k \# a')$ in $F_1(N)$. Indeed,

$$(n \otimes a)\kappa_{yz}^x(k \# a') = n \otimes \kappa_{yz}^x(k \# a')(a) = n \otimes (a \cdot k)a' = (n \otimes a)(k \# a').$$

If A is a dual K -Galois category extension of $B = A^K$, then R is an isomorphism of categories. The two other assumptions imply that F_1 is an equivalence of categories, see Theorem 5.4. The fact that $F_1 = R \circ F_2$ implies that F_2 is a category equivalence. \square

Let us briefly state the left-handed versions of the results in this Section. For a left K -module category A , we have a cluster $\mathcal{B}' = A \# K$, $\mathcal{B}'_{yz} = A_{yz} \# K_{zx}$, with multiplication

$$(33) \quad (a \# k)(a' \# k') = a(k_{(1,z,u)} \cdot a') \# k_{(2,u,x)} k,$$

for $a \in A_{yz}$, $k \in K_{zx}$, $a' \in A_{zu}$, $k' \in K_{ux}$. The units are $1_y^x = 1_y \# 1_{yx} \in \mathcal{B}'_{yy}$. Let $B = A^K$. Then we have a canonical morphism of clusters

$$(34) \quad \kappa : \mathcal{B}' = A \# K \rightarrow \mathcal{A}' = \text{End}_B(A),$$

given by the formulas

$$\kappa_{yz}^x : A_{yz} \# K_{zx} \rightarrow \text{Hom}_{B_x}(A_{zx}, A_{yx}), \quad \kappa_{yz}^x(a \# k)(a') = a(k \cdot a').$$

We call A a dual (left) K -Galois category extension of $B = A^K$ if κ is an isomorphism of clusters.

7. DUALITY

7.1. The Koppinen smash product. In the previous Sections, we have introduced the notions of H -Galois category extension and dual K -Galois category extension, H being a (semi)-Hopf category, and K being a dual (semi)-Hopf category. In this Section, we discuss how these notions are connected via duality.

To a right H -comodule category A , we associate a k -linear cluster $\mathcal{C} = \#(H, A)$, which can be viewed as the multi-object version of the Koppinen smash product [13].

$$\mathcal{C}_{yz}^x = \#(H_{xz}, A_{zy}),$$

where $\#$ replaces Hom , to indicate that we have specially defined multiplication maps $\# : \mathcal{C}_{yz}^x \otimes \mathcal{C}_{zu}^x \rightarrow \mathcal{C}_{yu}^x$ defined as follows: for $g : H_{xz} \rightarrow A_{zy}$ and $g' : H_{xu} \rightarrow A_{uz}$, we have $g\#g' : H_{xu} \rightarrow A_{uy}$ given by the formula

$$(35) \quad (g\#g')(h) = g'(h_{(2)})_{[0]}g(h_{(1)}g'(h_{(2)})_{[1]}).$$

The units are the maps $i_y^x = \eta_y \circ \varepsilon_{xy} : H_{xy} \rightarrow A_{yy}$, $i_y^x(h) = \varepsilon_{xy}(h)1_y$. Verification of the associativity and unit conditions is left to the reader.

Proposition 7.1. *Let A be a right H -comodule category, and let $B = A^{\text{co}H}$. We have a morphism of clusters*

$$\delta : \mathcal{C} = \#(H, A) \rightarrow \mathcal{A}^{\text{op}} = {}_B\text{End}(A)^{\text{op}}.$$

$$\delta_{yz}^x : \mathcal{C}_{yz}^x = \#(H_{xz}, A_{zy}) \rightarrow \mathcal{A}_{yz}^{\text{op}x} = \mathcal{A}_{zy}^x = {}_{B_x}\text{Hom}(A_{xz}, A_{xy})$$

is given by the formula

$$(36) \quad \delta_{yz}^x(g)(a) = a_{[0]}g(a_{[1]}).$$

Proof. Let us first show that δ is multiplicative: for g and g' as above, we show that

$$\delta_{yu}^x(g\#g') = \delta_{yz}^x(g) \circ \delta_{zu}^x(g') \in {}_{B_x}\text{Hom}(A_{xu}, A_{xy}).$$

For $a \in A_{xu}$, we have that

$$\begin{aligned} (\delta_{yz}^x(g) \circ \delta_{zu}^x(g'))(a) &= \delta_{yz}^x(g)(a_{[0]}g(a_{[1]})) = a_{[0][0]}g(a_{[1]})_{[0]}g'(a_{[0][1]}g(a_{[1]})_{[1]}) \\ &= a_{[0]}g(a_{[2]})_{[0]}g'(a_{[1]}g(a_{[2]})_{[1]}) = a_{[0]}((g\#g')(a_{[1]})) = (\delta_{yu}^x(g\#g'))(a). \end{aligned}$$

δ preserves the units: $\delta_{yy}^x(i_y^x)$ is the identity map on A_{xy} since $\delta_{yy}^x(i_y^x)(a) = a_{[0]}i_y^x(a_{[1]}) = a_{[0]}\varepsilon_{xy}(a_{[1]})1_y = a$, for all $a \in A_{xy}$. \square

Proposition 7.2. *Assume that A is an H -Galois category extension of $B = A^{\text{co}H}$ (see Theorem 3.5). Then $\delta : \mathcal{C} = \#(H, A) \rightarrow \mathcal{A}^{\text{op}} = {}_B\text{End}(A)^{\text{op}}$ is an isomorphism of clusters.*

Proof. Consider the morphisms

$$\gamma_{xy} : H_{xy} \rightarrow A_{yx} \otimes_{B_x} A_{xy}, \quad \gamma_{xy}(h) = \sum_i l_i(h) \otimes_{B_x} r_i(h),$$

from condition (3) in Theorem 3.5. We will show that

$$\tilde{\delta}_{yz}^x : {}_{B_x}\text{Hom}(A_{xz}, A_{xy}) \rightarrow \#(H_{xz}, A_{zy}), \quad \tilde{\delta}_{yz}^x(\varphi)(h) = \sum_i l_i(h)\varphi(r_i(h))$$

is the inverse of δ_{yz}^x . For all $\varphi \in {}_{B_x}\text{Hom}(A_{xz}, A_{xy})$, $a \in A_{xz}$, $g \in \#(H_{xz}, A_{zy})$ and $h \in H_{xz}$, we have that

$$(\delta_{yz}^x \circ \tilde{\delta}_{yz}^x)(\varphi)(a) = a_{[0]}(\tilde{\delta}_{yz}^x(a_{[1]})) = \sum_i a_{[0]}l_i(a_{[1]})\varphi(r_i(a_{[1]})) \stackrel{(18)}{=} \varphi(a);$$

$$\begin{aligned} (\tilde{\delta}_{yz}^x \circ \delta_{yz}^x)(g)(h) &= \sum_i l_i(h)\delta_{yz}^x(g)(r_i(h)) = \sum_i l_i(h)r_i(h)_{[0]}g(r_i(h)_{[1]}) \\ &\stackrel{(22)}{=} \sum_i l_i(h_{(1)})r_i(h_{(1)})g(h_{(2)}) \stackrel{(23)}{=} \varepsilon_{xz}(h_{(1)})1_zg(h_{(2)}) = g(h). \end{aligned}$$

\square

Our next goal is to prove the converse of Proposition 7.2. Some additional finiteness conditions will be needed.

Proposition 7.3. *Let A be a right H -comodule category and let $B = A^{\text{co}H}$. Assume that the following conditions are satisfied*

- (1) *A locally finite as a left B -module;*
- (2) *H is locally finite;*
- (3) *δ_{xy}^x and δ_{yy}^x are bijective, for all $x, y \in X$.*

Then A is an H -Galois category extension of $B = A^{\text{co}H}$.

Proof. Let $\sum_j a_j^* \otimes_{B_x} a_j \in {}_{B_x}\text{Hom}(A_{xy}, B_x) \otimes_{B_x} A_{xy}$ be a finite dual basis of A_{xy} as a left B_x -module. We define $\gamma_{xy} : H_{xy} \rightarrow A_{yx} \otimes_{B_x} A_{xy}$ by the formula

$$(37) \quad \gamma_{xy}(h) = \sum_i l_i(h) \otimes_{B_x} r_i(h) = \sum_j \tilde{\delta}_{xy}^x(i_x \circ a_j^*)(h) \otimes_{B_x} a_j.$$

Recall that $i_x : B_x \rightarrow A_{xx}$, so that $i_x \circ a_j^* \in {}_{B_x}\text{Hom}(A_{xy}, A_{xx})$. We have to show that (17-18) are satisfied. (18) follows from the fact that $\tilde{\delta}_{xy}^x$ is a right inverse of δ_{xy}^x : for all $f \in {}_{B_x}\text{Hom}(A_{xy}, A_{xx})$ and $a \in A_{xy}$, we have

$$(38) \quad f(a) = ((\delta_{xy}^x \circ \tilde{\delta}_{xy}^x)(f))(a) = a_{[0]}(\tilde{\delta}_{xy}^x(f))(a_{[1]}),$$

hence

$$\begin{aligned} \sum_i a_{[0]} l_i(a_{[1]}) \otimes_{B_x} r_i(a_{[1]}) &\stackrel{(37)}{=} \sum_{i,j} a_{[0]} \tilde{\delta}_{xy}^x(i_x \circ a_j^*)(a_{[1]}) \otimes_{B_x} a_j \\ &\stackrel{(38)}{=} \sum_j a_j^*(a) 1_x \otimes_{B_x} a_j = 1_x \otimes_{B_x} a. \end{aligned}$$

Before we are able to prove (17), we need two observations. The first observation is that we can reformulate the right coaction on A in terms of the dual basis $\sum_l h_l^* \otimes h_l \in H_{xy}^* \otimes H_{xy}$ of H_{xy} . For all $a \in A_{xy}$, we have that

$$(39) \quad \rho(a) = \sum_l a_{[0]} \langle h_l^*, a_{[1]} \rangle \otimes h_l = \sum_l \delta_{yy}^x(\eta_y \circ h_l^*)(a) \otimes h_l.$$

The second observation is that δ is right A -linear in the following sense. For $g \in \text{Hom}(H_{xz}, A_{zy})$, $\varphi \in {}_{B_x}\text{Hom}(A_{xz}, A_{xy})$ and $a' \in A_{yu}$, we define $g \cdot a \in \text{Hom}(H_{xz}, A_{zu})$ and $\varphi \cdot a' \in {}_{B_x}\text{Hom}(A_{xz}, A_{xu})$ by right multiplication:

$$(g \cdot a')(h) = g(h)a' \text{ and } (\varphi \cdot a')(a) = \varphi(a)a'.$$

It is then easily computed that

$$\delta_{uz}^x(g \cdot a')(a) = a_{[0]}(g \cdot a')(a_{[1]}) = a_{[0]}g(a_{[1]})a' = \delta_{yz}^x(g)(a)a' = (\delta_{yz}^x(g) \cdot a')(a).$$

Now δ_{xy}^x and δ_{yy}^x are invertible, so we have, for $\varphi \in {}_{B_x}\text{Hom}(A_{xy}, A_{xx})$ and $a' \in A_{xy}$, that

$$(40) \quad \tilde{\delta}_{yy}^x(\varphi \cdot a') = \tilde{\delta}_{xy}^x(\varphi) \cdot a'.$$

For $h \in H_{xy}$, we compute that

$$\begin{aligned} L &:= \sum_i l_i(h) r_i(h)_{[0]} \otimes r_i(h)_{[1]} \\ &\stackrel{(37)}{=} \tilde{\delta}_{xy}^x(i_x \circ a_j^*)(h) a_{j[0]} \otimes a_{j[1]} \\ &\stackrel{(39)}{=} \sum_{j,l} \left(\tilde{\delta}_{xy}^x(i_x \circ a_j^*)(h) \right) \left(\delta_{yy}^x(\eta_y \circ h_l^*)(a_j) \right) \otimes h_l \end{aligned}$$

Now let $a' = \delta_{yy}^x(\eta_y \circ h_l^*)(a_j) \in A_{xy}$. Applying (40), we obtain that

$$L = \sum_{j,l} \tilde{\delta}_{yy}^x((i_x \circ a_j^*) \cdot a')(h) \otimes h_l.$$

For all $a'' \in A_{xy}$, we have that

$$\sum_j ((i_x \circ a_j^*) \cdot a')(a'') = \sum_j a_j^*(a'') 1_x \delta_{yy}^x(\eta_y \circ h_l^*)(a_j) = \delta_{yy}^x(\eta_y \circ h_l^*)(a''),$$

and

$$\begin{aligned} L &= \sum_l \left(\tilde{\delta}_{yy}^x(\delta_{yy}^x(\eta_y \circ h_l^*)) \right)(h) \otimes h_l \\ &= \sum_l (\eta_y \circ h_l^*)(h) \otimes h_l = \sum_l \langle h_l^*, h \rangle 1_y \otimes h_l = 1_y \otimes h. \end{aligned}$$

completing the proof of (17). \square

7.2. The smash product versus the Koppinen smash product. We briefly return to the classical situation, where X is a singleton. Consider a (finitely generated projective) bialgebra H coacting from the right on a k -module M . The usual way to define an action of the dual bialgebra H^* is via the formula

$$(41) \quad h^* \cdot m = \langle h^*, m_{[1]} \rangle m_{[0]}.$$

If $M = A$ is a right H -comodule algebra, then A is a left H^* -module algebra. Here we need the convolution product and the convolution coproduct on H^* . We can also consider the formula

$$(42) \quad m \cdot h^* = \langle h^*, m_{[1]} \rangle m_{[0]}.$$

It makes A into a right H^* -module algebra, now H^* equipped with anti-convolution product, but convolution coproduct.

Passing to the categorical situation where X is no longer a singleton, (41) no longer makes sense. There are two ways to fix this problem; one may use the antipode (if it exists), we come back to this in Remark 7.7. An alternative solution is to introduce some op-arguments. From now on, H is a locally finite semi-Hopf category, with corresponding dual semi-Hopf category K , as in Section 1.3. The proof of Proposition 7.4 is a direct verification.

Proposition 7.4. *Let A be a right H -module category. Then A^{op} is a left K^{op} -module category, with*

$$(43) \quad k \cdot a = \langle k, a_{[1]} \rangle a_{[0]},$$

for all $k \in K_{xy}$ and $a \in A_{xy}^{\text{op}} = A_{yx}$.

The associated smash product $\mathcal{B}' = A^{\text{op}} \# K^{\text{op}}$ is described as follows, see

$$(33): \mathcal{B}'_{yz} = A_{zy} \# K_{zx}, \text{ with multiplication } (a' \in A_{uz}, k' \in K_{ux})$$

$$(44) \quad (a \# k)(a' \# k') = \langle k_{(1,z,u)}, a'_{[1]} \rangle a'_{[0]} a \# k' k_{(2,u,x)}.$$

Observe that $\text{End}_{\mathcal{B}'^{\text{op}}}(A^{\text{op}}) = {}_B\text{End}(A)^{\text{op}} = \mathcal{A}^{\text{op}}$. Indeed,

$$\text{Hom}_{B_x}^{\text{op}}(A_{zx}^{\text{op}}, A_{yx}^{\text{op}}) = {}_{B_x}\text{Hom}(A_{xz}, A_{xy}) = \mathcal{A}_{yz}^x = \mathcal{A}_{yz}^{\text{op}x}.$$

Applying (34), we obtain a morphism of clusters $\kappa' : \mathcal{B}' \rightarrow \mathcal{A}^{\text{op}}$,

$$(45) \quad \kappa'_{yz} : A_{zy} \# K_{zx} \rightarrow {}_{B_x}\text{Hom}(A_{xz}, A_{xy}), \quad \kappa'_{yz}(a \# k)(a') = (k \cdot a')a.$$

Proposition 7.5. *Let H be a locally finite k -linear semi-Hopf category, and let A be a right H -comodule category. We have an isomorphism of clusters $\beta : \mathcal{B}' = A^{\text{op}} \# K^{\text{op}} \rightarrow \mathcal{C} = \#(H, A)$, which is such that the diagram*

$$\begin{array}{ccc} \mathcal{B}' = A^{\text{op}} \# K^{\text{op}} & \xrightarrow{\kappa'} & \mathcal{A}^{\text{op}} = {}_B\text{End}(A)^{\text{op}} \\ \beta \downarrow & \nearrow \delta & \\ \mathcal{C} = \#(H, A) & & \end{array}$$

commutes.

Proof. It is well-known that $\beta_{yz}^x : A_{zy} \# K_{zx} \rightarrow \text{Hom}(H_{xz}, A_{zy})$ and $\tilde{\beta}_{yz}^x : \text{Hom}(H_{xz}, A_{zy}) \rightarrow A_{zy} \# K_{zx}$ given by the formulas

$$(46) \quad \beta_{yz}^x(a \# k)(h) = \langle k, h \rangle a \text{ and } \tilde{\beta}_{yz}^x(f) = \sum_i f(h_i) \otimes k_i,$$

where $\sum_i h_i \otimes k_i$ is the dual basis of H_{xz} , are inverses. It is clear that $\beta_{yy}^x(1_y^A \# \varepsilon_{xz}) = \eta_y \circ \varepsilon_{xz}$. Let us show that β preserves the multiplication. Take $a \# k \in A_{zy} \# K_{zx}$, $a' \# k' \in A_{uz} \# K_{ux}$, and write $\beta_{yz}^x(a \# k) = g$, $\beta_{zu}^x(a' \# k') = g'$. For all $h \in H_{xu}$, we have that

$$\begin{aligned} \beta_{yu}^x((a \# k)(a' \# k'))(h) &\stackrel{(44,46)}{=} \langle k_{(1,z,u)}, a'_{[1]} \rangle \langle k' k_{(2,u,x)}, h \rangle a' a_{[0]} \\ &= \langle k_{(1,z,u)}, a'_{[1]} \rangle \langle k', h_{(2)} \rangle \langle k_{(2,u,x)}, h_{(1)} \rangle a' a_{[0]} \\ &= \langle k, h_{(1)} a'_{[1]} \rangle \langle k', h_{(2)} \rangle a' a_{[0]} \\ &\stackrel{(46)}{=} g'(h_{(2)})_{[0]} g(h_{(1)} g'(h_{(2)})_{[1]}) \stackrel{(35)}{=} (g \# g')(h). \end{aligned}$$

We are left to show that $\delta_{yz}^x \circ \beta_{yz}^x = \kappa'_{yz}$. For all $a' \in A_{xz}$, we have that

$$\begin{aligned} \delta_{yz}^x(\beta_{yz}^x(a \# k))(a') &\stackrel{(36)}{=} a'_{[0]} (\beta_{yz}^x(a \# k)(a'_{[1]})) \\ &\stackrel{(46)}{=} \langle k, a'_{[1]} \rangle a'_{[0]} a \stackrel{(45)}{=} \kappa'_{yz}(a \# k)(a'). \end{aligned}$$

□

We now summarize our results.

Theorem 7.6. *Let H be a locally finite semi-Hopf category, with dual $K = H^*$ and let A be a right H -comodule category. Then A^{op} is a right K^{op} -module category, see (43), and $A^{K^{\text{op}}} = A^{\text{co}H} = B$. The following assertions are equivalent.*

- (1) A is an H -Galois category extension of B , that is, $\text{can}_{xy}^z : A_{zx} \otimes_{B_x} A_{xy} \rightarrow A_{zy} \otimes H_{xy}$ is bijective, for all $x, y, z \in X$, see Theorem 3.5;
- (2) A^{op} is a dual left K^{op} -Galois category extension of $B = A^{K^{\text{op}}}$, that is, $\kappa' : \mathcal{B}' = A^{\text{op}} \# K^{\text{op}} \rightarrow \mathcal{A}^{\text{op}} = {}_B \text{End}(A)^{\text{op}}$ is an isomorphism of k -linear clusters;
- (3) $\delta : \mathcal{C} \rightarrow \#(H, A) \rightarrow \mathcal{A}^{\text{op}} = {}_B \text{End}(A)^{\text{op}}$ is an isomorphism of k -linear clusters;
- (4) κ'_{yx} and κ'_{yy} are bijective, for all $x, y \in X$;
- (5) δ_{yx}^x and δ_{yy}^x are bijective, for all $x, y \in X$.

Proof. (1) \Rightarrow (3): Proposition 7.2; (3) \Rightarrow (5) is trivial; (5) \Rightarrow (1): Proposition 7.3; (2) \Leftrightarrow (3) and (4) \Leftrightarrow (5): Proposition 7.5. \square

Remark 7.7. In the preceding Sections, we have provided a detailed account of the righthanded theory, ending with a brief description of the lefthanded theory at the end of each Section. It may come as a surprise that we have to switch from right to left in Theorem 7.6: in order to give alternative characterizations of A being an H -Galois category extension of B , we need a smash product obtained from a left action. If H has an antipode, then we can also work with a right action: A is a right K^{op} -module category, with action is given by the formula

$$(47) \quad a \leftarrow k = \langle k, S_{xy}(a_{[1]}) \rangle a_{[0]},$$

for all $k \in K_{xy}$ and $a \in A_{xy}$. The associated smash product $\bar{\mathcal{B}} = K^{\text{op}} \# A$ is described as follows: $\bar{\mathcal{B}}_{yz}^x = K_{xy} \# A_{yz}$, with multiplication ($k' \in K_{xz}$, $a' \in A_{zu}$)

$$(48) \quad (k \# a)(k' \# a') = k'_{(1,x,y)} k \# (a \leftarrow k'_{(2,y,z)}) a'.$$

Applying (32), we have a morphism of clusters $\bar{\kappa} : \bar{\mathcal{B}} \rightarrow {}_B \text{End}(A) = \mathcal{A}$,

$$\bar{\kappa}_{yz}^x : K_{xy} \# A_{yz} \rightarrow A_{yz}^x = {}_{B_x} \text{Hom}(A_{xy}, A_{xz}), \quad \bar{\kappa}_{yz}^x(k \# a)(a') = (a' \leftarrow k)a.$$

In Proposition 7.8, we will see that this brings nothing new: the smash products $\bar{\mathcal{B}}$ and \mathcal{B}' are anti-isomorphic.

Proposition 7.8. *Let H be a locally finite k -linear Hopf category, and let A be a right H -comodule category. We have an isomorphism of clusters $\alpha : \bar{\mathcal{B}} = K^{\text{op}} \# A \rightarrow \mathcal{B}'^{\text{op}} = (A^{\text{op}} \# K^{\text{op}})^{\text{op}}$ which is such that the diagram*

$$\begin{array}{ccc} \bar{\mathcal{B}} & \xrightarrow{\bar{\kappa}} & \mathcal{A} \\ \alpha \downarrow & \nearrow \kappa'^{\text{op}} & \\ \mathcal{B}'^{\text{op}} & & \end{array}$$

commutes. Consequently the equivalent statements of Theorem 7.6 are also equivalent to

- (6) A is a dual right K^{op} -Galois category extension of B , that is, $\bar{\kappa} : \bar{\mathcal{B}} \rightarrow {}_B\text{End}(A) = \mathcal{A}$ is an isomorphism of k -linear clusters;
- (7) $\bar{\kappa}_{xy}^x$ and $\bar{\kappa}_{yy}^x$ are bijective, for all $x, y \in X$.

Proof. α is defined as follows:

$$\alpha_{yz}^x : \bar{\mathcal{B}}_{yz}^x = K_{xy} \# A_{yz} \rightarrow \mathcal{B}'_{zy}{}^x = A_{yz} \# K_{yx}, \quad \alpha_{yz}^x(k \# a) = a \# S_{xy}(k).$$

From the fact that H is locally finite (see [3, Prop. 10.6]), it follows that the antipode of H is bijective, and this implies that α is bijective, with inverse given by the formula

$$(\tilde{\alpha}_{yz}^x)(a \# k) = S_{xy}^{-1}(k) \# a.$$

It is left to the reader to show that α is multiplicative, and $\tilde{\alpha}$ are bijective. Let us verify that the first triangle commutes. For all $x, y, z \in X$ the triangle

$$(49) \quad \begin{array}{ccc} \bar{\mathcal{B}}_{yz}^x & \xrightarrow{\bar{\gamma}_{yz}^x} & \mathcal{A}_{yz}^x \\ \alpha_{yz}^x \downarrow & \nearrow \kappa'_{zy}{}^x & \\ \mathcal{B}'_{zy}{}^x & & \end{array}$$

commutes: for all $k \in K_{xy}$, $a \in A_{yz}$ and $a' \in A_{xy}$, we have that

$$\begin{aligned} (\kappa'_{zy}{}^x \circ \alpha_{yz}^x)(k \# a)(a') &= \kappa'_{zy}{}^x(a \# S_{xy}(k))(a') \\ &= (S_{xy}(k) \cdot a')a = (a' \leftarrow k)a = \bar{\kappa}_{yz}^x(k \# a)(a'). \end{aligned}$$

□

7.3. The Koppinen smash product revisited. In Section 7.1, we introduced the Koppinen smash product, and gave its relationship to right faithfully projective descent data. Now we present an alternative version, related to left faithfully projective descent data. This Koppinen smash product is isomorphic to a smash product associated to a right H -module category in the sense of Section 6. We restrict to giving the main results, the proofs are similar to the proofs presented in Sections 7.1 and 7.2. We assume that H is locally finite semi-Hopf category with associated dual semi-Hopf category K , and that A is a right H -comodule category.

We have a k -linear cluster $\mathcal{C}' = \#'(H, A)$, defined componentwise as

$$\mathcal{C}'_{yz}{}^x = \#'(H_{yx}, A_{zy}),$$

with the following multiplication: for $g \in \#'(H_{yx}, A_{zy})$ and $g' \in \#'(H_{zx}, A_{uz})$, $g \#' g' \in \#'(H_{yx}, A_{uy})$ is given by the formula

$$(50) \quad (g \#' g')(h) = g'(g(h_{(2)})_{[1]}h_{(1)})g(h_{(2)})_{[0]},$$

for $h \in H_{yx}$. Our next observation is that (42) can be applied to construct a right K -module category: A^{op} is a right K -module category, with action given by the formula

$$(51) \quad a \cdot k = \langle k, a_{[1]} \rangle a_{[0]},$$

for all $k \in K_{xy}$ and $a \in A_{xy}^{\text{op}} = A_{yx}$. The associated smash product $\mathcal{B} = K \# A^{\text{op}}$ is described as follows: $\mathcal{B}_{yz}^x = K_{xy} \# A_{zy}$, with multiplication ($a' \in A_{uz}$, $k' \in K_{xz}$)

$$(52) \quad (k \# a)(k' \# a') = kk'_{(1,x,y)} \# a'(a \cdot k_{(2,y,z)}) = \langle k_{(2,y,z)}, a_{[1]} \rangle kk'_{(1,x,y)} \# a' a_{[0]}.$$

Now ${}_{B^{\text{op}}} \text{End}(A^{\text{op}}) = \text{End}_B(A)^{\text{op}} = \mathcal{A}'^{\text{op}}$. Applying (32), we have a morphism of clusters $\kappa : \mathcal{B} \rightarrow \text{End}_B(A)^{\text{op}} = \mathcal{A}'^{\text{op}}$,

$$\kappa_{yz}^x : \mathcal{B}_{yz}^x = K_{xy} \# A_{zy} \rightarrow \mathcal{A}'_{zy} = \text{Hom}_{B_x}(A_{yx}, A_{zx}), \quad \kappa_{yz}^x(k \# a)(a') = a(a' \cdot k).$$

Theorem 7.9. *Let H be a locally finite k -linear semi-Hopf category, and let A be a right H -comodule category. We have an isomorphism of clusters $\beta' : \mathcal{B} \rightarrow \mathcal{C}'$ and a morphism of clusters $\delta' : \mathcal{C}' \rightarrow \mathcal{A}'^{\text{op}}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{B} = K \# A^{\text{op}} & \xrightarrow{\kappa} & \mathcal{A}'^{\text{op}} = \text{End}_B(A)^{\text{op}} \\ \beta' \downarrow & \nearrow \delta' & \\ \mathcal{C}' = \#'(H, A) & & \end{array}$$

commutes. Also assume that A is locally finite as a right B -module. The following assertions are equivalent:

- (1) A is an H -Galois' category extension of $B = A^{\text{co}H}$;
- (2) A is a dual right K -Galois category extension of B , that is, κ is an isomorphism of clusters;
- (3) δ' is an isomorphism of clusters;
- (4) κ_{yx}^x and κ_{yy}^x are invertible, for all $x, y \in X$;
- (5) δ'_{yx} and δ'_{yy} are invertible, for all $x, y \in X$.

Proof. (sketch) $\delta'_{yz} : \mathcal{C}'_{yz} = \#'(H_{yx}, A_{zy}) \rightarrow \mathcal{A}'_{zy} = \text{Hom}_{B_x}(A_{yx}, A_{zx})$ is given by the formula

$$\delta'_{yz}(g)(a') = g(a'_{[1]})a'_{[0]},$$

for all $a' \in A_{yx}$. $\beta'_{yz} : K_{xy} \# A_{zy} \rightarrow \#'(H_{yx}, A_{zy})$ is given by the formula $\beta'_{yz}(k \# a)(h) = \langle k, h \rangle a$.

(1) \Rightarrow (3). It follows from Theorem 3.12 that there exists $\gamma'_{zx} : H_{zx} \rightarrow A_{zx} \otimes_{B_x} A_{zx}$ satisfying (28-29). The inverse of δ'_{yz} is given by the formula (with notation as in Theorem 3.12): $(\delta'_{yz})^{-1}(\varphi)(h) = \sum_i \varphi(l'_i(h))r'_i(h)$.

(5) \Rightarrow (1). Let $\sum_j a_j \otimes_{B_x} a_j^* \in A_{zx} \otimes_{B_x} \text{Hom}_{B_x}(A_{zx}, B_x)$ be the finite dual basis of A_{zx} as a right B_x -module. Observe that $i_x \circ a_j^* \in \text{Hom}_{B_x}(A_{zx}, A_{zx})$, and define γ'_{zx} as follows: $\gamma'_{zx}(h) = \sum_j a_j \otimes_{B_x} (\delta'_{zx})^{-1}(i_x \circ a_j^*)(h)$, for $h \in H_{zx}$. The proof of (29) is straightforward. (28) is more tricky, and depends on

the assumption that δ'_{zz} is invertible. We sketch the details. For $a' \in A_{uz}$, $g \in \#'(H_{yx}, A_{zy})$ and $\varphi \in \text{Hom}_{B_x}(A_{yx}, A_{zy})$, we define $a' \cdot g \in \#'(H_{yx}, A_{uy})$ and $a' \cdot \varphi \in \text{Hom}_{B_x}(A_{yx}, A_{uy})$ by left multiplication: $(a' \cdot g)(h) = a'g(h)$ and $(a' \cdot \varphi)(a) = a'\varphi(a)$. Obviously δ' is left A -linear, in the sense that $\delta'_{yu}(a' \cdot g) = a' \cdot \delta'_{yz}(g)$. This implies that

$$(53) \quad (\delta'_{zz})^{-1}(a' \cdot \varphi) = a' \cdot (\delta'_{zx})^{-1}(\varphi),$$

for $\varphi \in \text{Hom}_{B_x}(A_{zx}, A_{xx})$ and $a' \in A_{zx}$. Let $\sum_j h_l^* \otimes h_l$ be the finite dual basis of H_{zx} . Then we have for all $a \in A_{zx}$ that

$$(54) \quad \rho_{zx}(a) = \sum_l \delta'_{zz}(\eta_z \circ h_l^*)(a) \otimes h_l = \sum_l a^l \otimes h_l.$$

With this notation, we can show that

$$(55) \quad \sum_j a_j^l \cdot (i_x \circ a_j^*) = \delta'_{zz}(\eta_z \circ h_l^*).$$

Finally

$$\begin{aligned} \sum_i l'_i(h)_{[0]} r'_i(h) \otimes l'_i(h)_{[1]} &= \sum_j a_{j[0]} (\delta'_{zx})^{-1}(i_x \circ a_j^*)(h) \otimes a_{j[1]} \\ &\stackrel{(54)}{=} \sum_{j,l} \left(a_j^l \cdot ((\delta'_{zx})^{-1}(i_x \circ a_j^*)) \right)(h) \otimes h_l \\ &\stackrel{(53)}{=} \sum_{j,l} \left((\delta'_{zx})^{-1}(a_j^l \cdot (i_x \circ a_j^*)) \right)(h) \otimes h_l \\ &\stackrel{(55)}{=} \sum_l \left((\delta'_{zz})^{-1}(\delta'_{zz}(\eta_z \circ h_l^*)) \right)(h) \otimes h_l \\ &= \sum_l \langle h_l^*, h \rangle 1_z \otimes h_l = 1_z \otimes h, \end{aligned}$$

proving (28). \square

Remark 7.10. If H is a Hopf category, then A is a left K -module category, with action

$$k \dashv a = \langle k, S_{xy}(a_{[1]}) \rangle a_{[0]},$$

for $k \in K_{xy}$ and $a \in A_{xy}$. Proceeding as in Remark 7.7, we can add two more equivalent conditions to Theorem 7.9. Moreover, the conditions in Theorems 7.6 and 7.9 are equivalent, by Theorem 3.14.

8. HOPF-GALOIS EXTENSIONS AND GROUPOID GRADED ALGEBRAS

Let G be a groupoid, with underlying class of objects X . The unit element of G_{xx} is denoted as e_x . Then kG is a k -linear Hopf category, see [3, Ex. 3.4]. In the situation where X is a set, we can consider the groupoid algebra kG , which is the Hopf category kG in packed form: $kG = \bigoplus_{x,y \in X} kG_{xy}$, with multiplication extended linearly from the composition in G , where we put $\tau\sigma = 0$ if τ and σ cannot be composed. If X is finite, then kG has the unit

$$\sum_x e_x.$$

A G -grading on $M \in \mathcal{M}_k(X)$ consists of a direct sum decomposition

$$M_{xy} = \bigoplus_{\sigma \in G_{xy}} M_\sigma,$$

for all $x, y \in X$. If $m \in M_\sigma$, then m is said to be homogeneous of degree σ , written as $\deg(m) = \sigma$. $\mathcal{M}_k(X)^G$ is the category of G -graded objects of $\mathcal{M}_k(X)$. Its morphisms are degree preserving morphisms in $\mathcal{M}_k(X)$.

A G -graded k -linear category is a k -linear category A with a G -grading such that $1_x \in A_{e_x}$ and $A_\sigma A_\tau \subset A_{\sigma\tau}$, for all $x, y, z \in X$ and $\sigma \in G_{xy}$ and $\tau \in G_{yz}$. If $A_\sigma A_\tau = A_{\sigma\tau}$ for all σ and τ , then A is called a strongly G -graded k -linear category. If X is a (finite) set, then these definitions can be restated in packed form, and we recover definitions from [15], where a structure theorem for strongly graded algebras over a groupoid is presented. A G -graded right A -module is an object $M \in \mathcal{M}_k(X)^G$ with a right A -action such that $M_\sigma A_\tau \subset M_{\sigma\tau}$, for all $x, y, z \in X$ and $\sigma \in G_{xy}$ and $\tau \in G_{yz}$. $\mathcal{M}_k(X)_A^G$ is the category of G -graded right A -modules.

Proposition 8.1. *For a groupoid G , the categories $\mathcal{M}_k(X)^G$ and $\mathcal{M}_k(X)^{kG}$ are isomorphic. kG -comodule category structures on a k -linear category A correspond bijectively to G -gradings on A , and, in this situation, the categories $\mathcal{M}_k(X)_A^G$ and $\mathcal{M}_k(X)_A^{kG}$ are isomorphic.*

Proof. For $M \in \mathcal{M}_k(X)^G$, the maps $\rho_{xy} : M_{xy} \rightarrow M_{xy} \otimes kG_{xy}$ given by the formula $\rho_{xy}(m) = m \otimes \sigma$ if $\deg(m) = \sigma$, extended linearly, define a right kG -coaction on M .

For $M \in \mathcal{M}_k(X)^{kG}$ and $\sigma \in G_{xy}$, define

$$M_\sigma = \{m \in M_{xy} \mid \rho_{xy}(m) = m \otimes \sigma\}.$$

It is clear that $M_\sigma \cap M_\tau = \{0\}$ if $\sigma \neq \tau \in G_{xy}$. Since $\rho_{xy} : M_{xy} \rightarrow M_{xy} \otimes kG_{xy} = \bigoplus_{\sigma \in G_{xy}} M_\sigma \otimes \sigma$, we can write

$$\rho_{xy}(m) = \sum_{\sigma \in G_{xy}} m_\sigma \otimes \sigma,$$

with $m_\sigma \in M_\sigma$. Applying $M_{xy} \otimes \varepsilon_{xy}$ to both sides, we see that

$$m = \sum_{\sigma \in G_{xy}} m_\sigma.$$

The coassociativity of ρ entails that

$$\sum_{\sigma \in G_{xy}} \rho_{xy}(m_\sigma) \otimes \sigma = \sum_{\sigma \in G_{xy}} m_\sigma \otimes \sigma \otimes \sigma \in \bigoplus_{\sigma \in G_{xy}} M_\sigma \otimes kG_{xy} \otimes k\sigma.$$

Fixing $\tau \in G_{xy}$, and taking the projection of both sides onto the component $M_\tau \otimes kG_{xy} \otimes k\tau$, we find that $\rho_{xy}(m_\tau) = m_\tau \otimes \tau$, and $m_\tau \in M_\tau$, for all $\tau \in G_{xy}$. This proves that $m = \sum_{\sigma \in G_{xy}} m_\sigma \in \bigoplus_{\sigma \in G_{xy}} M_\sigma$.

The proof of other assertions is similar and is left to the reader. \square

Theorem 8.2. *Let G be a groupoid, and let A be a G -graded k -linear category. Let $B = A^{\text{cok}G}$ be the diagonal algebra with $B_x = A_{e_x}$. The following statements are equivalent.*

- (1) A is strongly graded;
- (2) $A_{\sigma^{-1}}A_\sigma = B_y$, for all $x, y \in X$ and $\sigma \in G_{xy}$;
- (3) The adjunction (F, G) from Proposition 3.2 is a pair of inverse equivalences;
- (4) A is a kG -Galois category extension of $B = A^{\text{cok}G}$, in the sense of Theorem 3.5.

Proof. (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3). Take $N \in \mathcal{D}_k(X)_B$ and $M \in \mathcal{M}_k(X)_A^G$. Recall from Proposition 3.2 that $F(N)_{xy} = N_x \otimes_{B_x} A_{xy}$ and $G(M)_x = M_x^{\text{cok}G_{xx}} = M_{e_x}$. It is obvious that

$$\eta_x^N : N_x \rightarrow F(N)_{e_x} = N_x \otimes_{B_x} A_{e_x}, \quad \eta_x^N(n) = n \otimes_{B_x} 1_x$$

is bijective, for all $x \in X$. We are done if we can show that

$$\varepsilon_{xy}^M : M_{e_x} \otimes_{B_x} A_{xy} \rightarrow M_{xy}, \quad \varepsilon_{xy}^M(m \otimes_{B_x} a) = ma$$

is bijective. Take $\sigma \in G_{xy}$ and $m \in M_\sigma \subset M_{xy}$. (2) implies that there exist $a_i \in A_{\sigma^{-1}} \subset A_{yx}$ and $a'_i \in A_\sigma \subset A_{xy}$ such that $\sum_i a_i a'_i = 1_y$. Then $ma_i \in M_\sigma A_{\sigma^{-1}} \subset M_{e_x}$, $\sum_i ma_i \otimes_{B_x} a'_i \in M_{e_x} \otimes_{B_x} A_{xy}$, and $\varepsilon_{xy}^M(\sum_i ma_i \otimes_{B_x} a'_i) = m$. This proves that ε_{xy}^M is surjective.

Finally take $\omega = \sum_j m_j \otimes_{B_x} c_j \in \text{Ker}(\varepsilon_{xy}^M) \subset M_{e_x} \otimes_{B_x} A_{xy}$. For each j , we have that

$$c_j = \sum_{\sigma \in G_{xy}} c_{j\sigma} \in \bigoplus_{\sigma \in G_{xy}} A_\sigma.$$

For all $\sigma \in G_{xy}$, we find that $0 = \varepsilon_{xy}^M(\omega)_\sigma = \sum_j m_j c_{j\sigma}$. Using the fact that $c_{j\sigma} a_i \in A_\sigma A_{\sigma^{-1}} = B_x$, we find that

$$\omega_\sigma = \sum_j m_j \otimes_{B_x} c_{j\sigma} = \sum_{i,j} m_j \otimes_{B_x} c_{j\sigma} a_i a'_i = \sum_{i,j} m_j c_{j\sigma} a_i \otimes_{B_x} a'_i = 0.$$

It follows that $\omega = 0$, and this shows that ε_{xy}^M is injective.

(3) \Rightarrow (4) follows from Proposition 3.3(2).

(4) \Rightarrow (1). It follows from Theorem 3.5(3) that there exist maps $\gamma_{xy} : G_{xy} \rightarrow A_{yx} \otimes_{B_x} A_{xy}$ satisfying (17-18). Take $\sigma \in G_{xy}$. (17) can be restated as

$$\sum_i \sum_{\tau \in G_{xy}} l_i(\sigma) r_i(\sigma)_\tau \otimes \tau = 1_y \otimes \tau \in \bigoplus_{\tau \in G_{xy}} A_{yy} \otimes k\tau.$$

Taking the projection of both sides to the component $A_{yy} \otimes k\sigma$, it follows that

$$\sum_i l_i(\sigma) r_i(\sigma)_\sigma \otimes \sigma = 1_y \otimes \sigma,$$

$$1_y = \sum_i l_i(\sigma) r_i(\sigma)_\sigma = \sum_i \sum_{\tau \in G_{yx}} l_i(\sigma)_\tau r_i(\sigma)_\sigma \in A_{yy} = \bigoplus_{\rho \in G_{yy}} A_\rho.$$

Taking the homogeneous components of degree e_y of both sides, we find that

$$\sum_i l_i(\sigma)_{\sigma^{-1}} r_i(\sigma)_\sigma = 1_y \in A_{\sigma^{-1}} A_\sigma.$$

This proves that $A_{\sigma^{-1}} A_\sigma = B_y$. Finally take $\tau \in G_{zx}$ and $a \in A_{\tau\sigma} \subset A_{zy}$. Then

$$a = a 1_y = \sum_i a l_i(\sigma)_{\sigma^{-1}} r_i(\sigma)_\sigma \in A_{\tau\sigma\sigma^{-1}} A_\sigma = A_\tau A_\sigma.$$

Thus $A_{\tau\sigma} = A_\tau A_\sigma$ and A is strongly graded. \square

Remark 8.3. Galois theory for finite groups acting on commutative extensions was introduced in [1], see also [9, 12] for an elegant presentation. It was already observed by Chase and Sweedler [7] that these Galois extensions appear as Hopf-Galois extensions over the Hopf algebra $H = (kG)^*$, the dual of the group algebra kG . One may also consider Hopf-Galois extensions over the group algebra kG itself, and these are precisely strongly graded algebras, an observation that was first made by Ulbrich in [20]. Theorem 8.2 is the proper generalization of Ulbrich's result. What is currently missing is a clear link to the classical theory, involving actions by groupoids, which would make the picture complete. It is true that we have a theory involving actions, see Sections 6 and 7, but this does not bring us what we would expect, since it involves actions by **dual** k -linear categories, while groupoids are ordinary k -linear categories. However, a Galois theory for groupoids acting (even partially) on algebras was developed recently in [4, 17]. The connection to our theory seems unclear, our plan is to investigate this in the future.

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