SOME PROPERTIES OF SEMIREFLEXIVITY

O. Cerbu[1] and D. Botnaru[2]

[1] *State University of Moldova, Str. Al. Mateevici 60, MD-2009 Chisinau, Moldova*
e-mail: olga.cerbu@gmail.com
[2] *Academy of Transport, Computer Science and Communications, Str. Muncesti 121A, Chisinau, Moldova*
e-mail: dumitru.botnaru@gmail.com

Abstract

We examine the properties of semireflexive product, the relations between semireflexive subcategories, the right product of two subcategories and bicategory structures. We construct the examples of semireflexive subcategories and some problems are formulated.

Introduction

Several monographs on the general theory of locally convex spaces are available in the literature, see for example [14]-[18]. Reflexive and semireflexive spaces are defined using the dual space. Different classes of semireflexive spaces have been studied by several authors, see for example [1], [11]-[13], [19], [20]. Actually the definition of a semireflexive space can be stated as follows: a locally convex space is semireflexive if and only if it is quasicomplete in the weak topology. This criterium allows a categorical formulation, leading to the definition of semireflexive product and subcategory, see Definition 4.1. In this paper, we study the properties of reflexive subcategories, the relation between the semireflexive product and the right product. We also look at some examples.

In Section 2, we examine the factorization problem of a reflector functor within a bicategory structure. The notion of k-functor is introduced in Section 3; examples of k-functors are given in Theorem 3.4. Then we introduce property $(\mathcal{SR}t)$, generalizing property $(\mathcal{SR})$ from [4]. These properties allow us to characterize semireflexive subcategories and to construct examples. The right product of two subcategories [8] is examined, and we discuss when the right product satisfies property $(\mathcal{SR}k)$ (see Theorem 3.13).

The construction from Section 4 plays a decisive role in the proof of our main result, Theorem 4.5, establishing necessary and sufficient conditions for a subcategory to be semireflexive. It shows:

1. A semireflexive subcategory can be presented as the semireflexive product of more pairs of subcategories (see [4, Problem 2.7]).

2. The property $(\mathcal{SR})$ of a subcategory $\mathcal{L}$ is equivalent to the property $(\mathcal{SR}t)$ of the subcategory $\mathcal{B} = \mathcal{B}(\mathcal{L})$.

3. As it is shown in [4, Theorem 2.12], property $(\mathcal{SR})$ of the subcategory $\mathcal{L}$ is not equivalent to the property $(\mathcal{SR})$ of the subcategory $\Gamma \in \overline{G}(\mathcal{L})$.

In Section 5, we give examples, draw some conclusions, and formulate a series of problems.

1 Preliminaries on bicategory structures

1.1. $\mathcal{C}_2\mathcal{V}$ will be the category of locally convex Hausdorff topological vector spaces. For details on bicategory structures (or factorization structures), we refer to [1, 7, 8]. In the category $\mathcal{C}_2\mathcal{V}$ we consider the following bicategory structures:
$(\mathcal{E}pi, \mathcal{M}_f)$=(the class of epimorphisms, the class of strict monomorphisms);
$(\mathcal{E}_u, \mathcal{M}_p)$=(the class of universal epimorphisms, the class of exact monomorphisms)=(the class of surjective mappings, the class of topological embeddings);
$(\mathcal{E}_p, \mathcal{M}_u)$=(the class of precise epimorphisms, the class of universal monomorphisms) [6, 9];
$(\mathcal{E}_f, \mathcal{M}ono)$=(the class of strict epimorphisms, the class of monomorphisms).

We will consider the following subcategories:
Π, the subcategory of complete spaces with weak topology [14];
$\mathcal{S}$, the subcategory of spaces with weak topology [14];
$s\mathcal{N}$, the subcategory of strict nuclear spaces [10];
$\mathcal{N}$, the subcategory of nuclear spaces [16];
$\mathcal{S}c$, the subcategory of Schwartz spaces [17];
Γ_0, the subcategory of complete spaces [14];
$q\Gamma_0$, the subcategory of quasicomplete spaces [18];
$s\mathcal{R}$, the subcategory of semireflexive spaces [14, 16, 17, 18];
$i\mathcal{R}$, the subcategory of inductive semireflexive spaces [1];
$\mathcal{M}$, the subcategory of spaces with Mackey topology [17];
The last subcategory is coreflective and the others are reflective.

Definition 1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms in a category $\mathcal{C}$. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary if $fg \in \mathcal{A}$ and $f \in \mathcal{B}$ implies $g \in \mathcal{A}$. In a similar way, we have the dual notion of $\mathcal{B}$-cohereditary class.

Definition 1.3. The composition of two classes of morphisms $\mathcal{A}$ and $\mathcal{B}$ in a category $\mathcal{C}$ is defined as follows:

$$\mathcal{A} \circ \mathcal{B} = \{ab \mid a \in \mathcal{A},\ b \in \mathcal{B} \text{ and the composition } ab \text{ exists}\}.$$

On the class $\mathbb{R}$ of non-zero reflective subcategories of $\mathcal{C}_2\mathcal{V}$, we consider the following order: $\mathcal{R}_1 \leq \mathcal{R}_2$ if $\mathcal{R}_1 \subset \mathcal{R}_2$. On the class of right bicategory structures we consider the order: $(\mathcal{P}_1, I_1) \leq (\mathcal{P}_2, I_2)$ if $\mathcal{P}_1 \subset \mathcal{P}_2$.
Let Π be the subcategory of the complete spaces with weak topology. The subcategory Π is the minimal element of the lattice $\mathbb{R}$. Let $\mathcal{R} \in \mathbb{R}$. Denote by $r^X : X \longrightarrow rX$ the $\mathcal{R}$-replique and by $\pi^X : X \longrightarrow \pi X$ the Π-replique of the object X of the category $\mathcal{C}_2\mathcal{V}$. Since $\Pi \subset \mathcal{R}$, we have

$$\pi^X = v^X r^X$$

for some morphism v^X. Denote by

$$\mathcal{U} = \{r^X \mid X \in \mid \mathcal{C}_2\mathcal{V} \mid\}, \quad \mathcal{V} = \{v^X \mid X \in \mid \mathcal{C}_2\mathcal{V} \mid\}.$$

We put:

$$(\mathcal{P}'', I'') = (\mathcal{P}''(\mathcal{R}), I''(\mathcal{R})) = (\mathcal{V}^{\urcorner}, \mathcal{V}^{\urcorner\llcorner}),$$
$$(\mathcal{P}', I') = (\mathcal{P}'(\mathcal{R}), I'(\mathcal{R})) = (\mathcal{U}^{\llcorner\urcorner}, \mathcal{U}^{\llcorner}).$$

Let $\mathbb{B}$ be the lattice of all bicategory structures in category $\mathcal{C}_2\mathcal{V}$ and $\mathbb{B}_u$ the subclass of bicategory structures $(\mathcal{P}, I)$ with the following properties:

a) $\mathcal{E}_p \subset \mathcal{P}$;

b) the class $\mathcal{P}$ is $\mathcal{M}_u$-hereditary.

Theorem 1.4. *[5] 1. The map*

$$\mathcal{R} \longmapsto (\mathcal{P}''(\mathcal{R}), I''(\mathcal{R}))$$

establishes a one-to-one correspondence between the lattices $\mathbb{R}$ *and* $\mathbb{B}_u$.
2. Let $\varepsilon\mathcal{R} = \{f \in \mathcal{E}pi \mid r(f) \in Iso\}$, *for a reflector functor* $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$. *Then*

$$\mathcal{P}''(\mathcal{R}) = (\varepsilon\mathcal{R}) \circ \mathcal{E}_p \ \ ; \ \ I''(\mathcal{R}) = (\varepsilon\mathcal{R})^{\llcorner} \cap \mathcal{M}_u.$$

2 The factorization of the reflector functor

2.1. Any bicategory structure $(\mathcal{P}, I)$ of the category $\mathcal{C}_2\mathcal{V}$ divides the class $\mathbb{R}$ of non-zero reflective subcategories into three classes (see [2, 21]):

a) The class $\mathbb{R}(\mathcal{P})$ of the $\mathcal{P}$-reflective subcategories;

b) The class $\mathbb{R}(I)$ of the I-reflective subcategories;

c) The class $\mathbb{R}_m = (\mathbb{R}\backslash(\mathbb{R}(\mathcal{P}) \cup \mathbb{R}(I))) \cup \{\mathcal{C}_2\mathcal{V}\}$ - the subcategories which are neither $\mathcal{P}$-reflective nor I-reflective (with the exception of the element $\mathcal{C}_2\mathcal{V}$).
All this classes have $\mathcal{C}_2\mathcal{V}$ itself as the maximal element.

Theorem 2.2. *[21, Theorems 1.3 and 2.2]*
1. The class $\mathbb{R}(\mathcal{P})$ *possesses the minimal element* $\overline{\mathcal{S}}$ *and*

$$\mathbb{R}(\mathcal{P}) = \{\mathcal{R} \in \mathbb{R} \mid \overline{\mathcal{S}} \subset \mathcal{R}\}.$$

2. Let $(I \cap \mathcal{E}pi, (I \cap \mathcal{E}pi)^{\perp})$ *be a right bicategory structure. Then* $\mathbb{R}(I)$ *possesses the minimal element* $\overline{\mathcal{A}}$ *and*

$$\mathbb{R}(I) = \{\mathcal{R} \in \mathbb{R} \mid \overline{\mathcal{A}} \subset \mathcal{R}\}.$$

2.3. We mention, cf. [7], that a class of bicategory structures on $\mathcal{C}_2\mathcal{V}$ satisfying the properties of Theorem 2.2 can be constructed from the injective objects of $\mathcal{C}_2\mathcal{V}$ (see [15]).

2.4. In the case of the bicategory structure $(\mathcal{E}_u, \mathcal{M}_p)$ we have the following division of the lattice $\mathbb{R}$ in three complete sublattices:

a) The sublattice $\mathbb{R}_b$ of the $\mathcal{E}_u$-reflective subcategories. A $\mathcal{R}$ $\mathcal{E}_u$-reflective subcategory is characterized by the fact that the $\mathcal{R}$-replique of every object of the category $\mathcal{C}_2\mathcal{V}$ is a bijection. Another characterization is the following:

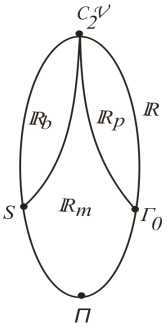
$$\mathbb{R}(\mathcal{E}_u) = \mathbb{R}_b = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\}.$$

b) The sublattice $\mathbb{R}_p$ of the $\mathcal{M}_p$-reflective subcategories, meaning the class of those reflective subcategories $\mathcal{R}$ for which $\mathcal{R}$-replique for any object of category $\mathcal{C}_2\mathcal{V}$ is topological embedding:

$$\mathbb{R}(I_p) = \mathbb{R}_p = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\}.$$

c)
$$\mathbb{R}_m = (\mathbb{R}\backslash(\mathbb{R}_b \cup \mathbb{R}_p)) \cup \{\mathcal{C}_2\mathcal{V}\}.$$

$\mathbb{R}_m$ is also a complete sublattice with first element Π and the last element $\mathcal{C}_2\mathcal{V}$.



2.5. Let $(\mathcal{P}, I)$ be a bicategory structure in the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{L}$ a non-zero reflective subcategory. For any object X of the category $\mathcal{C}_2\mathcal{V}$ let $l^X : X \longrightarrow lX$ bee the $\mathcal{L}$-replique, and

$$l^X = i^X p^X \tag{1}$$

the $(\mathcal{P}, I)$ - factorization of l^X.

$$\begin{tikzcd}
X \arrow[r, "p^X"] \arrow[rr, bend right, "l^X"'] & bX \arrow[r, "i^X"] & lX
\end{tikzcd}$$

Let $\mathcal{B} = \mathcal{B}(\mathcal{L})$ be the full subcategory of the category $\mathcal{C}_2\mathcal{V}$ consisting of objects isomorphic to bX where $X \in |\mathcal{C}_2\mathcal{V}|$. The subcategory $\mathcal{B}$ is $\mathcal{P}$-reflective, and $p^X : X \longrightarrow bX$ is the $\mathcal{B}$-replique of X.

Let $\mathcal{A}'' = \mathcal{A}''(\mathcal{L})$ be the full subcategory of all objects A satisfying the property:
For every $X \in \mathcal{C}_2\mathcal{V}$, every morphism $f : bX \longrightarrow A$ factorizes through i^X:

$$f = gi^X$$

for some morphism g.

$$\begin{tikzcd}
X \arrow[r, "p^X"] & bX \arrow[r, "i^X"] \arrow[d, "f"] & lX \arrow[dl, dashed, "g"] \\
 & A &
\end{tikzcd}$$

$\mathcal{A}''$ is closed under products and $\mathcal{M}_f$-subobjects. So it is reflective, and $i^X : bX \longrightarrow lX$ is the $\mathcal{A}''$-replique of the bX.

Let $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$, $b : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{B}$ and $a'' : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{A}''$ be the respective reflector functors. Then

$$l = a''b. \tag{2}$$

Let $G(\mathcal{L})$ be the class consisting of pairs $(\mathcal{A}, a)$, where $\mathcal{A}$ is an I-reflective subcategory of $\mathcal{C}_2\mathcal{V}$, with reflector functor $a : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{A}$. From the above remarks, we note that the these pairs satisfy the relation $l = ab$.

Let $\overline{G}(\mathcal{L})$ be the class consisting of pairs $(\mathcal{A}, a)$, where $\mathcal{A}$ is a reflective subcategory of $\mathcal{C}_2\mathcal{V}$, with reflector functor a satisfying $l = ab$.

It is possible that $G(\mathcal{L})$ is empty. We also mention that $G(\mathcal{L}) = \overline{G}(\mathcal{L}) \cap \mathbb{R}(I)$. In [2, 21] sufficient conditions for completeness of the lattice $G(\mathcal{L})$ are given.

Theorem 2.6. *Let $(\mathcal{P}, I)$ be a bicategory structure in the category $\mathcal{C}_2\mathcal{V}$, so that $(I \cap \mathcal{E}pi, (I \cap \mathcal{E}pi)^{\perp})$ is a right bicategory structure. Then for every $\mathcal{L} \in \mathbb{R}$ we have:*

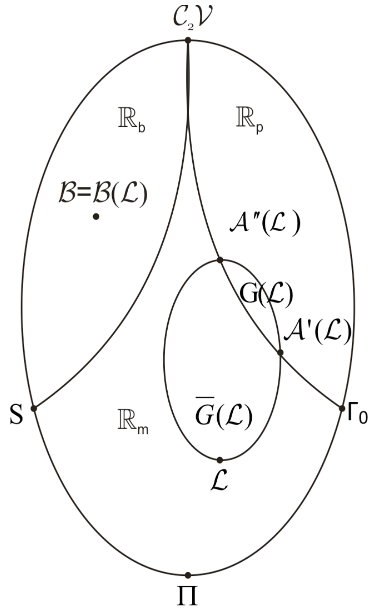
1. *$\mathcal{A}''(\mathcal{L}) \in G(\mathcal{L})$;*
2. *the subcategory $\mathcal{A}' = \mathcal{A}'(\mathcal{L}) = \cap\{\mathcal{R} \mid \mathcal{R} \in G(\mathcal{L})\}$ belongs to $\mathbb{R}(I)$;*
3. *$G(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{A}' \subset \mathcal{R} \subset \mathcal{A}''\}$.*

The class $\overline{G}(\mathcal{L})$ is easier to describe.

Theorem 2.7. *For any bicategory structure $(\mathcal{P}, I)$ we have:*

$$\overline{G}(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{L} \subset \mathcal{R} \subset \mathcal{A}''\}.$$

2.8. Let us examine in detail the situation when $(\mathcal{P}, I) = (\mathcal{E}_u, \mathcal{M}_p)$. For every element $\mathcal{L} \in \mathbb{R}_b$, we have $\mathcal{B}(\mathcal{L}) = \mathcal{L}$, $G(\mathcal{L}) = \{\mathcal{C}_2\mathcal{V}\}$, and $\overline{G}(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{L} \subset \mathcal{R}\}$. If $\mathcal{L} \in \mathcal{R}_p$, then $\mathcal{B}(\mathcal{L}) = \mathcal{C}_2\mathcal{V}$, $G(\mathcal{L}) = \{\mathcal{L}\}$, and $\overline{G}(\mathcal{L}) = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{L} \subset \mathcal{R}\}$.
The remaining case is the nontrivial one: $\mathcal{L} \in \mathbb{R}_m$.



Definition 2.9. Let $(\mathcal{P}, I)$ be a bicategory structure of the category $\mathcal{C}_2\mathcal{V}$. We will say that two reflective subcategories $\mathcal{R}$ and Γ of the category $\mathcal{C}_2\mathcal{V}$ form a $(\mathcal{P}, I)$-pair if $\mathcal{R}$ is a $\mathcal{P}$-reflective subcategory, and the Γ-replique of any object of $\mathcal{R}$ belongs to I.

In particular, if $\mathcal{B} = \mathcal{B}(\mathcal{L})$ and $\Gamma \in \overline{G}(\mathcal{L})$, then $(\mathcal{B}, \Gamma)$ is a $(\mathcal{P}, I)$-pair of reflective subcategories.

3 *k*-Functors

Definition 3.1. A functor $t : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{C}_2\mathcal{V}$ is called a *k*-functor if

$$t(E,u) = (E, t(u)), \quad u \le f(u)$$

for every object (E,u).

3.2. Every non-zero coreflector functor $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ in composition with the embedding functor $i : \mathcal{K} \longrightarrow \mathcal{C}_2\mathcal{V}$ is a k-functor.

3.3. Let $\mathcal{R}$ a reflective, resp. $\mathcal{K}$ a coreflective subcategory of $\mathcal{C}_2\mathcal{V}$. For every $X \in \mathcal{C}_2\mathcal{V}$, let $r^X : X \longrightarrow rX$, resp. $k^{rX} : krX \longrightarrow rX$ be the $\mathcal{R}$-replique of X, resp. the $\mathcal{K}$-coreplique of rX. We construct a pullback of r^X and k^{rX}:

$$r^X t^X = k^{rX} u^X. \tag{3}$$

$$\begin{tikzcd}
tX \arrow[r, dashed, "u^X"] \arrow[d, dashed, "t^X"'] & krX \arrow[d, "k^{rX}"] \\
X \arrow[r, "r^X"] & rX
\end{tikzcd}$$

Theorem 3.4. *We have a k-functor* $t : \mathcal{C}_2\mathcal{V} \to \mathcal{C}_2\mathcal{V}$*, called the k-functor generated by the reflective subcategory* $\mathcal{R}$ *and the coreflective subcategory* $\mathcal{K}$

Proof. First, since k^{rX} is a bijective map, so is t^X. Therefore the objects X and tX have the same vector spaces as support and the topology on tX is stronger than the topology on X.
Now we define the functor t on the morphisms. Fix a morphism $f : X \longrightarrow Y$ in $\mathcal{C}_2\mathcal{V}$. We examine the pullbacks described above corresponding to X and Y.

$$\begin{tikzcd}
tX \arrow[rr, "u^X"] \arrow[dr, dashed, "g=t(f)"] \arrow[dd, "t^X"'] & & krX \arrow[dr, dashed, "f_2"] \arrow[dd, "k^{rX}" near start] & \\
& tY \arrow[rr, "u^Y" near start] \arrow[dd, "t^Y" near end] & & krY \arrow[dd, "k^{rY}"] \\
X \arrow[rr, "r^X" near end] \arrow[dr, "f"'] & & rX \arrow[dr, dashed, "f_1"] & \\
& Y \arrow[rr, "r^Y"] & & r^Y
\end{tikzcd}$$

In particular, we have the pullback constructed from Y:

$$r^Y t^Y = k^{rY} u^Y. \tag{4}$$

For the morphism $r^Y f$, there exists a unique morphism f_1 such that

$$r^Y f = f_1 r^X. \tag{5}$$

For the morphism $f_1 k^{rX}$, there exists a unique morphism f_2 such that

$$f_1 k^{rX} = k^{rY} f_2. \tag{6}$$

We now have

$$r^Y f t^X \overset{(5)}{=} f_1 r^X t^X \overset{(3)}{=} f_1 k^{rX} u^X \overset{(6)}{=} k^{rY} f_2 u^X,$$

or

$$r^Y (f t^X) = k^{rY} (f_2 u^X). \tag{7}$$

Taking into account that (4) is a pullback, it follows from (7) that there exists a unique morphism g such that

$$\begin{aligned} ft^X &= t^Y g; \qquad (8) \\ f_2 u^X &= u^Y g. \qquad (9) \end{aligned}$$

We define $g = t(f)$. In (8) t^Y is a mono. So we deduce that the morphism g which verifies the equality (8) is unique. It follows that $t(1) = 1$ and $t(fh) = t(f)t(h)$. This completes the proof. □

Remark 3.5. We mention that a k-functor is not always a coreflector functor, since a k-functor is not necessarily idempotent.

3.6. Let $t : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{C}_2\mathcal{V}$ be a k-functor, and $\mathcal{R}$ a reflective subcategory. We consider the following property $(\mathcal{S}\mathcal{R}t)$:

$(\mathcal{S}\mathcal{R}t)$ If $(E, u) \in |\mathcal{R}|$ and v is a locally convex topology on the vector space E such that $u \leq v \leq t(u)$, then $(E, v) \in |\mathcal{R}|$.

Remark 3.7. 1. Property $(\mathcal{S}\mathcal{R})$ from [4] may be written as condtion $(\mathcal{S}\mathcal{R}m)$, where $m : \mathcal{C}_2\mathcal{V} \to \mathcal{M}$ is the coreflector functor, and $\mathcal{M}$ is the coreflective subcategory of spaces with the Mackey topology.
2. $(\mathcal{S}\mathcal{R}t)$ can be reformulated in a categorical way:

$(\mathcal{S}\mathcal{R}t)$ If $X \in |\mathcal{R}|$ and $f : Y \to X$ is a monomorphism such that $t^X = fg$ for some morphisms G, then $Y \in |\mathcal{R}|$.

$$\begin{tikzcd} & tX \arrow[dl, "g"'] \arrow[d, "t^X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

From the fact that t^X is bijective, it easily follows that f is bijective. Then it follows that g is also bijective.

We will now examine property $(\mathcal{S}\mathcal{R})$ for elements of $\mathbb{R}_b$, $\mathbb{R}_p$ and $\mathbb{R}_m$.

Theorem 3.8. *1. ([4, Theorem 2.12]) Every element of* $\mathbb{R}_p$ *satisfies* $(\mathcal{S}\mathcal{R})$.
2. If $\mathcal{L} \in \mathbb{R}_p$ *satisfies* $(\mathcal{S}\mathcal{R})$*, then* $\mathcal{L} = \mathcal{C}_2\mathcal{V}$.

Proof. 2. Take an arbitrary $X \in |\mathcal{C}_2\mathcal{V}|$, and its $\mathcal{L}$-replique $l^X : X \longrightarrow lX$. It follows from the hypothesis that $l^X \in \mathcal{E}_u$, so $l^X \in \mathcal{E}_u \cap \mathcal{M}_u$, since every non-zero relective subcategory is monoreflective, and also $\mathcal{M}_u$-reflective. Therefore $lX \in |\mathcal{L}|$ and $l^X \in \mathcal{E}_u \cap \mathcal{M}_u$. Then it follows from the hypothesis that $X \in |\mathcal{L}|$. □

3.9. We now formulate two problems.

A. Describe the elements of the lattice $\mathcal{R}$ that satisfy $\mathcal{S}\mathcal{R}$.

B. Describe the elements of the lattice $\mathcal{R}_m$ that satisfy $\mathcal{S}\mathcal{R}$.

Observe that Theorem 3.8 reduces problem A to problem B.

3.10. Let $\mathcal{K}$ be a coreflective subcategory and $\mathcal{R}$ a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ with respective functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$. For any object $X \in |\mathcal{C}_2\mathcal{V}|$, let $k^X : X \longrightarrow kX$, $r^X : X \longrightarrow rX$ be the $\mathcal{K}$-coreplique and the $\mathcal{R}$-replique, and let $k^{rX} : krX \longrightarrow rX$ be the $\mathcal{K}$-coreplique of the object rX. Then

$$r^X k^X = k^{rX} k(r^X). \tag{10}$$

On the morphisms k^X and $k(r^X)$ we construct the copullback

$$n^X k^X = g k(r^X). \tag{11}$$

From (10), it follows that there exists a morphism u^X such that

$$\begin{aligned} u^X n^X &= r^X; \quad (12) \\ u^X g &= k^{rX}. \quad (13) \end{aligned}$$

Since $k^X \in \mathcal{E}_u$, and (11) is a copullback, it follows that $g \in \mathcal{E}_u$ and from (13) it results that $g \in \mathcal{M}ono$. Therefore g is a bijective map. Since k^{rX} is also a bijective map, we conclude that u^X is bijective too. Then from the fact that u^X is *mono* and from equality (13) it follows that g is the $\mathcal{K}$-coreplique of the object nX.

$$\begin{tikzcd}
kX \arrow[rr, "k(r^X)"] \arrow[dd, "k^X"'] & & krX \arrow[dl, dashed, "g=k^{nX}"'] \arrow[dd, "k^{rX}"] \\
& nX \arrow[dr, dashed, "u^X"] & \\
X \arrow[ur, dashed, "n^X"] \arrow[rr, "r^X"'] & & rX
\end{tikzcd} \tag{14}$$

We denote by $\mathcal{K} \times_d \mathcal{R}$ the full subcategory of $\mathcal{C}_2\mathcal{V}$ consisting of objects isomorphic to objects of the form nX.

Definition 3.11. [8] The subcategory $\mathcal{K} \times_d \mathcal{R}$ is called the right product of the subcategories $\mathcal{K}$ and $\mathcal{R}$.

Theorem 3.12. *[8, Theorem 2.5] For $\mathcal{V} = \mathcal{K} \times_d \mathcal{R}$, the following statements are equivalent.*

1. *$\mathcal{V}$ is a reflective subcategory of $\mathcal{C}_2\mathcal{V}$;*
2. *for any $X \in |\mathcal{C}_2\mathcal{V}|$, the morphism n^X is the $\mathcal{V}$-replique of X;*
3. *for any $X \in |\mathcal{C}_2\mathcal{V}|$, the morphism n^X is an epi;*
4. *for any $X \in |\mathcal{C}_2\mathcal{V}|$, the morphism u^X is the $\mathcal{R}$-replique of nX;*
5. *$X \in |\mathcal{V}| \iff r^X k^X$ is the $\mathcal{K}$-replique of rX.*

The right product $\mathcal{B} = \mathcal{K} \times_d \mathcal{R}$ of two subcategories is often a reflective subcategory, see [8, Theorem 2.5] and [3, Theorem 5.2].

Theorem 3.13. *Let $\mathcal{K}$ be a non-zero coreflective and $\mathcal{R}$ a non-zero reflective subcategory of $\mathcal{C}_2\mathcal{V}$, and assume that the right product $\mathcal{B} = \mathcal{K} \times_d \mathcal{R}$ is a reflective subcategory. If the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is a monofunctor, then $\mathcal{B}$ satisfies $(\mathcal{S}\mathcal{R}k)$, where $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ is a coreflector functor.*

Proof. In [3, Sec. 2], it was examined when a reflector functor is a monofunctor. We examine the diagram (14), and let

$$g = fh \tag{15}$$

be two bimorphisms, so $f, g \in \mathcal{E}_u \cap \mathcal{M}ono$. Since $k^X \in \mathcal{E}_u$ and

$$n^X k^X = gk(r^X) \tag{16}$$

is a copullback, it follows that $g \in \mathcal{E}_u$. Furthermore $k^{rX} \in \mathcal{M}ono$, and from the equality

$$k^{rX} = u^X g, \tag{17}$$

it follows that g is a mono, and therefore the morphisms k^{rX}, g and u^X are bijective. It follows from Theorem 3.12 that u^X is the $\mathcal{R}$-replique of nX, and $g = k^{nX}$ is the $\mathcal{K}$-coreplique of nX. Since f is a mono it follows that h is the $\mathcal{K}$-coreplique of the Y, $h = k^Y$. We have the following commutative square

$$\begin{aligned} r^Y k^Y &= k^{rY} k(r^Y) && (18) \\ u^X f &= r(f) r^Y && (19) \end{aligned}$$

It follows from the assumptions that $r(f)$ is a *mono*. Then $r^Y k^Y$ is the $\mathcal{K}$-coreplique of rY. Therefore $k(r^Y)$ and, a fortiori, n^Y is also an *iso*, and it follows that $Y \in |\mathcal{B}|$.

$$\begin{tikzcd}
kX \arrow[rrr, "k(r^X)"] \arrow[dddd, "k^X"'] & & & krX \arrow[rrr, dashed, "k(r^Y)"] \arrow[dddll, dashed, "g=k^{nX}"'] \arrow[ddl, dashed, "h=k^Y"] \arrow[dddd, "k^{tX}"] & & & krY \arrow[dll, dashed, "k^{nY}"] \arrow[ddl, dashed, "k^{rY}"] \\
& & & & nY \arrow[dr, dashed, "u^Y"] & & \\
& & Y \arrow[urr, dashed, "n^Y"] \arrow[rrr, "r^Y"] \arrow[dl, dashed, "f"] & & & rY \arrow[ddll, dashed, "r(f)"'] & \\
& nX \arrow[drr, dashed, "u^X"'] & & & & & \\
X \arrow[ur, dashed, "n^X"] \arrow[rrr, "r_X"'] & & & rX & & &
\end{tikzcd}$$

□

4 Semireflexive product of two subcategories

Definition 4.1. [3]

1. Let $\mathcal{R}$ be a reflective subcategory, and $\mathcal{A}$ a subcategory of the category $\mathcal{C}$. $X \in |\mathcal{C}|$ is called $(\mathcal{R}, \mathcal{A})$-semireflexive, if its $\mathcal{R}$-replique belongs to $\mathcal{A}$.

2. The full subcategory $\mathcal{L}$ of all $(\mathcal{R},\mathcal{A})$-semireflexive objects is called the semireflexive product of the subcategories $\mathcal{R}$ and $\mathcal{A}$, and is denoted by $\mathcal{L}=\mathcal{R}\times_{sr}\mathcal{A}$.

3. The subcategory $\mathcal{L}\in\mathbb{R}_m$ of the category $\mathcal{C}_2\mathcal{V}$ is called semireflexive if there exists a reflective subcategory $\mathcal{R}\in\mathbb{R}_b$ and a reflective subcategory $\Gamma\in\mathbb{R}_p$ of the category $\mathcal{C}_2\mathcal{V}$ so that $\mathcal{L}=\mathcal{R}\times_{sr}\Gamma$.

Remark 4.2. The respective conditions from the definition of the semireflexive subcategories have been imposed to exclude the trivial cases. Since for every subcategory $\mathcal{L}$ of the category $\mathcal{C}$ can be presented as

$$\mathcal{L}=\mathcal{C}\times_{sr}\mathcal{L}.$$

4.3. Let $(\mathcal{P},I)$ be a bicategory structure in the category $\mathcal{C}_2\mathcal{V}$, and $\mathcal{L}$ a non-zero reflective subcategory. Assume that the $(\mathcal{P},I)$-factorization of the reflector functor $l:\mathcal{C}_2\mathcal{V}\longrightarrow\mathcal{L}$ generates the $\mathcal{P}$-reflective subcategory $\mathcal{B}=\mathcal{B}(\mathcal{L})$ and the lattice $\overline{G}(\mathcal{L})$. Let $\Gamma\in\overline{G}(\mathcal{L})$. Now we consider the following conditions.

A. $\mathcal{L}=\mathcal{B}\times_{sr}\Gamma$, where $\mathcal{B}=\mathcal{B}(\mathcal{L})$ and $\Gamma\in\overline{G}(\mathcal{L})$.

B. There exists a $(\mathcal{P},I)$-pair of reflective subcategories $\mathcal{R}$ and $\mathcal{T}$ of the category $\mathcal{C}_2\mathcal{V}$ such that $\mathcal{L}=\mathcal{R}\times_{sr}\mathcal{T}$.

C. The subcategory $\mathcal{L}$ is closed under taking $(\mathcal{P}\cap\mathcal{M}_u)$-subobjects.

D. The subcategory $\mathcal{L}$ verifies the condition $(\mathcal{SR})$ - the subcategory $\mathcal{L}$ is closed under $(\mathcal{E}_u\cap\mathcal{M}_u)$-subobjects.

E. The subcategory $\mathcal{B}=\mathcal{B}(\mathcal{L})$ verifies the condition $(\mathcal{SR}t)$ for the k-functor $t:\mathcal{C}_2\mathcal{V}\longrightarrow\mathcal{C}_2\mathcal{V}$ generated by the reflective subcategory $\Gamma\in\overline{G}(\mathcal{L})$ and the coreflective subcategory $\mathcal{M}$ of the spaces with the Mackey topology.

Lemma 4.4. *1. If the conditions A-E of (4.3) are satisfied, then*

$$\mathcal{L}\subset\mathcal{B}\times_{sr}\Gamma$$

where $\mathcal{B}=\mathcal{B}(\mathcal{L})$ *and* $\Gamma\in\overline{G}(\mathcal{L})$.
2. For objects of the subcategory $\mathcal{L}$ $(\mathcal{L}\subset\mathcal{B}(\mathcal{L}))$, *the properties* $(\mathcal{SR}t)$ *and* $(\mathcal{SR})$*are equivalent.*

Proof. 1. Let $X\in\mid\mathcal{L}\mid$. Then $X\in\mid\mathcal{B}\cap\Gamma\mid$, since $\mathcal{B}\cap\Gamma=\mathcal{L}$. $\mathcal{B}$-replique of the object X is an *iso*, and $bX\in\mid\Gamma\mid$.
2. Let $X\in\mid\mathcal{L}\mid$. Since $\mathcal{L}\subset\Gamma$ the Γ-replique of the object X $g^X:X\longrightarrow gX$ is an *iso*.
Let $m^{lX}:mgX\longrightarrow gX$ be the $\mathcal{M}$-replique of the object gX, and

$$g^Xt^X=m^{gX}u^X$$

the copullback constructed on the morphisms g^X and m^{gX}. Since g^X is an *iso* it follows that u^X is also an *iso*. Therefore $tX\in\mid\Gamma\mid\Leftrightarrow mgX\in\mid\Gamma\mid$. This way X has the $(\mathcal{SR}t)$ property if and only if gX has the $(\mathcal{SR})$ property. Because the X and gX objects are isomorphic, we obtain that the object X from the subcategory $\mathcal{L}$ has the $(\mathcal{SR}t)$ property if and onl if it has the $\mathcal{SR}$ property. ☐

Theorem 4.5. *With notation as in (4.3), we have the following implications.*

1. *$C \Longrightarrow A \Longrightarrow B$.*
2. *If the that class $\mathcal{P}$ is $\mathcal{M}_u$-hereditary, then $B \Longrightarrow C$.*
3. *If $\mathcal{P} \subset \mathcal{E}_u$, then $E \Longrightarrow D \Longrightarrow C$.*
4. *If I is right stable, then $D \Longrightarrow E$.*

$$\begin{tikzcd}
A \arrow[rr, Rightarrow] & & B \arrow[d, Rightarrow, "\text{The class } \mathcal{P} \text{ is } \mathcal{M}_u\text{-hereditary}"] \\
D \arrow[rr, Rightarrow, "\mathcal{P}\subset\mathcal{E}_u"] \arrow[d, Rightarrow, shift right, "\text{The class } I \text{ is right stable}"'] & & C \arrow[ull, Rightarrow] \\
E \arrow[u, Rightarrow, shift right, "\mathcal{P}\subset\mathcal{E}_u"'] \arrow[urr, Rightarrow, "\mathcal{P}\subset\mathcal{E}_u"'] & &
\end{tikzcd}$$

Proof. $A \Longrightarrow B$ is obvious.

$C \Longrightarrow A$.It follows from Lemma 4.4 that $\mathcal{L} \subset \mathcal{B} \times_{sr} \Gamma$. We will prove the converse inclusion. Let $X \in |\mathcal{B} \times_{sr} \Gamma|$. Then $b^X : X \longrightarrow bX \in (\mathcal{P} \cap \mathcal{M}_u)$, and $bX \in |\Gamma|$, hence $bX \in |\mathcal{L}|$. So X is a $(\mathcal{P} \cap \mathcal{M}_u)$-subobject of $bX \in |\mathcal{L}|$, and $X \in |\mathcal{L}|$.

$B \Longrightarrow C$. Let $f : Y \longrightarrow X \in (\mathcal{P} \cap \mathcal{M}_u)$ and $X \in |\mathcal{L}|$. We examine the $\mathcal{R}$-repliques of the objects X and Y: $r^X : X \longrightarrow rX$ and $r^Y : Y \longrightarrow rY$. Let $t^{rY} : rY \longrightarrow trY$ be the $\mathcal{T}$-replique of the object rY. Than $r^X \in \mathcal{P}$, $t^{rY} \in I$. Since $X \in |\mathcal{L}|$ we deduce that $rX \in |\mathcal{T}|$. Therefore

$$r^X f = g r^Y \tag{20}$$

for some morphism g, and

$$g = h t^{rY} \tag{21}$$

for some morphism h. Since

$$h t^{rY} r^Y = r^X f \in \mathcal{M}_u, \tag{22}$$

$t^{rY} r^Y \in \mathcal{E}pi$ and the class $\mathcal{M}_u$ is $\mathcal{E}pi$-cohereditary, it follows that $h \in \mathcal{M}_u$. In (22), $r^X f \in (\mathcal{P} \cap \mathcal{M}_u)$, meaning that $h t^{rY} r^Y \in \mathcal{P}$ and $h \in \mathcal{M}_u$. The hypothesis implies that the class $\mathcal{P}$ is $\mathcal{M}_u$-hereditary. So $t^{rY} r^Y \in \mathcal{P}$, and this implies that $t^{rY} \in \mathcal{P}$. Finally $t^{rY} \in (\mathcal{P} \cap I) = Iso$. Therefore $rY \in |\mathcal{T}|$, so $Y \in |\mathcal{L}|$.

$$\begin{tikzcd}
Y \arrow[r, "r^Y"] \arrow[d, "f"] & rY \arrow[r, "t^{rY}"] \arrow[d, dashed, "g"] & trY \arrow[dl, dashed, "h"] \\
X \arrow[r, "r^X"] & rX &
\end{tikzcd}$$

$E \Longrightarrow D$ follows from Lemma 4.4.

$D \Longrightarrow C$. It is obvious that $(\mathcal{P} \cap \mathcal{M}_u) \subset (\mathcal{E}_u \cap \mathcal{M}_u)$.

$D \Longrightarrow E$. Let $X \in |\mathcal{B}|$. We construct the object tX. Let $g^X : X \longrightarrow gX$ be the Γ-replique, $m^{gX} : mgX \longrightarrow gX$ the $\mathcal{M}$-coreplique of gX, and

$$g^X t^X = m^{gX} u^X \tag{23}$$

the pullback constructed on the morphisms g^X and m^{gX}. Let $f : Y \longrightarrow X$ be a *mono* so that

$$t^X = fg \tag{24}$$

for some morphism g. We have to show that $Y \in |\mathcal{B}|$.
On the morphisms u^X and g we construct the copullback

$$u'g = g'u^X. \tag{25}$$

$$\begin{tikzcd}
tX \arrow[rr, dashed, "u^X"] \arrow[d, "g"] \arrow[dd, bend right, "t^X"'] & & mgX \arrow[dl, dashed, "g'"'] \arrow[dd, "m^{gX}"] \\
Y \arrow[r, dashed, "u'"] \arrow[d, "f"] & T \arrow[dr, dashed, "h"] & \\
X \arrow[rr, "g^X"] & & gX
\end{tikzcd}$$

As we have mentioned, $u^X \in I$. Moreover, we have

$$m^{gX} = hg' \ ; \ g^X f = hu' \tag{26}$$

for some morphism h. In the copullback (25), $u^X \in I$, hence $u' \in I$, since I is right stable. We also have that $g \in \mathcal{E}_u \cap \mathcal{M}_u$, hence $g' \in \mathcal{E}_u \cap \mathcal{M}_u$, and $h \in \mathcal{E}_u \cap \mathcal{M}_u$. According to the hypothesis, $X \in| \mathcal{B} |$, so its Γ-replique is also $\mathcal{L}$-replique. Therefore $gX \in |\mathcal{L}|$. It follows that $T \in |\mathcal{L}| \subset |\mathcal{B}|$, and Y, as an I-subobject of T, belongs to the subcategory $\mathcal{B}$. □

Theorem 4.6. *In the case when* $(\mathcal{P}, I) = (\mathcal{E}_u, \mathcal{M}_p)$ *the conditions A-E in (4.3) are equivalent.*

5 Examples, conclusions, problems

5.1. Let $q\Gamma_0$ be a subcategory of the category of quasicomplete spaces, and $s\mathcal{R}$ the subcategory consisting the semireflexive spaces [18]. Then

$$\mathcal{S} \times_{sr} (q\Gamma_0) = s\mathcal{R}.$$

and

$$\mathcal{N} \times_{sr} (q\Gamma_0) = s\mathcal{R},$$

where $\mathcal{S}$ is the subcategory of space with weak topology [14], and $\mathcal{N}$ is the subcategory of nuclear spaces.

5.2. For the subcategory $\mathcal{S}c$ of the Schwartz spaces and the subcategory Γ_0 of the complete spaces we have

$$\mathcal{S}c \times_{sr} \Gamma_0 = \mathcal{K} \times_d (\mathcal{S}c \cap \Gamma_0) = i\mathcal{R},$$

where $i\mathcal{R}$ is the subcategory of the semireflexive inductive spaces ([1, Theorem 1.5]), and $\mathcal{K}$ is the coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$, which forms with the subcategory $\mathcal{S}c$ a pair of conjugate subcategories (see [3]).

5.3. The subcategory Π of the complete spaces with the weak topology is semireflexive. For the case $(\mathcal{P}, I) = (\mathcal{E}_u, \mathcal{M}_p)$, we have $\mathcal{B}(\Pi) = \mathcal{S}$, the subcategory of spaces with the weak topology, $\mathcal{A}'(\Pi) = \Gamma_0$, and $\mathcal{A}''(\Pi)$ contains all the normed spaces. From that results $G(\Pi)$ is a proper class.

For more examples, we refer to [3, 4].

5.4. Condition D from (4.3) indicates the fact that the semireflexivity of a subcategory is independent of the bicategory structure $(\mathcal{P}, I)$.

Definition 5.5. The subcategory $\mathcal{A}$ of the category $\mathcal{C}_2\mathcal{V}$ is called closed under extensions, if the following property holds: if $f : A \longrightarrow B \in \mathcal{E}pi \cap \mathcal{M}_p$ and $A \in |\mathcal{A}|$, then $B \in |\mathcal{A}|$.

Problem 5.6. Let $\mathcal{R}$ be a reflective subcategory closed under extensions, and $\mathcal{K}$ a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. When is the right product $\mathcal{K} \times_d \mathcal{R}$ of the subcategories $\mathcal{K}$ and $\mathcal{R}$ closed under extensions?

5.7. Let $\mathcal{B} = \mathcal{K} \times_d \mathcal{R}$ be a reflective subcategory (see [8, Theorem 2.5] and [3, Theorem 5.2]), let $\mathcal{B}$ be closed under extensions. Then for every $\Gamma \in \mathbb{R}_p$ we have that

$$\mathcal{B} \cap \Gamma = \mathcal{B} \times_{sr} \Gamma_1, \quad \Gamma_1 \in G(\mathcal{B} \cap \Gamma).$$

In view of Theorem 3.13, the subcategory $\mathcal{B}$ can satisfy condition $(\mathcal{S}\mathcal{R}k)$, with $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ the coreflector functor.

5.8. In many cases, semireflexive subcategories can be presented as the right product of some subcategories (see [3, Theorem 5.4]).

Problem 5.9. Is it true that every semireflexive subcategory can be written as the right product of two subcategories?

References

[1] I. A. Berezansky, Inductive reflexive locally convex spaces (Russian), *DAN SSSR*, **189**, No.1 (1968), 20–22.

[2] D. Botnaru, Composition and commutativity of reflexive functors (Russian), *Functional analysis* **21**, 59–71, Ul'yanovsk. Gos. Ped. Inst., Ul'yanovsk, 1983.

[3] D. Botnaru and O. Cerbu, Semireflexive product of two subcategories, to appear.

[4] D. Botnaru and O. Cerbu, Semireflexive product of two subcategories, in "Proceeding of the Sixth Congress of Romanian Mathematicians", Bucharest, 2007, I (2007), p. 5–19.

[5] D. Botnaru and O. Cerbu, *Semireflective subcategories*, to appear.

[6] D. Botnaru and V.B. Gysin, Stable monomorphisms in the category of separated locally convex spaces (Russian), Bul. Akad. Štiince RSS Moldoven, 1973, no. 1, 3–7.

[7] D. Botnaru , O. Cerbu and A. Ţurcanu, Bicategory structures generated by injective objects, *Romai J.* **3** (2007), 31–50.

[8] D. Botnaru and A. Ţurcanu, *Les produits de gauche et de droite de deux souscatégories*, Acta et Com., VIII, Chisinau, 2003, p.57-73.

[9] D. Botnaru and A. Ţurcanu, *On Giraux subcategories in locally convex spaces*, Romai J. **1** (2005), 7–30.

[10] B. S. Brudovsky, The additional nuclear topology, transformation of type s and the strict nuclear spaces, *Dokl. Acad. Nauk, SSSR*, **178** (1968), 271–273.

[11] J. Dazord and U. Jourlin, Sur quelques classes d' espaces localement convexes, *Publ. Dép. Math. (Lyon)* **8** (1971), 39–69.

[12] J. Dazord and U. Jourlin, Sur les prècompacts d'un espace localement convexe, *C. R. Acad. Sci. Paris Sr. A-B* **274** (1972), A463–A466.

[13] J. Dazord and U. Jourlin, Sur les prècompacts d'un espace localement convexe, *C. R. Acad. Sci. Paris Sr. A-B* **274** (1972), A550–A553.

[14] A. Grothendieck, "Topological vector spaces", Gordon and Breach, New York, London, Paris, 1973.

[15] V. P. Palamodov, Homological methods in the theory of locally convex Spaces (Russian), *Uspehi Mat. Nauk* **26** (1971), 3–65.

[16] A. Pietsch, "Nuclear locally convex spaces", *Ergebnisse Mathematik Grenzgebiete* **66**, Springer-Verlag, Berlin, 1972.

[17] A.P. Robertson, W.J. Robertson, "Topological vector spaces", *Cambridge Tracts Math.* **53**, Cambridge University Press, Cambridge, 1964.

[18] H.H. Schaefer, "Topological vector spaces", Macmillan, New York; Collier-Macmillan, London, 1973.

[19] V.S. Secovanov, α-reflexivity locally convex spaces (Russian), *Functional analysis Ul'yanovsk*, Operator theory, 1981, 111–117.

[20] V.S. Secovanov, About two generalizes concepts of reflexivity locally convex spaces (Russian), *Math. Notes* **35** (1984), 415–424.

[21] A. Ţurcanu, The factorization of reflective functors, *Bull. Inst. Politechn., Iasi* LII (LVII) (2007), 377–391.