A DIFFERENTIAL CALCULUS ON h-SUPERSPACE

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Abstract

In this work, a non-commutative differential calculus on the h -superspace is presented via a contraction of the q-superspace. An R matrix which satisfies both ungraded and graded Yang-Baxter equations is obtained.

Introduction

Quantum Groups are a generalization of the concept of classical groups. The theory of quantum groups has an important role in Mathematical Physics. A possible approach to quantum groups is obtained by deforming the coordinates of a linear space to be non-commuting objects. In this scheme the quantum group structure appears if one considers linear transformations which preserve the algebraic properties of the algebra of coordinates.

Noncommutative differential geometry of quantum groups was brought forward by Woronowicz [1]. Recently Wess and Zumino [2] have shown that a consistent quantum deformation of the differential calculus is satisfied by an R-matrix which can be any solution of the quantum Yang-Baxter equation.

h-deformation is known as Jordanian deformation. Aghamohammadi and his friends [3], using the q deformation of the plane by a singular matrix, obtained the h-deformation of the plane. In Ref. 4, it is developed a differential calculus on the quantum superplane, using a singular transformation. We shall now follow the approach.

The aim of this study, using the q-deformation of the coordinate functions on the 3-dimensional space, with a singular transformation, is to develop the h-superspace and set a non-commutative differential calculus on it. Therefore, first a $3x3$ singular matrix will be determined. Then, other steps are about differential calculus with the approach of Wess and Zumino.

In this work we denote q-deformed objects by primed quantities. Unprimed quantities represent transformed coordinates. As usual, we assume that even (bosonic) objects commute with everything and odd (grassmann) objects anti-commute among themselves.

1 Quantum h-superspace

Consider the quantum superspace introduced by Manin [5]. The commutation relations between the even coordinates x', y' and the odd coordinate θ' of the quantum superspace is in the form

$$x' y' - qy' x' = 0, \quad x' \theta' - q\theta' x' = 0, \quad y' \theta' - q\theta' y' = 0, \tag{1}$$

where q is a complex deformation parameter. We now introduce new coordinates x, y and θ, in terms of x', y' and θ' as

$$x' = x, \;\; y' = y, \;\; \theta' = \theta + \frac{h}{q-1} x \tag{2}$$

As in [3] This transformation is singular in the $q \to 1$ limit. Using (1), it is easy to verify that

$$x\,y = qy\,x, \;\; x\,\theta = q\theta\,x + hx^2, \;\; y\,\theta = q\theta\,y + (1+q)hy\,x, \tag{3}$$

where the new deformation parameter h commutes with the coordinates x and y. Also, since the Grassmann coordinate θ' satisfies $\theta'^{\,2} = 0$, one obtains that

$$\theta^2 = -h\theta\,x, \tag{4}$$

where h anti-commutes with θ and

$$h^2 = 0, \tag{5}$$

that is, the new deformation parameter h is a Grassmann number [6]. Taking the $q \to 1$ limit we obtain the following relations which define the $h-$superspace

$$x\,y = y\,x, \;\; x\,\theta = \theta\,x + hx^2, \;\; y\,\theta = \theta\,y + 2hy\,x, \;\; \theta^2 - h\theta\,x\,. \tag{6}$$

There is an important difference between the quantum superspace (6) and the one discussed in [7], associated to the Jordanian quantum super group $GL_h(2 \mid 1)$. The deformation parameter h in this work is a Grassmannian, and this is not the case in [7].

2 Relations between coordinates and differentials

To establish a non-commutative differential calculus on the quantum h -superspace, we assume that the commutataion relations between the coordinates and their differentials are in the following form

$$\begin{array}{rcl} x'\,dx' &=& Adx'\,x', \quad x'\,dy' = A_{11}dy'\,x' + A_{12}dx'y', \quad y'dx' = A_{21}dx'y' + A_{22}dy'x', \\ y'\,dy' &=& Bdy'\,y', \quad x'\,d\theta' = B_{11}d\theta'\,x' + B_{12}dx'\,\theta', \quad \theta'\,dx' = B_{21}dx'\,\theta' + B_{22}d\theta'x', \\ \theta'd\theta' &=& Cd\theta'\,\theta', \quad y'\,d\theta' = C_{11}d\theta'\,y' + C_{12}dy'\,\theta', \quad \theta'\,dy' = C_{21}dy'\,\theta' + C_{22}d\theta'\,y', \end{array} \tag{7}$$

We impose the following conditions on an exterior differential d:

$$d^2 = 0, \tag{8}$$

and the graded Leibnitz rule

$$d(fg) = (df\,)g + (-1)^{\widehat{f}}\,f(dg), \tag{9}$$

where $\widehat{f}$ is the Grassmann degree of f (recall that d should be odd), that is, $\widehat{f} = 0$ for even variables and $f = 1$ for odd variables. Considering the differentials of a function and differentiating (2), we find

$$dx' = dx, \;\; dy' = dy, \;\; d\theta' = d\theta - \frac{h}{q-1} dx\,. \tag{10}$$

Substituting (2) and (10) into (7), we find

$$
\begin{aligned}
x\,dx &= A\mathrm{d}x\,x, \quad x\,dy = A_{11}dy\,x + A_{12}dx\,y, \quad y\,dx = A_{21}dx\,y + A_{22}dy\,x,\\
y\,dy &= Bdy\,y, \quad x\,\mathrm{d}\theta = B_{11}d\theta\,x + B_{12}dx\,\theta + \frac{h}{q-1}(A - B_{11} - B_{12})dx\,x,\\
\theta\,dx &= B_{21}dx\,\theta + B_{22}d\theta\,x - \frac{h}{q-1}(A + B_{21} + B_{22})\,dx\,x,\\
\theta\,d\theta &= Cd\theta\,\theta - \frac{h}{q-1}\left[(B_{21} + B_{12} + C)\,dx\,\theta - (C - B_{11} - B_{12})d\theta\,x\right],\\
y\,d\theta &= C_{11}d\theta\,y + C_{12}dy\,\theta + \frac{h}{q-1}\left[(A_{21} - C_{11})\,dx\,y + (A_{22} - C_{12})dy\,x\right],\\
\theta\,dy &= C_{21}dy\,\theta + C_{22}d\theta\,y - \frac{h}{q-1}\left[(A_{11} + C_{21})\,dy\,x + (A_{12} + C_{22})dx\,y\right]\;.
\end{aligned}
\tag{11}
$$

The coefficients A, B, C, A_{ij}, B_{ij} and C_{ij} given in (11) can be related to q by the consistency of calculus. Thus we apply the exterior derivative d to the relation (4). From the consistency condition

$$
\begin{aligned}
0 &= d(xy - qyx)\;,\\
0 &= d(x\theta - q\theta x - hx^2)\;,\\
0 &= d(y\theta - q\theta y - (1+q)hyx)
\end{aligned}
\tag{12}
$$

we find

$$
\begin{aligned}
B_{11} &= q,\; B_{12} = q^2 - 1 \;\; B_{21} = -q \;\; B_{22} = 0\;,\\
C_{11} &= q,\; C_{12} = q^2 - 1 \;\; C_{21} = -q \;\; C_{22} = 0\;,\\
A_{11} &= q,\; A_{12} = q^2 - 1 \;\; A_{21} = q \;\; A_{22} = 0\;, C = 1
\end{aligned}
\tag{13}
$$

A and B are undetermined. We choose A and B equal to q^2, since this leads to the standard R-matrix in the $h \to 0$ limit. We are thus led to the following deformed relations containing q and h:

$$
\begin{aligned}
x\,dx &= q^2 dx\,x,\\
x\,dy &= qdy\,x + (q^2-1)dx\,y,\\
x\,d\theta &= qd\theta\,x + (q^2-1)dx\,\theta - hdx\,x,\\
y\,dx &= qdx\,y,\\
y\,dy &= q^2 dy\,y,\\
y\,d\theta &= qd\theta\,y + (q^2-1)dy\,\theta - (q+1)hdy\,x,\\
\theta\,dx &= -qdx\,\theta - qhdx\,x,\\
\theta\,dy &= -qdy\,\theta - (q+1)hdy\,x,\\
\theta\,d\theta &= d\theta\,\theta - h(qdx\,\theta + d\theta\,x)
\end{aligned}
\tag{14}
$$

Note that altough in the $q \to 1$ limit the transformations (2) and (10) are ill behaved, the resulting commutation relations are well defined:

$$\begin{aligned}
x\,dx &= dx\,x,\\
x\,dy &= dy\,x,\\
x\,d\theta &= d\theta\,x - hdx\,x,\\
y\,dx &= dx\,y,\\
y\,dy &= dy\,y,\\
y\,d\theta &= d\theta\,y - 2hdy\,x,\\
\theta\,dx &= -dx\,\theta - hdx\,x,\\
\theta\,dy &= -dy\,\theta - 2hdy\,x,\\
\theta\,d\theta &= d\theta\,\theta - h(dx\,\theta + d\theta\,x)
\end{aligned} \tag{15}$$

To find the commutation relations between the differentials dx, dy and $d\theta$, we apply the exterior derivative d to (11) and use the nilpotency of d (8). Then it is easy to see that

$$\begin{aligned}
dx\,dy &= -\frac{1}{q}dy\,dx,\\
dx\,d\theta &= \frac{1}{q}d\theta\,dx,\\
dy\,d\theta &= \frac{1}{q}d\theta\,dy + \frac{1}{q}(q+1)hdx\,dy,\\
dx^2 &= dy^2 = 0.
\end{aligned} \tag{16}$$

In the $q \to 1$ limit one has

$$\begin{aligned}
dx\,dy &= -dy\,dx,\\
dx\,d\theta &= d\theta\,dx,\\
dy\,d\theta &= d\theta\,dy + 2hdxdy,\\
dx^2 &= dy^2 = 0.
\end{aligned} \tag{17}$$

Again, the commutation relations (17) are well-defined.

3 Relations between derivatives and variables

To conclude, we introduce here the commutation relations between the coordinates of the $R_h(2 \mid 1)$ quantum 3d superspace and their partial derivatives.
To proceed, let us obtain the relations of the coordinates with their partial derivatives. We know that the exterior differential d can be expressed in the form

$$df = (dx'\,\partial_{x'} + dy'\,\partial_{y'} + d\theta'\,\partial_{\theta'})f. \tag{18}$$

where ∂_x denotes the partial derivative with respect to x , etc. From the chain rule, we obtain

$$\begin{aligned}
\partial_x &= \partial_{x'} + \frac{h}{q-1}\partial_{\theta'}\\
\partial_y &= \partial_{y'}\\
\partial_\theta &= \partial_{\theta'}
\end{aligned} \tag{19}$$

So the d operator satisfies the same equation for h-deformation:

$$df = (dx\partial_x + dy\partial_y + d\theta\partial_\theta)f. \tag{20}$$

We now wish to obtain the relations of partial derivatives with coordinate functions. Replacing f by xf in (20), we find

$$d(xf) = dxf + xdf = dxf + x(dx\partial_x + dy\partial_y + d\theta\partial_\theta)f.$$

On the other hand, we have

$$d(xf) = (dx\,\partial_x + dy\,\partial_y + d\theta\,\partial_\theta)(xf),$$

so we obtain that

$$\begin{aligned}
\partial_x\, x &= 1 + q^2 x\,\partial_x + (q^2-1)y\,\partial_y + (q^2-1)\theta\,\partial_\theta + hx\partial_\theta, \\
\partial_y\, x &= qx\,\partial_y, \\
\partial_\theta\, x &= qx\,\partial_\theta\,.
\end{aligned}$$

The other relations can be obtained in a similar way:

$$\begin{aligned}
\partial_x\, x &= 1 + q^2 x\,\partial_x + (q^2-1)y\,\partial_y + (q^2-1)\theta\,\partial_\theta + hx\partial_\theta, \\
\partial_y\, x &= qx\,\partial_y, \\
\partial_\theta\, x &= qx\,\partial_\theta, \partial_x y = qy\partial_x, \\
\partial_x y &= qy\partial_x, \\
\partial_y y &= 1 + q^2 y\partial_y + (q^2-1)\theta\partial_\theta + (1+q)hx\partial_\theta, \\
\partial_\theta y &= qy\partial_\theta, \\
\partial_x\theta &= q\theta\partial_x - qhx\partial_x - (1+q)hy\partial_y - qh\theta\partial_\theta, \\
\partial_y\theta &= q\theta\partial_y, \\
\partial_\theta\theta &= 1 - \theta\partial_\theta + hx\partial_\theta\,.
\end{aligned}$$

In the $q \to 1$ limit, we find

$$\begin{aligned}
\partial_x x &= 1 + x\partial_x + hx\partial_\theta, \quad \partial_y x = x\partial_y, \quad \partial_\theta x = x\partial_\theta\,, \\
\partial_x y &= y\partial_x, \quad \partial_y y = 1 + y\partial_y + 2hx\partial_\theta, \quad \partial_\theta y = y\partial_\theta, \\
\partial_x\theta &= \theta\partial_x - hx\partial_x - 2hy\partial_y - h\theta\partial_\theta, \quad \partial_y\theta = \theta\partial_y, \\
\partial_\theta\theta &= 1 - \theta\partial_\theta + hx\partial_\theta\,.
\end{aligned} \tag{21}$$

In order to obtain the h-deformed relations between partial derivatives, we have to use the nilpotency of the operator d: $0 = d^2 = d(dx\partial_x + dy\partial_y + d\theta\partial_\theta)$. We then find the relations:

$$\begin{aligned}
\partial_x\partial_y &= q^{-1}\partial_y\partial_x + q^{-1}(q+1)h\,\partial_y\partial_\theta, \\
\partial_x\partial_\theta &= q^{-1}\partial_\theta\partial_x, \quad \partial_y\partial_\theta = q^{-1}\partial_\theta\partial_y, \\
\partial_y\partial_\theta &= q^{-1}\partial_\theta\partial_y, \quad (\partial_\theta)^2 = 0\,.
\end{aligned} \tag{22}$$

These commutation relations are well-defined in the limit $q \to 1$:

$$\begin{aligned}
\partial_x\partial_y &= \partial_y\partial_x + 2h\,\partial_y\partial_\theta, \\
\partial_x\partial_\theta &= \partial_\theta\partial_x, \quad \partial_y\partial_\theta = \partial_\theta\partial_y, \\
\partial_y\partial_\theta &= \partial_\theta\partial_y, \quad (\partial_\theta)^2 = 0\,.
\end{aligned} \tag{23}$$

4 Relations between differentials and derivatives

Finally we compute the commutation relations between differentials and derivatives. We assume that they have the following form in terms of primed quantities derivatives. Applying (19,11,10), we obtain

$$\begin{aligned}
\partial_x dx &= q^{-2}dx\partial_x - q^{-2}dx\partial_\theta,\\
\partial_x dy &= q^{-1}dy\partial_x,\\
\partial_x d\theta &= q^{-1}d\theta\partial_x + q^{-1}dx\partial_x + q^{-1}(1+q)hdy\partial_y + q^{-1}hd\theta\partial_\theta,\\
\partial_y dx &= q^{-1}dx\partial_y,\\
\partial_y dy &= q^{-2}dy\partial_y + (q^{-2}-1)dx\partial_x - q^{-2}(1+q)hdx\partial_\theta,\\
\partial_y d\theta &= q^{-1}d\theta\partial_y,\\
\partial_\theta dx &= -q^{-1}dx\partial_\theta,\\
\partial_\theta dy &= -q^{-1}dy\partial_\theta,\\
\partial_\theta d\theta &= d\theta\partial_\theta + hdx\partial_\theta,
\end{aligned} \tag{24}$$

where we used that the exterior differential d (anti-) commutes with the differentials, that is,

$$d(dx) = -(dx)d, \;\; d(dy) = -(dy)d, \;\; d(d\theta) = (d\theta)d \tag{25}$$

and the relation

$$\partial_i(X^j dX^k) = \delta^i_j\, \delta^k_l\, dX^k \tag{26}$$

where $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_\theta$, $X^1 = x$, $X^2 = y$, $X^3 = \theta$. Again, the relations (24) are well-defined in the limit $q \to 1$:

$$\begin{aligned}
\partial_x dx &= dx\partial_x - dx\partial_\theta,\\
\partial_x dy &= dy\partial_x,\\
\partial_x d\theta &= d\theta\partial_x + dx\partial_x + 2hdy\partial_y + hd\theta\partial_\theta,\\
\partial_y dx &= dx\partial_y,\\
\partial_y dy &= dy\partial_y - 2hdx\partial_\theta,\\
\partial_y d\theta &= d\theta\partial_y,\\
\partial_\theta dx &= -dx\partial_\theta,\\
\partial_\theta dy &= -dy\partial_\theta,\\
\partial_\theta d\theta &= d\theta\partial_\theta + hdx\partial_\theta\ .
\end{aligned} \tag{27}$$

5 The R-matrix formalism

We will now compute the R-matrix satisfying the graded Yang-Baxter equations

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}; \tag{28}$$

$$\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12} = \widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23}. \tag{29}$$

To this end, we rewrite the commutation relations between variables and their differentials, see (14), in the following form

$$X^i dX^j = (-1)^{\widehat{i}(\widehat{j}+1)} K^{ji}_{\ \ kl} dX^k X^l. \tag{30}$$

where $K \in (C \otimes C)$. Comparing (30) with (14), we find

$$K_{h,q} = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q-q^{-1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 \\ 0 & h+q^{-1}h & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -q^{-1}h & 0 & q-q^{-1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -h-hq^{-1} & 0 & q-q^{-1} & 0 & 1 & 0 \\ 0 & 0 & -h & 0 & 0 & 0 & q^{-1}h & 0 & q^{-1} \end{pmatrix} \tag{31}$$

Now we define

$$K_h = \lim_{q \to 1} K_{h,q}, \tag{32}$$

$$K_h = \lim_{q \to 1} K_{h,q}, \tag{33}$$

where P is the super permutation matrix, that is,

$$P^{ij}{}_{kl} = (-1)^{ij} \delta^i_l \, \delta^j_k. \tag{34}$$

The matrix $\widehat{K}_h$ satifies equation (30) with the grading

$$\begin{aligned} (\widehat{K}_{12})^{abc}{}_{def} &= \widehat{K}^{ab}{}_{de} \delta^c_f \\ (\widehat{K}_{13})^{abc}{}_{def} &= (-1)^{b(c+f)} \, \widehat{K}^{ac}{}_{df} \, \delta^b_e \\ (\widehat{K}_{23})^{abc}{}_{def} &= (-1)^{a(b+c+e+f)} \, \widehat{K}^{bc}{}_{ef} \, \delta^a_d \end{aligned} \tag{35}$$

Also, the $R-$matrix

$$R_h = P\widehat{K}_h \tag{36}$$

obeys both the ungraded and the graded Yang-Baxter equations with the grading again given by (36). This is due to the odd character of h. As a consequence,

$$K_h = R_h^{-1} \tag{37}$$

so that K_h has the same properties as R_h .
Using the K_h matrix, we now formulate to the differential calculus on the h-superspace. The commutation relations between variables and their differentials are

$$X^i dX^j = (-1)^{\hat{i}(\hat{j}+1)} K^{ji}{}_{kl} \, dX^k X^l \, . \tag{38}$$

The commutatian relations between variables and derivatives are

$$\partial_j X^i = \delta^i{}_j + (-1)^{\widehat{\hat{i}\hat{j}}} K^{ik}{}_{lj} \, X^l \partial_k \, , \tag{39}$$

and the relations between differentials and derivatives are

$$\partial_j dX^i = (-1)^{\hat{i}(\hat{j}+1)} (K^{-1})^{ik}{}_{lj} \, dX^l \partial_k \, . \tag{40}$$

Note that the commutation relations between variables can be expressed using the K matrix as

$$X^i X^j = \widehat{K}^{ij}{}_{kl} \, X^k X^l \, , \tag{41}$$

and the relations between derivatives as

$$\partial_i \partial_j = \widehat{K}^{kl}{}_{ji} \, \partial_l \, \partial_k \, . \tag{42}$$

References

[1] S. Woronowicz, Differential calculus on compact matrix pseudogroups, *Comm. Math. Phys.* **122** (1989), 125–170.

[2] J. Wess and B. Zumino, Covariant Differential Calculus on the Quantum Hyperplane, *Nucl. Phys. B*, **18** (1990), 302–312.

[3] A. Aghamohammadi, M. Khorromi, and A. Shariati A., h-Deformation as a Contraction of q-Deformation, *J. Phys. A* **28** (1995), L225.

[4] S. Çelik, S. A. Çelik, and M. Arık, Differential Calculus on the h-superplane, *J. Math. Phys.* **39** (1998), 3426–3436.

[5] Y. Manin, Multiparametric quantum deformation of the general linear supergroup, *Comm. Math. Phys.* **123** (1989), 163–175.

[6] L. Dabrowski and P. Parashar, *Lett. Math. Phys.* **38** (1996), 163.

[7] N. Aizawa and R. Chakrabati, Noncommutative Geometry of Super-Jordanian $\mathrm{OS}_{p_h}(2\mid 1)$ Covariant Quantum Space, *J. Math. Phys.* **45** (2004), 1623.