ON THE CLASSIFICATION OF SEMISIMPLE HOPF ALGEBRAS: STRUCTURE AND APPLICATIONS

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Abstract

We survey the results obtained so far in the study of semisimple Hopf algebras. The role of Kaplansky's sixth conjecture to the development of this study is presented in detail. The connection between fusion categories and this study is also emphasized. In the final Section we present a survey on the relation between Hopf algebra extensions and finite depth theory.

Introduction

Hopf algebras appeared in the early 60's from two apparently different directions, algebraic topology and algebraic groups, that interacted afterwards. A nice survey about the beginnings of Hopf algebras is [2]. The publication of Sweedlers book [68] in 1969 was the starting point of the subject of Hopf algebras as an independent part of abstract algebra. With the appearance in 1987 of the paper by Drinfeld [15] and with the subsequent work by him and many others, the subject experienced a very radical change in terms of methods, examples and interaction with other parts of mathematics. The developments obtained in understanding the structure of Hopf algebras and their representation theory have been outstanding in this latter period, and they have been strongly related with the development of different areas of mathematics, like knot theory and topology, conformal field theory, ring theory, category theory, combinatorics, etc.

A nice survey on the subject of classification of semisimple Hopf algebras is contained in [56]. The table on page 5 of that paper resumes very nicely the results obtained until then. In the present survey, we add the new results obtained after the publication of this paper and also review the techniques of classification used until then. The main trend in this subject is to classify semisimple Hopf algebras in terms of group algebras and their duals, and twists of them.

In order to do that two main directions were followed until now. One of the methods consists of using the theory of Hopf algebra extensions (see Section 1.5) which started with the developments of Blattner, Montgomery, Schneider and others. The second direction is quite recent and involves the classification of the fusion category associated to a semisimple Hopf algebra. Fusion categories arise in several areas of mathematics and physics such as conformal field theory, operator algebras, representation theory of quantum groups, and others (see for example [3, 18] and their references). A fusion category is a semisimple rigid monoidal category

with finitely many simple objects and finite dimensional spaces of morphisms, such that the endomorphism algebra of the neutral object is the base field.
A systematic study of fusion categories was initiated by Etingof, Drinfeld, Gelaki, Nikshych, Müger and Ostrik. It has been an area of intensive research in recent years and many remarkable results have already been obtained, see [22, 23, 17, 63, 25]). Braided fusion categories are of a central importance. The Drinfeld center of a fusion category is braided and also modular. Almost all present classifications of fusion categories were made by first considering their braided center. It is also worthwhile mentioning that every braided fusion category gives rise to a solution of the Quantum Yang Baxter equation on each of its objects. Conversely any such solution gives rise to a braided fusion category, namely the category of corepresentations of the FRT construction [37, 65].
Kaplansky's lecture notes [36] are based on a course taught in Chicago in 1975, and contain a list of ten conjectures, some of them are still open. In Section 1, we will give a survey of recent progress on Kaplansky's sixth conjecture. In Section 2, we present the classification results of semisimple Hopf algebras obtained using mainly the first method mentioned above. In Section 3, we will discuss the classification results that can be obtained using the theory of fusion categories.
In Section 4, we investigate the relation between Hopf algebra extensions and finite depth of extensions of algebras. Finite depth has its roots in the classification of type II_1 subfactors. Ocneanu observed in the late eighties that especially depth two has extraordinary algebraic properties. This project in algebra was realized in stages starting with Szymanski and others, see [61, 62, 35] and the references therein). This algebraic theory uses results from Hopf-Galois extensions and weak Hopf algebras as well as Hopf algebroids. Besides a very short history of the subject in the last section we also present some recent results concerning depth two from [11, 12, 6].
Throughout this paper, k is an algebraically closed field of characteristic zero. We use Sweedler's notation $\Delta(x) = \sum x_1 \otimes x_2$ for the comultiplication of a Hopf algebra. For all other specific notation related to Hopf algebra theory, we refer to [50].

1 Kaplansky's sixth conjecture

Recall that a Hopf algebra is semisimple if it is semisimple as an algebra. Over a separably closed field of characteristic zero, a semisimple Hopf algebra is also cosemisimple, see [42], otherwise stated, the dual Hopf algebra H^* is semisimple. In this situation H is involutory, i.e $S^2 = \mathrm{id}$, see [41]. Also recall from [68] that a semisimple Hopf algebra is automatically finite dimensional. Recall that a semisimple algebra R is of *Frobenius type* if the dimension of any irreducible simple R-module divides the dimension of R. Then Kaplansky's sixth conjecture can be stated as follows.

Conjecture 1.1. *Every semisimple Hopf algebra is of Frobenius type.*

Let G be a finite group. Trivial examples of semisimple Hopf algebras are the group algebra kG and its dual Hopf algebra kG^*. Recall that the coalgebra structure on kG is given by $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$. For the dual algebra one has $\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v$ and $S(p_g) = p_{g^{-1}}$. Here $\{p_g\}_{g\in G}$ is the dual basis of the basis given by group elements.

Definition 1.2. A Hopf algebra H is called trivial if it is isomorphic to a group algebra or a dual of a group algebra.

Frobenius showed in 1958 that group algebras are of Frobenius type, we refer to [28] for a proof. Later Lorentz [40] and Zhu [70] independently proved the following result:

Theorem 1.3. *Let H be a semisimple Hopf algebra and M an irreducible module of H affording the character χ. If χ is central in H^*, then* $\dim_k M$ *divides* $\dim_k H$.

1.1 A class of modules that satisfy Kaplansky's conjecture

Let H be a finite dimensional semisimple Hopf algebra over k. Let $\mathcal{G}(H)$ be the Grothendieck group of the category of left H-modules. Then $\mathcal{G}(H)$ becomes a ring under the tensor product of modules and $C(H) = \mathcal{G}(H) \otimes_{\mathbb{Z}} k$ is a semisimple subalgebra of H^* [71]. Denote by $\mathrm{Irr}(H)$ the set of irreducible characters of H. Then $C(H)$ has a basis consisting of the irreducible characters $\chi \in \mathrm{Irr}(H)$. Also $C(H) = \mathrm{Cocom}(H^*)$, the space of cocommutative elements of H^*. By duality, the character ring of H^* is a semisimple subalgebra of H and under this identification it follows that $C(H^*) = \mathrm{Cocom}(H)$.
For $f \in H^*$ and $h \in H$ define $f \rightharpoonup h = \sum f(h_{(2)})h_{(1)}$ and $h \leftharpoonup f = \sum f(h_{(1)})h_{(2)}$.
The Drinfeld double $D(H)$ of H is defined as follows: $D(H) \cong H^{*cop} \otimes H$ as coalgebras. The multiplication on $D(H)$ is given by:

$$(g \otimes h)(f \otimes l) = \sum g(h_1 \rightharpoonup f \leftharpoonup S^{-1}h_3) \otimes h_2 l.$$

Its antipode is given by $S(f \otimes h) = S(h)S^{-1}(f)$. If H is finite dimensional semisimple over $\mathbb{C}$ then $D(H)$ is also semisimple and cosemisimple [50]. Consider the induced module from H to $D(H)$ given by $A_0 = D(H) \otimes_H k$. Then $A_0 \cong H^*$ where the action is given by $(a.f)(x) = f(\sum S^{-1}a_2 x a_1)$ and $g.f = gf$ for all $a,\ x \in H$ and $f,\ g \in A^*$, see [38].
The module A_0 can also be realized on H as following: $x.a = \sum x_1 a S(x_2)$ and $f.x = x \leftharpoonup S^{-1}f = \sum f(S^{-1}x_1)x_2$. It was proven by Zhu [72] that

$$\mathrm{End}_{D(H)}(A_0) = C(H)^{op}$$

and the isomorphism is given by $\chi \mapsto R_\chi$ where R_χ is right multiplication by χ on H^*. Therefore the map

$$C(H) \to \mathrm{End}_{D(H)}(H^*), \quad \chi \mapsto R_{S(\chi)}$$

is a ring isomorphism. Also the map

$$C(H) \to \mathrm{End}_{D(H)}(H), \quad \chi \mapsto (S(\chi) \rightharpoonup \ -)$$

is a ring isomorphism.
Let $E_1, E_2, \cdots, E_s$ be the set of central primitive idempotents of $C(H)$. From the above facts it follows that the homogeneous $D(H)$- components of H^* are given by H^*E_i for all $1 \le i \le s$.
Using the class equation (see [39] or [71]) it follows that the dimension of each simple submodule of A_0 divides the dimension of H.

1.2 The results of Etingof and Gelaki

Later Etingof and Gelaki [19] proved that the dimension of each simple $D(H)$ module divides the dimension of H. Their proof uses the theory of modular categories, in particular Vafa's formula for the fusion coefficients. A modular category is a fusion category with a nondegenarate

S-matrix. The nondegeneracy of this matrix allows one to define an action of the modular group $SL_2(\mathbb{Z})$ on the Grothendieck group of the category. Etingof and Gelaki used their result concerning the dimension of modules over Drinfeld doubles to prove Kaplansky's conjecture for any quasitriangular Hopf algebras. In order to do that they used that $D(H)$ projects onto H as Hopf algebras by $f \otimes h \mapsto f(R^1)R^2h$, where R is the R-matrix of H.

1.3 The dual version of Kaplansky's sixth conjecture

The dual version of Kaplansky's sixth conjecture is obtained by stating the conjecture for the dual Hopf algebra H^*.

Conjecture 1.4. *Let H be a semisimple Hopf algebra and V a finite dimensional comodule of H. Then $\dim_k V$ divides $\dim_k H$.*

Nichols and Richmond [57] proved this conjecture in the case where $\dim_k V = 2$. More generally they have shown the following theorem concerning cosemisimple Hopf algebras.

Theorem 1.5. *Let H be a cosemisimple Hopf algebra over k and let $C \subset H$ be a simple subcoalgebra of dimension 4 (this is equivalent to saying that H has an irreducible comodule of dimension 2). In this case, at least one of the following possibilities holds:*

1. *H has a group-like element g of order 2 such that $gC = C$;*

2. *H has a Hopf subalgebra of dimension 24, which contains a grouplike element g of order 2 such that $gC \neq C$;*

3. *H has a Hopf subalgebra of dimension 12 or 60;*

4. *H has a family $C_n : n \geq 1$ of simple subcoalgebras' such that $\dim C_n = n^2$, and for all $n \geq 2$, $C_nS(C_2) = C_{n-1} + C_{n+1}$ if n is even, and $C_nC_2 = C_{n-1} + C_{n+1}$ if n is odd.*

Note that statement 4. cannot hold if H is finite dimensional. On the other hand the freeness theorem [58] applied to statement 1. or 2. implies that $\dim_k H$ is even. In this case (of finite dimensional semisimple Hopf algebra H) an improvement Theorem 1.5 was given recently in [4], where the possible structures on the Hopf subalgebra of H generated by $CS(C)$ are described.
Using the same techniques as in [57] the author proved the following result.

Theorem 1.6. *[8]. Let H be a cosemisimple Hopf algebra over an algebraically closed field. Assume that H contains a simple subcoalgebra C of dimension 9 and has no simple subcoalgebras of even dimension. Then one of the following conditions must hold:*

1. *H contains a grouplike element of order 2 or 3;*

2. *H has two families of subcoalgebras $\{C_{2n+1} \mid n \geq 1\}$ and $\{D_{2n+1} \mid n \geq 1\}$ with $\dim C_{2n+1} = \dim D_{2n+1} = (2n+1)^2$ and*

$$C_{2n+1}C_3 = C_{2n-1} + D_{2n+1} + C_{2n+3}.$$

In the finite dimensional case the assumption that H has no simple subcoalgebras of even dimension is equivalent to $\dim_k H$ being odd [38].
As before, if H is finite dimensional semisimple, then statement 2. in Theorem 1.6 cannot hold, and the freeness theorem [58] implies the following result. For a different proof, we refer to [38].

Theorem 1.7. *Let H be a semisimple Hopf algebra of odd dimension and V an irreducible comodule over H with* $\dim_k V = 3$. *Then* $\dim_k V$ *divides* $\dim_k H$.

1.4 Crossed products and semisolvable Hopf algebras

Let H be a Hopf algebra and A be an algebra. A weak action of H on A is a map $H \otimes A \to A$ denoted by $h \otimes a \mapsto h.a$ such that $h.(ab) = (h_1.a)(h_2.b)$, $h.1 = \varepsilon(h)1$ and $1.a = a$ for all $a, b \in A$ and $h \in H$. Suppose that $\sigma : H \otimes H \to A$ is a convolution invertible 2-cocycle compatible with the weak action (see [14] or [5]), this means:

$$h.(l.a) = \sum \sigma(h_1, l_1)(h_2 l_2.a)\sigma^{-1}(h_3, l_3);$$
$$\left(\sum h_1.\sigma(l_1, m_1)\right)\sigma(h_2, l_2 m_2) = \sum \sigma(h_1, l_1)\sigma(h_2 l_2, m),$$
$$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1.$$

The crossed product algebra $A\#_\sigma H$ is the vector space $A \otimes H$ together with the multiplication

$$(a\#_\sigma h)(b\#_\sigma l) = a(h_1.b)\sigma(h_2, l_1)\#_\sigma h_3 l_2.$$

In the case where $H = kG$, we recover the classical construction of a group algebra and an algebra. Conjecture 1.8 can be seen as a generalization of Kaplansky's conjecture. It was shown in the case where H is trivial by Montgomery and Witherspoon, see [51].

Conjecture 1.8. *If A is a finite dimensional semisimple finite dimensional algebra of Frobenius type and H is a semisimple Hopf algebra then the crossed product $A\#_\sigma H$ is of Frobenius type.*

1.5 Semisolvable Hopf algebras

Let H be a finite dimensional Hopf algebra over a field k. Recall that a Hopf subalgebra $A \subseteq H$ is called *normal* if $h_1 A \mathcal{S}(h_2) \subseteq A$, for all $h \in H$. If H does not contain proper normal Hopf subalgebras then it is called *simple*. If $A \subseteq H$ is a normal Hopf subalgebra then the structure of H can be reconstructed from A and the one of the quotient quotient Hopf algebra $\overline{H} = H/HA^+$. Lower and upper semisolvable Hopf algebras were introduced by Montgomery and Witherspoon in [51].
By definition, H is called **lower semisolvable** if there exists a chain of Hopf subalgebras

$$H_{n+1} = k \subset H_n \subset \cdots \subset H_1 = H$$

such that H_{i+1} is a normal Hopf subalgebra of H_i, for all i, and all factors $\bar{H}_i := H_{i+1}/H_{i+1}H_i^+$ are trivial. Dually, H is called **upper semisolvable** if there exists a chain of quotient Hopf algebras

$$H(0) = H \to H(1) \to \cdots \to H(n) = k$$

such that each of the maps $\pi_i :\ H(i-1) \to H(i)$ is normal, and all factors $H_i := H_{(i-1)}^{co\pi_i}$ are trivial. For the definition of the coinvariants space $H_{(i-1)}^{co\pi_i}$ see [50].

Obviously H is upper semisolvable if and only if H^* is lower semisolvable [51]. In this case H can be obtained from group algebras and their duals by a finite number of extensions and therefore H is semisimple.
Using Clifford theory for graded algebras the authors of the previous paper proved the generalized Kaplansky's conjecture when H is a trivial Hopf algebra. In particular they showed that any semisimple Hopf algebra of dimension p^n is upper and lower semisolvable and therefore satisfies Kaplansky's sixth conjecture. Indeed, H contains a non-trivial central group-like element [46]; using induction, one can see that H is both upper and lower semisolvable [51]. Also, if $\dim H = pq^2$, where $p \neq q$ are prime numbers, then it was shown in [53, 54, 55] that, under the assumption that H and H^* are both of Frobenius type, either H or H^* contains a non-trivial central group-like element. This implies that these Hopf algebras are also semisolvable, since semisimple Hopf algebras of dimension p, pq and q^2 are trivial. In [55], it is shown that all semisimple Hopf algebras of dimension $pq^2 < 100$ are of Frobenius type, and, a fortiori, semisolvable.
However, a non-trivial semisimple Hopf algebra H is not necessarily semisolvable. The smallest known example of a simple non-trivial semisimple Hopf algebra H has dimension 60 and was constructed by Nikshych in [59]: in this case H is a cocycle twist of the group algebra of the simple group $\mathbb{A}_5$.
Moreover, it was shown in the same paper [59] that if G is a finite simple group and $\phi \in kG \otimes kG$ is a non-trivial invertible pseudo 2-cocycle, then the twisted group algebra $(kG)_\phi$ is a non-trivial semisimple Hopf algebra, which is simple as a Hopf algebra.

1.6 Cotriangular Hopf algebras

A Hopf algebra is called triangular if it is quasitriangular and the squared braiding induced by its R-matrix is the identity. There is a dual notion of cotriangular Hopf algebra, and a finite dimensional Hopf algebra is triangular if and only if its dual is cotriangular. Using fusion categories Etingof and Gelaki classified all triangular Hopf algebras [19]. They are Drinfeld twists of group algebras (see [59] for the definition and main properties of a Drinfeld twist). Thus their dual Hopf algebras are cocycle twists of dual group algebras. Using this fact the representation theory of these Hopf algebras was completely classified in [21]. In particular it shown that they satisfy Kaplansky's sixth conjecture.

2 Classification results using extensions of Hopf algebras

A nice survey on the classification of semisimple Hopf algebras is presented in [56]. The table on page 5 of this paper resumes very nicely the results obtained until then. In the present survey we add the new results obtained after the publication of [56] and also review the techniques of classification used until now. The main trend is to classify Hopf algebras in terms of group algebras, their dual Hopf algebras or twists of them. This was done mainly using two different approaches. One is similar to the one used in group theory and it consists in classifying extensions of smaller objects. A general theory of extensions can be found for example in [1]. The second method is to classify first the fusion category associated to a semisimple finite dimensional Hopf algebra. We will talk about this method in Section 3.
If $A \subseteq H$ is a normal Hopf subalgebra then the structure of H can be reconstructed from A and the one of the quotient quotient Hopf algebra $\overline{H} = H/HA^+$. It is known that in this case H is

isomorphic to a bicrossed product $H \simeq A\#^{\rho,\tau}_{\rightharpoonup,\sigma}\overline{H}$, where $(\rightharpoonup, \sigma, \rho, \tau)$ is a *compatible datum*; see for instance [1, 47, 48].
This fact implies that, when trying to classify Hopf algebras of a given finite dimension, it is important to decide whether the Hopf algebra is simple or not. If the Hopf algebra is not simple, then it can be reconstructed from two Hopf algebras of smaller dimension and a compatible datum.
The early classification problems consisted in classifying Hopf algebras of a given dimension. Kaplansky's eight conjecture is the following.

Conjecture 2.1. *A finite dimensional semisimple Hopf algebra of prime dimension p is commutative and cocommutative.*

Zhu [71] proved this conjecture using the class equation. The proof is similar to an old result of Kac concerning the representation ring of what today are called Kac algebras [31]. The main ingredient of the proof is the following generalization of the class equation for groups to Hopf algebras.

Theorem 2.2. *[31, 71, 39] The character ring $R_k(H)$ of a semisimple Hopf algebra H is also semisimple. Moreover, for each primitive idempotent e of $R_k(H)$, we have that* $\dim(eH^*)$ *divides* $\dim H$.

For dimension pq with $p \neq q$, we have the following result.

Theorem 2.3. *[44, 19, 26, 27] Let $p \neq q$ be prime numbers. Every semisimple and cosemisimple Hopf algebra of dimension pq is trivial.*

Alternative proofs of Theorem 2.3 are given in [67]. Masuoka settled down the case p^2.

Theorem 2.4. *[43] Let p be a prime number. Then all the Hopf algebras of dimension p^2 are trivial.*

For dimension p^3 we have the following results:

Theorem 2.5. *1) [30] The only nontrivial semisimple Hopf algebra of dimension* 8 *is the Kac algebra.*
2) [43] For an odd prime number p, there are $p+8$ isomorphism classes of semisimple Hopf algebras.

In dimension pq^2 the results of Natale can be summarized as follows (see also [49]):

Theorem 2.6. *[54, 53] Let p and q be distinct primes, and let H be a semisimple Hopf algebra of dimension pq^2. Then H is not simple if and only if H or H^* contains a nontrivial central grouplike element. These equivalent conditions are satisfied in the following cases:*

1. $p = 2$ *or* $p = 3$.
2. $p^2 < q$.
3. $p > q^4$ *and* $p \neq 1 \ (mod q)$.
4. *H and H^* are both of Frobenius type; this is satisfied if if* $\dim H < 100$.
5. *H or H^* is of Frobenius type and $p < q$.*

3 Classification results using fusion categories

3.1 Monoidal categories from Hopf algebras

A classical example of a fusion category is the category of finite dimensional representations of a semisimple Hopf algebra. This is a very large class since any fusion category which admits a fiber functor is the category of representation of a certain Hopf algebra, see [18] or [66]. The center of a such category corresponds to the center of the Drinfeld double associated to the Hopf algebra [37].

The general study of fusion categories has led to interesting applications to Hopf algebras. For example, as we already mentioned in Section 1, the long standing Kaplansky's sixth conjecture regarding the dimension of a simple module over a semisimple Hopf algebra was partially solved using the theory of fusion categories.

The conjecture was proven in a more generality in [22] for group theoretical fusion categories (see Definition 3.1). For several years it was an open question whether all fusion categories with a fiber functor are group theoretical. Nikshych [60] found a Hopf algebra whose representation category is not group theoretical. This brought up the study of a more general class of fusion categories that are called weakly group theoretical categories.

Besides its progress to Kaplansky's conjecture, the study of fusion categories also contributed to the classification of finite dimensional semisimple Hopf algebras. As we mentioned in the second section, for certain dimensions the classification is complete. For example, semisimple Hopf algebras of dimension p, p^2, p^3, pq and pq^2 are classified completely in [43, 45, 55, 54, 53] and other papers. In dimension pq^3 and pqr, classification up to a twist is given in [23].

Semisolvable Hopf algebras were introduced by Montgomery and Witherspoon. The fact that they satisfy Kaplansky's conjecture has been a motivation for many researchers to classify them. Etingof has shown that there are weakly group theoretical categories which are not semisolvable and vice versa.

Definition 3.1. [22] A rigid semisimple k-linear tensor category $\mathcal{C}$ with finitely many simple objects and finite dimensional spaces of morphisms is called a fusion category.

The Grothendieck group of $\mathcal{C}$ becomes a ring via the tensor functor of the category $\mathcal{C}$. The category Vec of finite dimensional vector spaces over k has a natural structure of a monoidal category where the tensor functor is just the usual tensor product. This category is semisimple and rigid. The unit object is a one dimensional space k with fixed basis. The unit object is irreducible.

The category $\mathrm{Rep}(H)$ of finite dimensional modules is a fusion category with the tensor product structure obtained via the pullback of the comultiplication. In this case it is obvious that the forgetful functor $F : \mathrm{Rep(H)} \to \mathrm{Vec}$ is monoidal. Conversely any fusion category $\mathcal{C}$ that admits a fiber functor $F : \mathcal{C} \to \mathrm{Vec}$ is the category of representations of a finite dimensional Hopf algebra. The corresponding Hopf algebra is $\mathrm{End}_k(F)$.

We now present some other examples of tensor and fusion categories (see also [63]).

1. Let G be an affine group scheme over k. Then the category $\mathrm{Rep}(G)$ of finite dimensional rational representations of G has a natural structure of a rigid monoidal category with irreducible unit object, which is not semisimple in general. The functor forgetting the G-action has a natural structure of a fiber functor.

2. Let A be a semisimple abelian category. The category $\mathrm{Fun}(A,A)$ of endofunctors has a structure of a monoidal category with tensor product induced by composition of functors. This category is semisimple and rigid (duality is given by taking the adjoint functor). Its unit object is not irreducible if A has at least two non-isomorphic irreducible objects.

3. If the characteristic of the base field k is different from 2, then there are precisely two categories with based ring isomorphic to $K_0(\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z}))$, namely $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})_{tw}$. In fact, in such a category there is only one nontrivial associativity constraint (for triple product of nonunit object) and in the category $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})_{tw}$ it differs by sign from the one in $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$. Both categories are rigid. The category $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})_{tw}$ has no fiber functor.

4. More generally, let G be a finite group and consider the category $\mathcal{C}_G$ with (isomorphism classes of) simple objects X_g parameterized by G and the tensor product functor given by $X_{g_1}.X_{g_2} = X_{g_1 g_2}$. The monoidal structures on the category $\mathcal{C}_G$ are parameterized by the group $H^3(G,k.)$, see e.g. [69]. These categories are called *pointed.*

Definition 3.2. A module category over a monoidal category $\mathcal{C}$ is a category $\mathcal{M}$ together with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural associativity and unit isomorphisms $m_{X,Y,M}$: $(X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$, $l_M : 1 \otimes M \to M$ for any $X,Y \in \mathcal{C}, M \in \mathcal{M}$ satisfying a pentagon and a triangle axiom (compatibility with the structure of $\mathcal{C}$).

The Grothendieck group $K_0(\mathcal{M})$ of a module category $\mathcal{M}$ over a monoidal category $\mathcal{C}$ is a module over the ring $K_0(\mathcal{C})$.
An algebra in a monoidal category $\mathcal{C}$ is an object A of $\mathcal{C}$ endowed with a multiplication morphism $m : A \otimes A \to A$ and a unit morphism $e : 1 \to A$ which are morphisms in $\mathcal{C}$ and associativity and unit axioms are compatible with the associativity and unit structure of $\mathcal{C}$ (see for example [7]). Denote by $\mathrm{Mod}_{\mathcal{C}}(A)$ the category of A-modules in $\mathcal{C}$.

Theorem 3.3. *[63] Let $\mathcal{M}$ be a semisimple indecomposable module category over $\mathcal{C}$. Then there exists a semisimple indecomposable algebra $A \in \mathcal{C}$ such that the module categories $\mathcal{M}$ and $\mathrm{Mod}_{\mathcal{C}}(A)$ are equivalent.*

Definition 3.4. [63] Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are *Morita equivalent* if there is an indecomposable module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathcal{D} \cong \mathcal{C}^*_{\mathcal{M}}$.

Definition 3.5. [22] A fusion category $\mathcal{C}$ is *group theoretical* if it is Morita equivalent to a pointed category.

Definition 3.6. [25] A fusion category $\mathcal{C}$ is *nilpotent* if there is a sequence of fusion categories $\mathcal{C}_0 = \mathrm{Vec}$, $\mathcal{C}_1, \cdots, \mathcal{C}_n = \mathcal{C}$ and a sequence of finite groups $G_1, \cdots, G_n$ such that $\mathcal{C}_i$ is obtained from $\mathcal{C}_{i-1}$ by a G_i-extension, that is, $\mathcal{C}_i$ is faithfully graded by G_i with trivial component $\mathcal{C}_{i-1}$.

Definition 3.7. [23] A fusion category $\mathcal{C}$ is *weakly group theoretical* if it is Morita equivalent to a nilpotent fusion category.

We will say that $\mathcal{C}$ is *cyclically nilpotent* if the groups G_i can be chosen to be cyclic, or, equivalently, cyclic of prime order.

Definition 3.8. [23] A fusion category $\mathcal{C}$ is *solvable* if the following two equivalent conditions are satisfied:

1. $\mathcal{C}$ is Morita equivalent to a cyclically nilpotent fusion category;

2. there is a sequence of fusion categories $\mathcal{C}_0 = \text{Vec}, \mathcal{C}_1, \cdots, \mathcal{C}_n = \mathcal{C}$ and a sequence $G_1, \cdots, G_n$ of cyclic groups of prime order such that $\mathcal{C}_i$ is obtained from $\mathcal{C}_{i-1}$ either as aG_i-extension or as a G_i-equivariantization.

Thus, the class of weakly group-theoretical categories contains the classes of solvable and group-theoretical categories, that is, those Morita equivalent to pointed categories, see [22].

Remark 1. [22] The notion of a solvable fusion category is closely related to the notions of upper and lower solvable and semisolvable Hopf algebras introduced by Montgomery and Witherspoon [51]. However, we would like to point out that a semisimple Hopf algebra H such that $\text{Rep}(H)$ is solvable in the above sense is not necessarily upper or lower semisolvable in the sense of [51]. For example, Galindo and Natale [24] constructed self-dual Hopf algebras without nontrivial normal Hopf subalgebras as twisting deformations of solvable groups. Clearly, the representation category of such a Hopf algebra is solvable. It is also easy to construct an example of an upper and lower solvable semisimple Hopf algebra H, such that $\text{Rep}(H)$ is not solvable. For this, it suffices to take the Kac algebra associated to the exact factorization of groups $A_5 = A_4\mathbb{Z}/5\mathbb{Z}$.

We now turn to the classification of semisimple Hopf algebras of dimension pqr.

Theorem 3.9. *[23] Let $p < q < r$ be prime numbers. Any integral fusion category $\mathcal{C}$ of dimension pqr is group-theoretical.*

Recall that a fusion category is called integral fusion category if the Frobenius-Perron dimension of any of its simple objects is an integer. Fusion categories with fiber functors are integral since the Frobenius-Perron dimension of a simple object coincides with the vector space dimension. Using Theorem 3.9 and Lemma 3.11, the authors of [23] were able to classify semisimple Hopf algebras of dimension pqr.

Corollary 3.10. *Let H be a semisimple Hopf algebra of dimension pqr, where $p < q < r$ are prime numbers. Then there exists a finite group G of order pqr and an exact factorization $G = KL$ of G into a product of subgroups, such that H is the split abelian extension $H(G,K,L,1,1,1) = C[K] \otimes Fun(L)$ associated to this factorization.*

Lemma 3.11. *Let H be a group-theoretical semisimple Hopf algebra of square-free dimension. Then H is a split abelian extension of the form $H(G,K,L,1,1,1)$.*

Using the theory of fusion category semisimple Hopf algebras of dimension pq^2 were completely classified in [23] improving the results of Natale mentioned in the previous section.

Proposition 3.12. *[29] Every semisimple Hopf algebra H of dimension pq^2 is group-theoretical.*

Corollary 3.13. *[23] A semisimple Hopf algebra of dimension pq^2 is either a Kac algebra, or a twisted group algebra (by a twist corresponding to the subgroup $(\mathbb{Z}/q\mathbb{Z})^2$)), or the dual of a twisted group algebra.*

3.2 Triangular Hopf algebras

Recall that a triangular Hopf algebra with universal R-matrix R is called triangular if $R_{21} = R^{-1}$, where R_{21} is the tensor twist of R.
The classification of finite-dimensional semisimple triangular Hopf algebras H over an algebraically closed field k of characteristic 0 was done in [20]. There it was proved that if H is semisimple then it is obtained from the group algebra $k[G]$ of a finite group G by twisting its comultiplication in the sense of Drinfeld [16]. The proof of this theorem essentially relies on the following theorem of Deligne on Tannakian categories.

Theorem 3.14. *[13, Theorem 7.1] Let $\mathcal{C}$ be a semisimple rigid symmetric tensor category with finitely many irreducible objects, over an algebraically closed field k, in which categorical dimensions of objects are nonnegative integers. Then for a suitable finite group G there exists an equivalence of symmetric rigid tensor categories $F : \mathcal{C} \to \mathrm{Rep}(G)$ (where $\mathrm{Rep}(G)$ is the category of finite dimensional representations of G).*

Deligne's Theorem can be applied since the representation category $\mathrm{Rep}(H)$ of H has this property (possibly after modifying its commutativity constraint). Using [20, Theorem 2.1] and Movshev's theory of on twisting in finite groups [52], the same authors completely classify semisimple triangular Hopf algebras over an arbitrary algebraically closed field.

4 Finite Depth extensions of semisimple Hopf algebras

A finite Jones index subfactor is a Frobenius extension, where the conditional expectation and Pimsner-Popa orthonormal bases are the Frobenius coordinate system. A ring inclusion is depth two if the centralizers of the based ring in the first three terms of the Jones tower form a basic construction. Higher depth is also a notion from the classification of subfactors which describes where in the derived tower of centralizers there occurs three successive algebras forming a basic construction.
For an extension of algebras $C \subset B$ the sequence of inclusions

$$C \subset B \subset A \subset A_1 \subset \cdots \subset A_n \subset \cdots$$

is called the Jones tower of iterated right endomorphism rings. Here $A_0 = A = \mathrm{End}_A B, \cdots, A_{n+1} = \mathrm{End}_{A_{n-1}} A_n$.
Important properties of a ring inclusion sometimes pass up to the next term of the Jones tower. Some of these properties are the Frobenius property, the index of a Frobenius extension, or the Hattori-Stallings rank of the underlying projective module. Sometimes two properties are dual with respect to this shift of levels such as separability and splitness of an extension. For example, under suitable conditions, a split extension leads to separable endomorphism ring extension, and vice versa, a separable extension leads to a split extension. A certain compatible Frobenius homomorphism, (also called Markov trace) passes up in an endomorphism ring theorem as well. These four properties mentioned above where used by Jones in topology by considering an iterated endomorphism ring construction of a tower over a special split, separable Frobenius algebra extensions (type II_1 subfactors). This construction produced a countable set of idempotents that satisfy braidlike relations, which together with a Markov trace lead to the first knot and link polynomials since the classical Alexander polynomial (see [32]).

The algebraic version of the embedding theorem for subfactors is due to Nikshych and Vainerman [61]: an inclusion of algebras is depth n if the based ring is of depth 2 inside the $n-2$-th term of the iterated Jones tower. This theorem shows the importance of depth two in the algebraic setting.
The fact that a depth 2 inclusion of groups is equivalent to the normality of the subgroup was shown by Kadison and Külshammer [34]. It was shown that this result also holds for any extension of semisimple Hopf algebras ([10]). Recently this result was proved for any extension of Hopf algebras not necessarily semisimple [6]. Depth two extensions of multimatrix algebras coincide with normal extensions of semisimple rings introduced by Rieffel [64].

Theorem 4.1. *[12] The inclusion $B \subseteq A$ is of depth 2 if and only if B is normal in A (in Rieffel's sense).*

4.1 Depth three extension

The notion of depth three tower served to give an algebraic definition of finite depth, originally an analytic notion in subfactor theory. During this project we would like to characterize depth three towers of Hopf algebras. Some initial steps were done in the aforementioned paper. Depth three tower of groups were studied in [11]. For H a subgroup of G let

$$N_G(H) = \cap_{g \in G}\, {}^gH$$

be the largest subgroup of H which is normal in G. (Here ${}^gH = gHg^{-1}$.)

Theorem 4.2. *[11] A tower $G \supseteq N \supseteq H$ of groups is depth three if and only if $H \subset N_G(N)$.*

Let $B \subset A$ be an arbitrary extension of finite dimensional Hopf algebras. Define core($\mathbf{B}$) as the largest Hopf subalgebra of B which is normal in A. It is easy to see that core(A) always exists (see also [9]). If $H \subset G$ is a group inclusion with $A = kG$ and $B = kH$ note that $\mathrm{core}(B) = kN_G(H)$.
For an arbitrary tower of semisimple Hopf algebras, we have, so far, the following result.

Theorem 4.3. *[11] Suppose that $A \supseteq B \supseteq C$ is a tower of semisimple Hopf algebras. If $C \subset$ core(B) then the tower is depth-3.*

4.2 Depth three Frobenius extension

A Frobenius extension $A \subset B$ is defined to be *depth three* if the tower of subalgebras

$$\lambda(B) \subseteq \lambda(A) \subseteq E$$

is right depth three. Here E is the endomorphism ring $E = \mathrm{End}A_B$ and $\lambda : A \to E$ is given by $\lambda(a)(x) = ax$ $(x, a \in A)$. By [33, Theorem 3.1] the given tower is left depth three if and only if the tower is right depth three.
The definitions and first properties of depth two and three extensions are introduced in detail in [33].
Depth three extensions of Hopf algebras are studied in [12]. It is shown that an extension is depth three if and only if the relation (on the set of irreducible modules of B) introduced by Rieffel [64] is an equivalence relation. Also a sufficient condition for a Hopf algebra extension to be of higher depth n is also given in the same paper.

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