

More properties of Yetter-Drinfeld modules over quasi-Hopf algebras

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ABSTRACT. We generalize various properties of Yetter-Drinfeld modules over Hopf algebras to quasi-Hopf algebras. The dual of a finite dimensional Yetter-Drinfeld module is again a Yetter-Drinfeld module. The algebra H_0 in the category of Yetter-Drinfeld modules that can be obtained by modifying the multiplication in a proper way is quantum commutative. We give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules, and deduce the existence and uniqueness of integrals from it.

1. Introduction

The motivation for studying Yetter-Drinfeld modules over quasi-Hopf algebras is the same as for Hopf algebras. It is well known that for any finite dimensional Hopf algebra H the category of Yetter-Drinfeld modules ${}_H\mathcal{YD}^H$ is isomorphic to the category of modules over the quantum double $D(H)$. From a categorical point of view, the quantum double $D(H)$ arises by considering the center $\mathcal{Z}({}_H\mathcal{M})$ of the monoidal category ${}_H\mathcal{M}$ of left H -modules. More precisely, one has $\mathcal{Z}({}_H\mathcal{M}) \simeq_{D(H)} \mathcal{M}$ if H is finite dimensional. Actually, the category of Yetter-Drinfeld modules appears as an intermediate step in the proof of this isomorphism: one first proves that $\mathcal{Z}({}_H\mathcal{M}) \simeq {}_H\mathcal{YD}^H$, and then ${}_H\mathcal{YD}^H \simeq_{D(H)} \mathcal{M}$, where the finite dimensionality is not needed in the proof of the first isomorphism, see [16] for full detail.

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Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [13]; a categorical interpretation is the following: a quasi-bialgebra H is an algebra with the additional structure that is needed to make the category of left H -modules, with the tensor product over k as tensor product and k as unit object into a monoidal category. The difference with a usual bialgebra is that we do not require that the associativity isomorphism coincides with the associativity in the category of vector spaces. A quasi-Hopf algebra is a quasi-bialgebra with additional structure making the category of finite dimensional H -modules into a monoidal category with duality. The center construction $\mathcal{Z}(\mathcal{C})$ can be applied to any monoidal category \mathcal{C} . Majid [19] computed the center of the category of left modules over a quasi-Hopf algebra H , and introduced the category of Yetter-Drinfeld modules over H . Hausser and Nill [14], [15] constructed the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra H , and proved that ${}^H\mathcal{YD}^H \simeq_{D(H)} \mathcal{M}$. Recently, Schauenburg [22] gave the equivalence between the category of Yetter-Drinfeld modules ${}^H\mathcal{YD}$ and the category ${}^H\mathcal{M}_H^H$ of Hopf bimodules. In [5], the relation between Yetter-Drinfeld modules and Radford's biproduct is studied. In [4], the rigidity of the category of Yetter-Drinfeld modules is investigated, as well as the relations between left, left-right, right-left and right Yetter-Drinfeld modules.

In this paper, which can be seen as a sequel to [4], we continue our investigations of properties of Yetter-Drinfeld modules. In Section 3, we show that the linear dual of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module.

It was shown in [7], [5] that the multiplication on H can be modified in such a way that we obtain an algebra in the category of left Yetter-Drinfeld modules. The main result of Section 4 is that H_0 is quantum commutative.

In Section 5, we will generalize Doi's results [12] about Hopf modules in the category of Yetter-Drinfeld modules to our situation: we give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules over a quasi-Hopf algebras, and we use this result to obtain the existence and uniqueness of integrals for a finite dimensional braided Hopf algebra in ${}^H\mathcal{YD}$. We apply this to the braided Hopf algebra considered in Section 4, in the case where H is finite dimensional and quasitriangular.

2. Preliminary results

2.1. Quasi-Hopf algebras

We work over a commutative field k . All algebras, linear spaces etc. will be over k ; unadorned \otimes means \otimes_k . Following Drinfeld [13], a quasi-bialgebra is a fourtuple $(H, \Delta, \varepsilon, \Phi)$, where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \rightarrow H \otimes H$ and $\varepsilon : H \rightarrow k$ are algebra homomorphisms satisfying the identities

$$(id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1}, \quad (1)$$

$$(id \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \quad (\varepsilon \otimes id)(\Delta(h)) = 1 \otimes h, \quad (2)$$

for all $h \in H$, and Φ has to be a normalized 3-cocycle, in the sense that

$$(1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi), \quad (3)$$

$$(id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1 \otimes 1. \quad (4)$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras [23] we use the notation $\Delta(h) = \sum h_1 \otimes h_2$. Since Δ is only quasi-coassociative we adopt the further notation

$$(\Delta \otimes id)(\Delta(h)) = \sum h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = \sum h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3 = \dots$$

$$\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum v^1 \otimes v^2 \otimes v^3 = \dots$$

A quasi-bialgebra H is called a quasi-Hopf algebra if there exists an anti-automorphism S of the algebra H and $\alpha, \beta \in H$ such that:

$$\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1\beta S(h_2) = \varepsilon(h)\beta, \quad (5)$$

$$\sum X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad \sum S(x^1)\alpha x^2\beta S(x^3) = 1, \quad (6)$$

for all $h \in H$. It is shown in [9] that the condition that the antipode is bijective follows automatically from the other axioms in the case where H is finite dimensional. Observe that the antipode of a quasi-Hopf algebra is determined uniquely up to a transformation $\alpha \mapsto U\alpha$, $\beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, where $U \in H$ is invertible. The axioms for a quasi-Hopf algebra imply that $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β , we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$ and $\varepsilon \circ S = \varepsilon$. The identities (2-4) also imply that

$$(\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1 \otimes 1. \quad (7)$$

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ we also have H^{op} , H^{cop} and $H^{\text{op,cop}}$ as quasi-Hopf algebras, where “op” means opposite multiplication and “cop” means opposite comultiplication. The reassociators of these three quasi-Hopf algebras are $\Phi_{\text{op}} = \Phi^{-1}$, $\Phi_{\text{cop}} = (\Phi^{-1})^{321}$, $\Phi_{\text{op,cop}} = \Phi^{321}$, the antipodes are $S_{\text{op}} = S_{\text{cop}} = (S_{\text{op,cop}})^{-1} = S^{-1}$, and the elements α, β are $\alpha_{\text{op}} = S^{-1}(\beta)$, $\beta_{\text{op}} = S^{-1}(\alpha)$, $\alpha_{\text{cop}} = S^{-1}(\alpha)$, $\beta_{\text{cop}} = S^{-1}(\beta)$, $\alpha_{\text{op,cop}} = \beta$ and $\beta_{\text{op,cop}} = \alpha$.

Recall next that the definition of a quasi-Hopf algebra is “twist coinvariant”, in the following sense. An invertible element $F \in H \otimes H$ is called a *gauge transformation* or *twist* if $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$. If H is a quasi-Hopf algebra and $F = \sum F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = \sum G^1 \otimes G^2$, then we can define a new quasi-Hopf algebra H_F by keeping the multiplication, unit, counit and antipode of H and replacing the comultiplication, antipode and the elements α and β by

$$\Delta_F(h) = F\Delta(h)F^{-1}, \quad (8)$$

$$\Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1), \quad (9)$$

$$\alpha_F = \sum S(G^1)\alpha G^2, \quad \beta_F = \sum F^1\beta S(F^2). \quad (10)$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. The corresponding statement for a quasi-Hopf algebra is the following: there exists a gauge transformation $f \in H \otimes H$ such that

$$f\Delta(S(h))f^{-1} = \sum(S \otimes S)(\Delta^{\text{cop}}(h)), \quad (11)$$

for all $h \in H$, where $\Delta^{\text{cop}}(h) = \sum h_2 \otimes h_1$. The element f can be computed explicitly. First set

$$\sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}), \quad (12)$$

$$\sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1) \quad (13)$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = \sum S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \text{ and } \delta = \sum B^1\beta S(B^4) \otimes B^2\beta S(B^3). \quad (14)$$

Then f and f^{-1} are given by the formulas

$$f = \sum(S \otimes S)(\Delta^{\text{op}}(x^1))\gamma\Delta(x^2\beta S(x^3)), \quad (15)$$

$$f^{-1} = \sum\Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{op}}(x^3)). \quad (16)$$

Moreover, f satisfies the following relations:

$$f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta. \quad (17)$$

Furthermore the corresponding twisted reassociator (see (9)) is given by

$$\Phi_f = \sum(S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1). \quad (18)$$

In a Hopf algebra H , we obviously have the identity

$$\sum h_1 \otimes h_2 S(h_3) = h \otimes 1, \text{ for all } h \in H.$$

We will need the generalization of this formula to the quasi-Hopf algebra setting. Following [14, 15], we define

$$p_R = \sum p_R^1 \otimes p_R^2 = \sum x^1 \otimes x^2 \beta S(x^3), \quad (19)$$

$$q_R = \sum q_R^1 \otimes q_R^2 = \sum X^1 \otimes S^{-1}(\alpha X^3) X^2, \quad (20)$$

$$p_L = \sum p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}(X^1 \beta) \otimes X^3, \quad (21)$$

$$q_L = \sum q_L^1 \otimes q_L^2 = \sum S(x^1)\alpha x^2 \otimes x^3. \quad (22)$$

We then have, for all $h \in H$,

$$\sum \Delta(h_1) p_R [1 \otimes S(h_2)] = p_R (h \otimes 1), \quad (23)$$

$$\sum [1 \otimes S^{-1}(h_2)] q_R \Delta(h_1) = (h \otimes 1) q_R, \quad (24)$$

$$\sum \Delta(h_2) p_L [S^{-1}(h_1) \otimes 1] = p_L (1 \otimes h), \quad (25)$$

$$\sum [S(h_1) \otimes 1] q_L \Delta(h_2) = (1 \otimes h) q_L, \quad (26)$$

and

$$(q_R \otimes 1)(\Delta \otimes id)(q_R)\Phi^{-1} = \sum [1 \otimes S^{-1}(X^3) \otimes S^{-1}(X^2)] \\ [1 \otimes S^{-1}(f^2) \otimes S^{-1}(f^1)](id \otimes \Delta)(q_R \Delta(X^1)), \quad (27)$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (15).

A quasi-Hopf algebra H is quasitriangular if there exists an element $R \in H \otimes H$ such that

$$(\Delta \otimes id)(R) = \sum \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi, \quad (28)$$

$$(id \otimes \Delta)(R) = \sum \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1}, \quad (29)$$

$$\Delta^{\text{cop}}(h)R = R\Delta(h), \text{ for all } h \in H, \quad (30)$$

$$(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1. \quad (31)$$

Here we used the following notation: if σ is a permutation of $\{1, 2, 3\}$, then we write $\Phi_{\sigma(1)\sigma(2)\sigma(3)} = \sum X^{\sigma^{-1}(1)} \otimes X^{\sigma^{-1}(2)} \otimes X^{\sigma^{-1}(3)}$; R_{ij} means R acting non-trivially on the i -th and j -th tensor factors of $H \otimes H \otimes H$.

It is shown in [10] that R is invertible. Furthermore, the element

$$u = \sum S(R^2 p^2) \alpha R^1 p^1, \quad (32)$$

with $p_R = \sum p^1 \otimes p^2$ defined as in (19), is invertible in H , and

$$u^{-1} = \sum X^1 R^2 p^2 S(S(X^2 R^1 p^1) \alpha X^3), \quad (33)$$

$$\varepsilon(u) = 1 \text{ and } S^2(h) = uhu^{-1}, \quad (34)$$

for all $h \in H$. Consequently the antipode S is bijective, so, as in the Hopf algebra case, the assumptions about invertibility of R and bijectivity of S can be dropped. Moreover, the R -matrix $R = \sum R^1 \otimes R^2$ satisfies the identity (see [1], [15], [10]):

$$f_{21} R f^{-1} = (S \otimes S)(R) \quad (35)$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (15), and $f_{21} = \sum f^2 \otimes f^1$.

2.2. Monoidal categories

A monoidal or tensor category is a sextuple $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$, where \mathcal{C} is a category, \otimes is a functor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (called the tensor product), $\underline{1}$ is an object of \mathcal{C} , and

$$a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$$

$$l_V : V \cong V \otimes \underline{1}; \quad r_V : V \cong \underline{1} \otimes V$$

are natural isomorphisms satisfying certain coherence conditions, see for example [16, 18, 20]. An object V of a monoidal category \mathcal{C} has a left dual if there exists an object V^* and morphisms $ev_V : V^* \otimes V \rightarrow \underline{1}$, $coev_V : \underline{1} \rightarrow V \otimes V^*$ in \mathcal{C} such that

$$l_V^{-1} \circ (id_V \otimes ev_V) \circ a_{V,V^*,V} \circ (coev_V \otimes id_V) \circ r_V = id_V, \quad (36)$$

$$r_{V^*}^{-1} \circ (ev_V \otimes id_{V^*}) \circ a_{V^*,V,V^*}^{-1} \circ (id_{V^*} \otimes coev_V) \circ l_{V^*} = id_{V^*}. \quad (37)$$

\mathcal{C} is called a rigid monoidal category if every object of \mathcal{C} has a dual.

A braided monoidal category is a monoidal category equipped with a commutativity natural isomorphism $c_{U,V} : U \otimes V \rightarrow V \otimes U$, compatible with the unit and the associativity.

In a braided monoidal category, we can define algebras, coalgebras, bialgebras and Hopf algebras. For example, a bialgebra $(B, \underline{m}, \underline{\eta}, \underline{\Delta}, \underline{\varepsilon})$ consists of $B \in \mathcal{C}$, a multiplication $\underline{m} : B \otimes B \rightarrow B$ which is associative up to the natural isomorphism a , and a unit $\underline{\eta} : \underline{1} \rightarrow B$ such that $\underline{m} \circ (\underline{\eta} \otimes id) = \underline{m} \circ (id \otimes \underline{\eta}) = id$. The properties of the comultiplication $\underline{\Delta}$ and the counit $\underline{\varepsilon}$ are similar. In addition, $\underline{\Delta} : B \rightarrow B \otimes B$ has to be an algebra morphism, where $B \otimes B$ is an algebra with multiplication $\underline{m}_{B \otimes B}$, defined as the composition

$$\begin{array}{ccc}
(B \otimes B) \otimes (B \otimes B) & \xrightarrow{a} & B \otimes (B \otimes (B \otimes B)) \\
& \xrightarrow{id \otimes a^{-1}} & B \otimes ((B \otimes B) \otimes B) \\
& \xrightarrow{id \otimes c \otimes id} & B \otimes ((B \otimes B) \otimes B) \\
& \xrightarrow{id \otimes a} & B \otimes (B \otimes (B \otimes B)) \\
& \xrightarrow{a^{-1}} & (B \otimes B) \otimes (B \otimes B) \\
& \xrightarrow{\underline{m} \otimes \underline{m}} & B \otimes B
\end{array} \tag{38}$$

A Hopf algebra B is a bialgebra with a morphism $\underline{S} : B \rightarrow B$ in \mathcal{C} (the antipode) satisfying the usual axioms $\underline{m} \circ (\underline{S} \otimes id) \circ \underline{\Delta} = \eta \circ \underline{\varepsilon} = \underline{m} \circ (id \otimes \underline{S}) \circ \underline{\Delta}$. It is known, see e.g. [21], that the antipode \underline{S} of a Hopf algebra B in a braided monoidal category \mathcal{C} is an antialgebra and anticoalgebra morphism, in the sense that

$$\underline{S} \circ \underline{m} = \underline{m} \circ (\underline{S} \otimes \underline{S}) \circ c_{B,B} \quad \text{and} \quad \underline{\Delta} \circ \underline{S} = c_{B,B} \circ (\underline{S} \otimes \underline{S}) \circ \underline{\Delta}. \tag{39}$$

Recall also that an algebra A in a braided monoidal category \mathcal{C} is called quantum commutative if $\underline{m} \circ c_{A,A} = \underline{m}$.

Assume that $(H, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra, and let U, V, W be left H -modules. We define a left H -action on $U \otimes V$ by

$$h \cdot (u \otimes v) = \sum h_1 \cdot u \otimes h_2 \cdot v.$$

We have isomorphisms $a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ in ${}_H\mathcal{M}$ given by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)). \tag{40}$$

The counit $\varepsilon : H \rightarrow k$ makes $k \in {}_H\mathcal{M}$, and the natural isomorphisms $\lambda : k \otimes H \rightarrow H$ and $\rho : H \otimes k \rightarrow H$ are in ${}_H\mathcal{M}$. With this structures, $({}_H\mathcal{M}, \otimes, k, a, \lambda, \rho)$ is a monoidal category.

If H is a quasi-Hopf algebra then the category of finite dimensional left H -modules is rigid; the left dual of V is V^* with the H -module structure given by $(h \cdot \varphi)(v) = \varphi(S(h) \cdot v)$, for all $v \in V$, $\varphi \in V^*$, $h \in H$ and with

$$ev_V(\varphi \otimes v) = \varphi(\alpha \cdot v), \quad coev_V(1) = \sum_{i=1}^n \beta \cdot v_i \otimes v^i, \tag{41}$$

where $\{v_i\}$ is a basis in V with dual basis $\{v^i\}$.

Now let H be a quasitriangular quasi-Hopf algebra, with R -matrix $R = \sum R^1 \otimes R^2$. For two left H -modules U and V , we define

$$c_{U,V} : U \otimes V \rightarrow V \otimes U$$

by

$$c_{U,V}(u \otimes v) = \sum R^2 \cdot v \otimes R^1 \cdot u \quad (42)$$

and then $({}_H\mathcal{M}, \otimes, k, a, \lambda, \rho, c)$ is a braided monoidal category (cf. [16] or [20]).

3. Yetter-Drinfeld modules and the quasi-Yang-Baxter equation

From [19], we recall the notion of Yetter-Drinfeld module over a quasi-bialgebra.

DEFINITION 3.1. Let H be a quasi-bialgebra with reassociator Φ . A left H -module M together with a left H -coaction

$$\lambda_M : M \rightarrow H \otimes M, \quad \lambda_M(m) = \sum m_{(-1)} \otimes m_{(0)}$$

is called a left Yetter-Drinfeld module if the following equalities hold, for all $h \in H$ and $m \in M$:

$$\begin{aligned} & \sum X^1 m_{(-1)} \otimes (X^2 \cdot m_{(0)})_{(-1)} X^3 \otimes (X^2 \cdot m_{(0)})_{(0)} \\ &= \sum X^1 (Y^1 \cdot m)_{(-1)_1} Y^2 \otimes X^2 (Y^1 \cdot m)_{(-1)_2} Y^3 \otimes X^3 \cdot (Y^1 \cdot m)_{(0)} \end{aligned} \quad (43)$$

$$\sum \varepsilon(m_{(-1)}) m_{(0)} = m \quad (44)$$

$$\sum h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = \sum (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)}. \quad (45)$$

The category of left Yetter-Drinfeld H -modules and k -linear maps that intertwine the H -action and H -coaction is denoted by ${}^H_H\mathcal{YD}$. In [19] it is shown that ${}^H_H\mathcal{YD}$ is a prebraided monoidal category. The forgetful functor ${}^H_H\mathcal{YD} \rightarrow {}_H\mathcal{M}$ is monoidal, and the coaction on the tensor product $M \otimes N$ of two Yetter-Drinfeld modules M and N is given by

$$\lambda_{M \otimes N}(m \otimes n) = \sum X^1 (x^1 Y^1 \cdot m)_{(-1)} x^2 (Y^2 \cdot n)_{(-1)} Y^3 \quad (46)$$

$$\otimes X^2 \cdot (x^1 Y^1 \cdot m)_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot n)_{(0)}. \quad (47)$$

The braiding is given by

$$c_{M,N}(m \otimes n) = \sum m_{(-1)} \cdot n \otimes m_{(0)}. \quad (48)$$

This braiding is invertible if H is a quasi-Hopf algebra [5], and its inverse is then given by

$$\begin{aligned} c_{M,N}^{-1}(n \otimes m) &= \sum y_1^3 X^2 \cdot (x^1 \cdot m)_{(0)} \\ &\otimes S^{-1}(S(y^1) \alpha y^2 X^1 (x^1 \cdot m)_{(-1)} x^2 \beta S(y_2^3 X^3 x^3)) \cdot n. \end{aligned} \quad (49)$$

Let (H, R) be a quasitriangular quasi-bialgebra. It is well-known (see for example [16]) that R satisfies the so-called quasi-Yang-Baxter equation in $H \otimes H \otimes H$:

$$R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi = \Phi_{321} R_{23} \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12}.$$

On the other hand, if H is a bialgebra and M is a left-right Yetter-Drinfeld module over H , with structures

$$\begin{aligned} H \otimes M &\rightarrow M, & h \otimes m &\mapsto h \cdot m; \\ M &\rightarrow M \otimes H, & m &\mapsto \sum m_{(0)} \otimes m_{(1)}, \end{aligned}$$

then the map $R_M : M \otimes M \rightarrow M \otimes M$, $R_M(m \otimes n) = \sum n_{(1)} \cdot m \otimes n_{(0)}$ is a solution in $\text{End}(M \otimes M \otimes M)$ of the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

see for instance [17].

We will show a similar result for quasi-bialgebras; first we define left-right Yetter-Drinfeld modules over quasi-bialgebras as follows

$${}_H\mathcal{YD}^H = \frac{H^{\text{cop}}}{H^{\text{cop}}}\mathcal{YD}.$$

This is stated more explicitly in the next definition.

DEFINITION 3.2. Let H be a quasi-bialgebra. A k -linear space M with a left H -action $h \otimes m \mapsto h \cdot m$, and a right H -coaction $M \rightarrow M \otimes H$, $m \mapsto \sum m_{(0)} \otimes m_{(1)}$ is called a left-right Yetter-Drinfeld module if the following relations hold, for all $m \in M$ and $h \in H$:

$$\begin{aligned} \sum (x^2 \cdot m_{(0)})_{(0)} \otimes (x^2 \cdot m_{(0)})_{(1)} x^1 \otimes x^3 m_{(1)} \\ = \sum x^1 \cdot (y^3 \cdot m)_{(0)} \otimes x^2 (y^3 \cdot m)_{(1)} y^1 \otimes x^3 (y^3 \cdot m)_{(1)} y^2 \end{aligned} \quad (50)$$

$$\sum \varepsilon(m_{(1)}) m_{(0)} = m \quad (51)$$

$$\sum h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = \sum (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} h_1. \quad (52)$$

PROPOSITION 3.3. Let H be a quasi-bialgebra and $M \in {}_H\mathcal{YD}^H$. The map $R = R_M : M \otimes M \rightarrow M \otimes M$, $R(m \otimes n) = \sum n_{(1)} \cdot m \otimes n_{(0)}$, is a solution of the quasi-Yang-Baxter equation

$$R_{12}\Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi = \Phi_{321}R_{23}\Phi_{231}^{-1}R_{13}\Phi_{213}R_{12} \quad (53)$$

on $\text{End}(M \otimes M \otimes M)$.

We considered R_{12}, Φ_{312} , etc. as elements in $\text{End}(M \otimes M \otimes M)$ by left multiplication, for example $R_{12}(l \otimes m \otimes n) = \sum R^1 \cdot l \otimes R^2 \cdot m \otimes n$, $\Phi_{312}(l \otimes m \otimes n) = \sum X^2 \cdot l \otimes X^3 \cdot m \otimes X^1 \cdot n$ etc.

Proof. ${}_H\mathcal{YD}^H$ is a prebraided category, hence the result is a consequence of the fact (see [16]) that the braiding satisfies the categorical version of the Yang-Baxter equation. A direct proof is also possible. For all $l, m, n \in M$, we compute that

$$\begin{aligned} R_{12}\Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi(l \otimes m \otimes n) \\ = \sum (Y^3 x^3 (X^3 \cdot n)_{(1)} X^2 \cdot m)_{(1)} Y^2 (x^2 \cdot (X^3 \cdot n)_{(0)})_{(1)} x^1 X^1 \cdot l \\ \otimes (Y^3 x^3 (X^3 \cdot n)_{(1)} X^2 \cdot m)_{(0)} \otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)_{(0)})_{(0)} \\ (50) = \sum (Y^3 x^3 (y^3 X^3 \cdot n)_{(1)} y^2 X^2 \cdot m)_{(1)} Y^2 x^2 (y^3 X^3 \cdot n)_{(1)} y^1 X^1 \cdot l \\ \otimes (Y^3 x^3 (y^3 X^3 \cdot n)_{(1)} y^2 X^2 \cdot m)_{(0)} \otimes Y^1 x^1 \cdot (y^3 X^3 \cdot n)_{(0)} \\ = \sum (n_{(1)2} \cdot m)_{(1)} n_{(1)1} \cdot l \otimes (n_{(1)2} \cdot m)_{(0)} \otimes n_{(0)} \\ (52) = \sum n_{(1)2} m_{(1)} \cdot l \otimes n_{(1)1} \cdot m_{(0)} \otimes n_{(0)} \end{aligned}$$

and

$$\begin{aligned}
 & \Phi_{321}R_{23}\Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}(l \otimes m \otimes n) \\
 &= \sum Y^3x^3(X^3 \cdot n)_{(1)}X^2m_{(1)} \cdot l \otimes Y^2(x^2 \cdot (X^3 \cdot n)_{(0)})_{(1)}x^1X^1 \cdot m_{(0)} \\
 & \quad \otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)_{(0)})_{(0)} \\
 (50) \quad &= \sum Y^3x^3(y^3X^3 \cdot n)_{(1)_2}y^2X^2m_{(1)} \cdot l \otimes Y^2x^2(y^3X^3 \cdot n)_{(1)_1}y^1X^1 \cdot m_{(0)} \\
 & \quad \otimes Y^1x^1 \cdot (y^3X^3 \cdot n)_{(0)} \\
 &= \sum n_{(1)_2}m_{(1)} \cdot l \otimes n_{(1)_1} \cdot m_{(0)} \otimes n_{(0)}
 \end{aligned}$$

and (53) follows. \square

We will now present a generalization of [17, Prop. 4.4.2], stating that the dual M^* of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module and that $R_{M^*} = R_M^*$.

First we define right-left Yetter-Drinfeld modules for quasi-bialgebras as follows:

$${}^H\mathcal{YD}_H = {}_{H^{op, cop}}\mathcal{YD}^{H^{op, cop}}.$$

More explicitly:

DEFINITION 3.4. Let H be a quasi-bialgebra. A k -linear space M with a right H -action $m \otimes h \mapsto m \cdot h$, and a left H -coaction $M \rightarrow H \otimes M$, $m \mapsto \sum m_{(-1)} \otimes m_{(0)}$ is called a right-left Yetter-Drinfeld module if the following relations hold, for all $m \in M$ and $h \in H$:

$$\begin{aligned}
 & \sum m_{(-1)}x^1 \otimes x^3(m_{(0)} \cdot x^2)_{(-1)} \otimes (m_{(0)} \cdot x^2)_{(0)} \\
 &= \sum y^2(m \cdot y^1)_{(-1)_1}x^1 \otimes y^3(m \cdot y^1)_{(-1)_2}x^2 \otimes (m \cdot y^1)_{(0)} \cdot x^3 \quad (54)
 \end{aligned}$$

$$\sum \varepsilon(m_{(-1)})m_{(0)} = m \quad (55)$$

$$\sum m_{(-1)}h_1 \otimes m_{(0)} \cdot h_2 = \sum h_2(m \cdot h_1)_{(-1)} \otimes (m \cdot h_1)_{(0)}. \quad (56)$$

For $M \in {}^H\mathcal{YD}_H$, we consider the map

$$R_M : M \otimes M \rightarrow M \otimes M, \quad R_M(m \otimes n) = \sum m \cdot n_{(-1)} \otimes n_{(0)}.$$

If we consider M as an object in ${}_{H^{op, cop}}\mathcal{YD}^{H^{op, cop}}$, then we obtain the same map R_M , so R_M is also a solution of the corresponding quasi-Yang-Baxter equation, which is obtained after replacing Φ by $\Phi_{op, cop} = \Phi^{321}$.

Now let M be a finite dimensional right-left Yetter-Drinfeld module. Then M^* is a left H -module, with action given by $(h \cdot m^*)(m) = m^*(m \cdot h)$, for all $h \in H, m \in M, m^* \in M^*$. We also define a k -linear map $M^* \rightarrow M^* \otimes H$, $m^* \mapsto \sum m^*_{(0)} \otimes m^*_{(1)}$, by the condition

$$\sum m^*_{(0)}(m)m^*_{(1)} = \sum m^*(m_{(0)})m_{(-1)} \quad (57)$$

for all $m \in M$. We can prove now the following result.

PROPOSITION 3.5. *Let H be a quasi-bialgebra, M a finite dimensional right-left Yetter-Drinfeld module. Then*

- (i) $M^* \in {}_H\mathcal{YD}^H$;

(ii) $R_{M^*} = R_M^*$.

Proof. (i) We prove that (50), (51), (52) are satisfied. For $m^* \in M^*$ and $m \in M$, we compute:

$$\begin{aligned}
& \sum (x^2 \cdot m_{(0)}^*)(m)(x^2 \cdot m_{(0)}^*)(1)x^1 \otimes x^3 m_{(1)}^* \\
(57) \quad &= \sum (x^2 \cdot m_{(0)}^*)(m_{(0)})m_{(-1)}x^1 \otimes x^3 m_{(1)}^* \\
&= \sum m_{(0)}^*(m_{(0)} \cdot x^2)m_{(-1)}x^1 \otimes x^3 m_{(1)}^* \\
(54) \quad &= \sum m^*((m \cdot y^1)_{(0)} \cdot x^3)y^2(m \cdot y^1)_{(-1)1}x^1 \otimes y^3(m \cdot y^1)_{(-1)2}x^2 \\
&= \sum (x^3 \cdot m^*)((m \cdot y^1)_{(0)})y^2(m \cdot y^1)_{(-1)1}x^1 \otimes y^3(m \cdot y^1)_{(-1)2}x^2 \\
(57) \quad &= \sum (x^3 \cdot m^*)(m \cdot y^1)y^2(x^3 \cdot m^*)(1)1x^1 \otimes y^3(x^3 \cdot m^*)(1)2x^2 \\
&= \sum (y^1 \cdot (x^3 \cdot m^*)(0))(m)y^2(x^3 \cdot m^*)(1)1x^1 \otimes y^3(x^3 \cdot m^*)(1)2x^2
\end{aligned}$$

so obtain (50). Now we compute:

$$\begin{aligned}
& \sum \varepsilon(m_{(1)}^*)m_{(0)}^*(m) = \sum \varepsilon(m_{(0)}^*)(m)m_{(1)}^* \\
(57) \quad &= \sum \varepsilon(m^*(m_{(0)})m_{(-1)}) = \sum m^*(\varepsilon(m_{(-1)})m_{(0)}) = m^*(m),
\end{aligned}$$

using (55) at the last step. Thus (51) holds. For $h \in H$, we compute:

$$\begin{aligned}
& \sum (h_1 \cdot m_{(0)}^*)(m)h_2 m_{(1)}^* = \sum m_{(0)}^*(m \cdot h_1)h_2 m_{(1)}^* \\
(57) \quad &= \sum m^*((m \cdot h_1)_{(0)})h_2(m \cdot h_1)_{(-1)} \\
(56) \quad &= \sum m^*(m_{(0)} \cdot h_2)m_{(-1)}h_1 \\
&= \sum (h_2 \cdot m^*)(m_{(0)})m_{(-1)}h_1 \\
&= \sum (h_2 \cdot m^*)(0)(m)(h_2 \cdot m^*)(1)h_1
\end{aligned}$$

and (52) follows.

(ii) We identify $(M \otimes M)^* = M^* \otimes M^*$, and we prove that R_{M^*} and R_M^* coincide as maps $M^* \otimes M^* \rightarrow M^* \otimes M^*$. For $m, n \in M$ and $m^*, n^* \in M^*$, we compute:

$$\begin{aligned}
R_{M^*}(m^* \otimes n^*)(m \otimes n) &= \sum (n_{(1)}^* \cdot m^*)(m)n_{(0)}^*(n) \\
&= \sum m^*(m \cdot n_{(1)}^*)n_{(0)}^*(n) \\
(57) \quad &= \sum m^*(m \cdot n_{(-1)}^*)n^*(n_{(0)}) \\
&= (m^* \otimes n^*)(R_M(m \otimes n)) \\
&= R_M^*(m^* \otimes n^*)(m \otimes n),
\end{aligned}$$

as needed. \square

4. The quantum commutativity of H_0

Let H be a Hopf algebra. It is well-known that H is an algebra in the monoidal category ${}^H_H\mathcal{YD}$, with left action and coaction given by

$$h \triangleright h' = \sum h_1 h' S(h_2), \quad \lambda(h) = \sum h_1 \otimes h_2.$$

Moreover, H is quantum commutative as an algebra in ${}^H_H\mathcal{YD}$, see for example [11]. We will now prove a similar result for quasi-Hopf algebras. Let H be a quasi-Hopf algebra. In [7], a new multiplication on H was introduced; this multiplication is given by the formula

$$h \circ h' = \sum X^1 h S(x^1 X^2) \alpha x^2 X_1^3 h' S(x^3 X_2^3) \quad (58)$$

for all $h, h' \in H$. β is a unit for this multiplication \circ . Let H_0 be the k -linear space H , with multiplication \circ , and left H -action given by

$$h \triangleright h' = \sum h_1 h' S(h_2). \quad (59)$$

Then H_0 is a left H -module algebra. In H_0 , we also define a left H -coaction, as follows

$$\begin{aligned} \lambda_{H_0}(h) &= \sum h_{(-1)} \otimes h_{(0)} \\ &= \sum X^1 Y_1^1 h_1 g^1 S(q^2 Y_2^2) Y^3 \otimes X^2 Y_2^1 h_2 g^2 S(X^3 q^1 Y_1^2), \end{aligned} \quad (60)$$

where $f^{-1} = \sum g^1 \otimes g^2$ and $q_R = \sum q^1 \otimes q^2$ are the elements defined by (16) and (19). Then H_0 is an algebra in ${}^H_H\mathcal{YD}$, see [5] for details. In Proposition 4.2, we will show that H_0 is quantum commutative. But first we need the following formulas, which are of independent interest. Recall that $q_R = \sum q^1 \otimes q^2$, $q_L, f = \sum f^1 \otimes f^2$ and $f^{-1} = \sum g^1 \otimes g^2$ are defined by (20), (22), (15) and (16).

LEMMA 4.1. *Let H be a quasi-Hopf algebra. Then we have*

$$\sum q^1 y^1 \otimes S(q^2 y^2) y^3 = 1 \otimes \alpha, \quad (61)$$

$$\Phi(\Delta \otimes id)(f^{-1}) = \sum g^1 S(X^3) f^1 \otimes g_1^2 G^1 S(X^2) f^2 \otimes g_2^2 G^2 S(X^1), \quad (62)$$

$$\sum S(g^1) \alpha g^2 = S(\beta), \quad \sum f^1 \beta S(f^2) = S(\alpha), \quad (63)$$

$$\sum S(q_2^2 X^3) f^1 \otimes S(q^1 X^1 \beta S(q_1^2 X^2) f^2) = (id \otimes S)(q_L), \quad (64)$$

Proof. (61) and (62) are a direct consequence of (19) and (18). (63) has been proved in [6, Lemma 2.6] and [10, Lemma 2.5]. We are left to prove (64). Using (27), we obtain:

$$(id \otimes \Delta)(q) = \sum (1 \otimes S^{-1}(x^3 g^2) \otimes S^{-1}(x^2 g^1))(q \otimes 1)(\Delta \otimes id)(q) \Phi^{-1}(id \otimes \Delta)(\Delta(x^1))$$

and, using the formula (see [8])

$$(\Delta \otimes id)(q) \Phi^{-1} = \sum Y^1 \otimes q^1 Y_1^2 \otimes S^{-1}(Y^3) q^2 Y_2^2,$$

we obtain

$$(id \otimes \Delta)(q) = \sum Q^1 Y^1 x_1^1 \otimes S^{-1}(x^3 g^2) Q^2 q^1 Y_1^2 x_{(2,1)}^1 \otimes S^{-1}(Y^3 x^2 g^1) q^2 Y_2^2 x_{(2,2)}^1 \quad (65)$$

where $q_R = \sum q^1 \otimes q^2 = \sum Q^1 \otimes Q^2$. Now we compute

$$\begin{aligned}
& \sum S(q_2^2 X^3) f^1 \otimes S(q^1 X^1 \beta S(q_1^2 X^2) f^2) \\
(65) &= \sum S(q^2 Y_2^2 x_{(2,2)}^1 X^3) Y^3 x^2 \otimes S(Q^1 Y^1 x_1^1 X^1 \beta S(Q^2 q^1 Y_1^2 x_{(2,1)}^1 X^2) x^3) \\
(1) &= \sum S(q^2 Y_2^2 X^3 x_2^1) Y^3 x^2 \otimes S(Q^1 Y^1 X^1 x_{(1,1)}^1 \beta S(Q^2 q^1 Y_1^2 X^2 x_{(1,2)}^1) x^3) \\
(5) &= \sum S(q^2 Y_2^2 X^3 x^1) Y^3 x^2 \otimes S(Q^1 Y^1 X^1 \beta S(Q^2 q^1 Y_1^2 X^2) x^3) \\
(3) &= \sum S(q^2 y^2 X_1^3 Y^2 x^1) y^3 X_2^3 Y^3 x^2 \otimes S(Q^1 X^1 Y_1^1 \beta S(Q^2 q^1 y^1 X^2 Y_2^1) x^3) \\
(5, 61) &= \sum S(X_1^3 x^1) \alpha X_2^3 x^2 \otimes S(Q^1 X^1 \beta S(Q^2 X^2) x^3) \\
(5, 7) &= \sum S(x^1) \alpha x^2 \otimes S(Q^1 \beta S(Q^2) x^3) \\
(19, 6) &= \sum S(x^1) \alpha x^2 \otimes S(x^3),
\end{aligned}$$

as needed. \square

We can prove now the main result of this Section.

PROPOSITION 4.2. *Let H be a quasi-Hopf algebra. Then H_0 is quantum commutative as an algebra in ${}^H_H \mathcal{YD}$, that is, for all $h, h' \in H$:*

$$h \circ h' = \sum (h_{(-1)} \triangleright h') \circ h_{(0)}.$$

Proof. For all $h, h' \in H$ we compute:

$$\begin{aligned}
& \sum (h_{(-1)} \triangleright h') \circ h_{(0)} \\
(60) &= \sum (X^1 Y_1^1 h_1 g^1 S(q^2 Y_2^2) Y^3 \triangleright h') \circ X^2 Y_2^1 h_2 g^2 S(X^3 q^1 Y_1^2) \\
(59, 58) &= \sum Z^1 X_1^1 Y_{(1,1)}^1 h_{(1,1)} g_1^1 S(q^2 Y_2^2)_1 Y_1^3 h' \\
& \quad S(x^1 Z^2 X_2^1 Y_{(1,2)}^1 h_{(1,2)} g_2^1 S(q^2 Y_2^2)_2 Y_2^3) \\
& \quad \alpha x^2 Z_1^3 X^2 Y_2^1 h_2 g^2 S(x^3 Z_2^3 X^3 q^1 Y_1^2) \\
(3, 5) &= \sum Z^1 Y_{(1,1)}^1 h_{(1,1)} g_1^1 S(q^2 Y_2^2)_1 Y_1^3 h' \\
& \quad S(Z^2 Y_{(1,2)}^1 h_{(1,2)} g_2^1 S(q^2 Y_2^2)_2 Y_2^3) \\
& \quad \alpha Z^3 Y_2^1 h_2 g^2 S(q^1 Y_1^2) \\
(11) &= \sum Z^1 [Y^1 h S(Y^2)]_{(1,1)} g_1^1 S(q^2)_1 Y_1^3 h' \\
& \quad S(Z^2 [Y^1 h S(Y^2)]_{(1,2)} g_2^1 S(q^2)_2 Y_2^3) \\
& \quad \alpha Z^3 [Y^1 h S(Y^2)]_2 g^2 S(q^1) \\
(1, 5) &= \sum Y^1 h S(Y^2) Z^1 g_1^1 S(q^2)_1 Y_1^3 h' S(Z^2 g_2^1 S(q^2)_2 Y_2^3) \alpha Z^3 g^2 S(q^1) \\
(62) &= \sum Y^1 h S(Y^2) g^1 S(X^3) f^1 S(q^2)_1 Y_1^3 h' \\
& \quad S(g_1^2 G^1 S(X^2) f^2 S(q^2)_2 Y_2^3) \alpha g_2^2 G^2 S(q^1 X^1)
\end{aligned}$$

$$\begin{aligned}
 (5, 63) &= \sum Y^1 h S(X^3 Y^2) f^1 S(q^2)_1 Y_1^3 h' S(q^1 X^1 \beta S(X^2) f^2 S(q^2)_2 Y_2^3) \\
 (11) &= \sum Y^1 h S(q_2^2 X^3 Y^2) f^1 Y_1^3 h' S(q^1 X^1 \beta S(q_1^2 X^2) f^2 Y_2^3) \\
 (64) &= \sum Y^1 h S(x^1 Y^2) \alpha x^2 Y_1^3 h' S(x^3 Y_2^3) \\
 (58) &= h \circ h'.
 \end{aligned}$$

□

5. Hopf modules in ${}^H_H\mathcal{YD}$. Integrals

Let H be a quasi-Hopf algebra. The aim of this Section is to define the space of integrals of a finite dimensional braided Hopf algebra in ${}^H_H\mathcal{YD}$, and to prove, following [24], [12], that it is an object of ${}^H_H\mathcal{YD}$, and that it has dimension 1. We will apply our results to the braided Hopf algebra associated to H , in the case where H is a quasitriangular quasi-Hopf algebra.

Let A be an algebra in a monoidal category \mathcal{C} . Recall that a right A -module M is an object $M \in \mathcal{C}$ together with a morphism $\omega_M : M \otimes A \rightarrow M$ in \mathcal{C} such that $\omega_M \circ (id_M \otimes \eta) = l_M^{-1}$ and the following diagram is commutative:

$$\begin{array}{ccccc}
 (M \otimes A) \otimes A & \xrightarrow{\omega_M \otimes id_A} & M \otimes A & \xrightarrow{\omega_M} & M \\
 \downarrow a_{M,A,A} & & & & \uparrow \omega_M \\
 M \otimes (A \otimes A) & \xrightarrow{id_M \otimes m} & M \otimes A & &
 \end{array}$$

Clearly A itself is a right A -module, by right multiplication. Right comodules over a coalgebra C in \mathcal{C} can be defined in a similar way: we need $N \in \mathcal{C}$ together with a morphism $\rho_N : N \rightarrow N \otimes C$ in \mathcal{C} such that $(id_N \otimes \varepsilon) \circ \rho_N = l_N$ and the following diagram is commutative:

$$\begin{array}{ccccc}
 N & \xrightarrow{\rho_N} & N \otimes C & \xrightarrow{\rho_N \otimes id_C} & (N \otimes C) \otimes C \\
 \downarrow \rho_N & & & & \downarrow a_{N,C,C} \\
 N \otimes C & \xrightarrow{id_N \otimes \underline{\Delta}} & N \otimes (C \otimes C) & &
 \end{array}$$

C itself is a right C -comodule via the comultiplication $\underline{\Delta}$.

From [3], [21], [24], we recall the following.

DEFINITION 5.1. Let B be a bialgebra in a braided category \mathcal{C} . A right B -Hopf module is a triple (M, ω_M, ρ_M) , where (M, ω_M) is a right B -module and (M, ρ_M) is a right B -comodule such that $\rho_M : M \rightarrow M \otimes B$ is right B -linear. The B -module structure $\omega_{M \otimes B} : (M \otimes B) \otimes B \rightarrow M \otimes B$ on $M \otimes B$ is given by the following

composition:

$$\begin{array}{ccc}
(M \otimes B) \otimes B & \xrightarrow{id_{M \otimes B} \otimes \Delta} & (M \otimes B) \otimes (B \otimes B) \\
& \xrightarrow{a_{M, B, B \otimes B}} & M \otimes (B \otimes (B \otimes B)) \\
& \xrightarrow{id_M \otimes a_{B, B, B}^{-1}} & M \otimes ((B \otimes B) \otimes B) \\
& \xrightarrow{id_M \otimes (c_{B, B} \otimes id_B)} & M \otimes ((B \otimes B) \otimes B) \\
& \xrightarrow{id_M \otimes a_{B, B, B}} & M \otimes (B \otimes (B \otimes B)) \\
& \xrightarrow{a_{M, B, B \otimes B}^{-1}} & (M \otimes B) \otimes (B \otimes B) \\
& \xrightarrow{\underline{\omega}_M \otimes \underline{m}} & M \otimes B
\end{array} \tag{66}$$

\mathcal{M}_B^B will denote the category of right B -Hopf modules and morphisms in \mathcal{C} preserving the B -action and the corresponding B -coaction.

We can consider algebras, coalgebras, bialgebras and Hopf algebras in the braided category ${}^H_H\mathcal{YD}$ over a quasi-Hopf algebra H . More precisely, an algebra B in ${}^H_H\mathcal{YD}$ is an object $B \in {}^H_H\mathcal{YD}$ such that

- B is a left H -module algebra, i.e. B has a multiplication \underline{m} and a usual unit 1_B satisfying the following conditions:

$$(ab)c = \sum (X^1 \cdot a)[(X^2 \cdot b)(X^3 \cdot c)], \tag{67}$$

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B, \tag{68}$$

for all $a, b, c \in B$ and $h \in H$.

- B is a quasi-comodule algebra, that is, the multiplication \underline{m} and the unit η of B intertwine the H -coaction λ_B . By (47) this means:

$$\begin{aligned}
\lambda_B(bb') &= \sum X^1(x^1Y^1 \cdot b)_{(-1)}x^2(Y^2 \cdot b')_{(-1)}Y^3 \\
&\quad \otimes [X^2 \cdot (x^1Y^1 \cdot b)_{(0)}][X^3x^3 \cdot (Y^2 \cdot b')_{(0)}],
\end{aligned} \tag{69}$$

for all $b, b' \in B$, and

$$\lambda_B(1_B) = 1_H \otimes 1_B. \tag{70}$$

$M \in {}^H_H\mathcal{YD}$ is a right B -module if there exists a morphism $\underline{\omega}_M : M \otimes B \rightarrow M$ in ${}^H_H\mathcal{YD}$ (we will denote $\underline{\omega}_M(m \otimes b) := m \leftarrow b$) such that

$$m \leftarrow 1_B = m, \quad (m \leftarrow b) \leftarrow b' = \sum (X^1 \cdot m) \leftarrow [(X^2 \cdot b)(X^3 \cdot b')] \tag{71}$$

for all $m \in M$, $b, b' \in B$. The fact that $\underline{\omega}_M$ is a morphism in ${}^H_H\mathcal{YD}$ means (see (47))

$$h \cdot (m \leftarrow b) = \sum (h_1 \cdot m) \leftarrow (h_2 \cdot b), \tag{72}$$

$$\begin{aligned}
\lambda_M(m \leftarrow b) &= \sum X^1(x^1Y^1 \cdot m)_{(-1)}x^2(Y^2 \cdot b)_{(-1)}Y^3 \\
&\quad \otimes [X^2 \cdot (x^1Y^1 \cdot m)_{(0)}] \leftarrow [X^3x^3 \cdot (Y^2 \cdot b)_{(0)}]
\end{aligned} \tag{73}$$

for all $m \in M$, $b \in B$.

Similarly, $B \in {}^H_H\mathcal{YD}$ is a coalgebra if

- B is a left H -module coalgebra, i.e. B has a comultiplication $\underline{\Delta}_B : B \rightarrow B \otimes B$ (we will denote $\underline{\Delta}(b) = \sum b_{\underline{1}} \otimes b_{\underline{2}}$) and a usual counit $\underline{\varepsilon}_B$ such that:

$$\sum X^1 \cdot b_{(\underline{1}, \underline{1})} \otimes X^2 \cdot b_{(\underline{1}, \underline{2})} \otimes X^3 \cdot b_{\underline{2}} = \sum b_{\underline{1}} \otimes b_{(\underline{2}, \underline{1})} \otimes b_{(\underline{2}, \underline{2})}, \quad (74)$$

$$\underline{\Delta}_B(h \cdot b) = \sum h_1 \cdot b_{\underline{1}} \otimes h_2 \cdot b_{\underline{2}}, \quad \underline{\varepsilon}_B(h \cdot b) = \varepsilon(h)\underline{\varepsilon}_B(b), \quad (75)$$

for all $h \in H, b \in B$, where we use the same notation for the quasi-coassociativity of $\underline{\Delta}_B$ as in Section 2.

- B is a quasi-comodule coalgebra, i.e. the comultiplication $\underline{\Delta}_B$ and the counit $\underline{\varepsilon}_B$ intertwine the H -coaction λ_B . Explicitly, for all $b \in B$ we must have that:

$$\begin{aligned} \sum b_{(-1)} \otimes b_{(0)\underline{1}} \otimes b_{(0)\underline{2}} &= \sum X^1(x^1Y^1 \cdot b_{\underline{1}})_{(-1)}x^2(Y^2 \cdot b_{\underline{2}})_{(-1)}Y^3 \\ &\otimes X^2 \cdot (x^1Y^1 \cdot b_{\underline{1}})_{(0)} \otimes X^3x^3 \cdot (Y^2 \cdot b_{\underline{2}})_{(0)}, \end{aligned} \quad (76)$$

and

$$\sum \varepsilon_B(b_{(0)})b_{(-1)} = \varepsilon_B(b)1. \quad (77)$$

A right B -comodule in ${}^H_H\mathcal{YD}$ is an object $M \in {}^H_H\mathcal{YD}$ together with a morphism $\underline{\rho}_M : M \rightarrow M \otimes B$ in ${}^H_H\mathcal{YD}$ (we will denote $\underline{\rho}_M(m) = \sum m_{(0)} \otimes m_{(\underline{1})}$) for all $m \in M$ such that the following relations hold, for all $m \in M$:

$$\sum X^1 \cdot m_{(\underline{0}, \underline{0})} \otimes X^2 \cdot m_{(\underline{0}, \underline{1})} \otimes X^3 \cdot m_{(\underline{1})} = \sum m_{(\underline{0})} \otimes m_{(\underline{1})\underline{1}} \otimes m_{(\underline{1})\underline{2}}, \quad (78)$$

$$\sum \underline{\varepsilon}(m_{(\underline{1})})m_{(\underline{0})} = m, \quad (79)$$

where we will denote

$$(\underline{\rho}_M \otimes id_B)(\underline{\rho}_M(m)) = \sum m_{(\underline{0}, \underline{0})} \otimes m_{(\underline{0}, \underline{1})} \otimes m_{(\underline{1})} \text{ etc.}$$

The fact that $\underline{\rho}_M$ is a morphism in ${}^H_H\mathcal{YD}$ means that (see (47))

$$\underline{\rho}_M(h \cdot m) = \sum h_1 \cdot m_{(\underline{0})} \otimes h_2 \cdot m_{(\underline{1})}, \quad (80)$$

and

$$\begin{aligned} \sum m_{(-1)} \otimes m_{(0)(\underline{0})} \otimes m_{(0)(\underline{1})} &= \sum X^1(x^1Y^1 \cdot m_{(\underline{0})})_{(-1)}x^2(Y^2 \cdot m_{(\underline{1})})_{(-1)}Y^3 \\ &\otimes X^2 \cdot (x^1Y^1 \cdot m_{(\underline{0})})_{(0)} \otimes X^3x^3 \cdot (Y^2 \cdot m_{(\underline{1})})_{(0)}, \end{aligned} \quad (81)$$

for all $h \in H$ and $m \in M$.

Now, a bialgebra $B \in {}^H_H\mathcal{YD}$ is an algebra and a coalgebra in ${}^H_H\mathcal{YD}$ such that $\underline{\Delta}_B$ is an algebra morphism, i.e. $\underline{\Delta}_B(1_B) = 1_B \otimes 1_B$ and, by (38) and (48), for all $b, b' \in B$ we have that:

$$\begin{aligned} \underline{\Delta}_B(bb') &= \sum [y^1X^1 \cdot b_{\underline{1}}][y^2Y^1(x^1X^2 \cdot b_{\underline{2}})_{(-1)}x^2X_1^3 \cdot b'_{\underline{1}}] \\ &\otimes [y_1^3Y^2 \cdot (x^1X^2 \cdot b_{\underline{2}})_{(0)}][y_2^3Y^3x^3X_2^3 \cdot b'_{\underline{2}}]. \end{aligned} \quad (82)$$

If $B \in {}^H_H\mathcal{YD}$ is a bialgebra then $M \in {}^H_H\mathcal{YD}$ is a right B -Hopf module if M is a right B -module (as above, we will denote $\underline{\omega}_M(m \otimes b) = m \leftarrow b$) and a right B -comodule

such that the right B -coaction on M , $\rho_M : M \rightarrow M \otimes B$, is right B -linear, which means that the following relation holds, for all $m \in M$ and $b \in B$ (see (66)):

$$\begin{aligned} \rho_M(m \leftarrow b) &= \sum (y^1 X^1 \cdot m_{(0)}) \leftarrow [y^2 Y^1 (x^1 X^2 \cdot m_{(\underline{1})})_{(-1)} x^2 X_1^3 \cdot b_{\underline{1}}] \\ &\quad \otimes [y_1^3 Y^2 \cdot (x^1 X^2 \cdot m_{(\underline{1})})_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b_{\underline{2}}]. \end{aligned} \quad (83)$$

Finally, a bialgebra B in ${}^H_H\mathcal{YD}$ is a braided Hopf algebra if there exists a morphism $\underline{S} : B \rightarrow B$ in ${}^H_H\mathcal{YD}$ such that $\sum \underline{S}(b_{\underline{1}}) b_{\underline{2}} = \sum b_{\underline{1}} \underline{S}(b_{\underline{2}}) = \underline{\varepsilon}(b) 1_B$, for all $b \in B$. Since \underline{S} is a morphism in ${}^H_H\mathcal{YD}$, we have that

$$\underline{S}(h \cdot b) = h \cdot \underline{S}(b) \quad \text{and} \quad \sum \underline{S}(b)_{(-1)} \otimes \underline{S}(b)_{(0)} = \sum b_{(-1)} \otimes \underline{S}(b_{(0)}), \quad (84)$$

for all $h \in H$, $b \in B$. Also, by (39) and (48) we obtain that

$$\underline{S}(bb') = \sum [b_{(-1)} \cdot \underline{S}(b')] \underline{S}(b_{(0)}) \quad \text{and} \quad \underline{\Delta}(\underline{S}(b)) = \sum b_{\underline{1}(-1)} \cdot \underline{S}(b_{\underline{2}}) \otimes \underline{S}(b_{\underline{1}(0)}), \quad (85)$$

for all $b, b' \in B$.

The first step to prove the existence and uniqueness of integrals in a finite dimensional braided Hopf algebra is the structure theorem for Hopf modules. To this end we need first the following result.

LEMMA 5.2. *Let H be a quasi-bialgebra, B a bialgebra in ${}^H_H\mathcal{YD}$ and $N \in {}^H_H\mathcal{YD}$. Then $N \otimes B \in \mathcal{M}_B^B$ with following action $\omega_{N \otimes B} : (N \otimes B) \otimes B \rightarrow N \otimes B$ and coaction $\rho_{N \otimes B} : N \otimes B \rightarrow (N \otimes B) \otimes B$ given by*

$$(n \otimes b) \prec b' = \sum X^1 \cdot n \otimes [(X^2 \cdot b)(X^3 \cdot b')], \quad (86)$$

$$\rho_{N \otimes B}(n \otimes b) := \sum x^1 \cdot n \otimes x^2 \cdot b_{\underline{1}} \otimes x^3 \cdot b_{\underline{2}}, \quad (87)$$

for all $n \in N$ and $b, b' \in B$.

Proof. ${}^H_H\mathcal{YD}$ is a braided category, so $N \otimes B \in {}^H_H\mathcal{YD}$. It is not hard to see that (1) and (67) imply that $\omega_{N \otimes B}$ is left H -linear. It intertwines also the corresponding H -coaction. Indeed, by (47), the left H -coaction on $(N \otimes B) \otimes B$ is given by

$$\begin{aligned} &\lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b') \\ &= \sum Z^1 X^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} x^2 (Y^2 y_2^1 T_2^1 \cdot b)_{(-1)} Y^3 y^2 (T^2 \cdot b')_{(-1)} T^3 \\ &\quad \otimes Z_1^2 X^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \otimes Z_2^2 X^3 x^3 \cdot (Y^2 y_2^1 T_2^1 \cdot b)_{(0)} \otimes Z^3 y^3 \cdot (T^2 \cdot b')_{(0)}, \end{aligned}$$

for all $n \in N$, $b, b' \in B$. Therefore:

$$\begin{aligned}
& (id_H \otimes \underline{\omega}_{N \otimes B}) \circ \lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b') \\
(86) \quad &= \sum Z^1 X^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} x^2 (Y^2 y_2^1 T_2^1 \cdot b)_{(-1)} \\
& \quad Y^3 y^2 (T^2 \cdot b')_{(-1)} T^3 \otimes W^1 Z_1^2 X^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \\
& \quad \otimes [(W^2 Z_2^2 X^3 x^3 \cdot (Y^2 y_2^1 T_2^1 \cdot b)_{(0)}) [(W^3 Z^3 y^3 \cdot (T^2 \cdot b')_{(0)})]] \\
(3, 45, 67) \quad &= \sum Z^1 (X_1^1 x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} X_2^1 x^2 (Y^2 y_2^1 T_2^1 \cdot b)_{(-1)} \\
& \quad Y^3 y^2 (T^2 \cdot b')_{(-1)} T^3 \otimes Z^2 \cdot (X_1^1 x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \\
& \quad \otimes Z^3 \cdot [(X^2 x^3 \cdot (Y^2 y_2^1 T_2^1 \cdot b)_{(0)}) (X^3 y^3 \cdot (T^2 \cdot b')_{(0)})] \\
(3) \text{ twice, (45)} \quad &= \sum Z^1 (x^1 Y^1 T_1^1 \cdot n)_{(-1)} x^2 X^1 (y^1 Y^2 T_2^1 \cdot b)_{(-1)} y^2 \\
& \quad (Y_1^3 T^2 \cdot b')_{(-1)} Y_2^3 T^3 \otimes Z^2 \cdot (x^1 Y^1 T_1^1 \cdot n)_{(0)} \\
& \quad \otimes Z^3 x^3 \cdot [(X^2 \cdot (y^1 Y^2 T_2^1 \cdot b)_{(0)}) (X^3 y^3 \cdot (Y_1^3 T^2 \cdot b')_{(0)})] \\
(3, 69) \quad &= \sum Z^1 (x^1 Y^1 T_1^1 \cdot n)_{(-1)} x^2 [(Y_1^3 T^2 \cdot b) (Y_2^3 T^3 \cdot b')]_{(-1)} Y^3 \\
& \quad \otimes Z^2 \cdot (x^1 Y^1 T_1^1 \cdot n)_{(0)} \otimes Z^3 x^3 \cdot [(Y_1^3 T^2 \cdot b) (Y_2^3 T^3 \cdot b')]_{(0)} \\
(47, 86) \quad &= \sum \lambda_{N \otimes B}(T^1 \cdot n \otimes (T^2 \cdot b) (T^3 \cdot b')) \\
&= \lambda_{N \otimes B} \circ \underline{\omega}_{N \otimes B}((n \otimes b) \otimes b')
\end{aligned}$$

for all $n \in N$ and $b, b' \in B$. In a similar way, it can be proved that the map $\underline{\rho}_{N \otimes B}$ is a morphism in $\frac{H}{H} \mathcal{YD}$, we leave it to the reader to verify the details.

Using (67) and (3), it easily follows that $N \otimes B$ is a right B -module. Also, it is not hard to see that (74), (75) and (3) imply that $N \otimes B$ is a right B -comodule. It remains only to show that $\underline{\rho}_{N \otimes B}$ is right B -linear. By (66), we have that the right B -module structure of $(N \otimes B) \otimes B$ is given by

$$\begin{aligned}
& [(n \otimes b) \otimes b'] \bullet b'' \\
&= \sum [Z^1 y_1^1 X_1^1 \cdot n \otimes (Z^2 y_2^1 X_2^1 \cdot b) (Z^3 y^2 Y^1 (x^1 X^2 \cdot b')_{(-1)} x^2 X_1^3 \cdot b''_{\underline{1}})] \\
& \quad \otimes [y_1^3 Y^2 \cdot (x^1 X^2 \cdot b')_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b''_{\underline{2}}],
\end{aligned}$$

for all $n \in N$ and $b, b', b'' \in B$. This allows us to compute, for any $n \in N$ and $b, b' \in B$, that:

$$\begin{aligned}
\underline{\rho}_{N \otimes B}(n \otimes b) \bullet b' &= \sum [(z^1 \cdot n \otimes z^2 \cdot b_{\underline{1}}) \otimes z^3 \cdot b_{\underline{2}}] \bullet b' \\
&= \sum [Z^1 y_1^1 X_1^1 z^1 \cdot n \\
& \quad \otimes (Z^2 y_2^1 X_2^1 z^2 \cdot b_{\underline{1}}) (Z^3 y^2 Y^1 (x^1 X^2 z^3 \cdot b_{\underline{2}})_{(-1)} x^2 X_1^3 \cdot b'_{\underline{1}})] \\
& \quad \otimes [y_1^3 Y^2 \cdot (x^1 X^2 z^3 \cdot b_{\underline{2}})_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b'_{\underline{2}}]
\end{aligned}$$

$$\begin{aligned}
(3) &= \sum [Z^1 y_1^1 z^1 X^1 \cdot n \otimes (Z^2 y_2^1 z^2 T^1 X_1^2 \cdot b_1) \\
&\quad (Z^3 y^2 Y^1 (x^1 z_1^3 T^2 X_2^2 \cdot b_2)_{(-1)} x^2 z_{(2,1)}^3 T_1^3 X_1^3 \cdot b'_1)] \\
&\quad \otimes [y_1^3 Y^2 \cdot (x^1 z_1^3 T^2 X_2^2 \cdot b_2)_{(0)}] [y_2^3 Y^3 x^3 z_{(2,2)}^3 T_2^3 X_2^3 \cdot b'_2] \\
(1, 45) &= \sum [Z^1 y_1^1 z^1 X^1 \cdot n \otimes (Z^2 y_2^1 z^2 T^1 X_1^2 \cdot b_1) \\
&\quad (Z^3 y^2 Y^1 z_{(1,1)}^3 (x^1 T^2 X_2^2 \cdot b_2)_{(-1)} x^2 T_1^3 X_1^3 \cdot b'_1)] \\
&\quad \otimes [y_1^3 Y^2 z_{(1,2)}^3 \cdot (x^1 T^2 X_2^2 \cdot b_2)_{(0)}] [y_2^3 Y^3 z_2^3 x^3 T_2^3 X_2^3 \cdot b'_2] \\
(1, 3, 67) &= \sum \{y^1 X^1 \cdot n \otimes y^2 \cdot [(z^1 T^1 X_1^2 \cdot b_1) \\
&\quad (z^2 Y^1 (x^1 T^2 X_2^2 \cdot b_2)_{(-1)} x^2 T_1^3 X_1^3 \cdot b'_1)]\} \\
&\quad \otimes y^3 \cdot \{[z_1^3 Y^2 \cdot (x^1 T^2 X_2^2 \cdot b_2)_{(0)}] [z_2^3 Y^3 x^3 T_2^3 X_2^3 \cdot b'_2]\} \\
(75, 82) &= \sum \{y^1 X^1 \cdot n \otimes y^2 \cdot [(X^2 \cdot b)(X^3 \cdot b')]\}_1 \otimes y^3 \cdot [(X^2 \cdot b)(X^3 \cdot b')]\}_2 \\
(87, 86) &= \sum \rho_{N \otimes B}(X^1 \cdot n \otimes (X^2 \cdot b)(X^3 \cdot b')) = \rho_{N \otimes B}((n \otimes b) \prec b'),
\end{aligned}$$

as needed. \square

Our next result is the Fundamental Theorem for Hopf modules in the braided monoidal category ${}^H_H\mathcal{YD}$, generalizing [12, Theorem 1].

THEOREM 5.3. *Let H be a quasi-Hopf algebra, B a Hopf algebra in ${}^H_H\mathcal{YD}$ and $M \in \mathcal{M}_B^B$.*

- (i) $M^{\text{co}B} = \{m \in M \mid \rho_M(m) = m \otimes 1_B\} \in {}^H_H\mathcal{YD}$.
- (ii) For all $m \in M$, we have that $P(m) = \sum m_{(0)} \leftarrow \underline{S}(m_{(1)}) \in M^{\text{co}B}$.
- (iii) $\rho_M(n \leftarrow b) = \sum (x^1 \cdot n) \leftarrow (x^2 \cdot b_1) \otimes x^3 \cdot b_2$ and $P(n \leftarrow b) = \underline{\varepsilon}(b)n$, for all $n \in M^{\text{co}B}$ and $b \in B$.
- (iv) The map

$$F : M^{\text{co}B} \otimes B \rightarrow M, \quad F(n \otimes b) = n \leftarrow b,$$

is an isomorphism of Hopf modules in ${}^H_H\mathcal{YD}$, with inverse G given by

$$G(m) = \sum P(m_{(0)}) \otimes m_{(1)}.$$

Proof. (i) If $n \in M^{\text{co}B}$, then $\rho_M(h \cdot n) = \sum h_1 \cdot n \otimes h_2 \cdot 1_B = h \cdot n \otimes 1_B$, by (72) and (67). This shows that $M^{\text{co}B}$ is an H -submodule of M . On the other hand, for any $n \in N$ we have

$$\begin{aligned}
&\sum n_{(-1)} \otimes n_{(0)(0)} \otimes n_{(0)(1)} \\
(81) &= \sum X^1 (x^1 Y^1 \cdot n)_{(-1)} x^2 (Y^2 \cdot 1_B)_{(-1)} Y^3 \\
&\quad \otimes X^2 \cdot (x^1 Y^1 \cdot n)_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot 1_B)_{(0)} \\
(67) \text{ twice, (70)} &= \sum n_{(-1)} \otimes n_{(0)} \otimes 1_B.
\end{aligned}$$

Thus, $\rho_M(n) = \sum n_{(-1)} \otimes n_{(0)} \in H \otimes M^{\text{co}B}$ which means that $M^{\text{co}B}$ is a left H -quasi-subcomodule of M . It follows from the above arguments that $M^{\text{co}B} \in {}^H_H\mathcal{YD}$.

(ii) For any $m \in M$, we have that

$$\begin{aligned}
 \rho_M(P(m)) &= \sum \rho_M(m_{(0)} \leftarrow \underline{S}(m_{(1)})) \\
 (83) &= \sum (y^1 X^1 \cdot m_{(0,0)} \leftarrow [y^2 Y^1(x^1 X^2 \cdot m_{(0,1)})_{(-1)} x^2 X_1^3 \cdot \underline{S}(m_{(1)})_{\underline{1}}] \\
 &\quad [y_1^3 Y^2 \cdot (x^1 X^2 \cdot m_{(0,1)})_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot \underline{S}(m_{(1)})_{\underline{2}}]) \\
 (78, 84, 85) &= \sum (y^1 \cdot m_{(0)} \leftarrow \\
 &\quad [y^2 Y^1(x^1 \cdot m_{(1,1)})_{(-1)} x^2 \underline{S}(m_{(1,2,1)})_{(-1)} \cdot \underline{S}(m_{(1,2,2)})] \\
 &\quad \otimes y^3 \cdot \{[Y^2 \cdot (x^1 \cdot m_{(1,1)})_{(0)}] [Y^3 x^3 \cdot \underline{S}(m_{(1,2,1)})_{(0)}]\}) \\
 (69) &= \sum (y^1 \cdot m_{(0)} \leftarrow \\
 &\quad y^2 [(x^1 \cdot m_{(1,1)})(x^2 \cdot \underline{S}(m_{(1,2,1)}))]_{(-1)} x^3 \cdot \underline{S}(m_{(1,2,2)}) \\
 &\quad \otimes y^3 \cdot [(x^1 \cdot m_{(1,1)})(x^2 \cdot \underline{S}(m_{(1,2,1)}))]_{(0)}) \\
 (84, 74, 67) &= \sum (y^1 \cdot m_{(0)} \leftarrow (y^2 \cdot \underline{S}(m_{(1)})) \otimes y^3 \cdot 1_B = P(m) \otimes 1_B.
 \end{aligned}$$

(iii) For all $n \in N$ and $b \in B$, we compute, using (83),

$$\begin{aligned}
 \rho_M(n \leftarrow b) &= \sum (y^1 X^1 \cdot n \leftarrow [y^2 Y^1(x^1 X^2 \cdot 1_B)_{(-1)} x^2 X_1^3 \cdot b_{\underline{1}}] \\
 &\quad \otimes [y_1^3 Y^2 \cdot (x^1 X^2 \cdot 1_B)_{(0)}] [y_2^3 Y^3 x^3 X_2^3 \cdot b_{\underline{2}}]) \\
 (67, 70) &= \sum (y^1 \cdot n \leftarrow (y^2 \cdot b_{\underline{1}}) \otimes y^3 \cdot b_{\underline{2}}.
 \end{aligned}$$

For all $n \in M^{\text{co}B}$, we find

$$\begin{aligned}
 P(n \leftarrow b) &= \sum [(y^1 \cdot n) \leftarrow (y^2 \cdot b_{\underline{1}})] \leftarrow \underline{S}(y^3 \cdot b_{\underline{2}}) \\
 (71, 84) &= \sum n \leftarrow b_{\underline{1}} \underline{S}(b_{\underline{2}}) = \varepsilon(b)n \leftarrow 1_B = \varepsilon(b)n.
 \end{aligned}$$

(iv) By (i) and Lemma 5.2, we obtain that $M^{\text{co}B} \otimes B \in \mathcal{M}_B^B$. It follows from (72) that F is left H -linear. It also intertwines the corresponding left H -coaction by (47) and (73). Now we will prove that F and G are inverses. For all $m \in M$, we have

$$\begin{aligned}
 FG(m) &= \sum P(m_{(0)}) \leftarrow m_{(1)} \\
 (71) &= \sum (X^1 \cdot m_{(0,0)} \leftarrow [(X^2 \cdot \underline{S}(m_{(0,1)}))(X^3 \cdot m_{(1)})] \\
 (84, 78, 79) &= \sum m_{(0)} \leftarrow \underline{S}(m_{(1,1)}) m_{(1,2)} = m \leftarrow 1_B = m.
 \end{aligned}$$

Similarly, for any $n \in M^{\text{co}B}$ and $b \in B$, we compute

$$\begin{aligned}
 GF(n \otimes b) &= \sum P((n \leftarrow b)_{(0)}) \otimes (n \leftarrow b)_{(1)} \\
 (iii) &= \sum P((x^1 \cdot n) \leftarrow (x^2 \cdot b_{\underline{1}})) \otimes x^3 \cdot b_{\underline{2}} \\
 (iii), (75) &= \sum P(n) \otimes b = n \otimes b.
 \end{aligned}$$

We are left to show that F is a morphism in \mathcal{M}_B^B . It is not hard to see that (86) and (71) imply that F is right B -linear. Also, (iii) implies that

$$\underline{\rho}_M \circ F(n \otimes b) = (F \otimes id_B) \circ \underline{\rho}_{M \circ B \otimes B}(n \otimes b) = \sum (x^1 \cdot n) \leftarrow (x^2 \cdot b_1) \otimes x^3 \cdot b_2,$$

for all $n \in N$ and $b \in B$, and this finishes the proof. \square

Let H be a quasi-Hopf algebra, and let ${}^H_H\mathcal{YD}^{\text{fd}}$ be the category of finite dimensional left Yetter-Drinfeld modules over H . If $M \in {}^H_H\mathcal{YD}^{\text{fd}}$, then $M^* \in {}^H_H\mathcal{YD}^{\text{fd}}$ (cf. [4]). The action and coaction are given by

$$(h \cdot m^*)(m) = m^*(S(h) \cdot m) \quad (88)$$

$$\begin{aligned} \lambda_{M^*}(m^*) &= \sum m^*_{(-1)} \otimes m^*_{(0)} = \sum_{i=1}^n \langle m^*, f^2 \cdot (g^1 \cdot {}_i m)_{(0)} \rangle \\ &\quad S^{-1}(f^1(g^1 \cdot {}_i m)_{(-1)} g^2) \otimes {}_i m \end{aligned} \quad (89)$$

for all $h \in H$, $m^* \in M^*$, $m \in M$. Here $f = \sum f^1 \otimes f^2$ is the twist defined in (15), $({}_i m)_{i=\overline{1,n}}$ is a basis of M and $({}^i m)_{i=\overline{1,n}}$ its dual basis. Moreover, ${}^H_H\mathcal{YD}^{f.d.}$ is a rigid monoidal category. For each object $M \in {}^H_H\mathcal{YD}^{\text{fd}}$, the evaluation and coevaluation maps (ev_M and $coev_M$, respectively) are given by (41).

In addition, if $B \in {}^H_H\mathcal{YD}^{\text{fd}}$ is a Hopf algebra, then B^* is a Hopf algebra in ${}^H_H\mathcal{YD}^{f.d.}$. The structure is the following.

- The multiplication and unit are given by

$$\begin{aligned} (\varphi * \psi)(b) &= \langle \varphi, f^2 \tilde{q}_2^2 Y^3 S^{-1}(\tilde{q}^1 Y^1(p^1 \cdot b_2)_{(-1)} p^2) \cdot b_1 \rangle \\ &\quad \langle \psi, f^1 \tilde{q}_1^2 Y^2 \cdot (p^1 \cdot b_2)_{(0)} \rangle, \end{aligned} \quad (90)$$

$$1_{B^*} = \underline{\varepsilon} \quad (91)$$

for all $\varphi, \psi \in B^*$, $b \in B$, where $q_L = \sum \tilde{q}^1 \otimes \tilde{q}^2$ and $p_R = \sum p^1 \otimes p^2$ are the elements defined in (21) and (19).

- the comultiplication and counit are given by the formulas

$$\underline{\Delta}_{B^*}(\varphi) = \sum_{i,j=1}^n \langle \varphi, [(g^1 \cdot {}_j b)_{(-1)} g^2 \cdot {}_i b] (g^1 \cdot {}_j b)_{(0)} \rangle {}^i b \otimes {}^j b \quad (92)$$

$$\underline{\varepsilon}_{B^*}(\varphi) = \varphi(1_B), \quad (93)$$

for any $\varphi \in B^*$, where $f^{-1} = \sum g^1 \otimes g^2$ was defined in (16), $({}_i b)_{i=\overline{1,n}}$ is a basis of B and $({}^i b)_{i=\overline{1,n}}$ the corresponding dual basis of B^* .

- the antipode is given by

$$\underline{S}_{B^*} = \underline{S}^*, \text{ i. e. } \underline{S}_{B^*}(\varphi) = \varphi \circ \underline{S}, \quad (94)$$

for all $\varphi \in B^*$.

PROPOSITION 5.4. *Let $B \in {}^H_H\mathcal{YD}^{\text{fd}}$ a Hopf algebra. Then B^* is a right B -Hopf module, with structure:*

$$\langle \varphi \leftarrow b, b' \rangle = \sum \langle \varphi, [(U^1 \cdot b)_{(-1)} U^2 \cdot b'] \underline{S}((U^1 \cdot b)_{(0)}) \rangle, \quad (95)$$

$$\rho_{B^*}(\varphi) = \sum_{i=1}^n (S(\tilde{p}^1) \cdot {}_i b)_{(-1)} \cdot [{}^i b * (\tilde{p}^2 \cdot \varphi)] \otimes (S(\tilde{p}^1) \cdot {}_i b)_{(0)}, \quad (96)$$

for all $\varphi \in B^*$, $b, b' \in B$, where

$$U = \sum U^1 \otimes U^2 := \sum g^1 S(q^2) \otimes g^2 S(q^1), \quad (97)$$

$p_L = \sum \tilde{p}^1 \otimes \tilde{p}^2$, $q_R = \sum q^1 \otimes q^2$ and $f^{-1} = \sum g^1 \otimes g^2$ are the elements defined by (21), (19) and (16), and $\{{}_i b\}_{i=\overline{1,n}}$ is a basis of B with corresponding dual basis $\{{}^i b\}_{i=\overline{1,n}}$. Moreover,

$$B^{*\text{co}B} = \{\Lambda \in B^* \mid \sum (\tilde{p}^1 \cdot \varphi) * (\tilde{p}^2 \cdot \Lambda) = \varphi(1_B)\Lambda \text{ for all } \varphi \in B^*\}.$$

Proof. If B is a Hopf algebra in a braided rigid monoidal category \mathcal{C} , then B^* is a right Hopf B -module, as follows.

- the right B -module structure $\leftarrow: B^* \otimes B \rightarrow B$ on B^* is the composition

$$\begin{array}{ccc} B^* \otimes B & \xrightarrow{l_{B^* \otimes B}} & (B^* \otimes B) \otimes \underline{1} \\ & \xrightarrow{(id_{B^*} \otimes \underline{S}) \otimes coev_B} & (B^* \otimes B) \otimes (B \otimes B^*) \\ & \xrightarrow{a_{B^* \otimes B, B, B^*}^{-1}} & ((B^* \otimes B) \otimes B) \otimes B^* \\ & \xrightarrow{a_{B^*, B, B} \otimes id_{B^*}} & (B^* \otimes (B \otimes B)) \otimes B^* \\ & \xrightarrow{(id_{B^*} \otimes \underline{m}) \otimes id_{B^*}} & (B^* \otimes B) \otimes B^* \\ & \xrightarrow{ev_B \otimes id_{B^*}} & \underline{1} \otimes B^* \\ & \xrightarrow{r_{B^*}^{-1}} & B^* \end{array} \quad (98)$$

- the right B -comodule structure $\rho_{B^*}: B^* \rightarrow B^* \otimes B$ on B^* is the composition

$$\begin{array}{ccc} B^* & \xrightarrow{r_{B^*}} & \underline{1} \otimes B^* & \xrightarrow{coev_B \otimes id_{B^*}} & (B \otimes B^*) \otimes B^* \\ & \xrightarrow{a_{B, B^*, B^*}} & B \otimes (B^* \otimes B^*) & \xrightarrow{id_B \otimes \underline{m}_{B^*}} & B \otimes B^* \\ & \xrightarrow{c_{B, B^*}} & B^* \otimes B. & & \end{array} \quad (99)$$

Let $\gamma = \sum \gamma^1 \otimes \gamma^2$ and $f^{-1} = \sum g^1 \otimes g^2$ be the elements defined in (14) and (16). By (98), we have, for all $\varphi \in B^*$ and $b, b' \in B$:

$$\begin{aligned} & \langle \varphi \leftarrow b, b' \rangle \\ &= \sum \langle \varphi, S(X^1 p_1^1) \alpha \cdot [((X^2 p_2^1 \cdot \underline{S}(b))_{(-1)} X^3 p^2 \cdot b')(X^2 p_2^1 \cdot \underline{S}(b))_{(0)}] \rangle \\ (67, 45) \quad &= \sum \langle \varphi, [(S(X^1 p_1^1))_1 \alpha_1 X^2 p_2^1 \cdot \underline{S}(b))_{(-1)} S(X^1 p_1^1)_2 \alpha_2 X^3 p^2 \cdot b'] \\ & \quad (S(X^1 p_1^1)_1 \alpha_1 X^2 p_2^1 \cdot \underline{S}(b))_{(0)} \rangle \end{aligned}$$

$$\begin{aligned}
(17, 11) &= \sum \langle \varphi, [(g^1 S(X_2^1 p_{(1,2)}^1) \gamma^1 X^2 p_2^1 \cdot \underline{S}(b))_{(-1)} g^2 S(X_1^1 p_{(1,1)}^1) \gamma^2 X^3 p^2 \cdot b'] \\
&\quad (g^1 S(X_2^1 p_{(1,2)}^1) \gamma^1 X^2 p_2^1 \cdot \underline{S}(b))_{(0)} \rangle \\
(14, 3, 5) &= \sum \langle \varphi, [g^1 S(Y^2 p_{(1,2)}^1) \alpha Y^3 p_2^1 \cdot \underline{S}(b))_{(-1)} g^2 S(Y^1 p_{(1,1)}^1) \alpha p^2 \cdot b'] \\
&\quad (g^1 S(Y^2 p_{(1,2)}^1) \alpha Y^3 p_2^1 \cdot \underline{S}(b))_{(0)} \rangle \\
(1, 5, 6) &= \sum \langle \varphi, [(g^1 S(Y^2) \alpha Y^3 \cdot \underline{S}(b))_{(-1)} g^2 S(Y^1) \cdot b'] \\
&\quad (g^1 S(Y^2) \alpha Y^3 \cdot \underline{S}(b))_{(0)} \rangle \\
(19, 97, 84) &= \sum \langle \varphi, [\underline{S}(U^1 \cdot b)_{(-1)} U^2 \cdot b'] \underline{S}(U^1 \cdot b)_{(0)} \rangle \\
(84) &= \sum \langle \varphi, [(U^1 \cdot b)_{(-1)} U^2 \cdot b'] \underline{S}((U^1 \cdot b)_{(0)}) \rangle
\end{aligned}$$

which is just (95). (96) follows easily by (99), the details are left to the reader. Finally, by (99) we have

$$\begin{aligned}
\Lambda \in B^{*co(B)} &\iff \rho_{B^*}(\Lambda) = \Lambda \otimes 1_B \\
&\iff c_{B, B^*}^{-1} \circ \rho_{B^*}(\Lambda) = c_{B, B^*}^{-1}(\Lambda \otimes 1_B) \\
&\iff \sum_{i=1}^n S(\tilde{p}^1) \cdot {}_i b \otimes {}^i b * (\tilde{p}^2 \cdot \Lambda) = 1_B \otimes \Lambda \\
&\iff \sum (\tilde{p}^1 \cdot \varphi) * (\tilde{p}^2 \cdot \Lambda) = \varphi(1_B) \Lambda, \quad \text{for all } \varphi \in B^*.
\end{aligned}$$

□

We define the space of left integrals by $I_l(B^*) = B^{*co(B)}$. From the Fundamental Theorem for Hopf modules, we then obtain.

COROLLARY 5.5. *Let H be a quasi-Hopf algebra and B a finite dimensional Hopf algebra in ${}^H_H\mathcal{YD}$. Then $I_l(B^*) \otimes B \simeq B^*$ as right B -Hopf modules. In particular, $\dim_k(I_l(B^*)) = 1$.*

Now, let H be a quasi-Hopf algebra and H_0 the H -module algebra described in Section 4. If (H, R) is quasitriangular, then H_0 is a Hopf algebra in ${}^H_H\mathcal{YD}$, see [5]. The additional structure is the following.

$$\lambda_{H_0}(h) = \sum R^2 \otimes R^1 \triangleright h, \quad (100)$$

$$\underline{\Delta}(h) = \sum h_1 \otimes h_2 \quad (101)$$

$$= \sum x^1 X^1 h_1 g^1 S(x^2 R^2 y^3 X_2^3) \otimes x^3 R^1 \triangleright y^1 X^2 h_2 g^2 S(y^2 X_1^3), \quad (102)$$

$$\underline{\varepsilon}(h) = \varepsilon(h), \quad (103)$$

$$\underline{S}(h) = \sum X^1 R^2 p^2 S(q^1 (X^2 R^1 p^1 \triangleright h) S(q^2) X^3), \quad (104)$$

for all $h \in H$, where $R = \sum R^1 \otimes R^2$ and $f^{-1} = \sum g^1 \otimes g^2$, $p_R = \sum p^1 \otimes p^2$ and $q_R = \sum q^1 \otimes q^2$ are the elements defined by (16), (19) and (20). By the above arguments, if H is a finite dimensional Hopf algebra, then H_0^* is also a Hopf algebra

in ${}^H_H\mathcal{YD}$, with structure

$$(\varphi * \Psi)(h) = \sum \langle \varphi, f^2 \bar{R}^2 \triangleright h_{\underline{1}} \rangle \langle \Psi, f^1 \bar{R}^1 \triangleright h_{\underline{2}} \rangle \quad (105)$$

$$= \sum \langle \varphi, f^2 \triangleright Y^2 \bar{R}^2 X^1 x_1^1 h_1 g^1 S(Y^3 x^3) \rangle \\ \langle \Psi, f^1 Y^1 \bar{R}^1 \triangleright X^2 x_2^1 h_2 g^2 S(X^3 x^2) \rangle, \quad (106)$$

$$1_{H_0^*} = \varepsilon, \quad (107)$$

$$\underline{\Delta}_{H_0^*}(\varphi) = \sum_{i,j=1}^n \langle \varphi, (R^2 g^2 \triangleright_i e)(R^1 g^1 \triangleright_j e) \rangle^i e \otimes^j e, \quad (108)$$

$$\varepsilon_{H_0^*}(\varphi) = \varphi(\beta), \quad (109)$$

$$\underline{S}_{H_0^*}(\varphi) = v \circ \underline{S}, \quad (110)$$

for all $h \in H$ and $\varphi \in H^*$, where $R^{-1} = \sum \bar{R}^1 \otimes \bar{R}^2$, $\{e_i\}_{i=\overline{1,n}}$ is a basis of H and $\{e^i\}_{i=\overline{1,n}}$ the corresponding dual basis of H^* . In this particular case we have

$$I_l(H_0^*) = \{\Lambda \in H^* \mid \sum \Lambda(S(\tilde{p}^2) f^1 \bar{R}^1 \triangleright h_{\underline{2}}) S(\tilde{p}^1) f^2 \bar{R}^2 \triangleright h_{\underline{1}} = \Lambda(b)\beta, \text{ for all } h \in H\}.$$

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