More properties of Yetter-Drinfeld modules over quasi-Hopf algebras

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Abstract. We generalize various properties of Yetter-Drinfeld modules over Hopf algebras to quasi-Hopf algebras. The dual of a finite dimensional Yetter-Drinfeld module is again a Yetter-Drinfeld module. The algebra $H_0$ in the category of Yetter-Drinfeld modules that can be obtained by modifying the multiplication in a proper way is quantum commutative. We give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules, and deduce the existence and uniqueness of integrals from it.

1. Introduction

The motivation for studying Yetter-Drinfeld modules over quasi-Hopf algebras is the same as for Hopf algebras. It is well known that for any finite dimensional Hopf algebra $H$ the category of Yetter-Drinfeld modules $H\,\mathcal{YD}^H$ is isomorphic to the category of modules over the quantum double $D(H)$. From a categorical point of view, the quantum double $D(H)$ arises by considering the center $Z(H,\mathcal{M})$ of the monoidal category $H\mathcal{M}$ of left $H$-modules. More precisely, one has $Z(H,\mathcal{M}) \simeq D(H)\mathcal{M}$ if $H$ is finite dimensional. Actually, the category of Yetter-Drinfeld modules appears as an intermediate step in the proof of this isomorphism: one first proves that $Z(H,\mathcal{M}) \simeq H\,\mathcal{YD}^H$, and then $H\,\mathcal{YD}^H \simeq D(H)\mathcal{M}$, where the finite dimensionality is not needed in the proof of the first isomorphism, see [16] for full detail.

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Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [13]; a categorical interpretation is the following: a quasi-bialgebra $H$ is an algebra with the additional structure that is needed to make the category of left $H$-modules, with the tensor product over $k$ as tensor product and $k$ as unit object into a monoidal category. The difference with a usual bialgebra is that we do not require that the associativity isomorphism coincides with the associativity in the category of vector spaces. A quasi-Hopf algebra is a quasi-bialgebra with additional structure making the category of finite dimensional $H$-modules into a monoidal category with duality. The center construction $Z(C)$ can be applied to any monoidal category $C$. Majid [19] computed the center of the category of left modules over a quasi-Hopf algebra $H$, and introduced the category of Yetter-Drinfeld modules over $H$. Hausser and Nill [14], [15] constructed the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra $H$, and proved that $H\mathcal{YD}H \simeq D(H)M$. Recently, Schauenburg [22] gave the equivalence between the category of Yetter-Drinfeld modules $H\mathcal{YD}$ and the category $H\mathcal{HM}$ of Hopf bimodules. In [5], the relation between Yetter-Drinfeld modules and Radford’s biproduct is studied. In [4], the rigidity of the category of Yetter-Drinfeld modules is investigated, as well as the relations between left, left-right, right-left and right Yetter-Drinfeld modules.

In this paper, which can be seen as a sequel to [4], we continue our investigations of properties of Yetter-Drinfeld modules. In Section 3, we show that the linear dual of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module. It was shown in [7], [5] that the multiplication on $H$ can be modified in such a way that we obtain an algebra in the category of left Yetter-Drinfeld modules. The main result of Section 4 is that $H_0$ is quantum commutative. In Section 5, we will generalize Doi’s results [12] about Hopf modules in the category of Yetter-Drinfeld modules to our situation: we give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules over a quasi-Hopf algebras, and we use this result to obtain the existence and uniqueness of integrals for a finite dimensional braided Hopf algebra in $H\mathcal{YD}$. We apply this to the braided Hopf algebra considered in Section 4, in the case where $H$ is finite dimensional and quasitriangular.

2. Preliminary results

2.1. Quasi-Hopf algebras

We work over a commutative field $k$. All algebras, linear spaces etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [13], a quasi-bialgebra is a fourtuple $(H, \Delta, \varepsilon, \Phi)$, where $H$ is an associative algebra with unit, $\Phi$ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra homomorphisms satisfying the identities

\begin{align}
(id \otimes \Delta)(\Delta(h)) &= \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1}, \\
(id \otimes \varepsilon)(\Delta(h)) &= h \otimes 1, \quad (\varepsilon \otimes id)(\Delta(h)) = 1 \otimes h,
\end{align}

(1) (2)
for all \( h \in H \), and \( \Phi \) has to be a normalized 3-cocycle, in the sense that
\[
(1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi),
\]
\[
(id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1 \otimes 1.
\]
The map \( \Delta \) is called the coproduct or the comultiplication, \( \varepsilon \) the counit and \( \Phi \) the reassociator. As for Hopf algebras \([23]\) we use the notation \( \Delta(\varepsilon) = \sum h_1 \otimes h_2 \).
\( \Delta \) is only quasi-coassociative we adopt the further notation
\[(\Delta \otimes id)(\Delta(\varepsilon)) = \sum h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(\varepsilon)) = \sum h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},\]
for all \( h \in H \). We will denote the tensor components of \( \Phi \) by capital letters, and the ones of \( \Phi^{-1} \) by small letters, namely
\[
\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3 = \ldots
\]
\[
\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum v^1 \otimes v^2 \otimes v^3 = \ldots
\]
a quasi-bialgebra \( H \) is called a quasi-Hopf algebra if there exists an anti-automorphism \( S \) of the algebra \( H \) and \( \alpha, \beta \in H \) such that:
\[
\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1\beta S(h_2) = \varepsilon(h)\beta,
\]
\[
\sum X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad \sum S(x^1)\alpha x^2\beta S(x^3) = 1,
\]
for all \( h \in H \). It is shown in \([9]\) that the condition that the antipode is bijective follows automatically from the other axioms in the case where \( H \) is finite dimensional. Observe that the antipode of a quasi-Hopf algebra is determined uniquely up to a transformation \( \alpha \mapsto U\alpha, \beta \mapsto \beta U^{-1}, S(h) \mapsto US(h)U^{-1} \), where \( U \in H \) is invertible. The axioms for a quasi-Hopf algebra imply that \( \varepsilon(\alpha)\varepsilon(\beta) = 1 \), so, by rescaling \( \alpha \) and \( \beta \), we may assume without loss of generality that \( \varepsilon(\alpha) = \varepsilon(\beta) = 1 \) and \( \varepsilon \circ S = \varepsilon \).
\( \varepsilon \otimes id \otimes id)(\Phi) = (id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1 \otimes 1.\]
Together with a quasi-Hopf algebra \( H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta) \) we also have \( H^{op}, H^{cop} \) and \( H^{op,cop} \) as quasi-Hopf algebras, where “\( op \)” means opposite multiplication and “\( cop \)” means opposite comultiplication. The reassociators of these three quasi-Hopf algebras are \( \Phi_{op} = \Phi^{-1}, \Phi_{cop} = (\Phi^{-1})^{321}, \Phi_{op,cop} = \Phi^{321}, \) the antipodes are \( S_{op} = S_{cop} = (S_{op,cop})^{-1} = S^{-1} \), and the elements \( \alpha, \beta \) are \( \alpha_{op} = S^{-1}(\beta), \beta_{op} = S^{-1}(\alpha), \alpha_{cop} = S^{-1}(\alpha), \beta_{cop} = S^{-1}(\beta), \alpha_{op,cop} = \beta \) and \( \beta_{op,cop} = \alpha \).
Recall next that the definition of a quasi-Hopf algebra is “twist co invariant”, in the following sense. An invertible element \( F \in H \otimes H \) is called a gauge transformation or twist if \( (s \otimes id)(F) = (id \otimes s)(F) = 1 \). If \( H \) is a quasi-Hopf algebra and \( F = \sum F_1 \otimes F_2 \in H \otimes H \) is a gauge transformation with inverse \( F^{-1} = \sum G_1 \otimes G_2 \), then we can define a new quasi-Hopf algebra \( H_F \) by keeping the multiplication, unit, counit and antipode of \( H \) and replacing the comultiplication, antipode and the elements \( \alpha \) and \( \beta \) by
\[
\Delta_F(h) = F \Delta(h) F^{-1},
\]
\[
\Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1),
\]
\[
\alpha_F = \sum S(G^1)\alpha G^2, \quad \beta_F = \sum F^1\beta S(F^2),
\]
It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. The corresponding statement for a quasi-Hopf algebra is the following: there exists a gauge transformation $f \in H \otimes H$ such that

$$f \Delta(S(h)) f^{-1} = \sum (S \otimes S)(\Delta^{\text{cop}}(h)), \quad (11)$$

for all $h \in H$, where $\Delta^{\text{cop}}(h) = \sum h_2 \otimes h_1$. The element $f$ can be computed explicitly. First set

$$\sum A_1 \otimes A_2 \otimes A_3 \otimes A_4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}), \quad (12)$$

$$\sum B_1 \otimes B_2 \otimes B_3 \otimes B_4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1) \quad (13)$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = \sum S(A_2)^{\alpha A_3} \otimes S(A_1)^{\alpha A_4} \quad \text{and} \quad \delta = \sum B_1^{\beta S(B_4)} \otimes B_2^{\beta S(B_3)} \quad (14)$$

Then $f$ and $f^{-1}$ are given by the formulas

$$f = \sum (S \otimes S)(\Delta^{\text{cop}}(x^1))\gamma \Delta(x^2 \beta S(x^3)), \quad (15)$$

$$f^{-1} = \sum \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{cop}}(x^3)). \quad (16)$$

Moreover, $f$ satisfies the following relations:

$$f \Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta. \quad (17)$$

Furthermore the corresponding twisted reassociator (see (9)) is given by

$$\Phi_f = \sum (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1). \quad (18)$$

In a Hopf algebra $H$, we obviously have the identity

$$\sum h_1 \otimes h_2 S(h_3) = h \otimes 1, \quad \text{for all } h \in H.$$

We will need the generalization of this formula to the quasi-Hopf algebra setting. Following [14, 15], we define

$$p_R = \sum p_R^1 \otimes p_R^2 = \sum x^1 \otimes x^2 \beta S(x^3), \quad (19)$$

$$q_R = \sum q_R^1 \otimes q_R^2 = \sum X^1 \otimes S^{-1}(\alpha X^3) X^2, \quad (20)$$

$$p_L = \sum p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}(X^1 \beta) \otimes X^3, \quad (21)$$

$$q_L = \sum q_L^1 \otimes q_L^2 = \sum S(x^1)\alpha x^2 \otimes x^3. \quad (22)$$

We then have, for all $h \in H$,

$$\sum \Delta(h_1)p_R[1 \otimes S(h_2)] = p_R(h \otimes 1), \quad (23)$$

$$\sum [1 \otimes S^{-1}(h_2)]q_R \Delta(h_1) = (h \otimes 1)q_R, \quad (24)$$

$$\sum \Delta(h_2)p_L[S^{-1}(h_1) \otimes 1] = p_L(1 \otimes h), \quad (25)$$

$$\sum [S(h_1) \otimes 1]q_L \Delta(h_2) = (1 \otimes h)q_L. \quad (26)$$
and

\[(q_R \otimes 1)(\Delta \otimes id)(q_R)\Phi^{-1} = \sum [1 \otimes S^{-1}(X^3) \otimes S^{-1}(X^2)] \]

\[1 \otimes S^{-1}(f^2) \otimes S^{-1}(f^1)](id \otimes \Delta)(q_R\Delta(X^1)), \tag{27}\]

where \(f = \sum f^1 \otimes f^2\) is the twist defined in (15).

A quasi-Hopf algebra \(H\) is quasitriangular if there exists an element \(R \in H \otimes H\) such that

\[(\Delta \otimes id)(R) = \sum \Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi, \tag{28}\]

\[(id \otimes \Delta)(R) = \sum \Phi_{231}R_{13}\Phi_{213}^{-1}R_{12}\Phi^{-1}, \tag{29}\]

\[\Delta^{\text{cop}}(h)R = R\Delta(h), \text{ for all } h \in H, \tag{30}\]

\[(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1. \tag{31}\]

Here we used the following notation: if \(\sigma\) is a permutation of \(\{1, 2, 3\}\), then we write \(\Phi_{\sigma(1)\sigma(2)\sigma(3)} = \sum X^{\sigma^{-1}(1)} \otimes X^{\sigma^{-1}(2)} \otimes X^{\sigma^{-1}(3)}\); \(R_{ij}\) means \(R\) acting non-trivially on the \(i\)-th and \(j\)-th tensor factors of \(H \otimes H \otimes H\).

It is shown in [10] that \(R\) is invertible. Furthermore, the element

\[u = \sum S(R^2p^2)\alpha R^1p^1, \tag{32}\]

with \(p_R = \sum p^1 \otimes p^2\) defined as in (19), is invertible in \(H\), and

\[u^{-1} = \sum X^1 R^1 p^2 S(SX^2 R^1 p^1 \alpha X^3), \tag{33}\]

\[\varepsilon(u) = 1 \text{ and } S^2(h) = uh u^{-1}, \tag{34}\]

for all \(h \in H\). Consequently the antipode \(S\) is bijective, so, as in the Hopf algebra case, the assumptions about invertibility of \(R\) and bijectivity of \(S\) can be dropped. Moreover, the \(R\)-matrix \(R = \sum R^1 \otimes R^2\) satisfies the identity (see [1], [15], [10]):

\[f_{21}Rf_{1}^{-1} = (S \otimes S)(R) \tag{35}\]

where \(f = \sum f^1 \otimes f^2\) is the twist defined in (15), and \(f_{21} = \sum f^2 \otimes f^1\).

### 2.2. Monoidal categories

A monoidal or tensor category is a sixtuple \((\mathcal{C}, \otimes, 1, a, l, r)\), where \(\mathcal{C}\) is a category, \(\otimes\) is a functor \(\mathcal{C} \times \mathcal{C} \to \mathcal{C}\) (called the tensor product), \(1\) is an object of \(\mathcal{C}\), and

\[a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)\]

\[l_U : V \cong V \otimes 1 ; \quad r_U : V \cong 1 \otimes V\]

are natural isomorphisms satisfying certain coherence conditions, see for example [16, 18, 20]. An object \(V\) of a monoidal category \(\mathcal{C}\) has a left dual if there exists an object \(V^*\) and morphisms \(ev_V : V^* \otimes V \to 1\) and \(\text{coev}_V : 1 \to V \otimes V^*\) in \(\mathcal{C}\) such that

\[l_V^{-1} \circ (id_V \otimes ev_V) \circ a_{V,V^*,V} \circ (\text{coev}_V \otimes id_V) \circ r_V = id_V, \tag{36}\]

\[r_V^{-1} \circ (ev_V \otimes id_V) \circ a_{V^*,V,V^*} \circ (id_{V^*} \otimes \text{coev}_V) \circ l_{V^*} = id_{V^*}. \tag{37}\]

\(\mathcal{C}\) is called a rigid monoidal category if every object of \(\mathcal{C}\) has a dual.

A braided monoidal category is a monoidal category equipped with a commutativity natural isomorphism \(c_{U,V} : U \otimes V \to V \otimes U\), compatible with the unit and the associativity.
In a braided monoidal category, we can define algebras, coalgebras, bialgebras and Hopf algebras. For example, a bialgebra \((B, m, \eta, \Delta, \varepsilon)\) consists of \(B \in \mathcal{C}\), a multiplication \(m : B \otimes B \rightarrow B\) which is associative up to the natural isomorphism \(a\), and a unit \(\eta : 1 \rightarrow B\) such that \(m \circ (\eta \otimes \text{id}) = m \circ (\text{id} \otimes \eta) = \text{id}\). The properties of the comultiplication \(\Delta\) and the counit \(\varepsilon\) are similar. In addition, \(\Delta : B \rightarrow B \otimes B\) has to be an algebra morphism, where \(B \otimes B\) is an algebra with multiplication \(m_{B \otimes B}\), defined as the composition

\[
(B \otimes B) \otimes (B \otimes B) \xrightarrow{a} B \otimes (B \otimes (B \otimes B)) \xrightarrow{\text{id} \otimes \varepsilon^{-1}} B \otimes (B \otimes B) \otimes B \\
\xrightarrow{\text{id} \otimes c \otimes \text{id}} B \otimes ((B \otimes B) \otimes B) \\
\xrightarrow{\varepsilon \otimes a} B \otimes (B \otimes (B \otimes B)) \\
\xrightarrow{a^{-1}} (B \otimes B) \otimes (B \otimes B) \\
\xrightarrow{m \otimes m} B \otimes B
\]

A Hopf algebra \(B\) is a bialgebra with a morphism \(S : B \rightarrow B\) (the antipode) satisfying the usual axioms \(m \circ (S \otimes \text{id}) \circ \Delta = \eta \otimes \varepsilon = m \circ (\text{id} \otimes S) \circ \Delta\). It is known, see e.g. [21], that the antipode \(S\) of a Hopf algebra \(B\) in a braided monoidal category \(\mathcal{C}\) is an antialgebra and anticoalgebra morphism, in the sense that

\[
S \circ m = m \circ (S \otimes S) \circ c_{B,B} \quad \text{and} \quad \Delta \circ S = c_{B,B} \circ (S \otimes S) \circ \Delta.
\]

Recall also that an algebra \(A\) in a braided monoidal category \(\mathcal{C}\) is called quantum commutative if \(m \circ c_{A,A} = m\).

Assume that \((H, \Delta, \varepsilon, \Phi)\) is a quasi-bialgebra, and let \(U, V, W\) be left \(H\)-modules. We define a left \(H\)-action on \(U \otimes V\) by

\[
h \cdot (u \otimes v) = \sum h_i \cdot u \otimes h_2 \cdot v.
\]

We have isomorphisms \(a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)\) in \(h\mathcal{M}\) given by

\[
a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)).
\]

The counit \(\varepsilon : H \rightarrow k\) makes \(k \in h\mathcal{M}\), and the natural isomorphisms \(\lambda : k \otimes H \rightarrow H\) and \(\rho : H \otimes k \rightarrow H\) are in \(h\mathcal{M}\). With this structures, \((h\mathcal{M}, \otimes, k, \lambda, \rho)\) is a monoidal category.

If \(H\) is a quasi-Hopf algebra then the category of finite dimensional left \(H\)-modules is rigid; the left dual of \(V\) is \(V^*\) with the \(H\)-module structure given by \((h \cdot \varphi)(v) = \varphi(S(h) \cdot v)\), for all \(v \in V, \varphi \in V^*\), \(h \in H\) and with

\[
ev_V(\varphi \otimes v) = \varphi(\alpha \cdot v), \quad \coev_V(1) = \sum_{i=1}^n \beta \cdot v_i \otimes v_i^n,
\]

where \(\{v_i\}\) is a basis in \(V\) with dual basis \(\{v^i\}\). Now let \(H\) be a quasitriangular quasi-Hopf algebra, with \(R\)-matrix \(R = \sum R^1 \otimes R^2\). For two left \(H\)-modules \(U\) and \(V\), we define

\[
c_{U,V} : U \otimes V \rightarrow V \otimes U
\]
by
\[ c_{U,V}(u \otimes v) = \sum R^2 \cdot v \otimes R^1 \cdot u \] (42)
and then \((H,M,\otimes,k,a,\lambda,\rho,c)\) is a braided monoidal category (cf. [16] or [20]).

3. Yetter-Drinfeld modules and the quasi-Yang-Baxter equation

From [19], we recall the notion of Yetter-Drinfeld module over a quasi-bialgebra.

**Definition 3.1.** Let \( H \) be a quasi-bialgebra with reassociator \( \Phi \). A left \( H \)-module \( M \) together with a left \( \lambda_M : M \rightarrow H \otimes M, \lambda_M(m) = \sum m_{(-1)} \otimes m_{(0)} \)

is called a left Yetter-Drinfeld module if the following equalities hold, for all \( h \in H \) and \( m \in M \):

\[
\sum X^1 m_{(-1)} \otimes (X^2 \cdot m_{(0)})_{(-1)} X^3 \otimes (X^2 \cdot m_{(0)})_{(0)} = \sum X^1 (Y^1 \cdot m)_{(-1)} Y^2 \otimes X^2 (Y^1 \cdot m)_{(-1)} Y^3 \otimes X^3 \cdot (Y^1 \cdot m)_{(0)} \quad (43)
\]

\[
\sum \varepsilon(m_{(-1)}) m_{(0)} = m \quad (44)
\]

\[
\sum h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = \sum (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} \quad (45)
\]

The category of left Yetter-Drinfeld \( H \)-modules and \( k \)-linear maps that intertwine the \( H \)-action and \( H \)-coaction is denoted by \( \frac{H}{H} YD \). In [19] it is shown that \( \frac{H}{H} YD \) is a prebraided monoidal category. The forgetful functor \( \frac{H}{H} YD \rightarrow H \mathcal{M} \) is monoidal, and the coaction on the tensor product \( M \otimes N \) of two Yetter-Drinfeld modules \( M \) and \( N \) is given by

\[
\lambda_{M \otimes N}(m \otimes n) = \sum X^1 (x^1 Y^1 \cdot m)_{(-1)} x^2 (Y^2 \cdot n)_{(-1)} Y^3 \otimes X^2 \cdot (x^1 Y^1 \cdot m)_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot n)_{(0)} \quad (46)
\]

\[
\otimes \quad (47)
\]

The braiding is given by

\[
c_{M,N}(m \otimes n) = \sum m_{(-1)} \cdot n \otimes m_{(0)} \quad (48)
\]

This braiding is invertible if \( H \) is a quasi-Hopf algebra [5], and its inverse is then given by

\[
c^{-1}_{M,N}(n \otimes m) = \sum y_1 x_2 \cdot (x^1 \cdot m)_{(0)} \otimes S^{-1}(S(y^0)gx^0 x^1 \cdot m)_{(-1)} x^2 \beta S(y^2 x^3 x^3) \cdot n \quad (49)
\]

Let \((H,R)\) be a quasitriangular quasi-bialgebra. It is well-known (see for example [16]) that \( R \) satisfies the so-called quasi-Yang-Baxter equation in \( H \otimes H \otimes H \):

\[
R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} = \Phi_{321} R_{23} \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \cdot (49)
\]

On the other hand, if \( H \) is a bialgebra and \( M \) is a left-right Yetter-Drinfeld module over \( H \), with structures

\[
H \otimes M \rightarrow M, \quad h \otimes m \mapsto h \cdot m;
\]

\[
M \rightarrow M \otimes H, \quad m \mapsto \sum m_{(0)} \otimes m_{(1)} ;
\]
then the map $R_M : M \otimes M \to M \otimes M$, $R_M(m \otimes n) = \sum n_{(1)} \cdot m \otimes n_{(0)}$ is a solution in $\text{End}(M \otimes M \otimes M)$ of the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

see for instance [17]. We will show a similar result for quasi-bialgebras; first we define left-right Yetter-Drinfeld modules over quasi-bialgebras as follows

$$H \mathcal{YD}^H = H^\text{cop} \mathcal{YD}.$$

This is stated more explicitly in the next definition.

**Definition 3.2.** Let $H$ be a quasi-bialgebra. A $k$-linear space $M$ with a left $H$-action $h \otimes m \mapsto h \cdot m$, and a right $H$-coaction $M \to M \otimes H$, $m \mapsto \sum m_{(0)} \otimes m_{(1)}$ is called a left-right Yetter-Drinfeld module if the following relations hold, for all $m \in M$ and $h \in H$:

$$\sum (x^2 \cdot m_{(0)})_{(0)} \otimes (x^2 \cdot m_{(0)})_{(1)} x^1 \otimes x^3 m_{(1)}$$

$$= \sum x^1 \cdot (y^3 \cdot m)_{(0)} \otimes x^2 (y^3 \cdot m)_{(1)} y^1 \otimes x^3 (y^3 \cdot m)_{(1)} y^2$$

$$\sum \varepsilon(m_{(1)}) m_{(0)} = m$$

$$\sum h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = \sum (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} h_1.$$  

**Proposition 3.3.** Let $H$ be a quasi-bialgebra and $M \in H \mathcal{YD}^H$. The map $R = R_M : M \otimes M \to M \otimes M$, $R(m \otimes n) = \sum n_{(1)} \cdot m \otimes n_{(0)}$, is a solution of the quasi-Yang-Baxter equation

$$R_{12} \Phi_{312} R_{13} \Phi_{132} R_{23} \Phi = \Phi_{321} R_{23} \Phi_{231} R_{13} \Phi_{213} R_{12}$$

on $\text{End}(M \otimes M \otimes M)$.

We considered $R_{12}, \Phi_{312},$ etc. as elements in $\text{End}(M \otimes M \otimes M)$ by left multiplication, for example $R_{12}(l \otimes m \otimes n) = \sum R^1 \cdot l \otimes R^2 \cdot m \otimes n$, $\Phi_{312}(l \otimes m \otimes n) = \sum X^3 \cdot l \otimes X^3 \cdot m \otimes X^1 \cdot n$ etc.

**Proof.** $H \mathcal{YD}^H$ is a prebraided category, hence the result is a consequence of the fact (see [16]) that the braiding satisfies the categorical version of the Yang-Baxter equation. A direct proof is also possible. For all $l, m, n \in M$, we compute that

$$R_{12} \Phi_{312} R_{13} \Phi_{132} R_{23} \Phi(l \otimes m \otimes n)$$

$$= \sum (Y^3 x^3(X^3 \cdot n)_{(1)} X^2 \cdot m)_{(1)} Y^2(x^2 \cdot (X^3 \cdot n)_{(0)})_{(1)} x^1 X^1 \cdot l$$

$$\otimes (Y^3 x^3(X^3 \cdot n)_{(1)} X^2 \cdot m)_{(0)} \otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)_{(0)})_{(1)} x^1 X^1 \cdot l$$

$$= \sum (Y^3 x^3(y^3 X^3 \cdot n)_{(1)} y^2 X^2 \cdot m)_{(1)} Y^2 x^2(y^3 X^3 \cdot n)_{(1)} y^1 X^1 \cdot l$$

$$\otimes (Y^3 x^3(y^3 X^3 \cdot n)_{(1)} y^2 X^2 \cdot m)_{(0)} \otimes Y^1 x^1 \cdot (y^3 X^3 \cdot n)_{(0)}$$

$$= \sum (n_{(1)} l \otimes m)_{(1)} (n_{(1)} h \otimes (n_{(1)} h \otimes m)_{(0)} \otimes n_{(0)}$$

$$= \sum n_{(1)} m_{(1)} \cdot l \otimes n_{(1)} m_{(0)} \otimes n_{(0)}$$
and
\[
\Phi_{321} R_{23} \Phi_{231} R_{12} l \otimes m \otimes n \\
= \sum Y^3 x^3 (X^3 \cdot n)(1) X^2 m(1) \cdot l \otimes Y^2 (x^2 \cdot (X^3 \cdot n)(0))(1) x^1 \cdot m(0) \\
\otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)(0))(0) \\
(50) = \sum Y^3 x^3 (y^3 X^3 \cdot n)(1) y^2 X^2 m(1) \cdot l \otimes Y^2 x^2 (y^3 X^3 \cdot n)(1) y^1 X^1 \cdot m(0) \\
\otimes Y^1 x^1 \cdot (y^3 X^3 \cdot n)(0) \\
= \sum n_{(1,2)} m_{(1)} \cdot l \otimes n_{(1,1)} \cdot m_{(0)} \otimes n_{(0)}
\]
and (53) follows. \(\square\)

We will now present a generalization of [17, Prop. 4.4.2], stating that the dual \(M^*\) of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module and that \(R_{M^*} = R_M^*\).

First we define right-left Yetter-Drinfeld modules for quasi-bialgebras as follows:
\[
\text{H}YDH = \text{H}^{\text{op},\cop} \text{YD}^{\text{H}^{\text{op},\cop}}.
\]

More explicitly:

**Definition 3.4.** Let \(H\) be a quasi-bialgebra. A \(k\)-linear space \(M\) with a right \(H\)-action \(m \otimes h \mapsto m \cdot h\), and a left \(H\)-coaction \(\varepsilon\) is a right-left Yetter-Drinfeld module if the following relations hold, for all \(m \in M\) and \(h \in H\):

\[
\sum m_{(-1)} x^1 \otimes x^3 (m_{(0)} \cdot x^2)(-1) \otimes (m_{(0)} \cdot x^2)(0) \\
= \sum y^2 (m \cdot y^1)(-1) x^1 \otimes y^3 (m \cdot y^1)(-1) x^2 \otimes (m \cdot y^1)(0) x^3 \quad (54)
\]

\[
\sum \varepsilon(m_{(-1)}) m_{(0)} = m \quad (55)
\]

\[
\sum m_{(-1)} h_1 \otimes m_{(0)} \cdot h_2 = \sum h_2 (m \cdot h_1)(-1) \otimes (m \cdot h_1)(0) \quad (56)
\]

For \(M \in \text{H}YDH\), we consider the map
\[
R_M : M \otimes M \to M \otimes M, \quad R_M (m \otimes n) = \sum m \cdot n_{(-1)} \otimes n_{(0)}.
\]

If we consider \(M\) as an object in \(\text{H}^{\text{op},\cop} \text{YD}^{\text{H}^{\text{op},\cop}}\), then we obtain the same map \(R_M\), so \(R_M\) is also a solution of the corresponding quasi-Yang-Baxter equation, which is obtained after replacing \(\Phi\) by \(\Phi_{\text{op},\cop} = \Phi_{321}\).

Now let \(M\) be a finite dimensional right-left Yetter-Drinfeld module. Then \(M^*\) is a left \(H\)-module, with action given by \((h \cdot m^*)(m) = m^*(m \cdot h)\), for all \(h \in H, m \in M, m^* \in M^*\). We also define a \(k\)-linear map \(M^* \to M^* \otimes H, m^* \mapsto \sum m^*_{(0)} \otimes m^*_{(1)}\), by the condition
\[
\sum m^*_{(0)} (m) m^*_{(1)} = \sum m^* (m_{(0)}) m_{(-1)} \quad (57)
\]
for all \(m \in M\). We can prove now the following result.

**Proposition 3.5.** Let \(H\) be a quasi-bialgebra, \(M\) a finite dimensional right-left Yetter-Drinfeld module. Then

(i) \(M^* \in \text{H}YD^H\);
we compute:

\[
R_{M^*} = R_M^*.
\]

**Proof.** (i) We prove that (50), (51), (52) are satisfied. For \( m^* \in M^* \) and \( m \in M \), we compute:

\[
\sum (x^2 \cdot m^*_{(0)})(m)(x^2 \cdot m^*_{(0)})(1)x^1 \otimes x^3 m^*_{(1)}
\]

\[
= \sum (x^2 \cdot m^*_{(0)})(m_{(0)})(m_{(-1)})x^1 \otimes x^3 m^*_{(1)}
\]

\[
= \sum m^*_{(0)}(m_{(0)} \cdot x^2)m_{(-1)}x^1 \otimes x^3 m^*_{(1)}
\]

\[
= \sum m^*_{(0)}((m \cdot y^1)_{(0)} \cdot x^3)y^2(m \cdot y^1)_{(-1)}x^1 \otimes y^3(m \cdot y^1)_{(-1)}x^2
\]

\[
= \sum (x^3 \cdot m^*)_{(0)}((m \cdot y^1)_{(0)})y^2(m \cdot y^1)_{(-1)}x^1 \otimes y^3(m \cdot y^1)_{(-1)}x^2
\]

\[
= \sum (x^3 \cdot m^*)_{(0)}(m \cdot y^1)y^2(x^3 \cdot m^*)_{(1)}x^1 \otimes y^3(x^3 \cdot m^*)_{(1)}x^2
\]

so obtain (50). Now we compute:

\[
\sum \varepsilon(m^*_{(1)}m^*_{(0)}(m) = \sum \varepsilon(m^*_{(0)}(m)m^*_{(1)})
\]

\[
= \sum \sum m^*(m_{(0)}m_{(-1)}) = \sum m^*(m_{(-1)}m_{(0)}) = m^*(m),
\]

using (55) at the last step. Thus (51) holds. For \( h \in H \), we compute:

\[
\sum (h \cdot m^*_{(0)})(m)h_2 m^*_{(1)} = \sum m^*(m \cdot h_1)h_2 m^*_{(1)}
\]

\[
= \sum m^*((m \cdot h_1)(0))h_2(m \cdot h_1)_{(-1)}
\]

\[
= \sum m^*(m_{(0)} \cdot h_2)m_{(-1)}h_1
\]

\[
= \sum (h_2 \cdot m^*)_{(0)}(m)(h_2 \cdot m^*)_{(1)}h_1
\]

and (52) follows.

(ii) We identify \((M \otimes M)^* = M^* \otimes M^*\), and we prove that \( R_{M^*} \) and \( R_M^* \) coincide as maps \( M^* \otimes M^* \to M^* \otimes M^* \). For \( m, n \in M \) and \( m^*, n^* \in M^* \), we compute:

\[
R_{M^*}(m^* \otimes n^*)(m \otimes n) = \sum (n^*_{(1)} \cdot m^*)(m)n_{(0)}^*(n)
\]

\[
= \sum m^*(m \cdot n^*)n_{(0)}^*(n)
\]

\[
= \sum m^*(m \cdot n_{(-1)}^*)n^*(n_{(0)})
\]

\[
= (m^* \otimes n^*)(R_M(m \otimes n))
\]

\[
= R_M^*(m^* \otimes n^*)(m \otimes n),
\]

as needed. \( \square \)
4. The quantum commutativity of \( H_0 \)

Let \( H \) be a Hopf algebra. It is well-known that \( H \) is an algebra in the monoidal category \( H \mathcal{YD} \), with left action and coaction given by

\[
h \triangleright h' = \sum h_1 h'S(h_2), \quad \lambda(h) = \sum h_1 \otimes h_2.
\]

Moreover, \( H \) is quantum commutative as an algebra in \( H \mathcal{YD} \), see for example [11]. We will now prove a similar result for quasi-Hopf algebras. Let \( H \) be a quasi-Hopf algebra. In [7], a new multiplication on \( H \) was introduced; this multiplication is given by the formula

\[
h \circ h' = \sum x^1 h S(x^1 X^2) \alpha x^2 X^3 h' S(x^3 X^2_3)
\]

for all \( h, h' \in H \). \( \beta \) is a unit for this multiplication \( \circ \). Let \( H_0 \) be the \( k \)-linear space \( H \), with multiplication \( \circ \), and left \( H \)-action given by

\[
h \triangleright h' = \sum h_1 h'S(h_2).
\]

Then \( H_0 \) is a left \( H \)-module algebra. In \( H_0 \), we also define a left \( H \)-coaction, as follows

\[
\lambda_{H_0}(h) = \sum h_{(-1)} \otimes h_{(0)} = \sum x^1 Y^1_1 h_1 g_1^1 S(q^2 Y^2_2) Y^3 \otimes X^2 Y^3_2 h_2 g_2^2 S(X^3 q^3 Y^3_3),
\]

where \( f^{-1} = \sum g^1 \otimes g^2 \) and \( q_R = \sum q^1 \otimes q^2 \) are the elements defined by (16) and (19). Then \( H_0 \) is an algebra in \( H \mathcal{YD} \), see [5] for details. In Proposition 4.2, we will show that \( H_0 \) is quantum commutative. But first we need the following formulas, which are of independent interest. Recall that \( q_R = \sum q^1 \otimes q^2 \), \( q_L = \sum f^1 \otimes f^2 \) and \( f^{-1} = \sum g^1 \otimes g^2 \) are defined by (20), (22), (15) and (16).

**Lemma 4.1.** Let \( H \) be a quasi-Hopf algebra. Then we have

\[
\sum q^1 y^1 \otimes S(q^2 y^2) y^3 = 1 \otimes \alpha,
\]

\[
\Phi(\Delta \otimes id)(f^{-1}) = \sum g^1 S(X^1) f^1 \otimes g_1^2 G^1 S(X^2) f^2 \otimes g_2^2 G^2 S(X^1),
\]

\[
\sum S(g^1) \alpha g^2 = S(\beta), \quad \sum f^1 \beta S(f^2) = S(\alpha),
\]

\[
\sum S(q^1_2 X^1 f^1) \otimes S(q^2 X^1 \beta S(q^1 X^2 f^2) = (id \otimes S)(q_L),
\]

\[
\sum S(q^1_2 X^1 f^1) \otimes S(q^2 X^1 \beta S(q^1 X^2 f^2) = (id \otimes S)(q_R).
\]

**Proof.** (61) and (62) are a direct consequence of (19) and (18). (63) has been proved in [6, Lemma 2.6] and [10, Lemma 2.5]. We are left to prove (64). Using (27), we obtain:

\[
(id \otimes \Delta)(q) = 1 \otimes S^{-1}(x^3 g^2) \otimes S^{-1}(x^2 g^1)(q \otimes 1)((\Delta \otimes id)(q) \Phi^{-1}(id \otimes \Delta)(\Delta(x^1))
\]

and, using the formula (see [8])

\[
(\Delta \otimes id)(q) \Phi^{-1} = \sum Y^1 \otimes q^1 Y^2_1 \otimes S^{-1}(Y^3 q^3 Y^3_2,
\]

we obtain

\[
(id \otimes \Delta)(q) = \sum Q^1 Y^1_1 x^1_1 \otimes S^{-1}(x^2 g^2) Q^2 q^1 Y^2_2 x^1_{(2,1)} \otimes S^{-1}(Y^3 x^2 g^1) q^2 Y^2_2 x^1_{(2,2)}
\]

(65)
where \( q_R = \sum q^1 \otimes q^2 = \sum Q^1 \otimes Q^2 \). Now we compute
\[
\sum S(q_2^2 X^3) f^1 \otimes S(q^1 X^1 \beta S(q_2^2 X^2)) f^2
\]
\[(55) = \sum S(q_2^2 x_2^{(1,2)} X^3) Y^3 x^2 \otimes S(Q^1 Y^1 x_1^1 X^1 \beta S(Q^2 q^1 Y^1_1 X^2 x_1^{(1,2)})) x^3
\]
\[(5) = \sum S(q_2^2 Y^3 x_2^{(1,2)}) Y^3 x^2 \otimes S(Q^1 Y^1 X^1 \beta S(Q^2 q^1 Y^1_2 x^2 x_1^{(1,2)})) x^3
\]
\[(3) = \sum S(q_2^2 Y^3 x_1^{(1,2)} x^2) \otimes S(Q^1 Y^1 X^1 \beta S(Q^2 q^1 Y^1_2 x^2 x_1^{(1,2)})) x^3
\]
\[(5, 6) = \sum S(x_1^1 x^2) \alpha x_2^2 \otimes S(Q^1 X^1 \beta S(Q^2 X^2))
\]
\[(5, 7) = \sum S(x^1) \alpha x_2^2 \otimes S(Q^1 \beta S(Q^2 x^3))
\]
\[(19, 6) = \sum S(x^1) \alpha x_2^2 \otimes S(x^3),
\]
as needed.

We can prove now the main result of this Section.

**Proposition 4.2.** Let \( H \) be a quasi-Hopf algebra. Then \( H_0 \) is quantum commutative as an algebra in \( \text{H}D \), that is, for all \( h, h' \in H \):

\[
h \circ h' = \sum (h_{(-1)} \circ h') \circ h_{(0)}.
\]

**Proof.** For all \( h, h' \in H \) we compute:
\[
\sum (h_{(-1)} \circ h') \circ h_{(0)}
\]
\[(60) = \sum (X^1 Y_1^1 h_1 g^1 S(q^2 Y_2^2) Y^3 \circ h') \circ X^2 Y_2^1 h_2 g^2 S(X^3 q^1 Y_1^2)
\]
\[(59, 58) = \sum Z^1 X^1 Y_1^1 h_{(1,1)} g_1^1 S(q^2 Y_2^2)_{1} Y^3 h'
\]
\[
S(x_{1}^1 X^2 Y_{1}^{(1,2)} h_{(1,2)} g_{1}^2 S(q^2 Y_2^2)_{2} Y_2^3)
\]
\[
\alpha x_2^2 Z^1 X^2 Y_1^1 Y_2^1 h_2 g^2 S(x^3 Z_2^1 X^3 q^1 Y_1^2)
\]
\[(3.5) = \sum Z^1 Y_1^1 h_{(1,1)} g_1^1 S(q^2 Y_2^2)_{1} Y^3 h'
\]
\[
S(Z^2 Y_1^1 h_{(1,2)} g_{1}^2 S(q^2 Y_2^2)_{2} Y_2^3)
\]
\[
\alpha Z^1 Y_2^1 h_2 g^2 S(q^1 Y_1^2)
\]
\[(11) = \sum Z^1 [Y^1 h S(Y^2)]_{(1,1)} g_1^1 S(q^2)_{1} Y^3 h'
\]
\[
S(Z^2 [Y^1 h S(Y^2)]_{(1,2)} g_{1}^2 S(q^2)_{2} Y_2^3)
\]
\[
\alpha Z^1 [Y^1 h S(Y^2)]_{2} g^2 S(q^1)
\]
\[(1, 5) = \sum Y^1 h S(Y^2) Z^1 g_1 S(q^2)_{1} Y^3 h' S(Z^2 g_{1}^2 S(q^2)_{2} Y_2^3) \alpha Z^1 g^2 S(q^1)
\]
\[(62) = \sum Y^1 h S(Y^2) g^1 S(X^2) f^1 S(q^2)_{1} Y_1^3 h'
\]
\[
S(g_1^2 G^1 S(X^2) f^2 S(q^2)_{2} Y_2^3) \alpha g^2 G^2 S(q^1 X^1)
\begin{align*}
(5,63) &= \sum Y^1 h S(X^3 Y^2) f^1 S(q_1^2) Y_1^3 h' S(q^1 X^1 \beta S(X^2) f^2 S(q_1^2 Y_2^3) \\
(11) &= \sum Y^1 h S(q_2^1 X^3 Y^2) f^1 Y_1^3 h' S(q^1 X^1 \beta S(q_1^2 X^2) f^2 Y_2^3) \\
(64) &= \sum Y^1 h S(x^1 Y^2) \alpha x^2 Y^3 h' S(x^3 Y_2^3) \\
(58) &= h \circ h'.
\end{align*}

5. Hopf modules in $H^H Y D$. Integrals

Let $H$ be a quasi-Hopf algebra. The aim of this Section is to define the space of integrals of a finite dimensional braided Hopf algebra in $H^H Y D$, and to prove, following [24], [12], that it is an object of $H^H Y D$, and that it has dimension 1. We will apply our results to the braided Hopf algebra associated to $H$, in the case where $H$ is a quasitriangular quasi-Hopf algebra.

Let $A$ be an algebra in a monoidal category $C$. Recall that a right $A$-module $M$ is an object $M \in C$ together with a morphism $\omega_M : M \otimes A \rightarrow M$ in $C$ such that $\omega_M \circ (id_M \otimes \eta) = l_M^A$ and the following diagram is commutative:

\[
(M \otimes A) \otimes A \xrightarrow{\omega_M \otimes id_A} M \otimes A \xrightarrow{\omega_M} M
\]

Clearly $A$ itself is a right $A$-module, by right multiplication. Right comodules over a coalgebra $C$ in $C$ can be defined in a similar way: we need $N \in C$ together with a morphism $\rho_N : N \rightarrow N \otimes C$ in $C$ such that $(id_N \otimes \varepsilon) \circ \rho_N = l_N$ and the following diagram is commutative:

\[
N \xrightarrow{\rho_N} N \otimes C \xrightarrow{\rho_N \otimes id_C} (N \otimes C) \otimes C
\]

$C$ itself is a right $C$-comodule via the comultiplication $\Delta$.

From [3], [21], [24], we recall the following.

Definition 5.1. Let $B$ be a bialgebra in a braided category $C$. A right $B$-Hopf module is a triple $(M, \omega_M, \rho_M)$, where $(M, \omega_M)$ is a right $B$-module and $(M, \rho_M)$ is a right $B$-comodule such that $\rho_M : M \rightarrow M \otimes B$ is right $B$-linear. The $B$-module structure $\omega_{M \otimes B} : (M \otimes B) \otimes B \rightarrow M \otimes B$ on $M \otimes B$ is given by the following
composition:

\[(M \otimes B) \otimes B \xrightarrow{id_M \otimes \Delta} (M \otimes B) \otimes (B \otimes B) \]
\[\xrightarrow{a_M \otimes B} \quad M \otimes (B \otimes (B \otimes B)) \]
\[\xrightarrow{id_M \otimes a^{-1}_M} \quad M \otimes ((B \otimes B) \otimes B) \]
\[\xrightarrow{id_M \otimes (c_B \otimes \varepsilon_B)} \quad M \otimes ((B \otimes B) \otimes B) \]
\[\xrightarrow{id_M \otimes a_M \otimes \eta_M} \quad M \otimes (B \otimes (B \otimes B)) \]
\[\xrightarrow{\Delta_M} \quad M \otimes B \]

\(\mathcal{M}_B^H\) will denote the category of right \(B\)-Hopf modules and morphisms in \(\mathcal{C}\) preserving the \(B\)-action and the corresponding \(B\)-coaction.

We can consider algebras, coalgebras, bialgebras and Hopf algebras in the braided category \(\mathcal{H}_H\mathcal{YD}\) over a quasi-Hopf algebra \(H\). More precisely, an algebra \(B\) in \(\mathcal{H}_H\mathcal{YD}\) is an object \(B \in \mathcal{H}_H\mathcal{YD}\) such that

- \(B\) is a left \(H\)-module algebra, i.e. \(B\) has a multiplication \(m\) and a usual unit \(1_B\) satisfying the following conditions:
  \[(ab)c = \sum (X^1 \cdot a)[(X^2 \cdot b)(X^3 \cdot c)], \quad (67)\]
  \[h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B, \quad (68)\]
  for all \(a, b, c \in B\) and \(h \in H\).

- \(B\) is a quasi-comodule algebra, that is, the multiplication \(m\) and the unit \(\eta\) of \(B\) intertwine the \(H\)-coaction \(\lambda_B\). By (47) this means:
  \[\lambda_B(b') = \sum X^1(x^1 Y^1 \cdot b)(-1) x^2(Y^2 \cdot b')(0) Y^3 \]
  \[\otimes [X^2 \cdot (x^1 Y^1 \cdot b)(0)][X^3 x^3 \cdot (Y^2 \cdot b')(0)], \quad (69)\]
  for all \(b, b' \in B\), and
  \[\lambda_B(1_B) = 1_H \otimes 1_B. \quad (70)\]

\(M \in \mathcal{H}_H\mathcal{YD}\) is a right \(B\)-module if there exists a morphism \(\varpi_m : M \otimes B \rightarrow M\) in \(\mathcal{H}_H\mathcal{YD}\). (we will denote \(\varpi_M(m \otimes b) := m \leftrightarrow b\)) such that

\[m \leftrightarrow 1_B = m, \quad (m \leftrightarrow b) \leftrightarrow b' = \sum (X^1 \cdot m) \leftrightarrow [(X^2 \cdot b)(X^3 \cdot b')] \quad (71)\]

for all \(m \in M, b, b' \in B\). The fact that \(\varpi_M\) is a morphism in \(\mathcal{H}_H\mathcal{YD}\) means (see (47))

\[h \cdot (m \leftrightarrow b) = \sum (h_1 \cdot m) \leftrightarrow (h_2 \cdot b), \quad (72)\]
\[\lambda_M(m \leftrightarrow b) = \sum X^1(x^1 Y^1 \cdot m)(-1) x^2(Y^2 \cdot b)(-1) Y^3 \]
\[\otimes [X^2 \cdot (x^1 Y^1 \cdot m)(0)][X^3 x^3 \cdot (Y^2 \cdot b)(0)], \quad (73)\]

for all \(m \in M, b \in B\).

Similarly, \(B \in \mathcal{H}_H\mathcal{YD}\) is a coalgebra if
- $B$ is a left $H$-module coalgebra, i.e. $B$ has a comultiplication $\Delta_B : B \to B \otimes B$ (we will denote $\Delta(b) = \sum b_1 \otimes b_2$) and a usual counit $\varepsilon_B$ such that:

$$
\sum X^1 \cdot b_{(-1)} \otimes X^2 \cdot b_{(0)} \otimes X^3 \cdot b_{(2)} = \sum b_1 \otimes b_1 \otimes b_2 \otimes b_2,
$$

$$
\Delta_B(h \cdot b) = \sum h_1 \cdot b_1 \otimes h_2 \cdot b_2, \quad \varepsilon_B(h \cdot b) = \varepsilon(h) \varepsilon_B(b),
$$

for all $h \in H, b \in B$, where we use the same notation for the quasi-coassociativity of $\Delta_B$ as in Section 2.

- $B$ is a quasi-comodule coalgebra, i.e. the comultiplication $\Delta_B$ and the counit $\varepsilon_B$ intertwine the $H$-coaction $\lambda_B$. Explicitly, for all $b \in B$ we must have that:

$$
\sum b_{(-1)} \otimes b_{(0)} \otimes b_{(1)} = \sum X^1(x^1 Y^1 \cdot b_1)(-1)x^2(Y^2 \cdot b_2)(-1)Y^3
\otimes X^1(x^1 Y^1 \cdot b_1)(0) \otimes X^2 x^3 \cdot (Y^2 \cdot b_2)(0),
$$

and

$$
\sum \varepsilon_B(b_{(0)}) b_{(-1)} = \varepsilon_B(b). \varepsilon_B(1).
$$

A right $B$-comodule in $H\mathcal{YD}$ is an object $M \in H\mathcal{YD}$ together with a morphism $\rho_M : M \to M \otimes B$ in $H\mathcal{YD}$ (we will denote $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ for all $m \in M$) such that the following relations hold, for all $m \in M$:

$$
\sum X^1 \cdot m_{(0)} \otimes X^2 \cdot m_{(0)} \otimes X^3 \cdot m_{(1)} = \sum m_{(0)} \otimes m_{(1)} \otimes m_{(2)},
$$

$$
\sum \varepsilon(m_{(1)}) m_{(0)} = m,
$$

where we will denote

$$
(\rho_M \otimes \text{id}_B)(\rho_M(m)) = \sum m_{(0)} \otimes m_{(1)} \otimes m_{(2)}, \quad \text{etc.}
$$

The fact that $\rho_M$ is a morphism in $H\mathcal{YD}$ means that (see (47))

$$
\rho_M(h \cdot m) = \sum h_1 \cdot m_{(0)} \otimes h_2 \cdot m_{(1)},
$$

and

$$
\sum m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} = \sum X^1(x^1 Y^1 \cdot m_{(0)})(-1) x^2(Y^2 \cdot m_{(1)})(-1)Y^3
\otimes X^1(x^1 Y^1 \cdot m_{(0)})(0) \otimes X^2 x^3 \cdot (Y^2 \cdot m_{(1)})(0),
$$

for all $h \in H$ and $m \in M$.

Now, a bialgebra $B \in H\mathcal{YD}$ is an algebra and a coalgebra in $H\mathcal{YD}$ such that $\Delta_B$ is an algebra morphism, i.e. $\Delta_B(1_B) = 1_B \otimes 1_B$ and, by (38) and (48), for all $b, b' \in B$ we have that:

$$
\Delta_B(b b') = \sum [y^1 X^1 \cdot b_1][y^2 Y^1(x^1 X^2 \cdot b_2)(-1) x^2 X^1 \cdot b_1'] \otimes [y^3 Y^2 \cdot (x^1 X^2 \cdot b_2)(0)][y^3 Y^3 x^3 X^1 \cdot b_2'],
$$

If $B \in H\mathcal{YD}$ is a bialgebra then $M \in H\mathcal{YD}$ is a right $B$-Hopf module if $M$ is a right $B$-module (as above, we will denote $\omega_M(m \otimes b) = m \leftarrow b$) and a right $B$-comodule
such that the right $B$-coaction on $M$, $\rho_M : M \to M \otimes B$, is right $B$-linear, which means that the following relation holds, for all $m \in M$ and $b \in B$ (see (66)):

$$\rho_M(m \otimes b) = \sum (y^1 X^1 \cdot m(0)) \leftarrow [y^2 Y^1 (x^1 X^2 \cdot m(1))(-1)x^2 X^1 \cdot b_1]$$

$$\otimes [y^2 Y^1 (x^1 X^2 \cdot m(1))(-1)]y^2 Y^2 x^3 X^2 \cdot b_2;$$

(83)

Finally, a bialgebra $B$ in $H^L_YD$ is a braided Hopf algebra if there exists a morphism $\mathcal{S} : B \to B$ in $H^L_YD$ such that $\sum \mathcal{S}(b_1)b_2 = \sum b_2 \mathcal{S}(b_2) = \varepsilon(b)1_B$, for all $b \in B$. Since $\mathcal{S}$ is a morphism in $H^L_YD$, we have that

$$\mathcal{S}(h \cdot b) = h \cdot \mathcal{S}(b) \text{ and } \sum \mathcal{S}(b)(-1) \otimes \mathcal{S}(b)(0) = \sum b(-1) \otimes \mathcal{S}(b(0)),$$

(84)

for all $h \in H$, $b \in B$. Also, by (39) and (48) we obtain that

$$\mathcal{S}(bb') = \sum [b_{(-1)} \cdot \mathcal{S}(b)] \mathcal{S}(b(0)) \text{ and } \Delta(\mathcal{S}(b)) = \sum b_{(-1)} \cdot \mathcal{S}(b_2) \otimes \mathcal{S}(b_{(0)}),$$

(85)

for all $b, b' \in B$.

The first step to prove the existence and uniqueness of integrals in a finite dimensional braided Hopf algebra is the structure theorem for Hopf modules. To this end we need first the following result.

**Lemma 5.2.** Let $H$ be a quasi-bialgebra, $B$ a bialgebra in $H^L_YD$ and $N \in H^L_YD$. Then $N \otimes B \in H^L_YD$ with following action $\varphi_{N \otimes B} : (N \otimes B) \otimes B \to N \otimes B$ and coaction $\rho_{N \otimes B} : N \otimes B \to (N \otimes B) \otimes B$ given by

$$(n \otimes b) \mapsto b' = \sum X^1 \cdot n \otimes [(X^2 \cdot b)(X^3 \cdot b')]$$

(86)

$$\varphi_{N \otimes B}(n \otimes b) := \sum x^1 \cdot n \otimes x^2 \cdot b_1 \otimes x^3 \cdot b_2,$$

(87)

for all $n \in N$ and $b, b' \in B$.

**Proof.** $H^L_YD$ is a braided category, so $N \otimes B \in H^L_YD$. It is not hard to see that (1) and (67) imply that $\varphi_{N \otimes B}$ is left $H$-linear. It intertwines also the corresponding $H$-coaction. Indeed, by (47), the left $H$-coaction on $(N \otimes B) \otimes B$ is given by

$$\lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b')$$

$$= \sum Z^1 X^1 (x^1 Y^1 y^1 T^1_1 \cdot n)(-1)x^2 (Y^2 y^1 T^1_1 \cdot b)(-1)Y^3 y^3 (T^1 \cdot b')(T^3)$$

$$\otimes Z^2 X^2 \cdot (x^1 Y^1 y^1 T^1_1 \cdot n)(0) \otimes Z^3 X^3 \cdot (Y^2 y^1 T^1_1 \cdot b)(0) \otimes Z^3 y^3 \cdot (T^1 \cdot b')(0),$$
for all $n \in N$, $b, b' \in B$. Therefore:

$$
(id_H \otimes \varphi_{N \otimes B}) \circ \lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b')
$$

(86)  
$$
= \sum Z^1 X^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)(-y^2 Y^2 y_2^1 T_2^1 \cdot b)(-1)
$$

$$
Y^3 y^2 (T^2 \cdot b')(-1) T^3 \otimes W^1 Z^1 x^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)(0)
$$

$$
\otimes \{[W^2 Z^1 x^3 \cdot (Y^1 y_1^1 T_1^1 \cdot b)]) ([W^3 Z^3 y^3 \cdot (T^2 \cdot b')])(0)\}
$$

(3, 45, 67)  
$$
= \sum Z^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)(-y^2 Y^2 y_2^1 T_2^1 \cdot b)(-1)
$$

$$
Y^3 y^2 (T^2 \cdot b')(-1) T^3 \otimes Z^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)(0)
$$

$$
\otimes Z^3 \cdot \{[X^2 x^3 \cdot (Y^1 y_1^1 T_1^1 \cdot b)]) ([Y^3 Z^3 y^3 \cdot (T^2 \cdot b')])(0)\}
$$

(3) twice, (45)  
$$
= \sum Z^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)(-x^2 X^2 (y^1 Y^2 T_2^1 \cdot b)(-1)) y^2
$$

$$
(Y^1 y^2)^2 (T^2 \cdot b')(-1) Y_2^3 T^3 \otimes Z^2 \cdot (x^1 Y^1 T_1^1 \cdot n)(0)
$$

$$
\otimes Z^3 x^3 \cdot \{[X^2 \cdot (y^1 Y^2 T_2^1 \cdot b)]) ([X^3 y^3 \cdot (Y_1^3 T^3 \cdot b')])(0)\}
$$

(3, 69)  
$$
= \sum Z^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)(-x^2 Y^1 \otimes (Y_2^3 T^3 \cdot b))(T^3 \cdot b')]
$$

$$
\otimes Z^2 \cdot (x^1 Y^1 T^1 \cdot n)(0) \otimes Z^3 x^3 \cdot \{[Y^3 T^3 \cdot b'])(Y_2^3 T^3 \cdot b')\}\}
$$

(47, 86)  
$$
= \sum \lambda_{N \otimes B} ((T \cdot n \otimes (T^2 \cdot b)(T^3 \cdot b')))
$$

$$
= \lambda_{N \otimes B} \circ \varphi_{N \otimes B}((n \otimes b) \otimes b')
$$

for all $n \in N$ and $b, b' \in B$. In a similar way, it can be proved that the map $\rho_{N \otimes B}$ is a morphism in $\mathcal{YD}$, we leave it to the reader to verify the details.

Using (67) and (3), it easily follows that $N \otimes B$ is a right $B$-module. Also, it is not hard to see that (74), (75) and (3) imply that $N \otimes B$ is a right $B$-comodule. It remains only to show that $\rho_{N \otimes B}$ is right $B$-linear. By (66), we have that the right $B$-module structure of $(N \otimes B) \otimes B$ is given by

$$
[(n \otimes b) \otimes b'] \cdot b''
$$

$$
= \sum [Z^1 y_1^1 X_1^1 \cdot n \otimes (Z^2 y_2^1 X_2^1 \cdot b)] (Z^3 y^2 Y^1 (x^1 Y^1 x^2 \cdot b')(-1) x^2 X_1^1 \cdot b'')
$$

$$
\otimes [y_1^1 Y^2 \cdot (x^1 Y^1 x^2 \cdot b')(0)] [y_2^1 Y^3 x^3 X_2^1 \cdot b''(0)]
$$

for all $n \in N$ and $b, b', b'' \in B$. This allows us to compute, for any $n \in N$ and $b, b' \in B$, that:

$$
\rho_{N \otimes B}(n \otimes b) \cdot b' = \sum [(z^1 \cdot n \otimes z^2 \cdot b_1) \otimes z^3 \cdot b_2] \cdot b'
$$

$$
= \sum [Z^1 y_1^1 X_1^1 z^1 \cdot n]
$$

$$
\otimes [Z^2 y_2^1 X_2^1 z^2 \cdot b_1] (Z^3 y^2 Y^1 (x^1 Y^1 x^2 \cdot z^3 \cdot b_2)(-1) x^2 X_1^1 \cdot b'')
$$

$$
\otimes [y_1^1 Y^2 \cdot (x^1 Y^1 x^2 \cdot z_3 \cdot b_2)(0)] [y_2^1 Y^3 x^3 X_2^1 \cdot b''(0)]
$$
Our next result is the Fundamental Theorem for Hopf modules in the braided monoidal category $\mathcal{H}^D$, generalizing [12, Theorem 1].

**Theorem 5.3.** Let $H$ be a quasi-Hopf algebra, $B$ a Hopf algebra in $\mathcal{H}^D$ and $M \in \mathcal{M}^B_D$.

(i) $M^{coB} = \{ m \in M \mid \rho_M(m) = m \otimes 1_B \} \in \mathcal{H}^D$.

(ii) For all $m \in M$, we have that $P(m) = \sum m_{(0)} \in S(m_{(1)}) \in M^{coB}$.

(iii) $\rho_M(n \leftarrow b) = \sum (x_1 \cdot n \leftarrow (x_2 \cdot b_2) \otimes x_3 \cdot b_2$ and $P(n \leftarrow b) = \xi(b)n$, for all $n \in M^{coB}$ and $b \in B$.

(iv) The map $F : M^{coB} \otimes B \to M, \ F(n \otimes b) = n \leftarrow b,$

is an isomorphism of Hopf modules in $\mathcal{H}^D$, with inverse $G$ given by

\[ G(m) = \sum P(m_{(0)}) \otimes m_{(1)}. \]

**Proof.** (i) If $n \in M^{coB}$, then $\rho_M(h \cdot n) = \sum h_1 \cdot n \otimes h_2 \cdot 1_B = h \cdot n \otimes 1_B$, by (72) and (67). This shows that $M^{coB}$ is an $H$-submodule of $M$. On the other hand, for any $n \in N$ we have

\[ \sum n_{(-1)} \otimes n_{(0)(2)} \otimes n_{(0)(1)} \]

(81) \[ = \sum X^1(x^1Y^1 \cdot n)_{(-1)} x^2(Y^2 \cdot 1_B)_{(-1)}Y^3 \]

\[ \otimes X^2 \cdot (x^1Y^1 \cdot n)_{(0)} \otimes X^3 \cdot (Y^2 \cdot 1_B)_{(0)} \]

(67) twice, (70) \[ = \sum n_{(-1)} \otimes n_{(0)} \otimes 1_B. \]

Thus, $\rho_M(n) = \sum n_{(-1)} \otimes n_{(0)} \in H \otimes M^{coB}$ which means that $M^{coB}$ is a left $H$-quasi-subcomodule of $M$. It follows from the above arguments that $M^{coB} \in \mathcal{H}^D$. 

\[ (3) = \sum [Z^1y^1z^1X^1 \cdot n \otimes (Z^2y^2z^2T^3X^2 \cdot b_2)] \]

\[ \otimes [y^2z^2 \cdot (x^2z^2T^3X^2 \cdot b_2)_{(0)}] \]

\[ \otimes [y^3z^3 \cdot (x^3z^3T^3X^3 \cdot b^2_3)] \]

\[ (1, 45) = \sum [Z^1y^1z^1X^1 \cdot n \otimes (Z^2y^2z^2T^3X^2 \cdot b_2)] \]

\[ \otimes [y^2z^2 \cdot (x^2z^2T^3X^2 \cdot b_2)_{(0)}] \]

\[ \otimes [y^3z^3 \cdot (x^3z^3T^3X^3 \cdot b^2_3)] \]

\[ (1, 3, 67) = \sum (y^1X^1 \cdot n \otimes y^2 \cdot (z^1T^3X^2 \cdot b_2)] \]

\[ \otimes y^3 \cdot (z^3T^3X^3 \cdot b_3)] \]

\[ (75, 82) = \sum (y^1X^1 \cdot n \otimes y^2 \cdot (z^2b)(X^3 \cdot b)] \]

\[ \otimes y^3 \cdot (z^3b)(X^3 \cdot b)] \]

\[ (87, 86) = \sum \mathcal{P} \otimes B(X^1 \cdot n \otimes (z^2b)(X^3 \cdot b) = \mathcal{P} \otimes B(n \otimes b) \rightleftharpoons b), \]

as needed. \[ \square \]
(ii) For any $m \in M$, we have that

$$
\rho_M(P(m)) = \sum \rho_M(m_{(0)} \leftarrow S(m_{(1)}))
$$

(83) = \sum \rho_M(m_{(0)}) \leftarrow [y^2 Y^1(x^1 X^2 \cdot m_{(0)}(1))(-1)x^2 X_3^3 \cdot S(m_{(1)}(1))]

$$
[y^3 Y^2 \cdot (x^1 X^2 \cdot m_{(0)}(0))][y^3 Y^3 x^3 X_2^3 \cdot S(m_{(1)}(1))]
$$

(78, 84, 85) = \sum \rho_M(m_{(0)}) \leftarrow

$$
[y^2 Y^2(x^1 \cdot m_{(1)}(2))(-1)x^2 S(m_{(1)}(2))(-1) \cdot S(m_{(1)}(2))]
$$

$$\otimes y^3 \cdot ([Y^2 \cdot (x^1 \cdot m_{(1)}(0))][Y^3 x^3 \cdot S(m_{(1)}(2))])
$$

(69) = \sum \rho_M(m_{(0)}) \leftarrow

$$
y^3[x^1 \cdot m_{(1)}(2)][x^2 \cdot S(m_{(1)}(2))])(-1)x^3 \cdot S(m_{(1)}(2))]
$$

$$\otimes y^3 \cdot ([x^1 \cdot m_{(1)}(2)][x^2 \cdot S(m_{(1)}(2))])
$$

(84, 74, 67) = \sum \rho_M(m_{(0)}) \leftarrow (y^2 \cdot S(m_{(1)})) \otimes y^3 \cdot 1_B = P(m) \otimes 1_B.

(iii) For all $n \in N$ and $b \in B$, we compute, using (83),

$$
\rho_M(n \leftarrow b) = \sum \rho_M(m_{(0)}) \leftarrow [y^2 Y^1(x^1 X^2 \cdot 1_B)(-1)x^2 X_3^3 \cdot b_1]
$$

$$\otimes [y^3 Y^2 \cdot (x^1 X^2 \cdot 1_B)(0)][y^3 Y^3 x^3 X_2^3 \cdot b_2]
$$

(67, 70) = \sum \rho_M(m_{(0)}) \leftarrow (y^2 \cdot b_1) \otimes y^3 \cdot b_2.

For all $n \in M^{co B}$, we find

$$
P(n \leftarrow b) = \sum [(y^1 \cdot n) \leftarrow (y^2 \cdot b_1)] \leftarrow S(y^3 \cdot b_2)
$$

(71, 84) = \sum n \leftarrow b_1 S(b_2) = \varepsilon(b)n \leftarrow 1_B = \varepsilon(b)n.

(iv) By (i) and Lemma 5.2, we obtain that $M^{co B} \otimes B \subset M_B^B$. It follows from (72) that $F$ is left $H$-linear. It also intertwines the corresponding left $H$-coaction by (47) and (73). Now we will prove that $F$ and $G$ are inverses. For all $m \in M$, we have

$$
FG(m) = \sum P(m_{(0)} \leftarrow m_{(1)}
$$

(71) = \sum (X^1 \cdot m_{(0)}(0)) \leftarrow [(X^2 \cdot S(m_{(1)}(1)))(X^3 \cdot m_{(1)}(1))]

(84, 78, 79) = \sum m_{(0)} \leftarrow S(m_{(1)}(1)) m_{(1)}(2) = m \leftarrow 1_B = m.

Similarly, for any $n \in M^{co B}$ and $b \in B$, we compute

$$
GF(n \otimes b) = \sum P((n \leftarrow b)_{(0)}) \otimes (n \leftarrow b)_{(1)}
$$

(iii) = \sum P((x^1 \cdot n) \leftarrow (x^2 \cdot b_1)) \otimes x^3 \cdot b_2

(iii), (75) = \sum P(n) \otimes b = n \otimes b.
We are left to show that $F$ is a morphism in $M^B_{H^d}$. It is not hard to see that (86) and (71) imply that $F$ is right $B$-linear. Also, (iii) implies that
\[
\mathcal{P}_M \circ F(n \otimes b) = (F \otimes id_B) \circ \mathcal{P}_{M \otimes n \otimes B}(n \otimes b) = \sum (x^1 \cdot n) \leftrightarrow (x^2 \cdot b_1) \otimes x^3 \cdot b_2,
\]
for all $n \in N$ and $b \in B$, and this finishes the proof. \hfill \Box

Let $H$ be a quasi-Hopf algebra, and let $\mathcal{H}^d$ be the category of finite dimensional left Yetter-Drinfeld modules over $H$. If $M \in \mathcal{H}^d$, then $M^\ast \in \mathcal{H}^d$ (cf. [4]). The action and coaction are given by
\[
(h \cdot m^\ast)(m) = m^\ast(S(h) \cdot m)
\]
\[
\lambda_M^\ast(m^\ast) = \sum m^\ast(-1) \otimes m^\ast(0) = \sum_f (m^\ast, f \cdot (g^1 \cdot m^\ast)(0))
\]
\[
S^{-1}(f^1(g^1 \cdot m^\ast)(-1)) \otimes f^2m
\]
for all $h \in H$, $m^\ast \in M^\ast$, $m \in M$. Here $f = \sum f^1 \otimes f^2$ is the twist defined in (15), $(\cdot m)_{i=1}^{\infty}$ is a basis of $M$ and $(\cdot m^\ast)_{i=1}^{\infty}$ its dual basis. Moreover, $\mathcal{H}^d$ is a rigid monoidal category. For each object $M \in \mathcal{H}^d$, the evaluation and coevaluation maps $(ev_M$ and $coev_M$, respectively) are given by (41).

In addition, if $B \in \mathcal{H}^d$ is a Hopf algebra, then $B^\ast$ is a Hopf algebra in $\mathcal{H}^d$. The structure is the following.

- The multiplication and unit are given by
  \[
  \langle \varphi \cdot \psi \rangle(b) = \langle \varphi, f^2q_3^2Y^3S^{-1}(q_1^2Y^{-1}(p^1 \cdot b_2)(-1)p^2) \cdot b_2 \rangle
  \]
  \[
  \langle \psi, f^1q_1^3Y^2 \cdot (p^1 \cdot b_2)(0) \rangle,
  \]
  \[
  1_{B^\ast} = \xi
  \]
  for all $\varphi, \psi \in B^\ast$, $b \in B$, where $q_1 = \sum q^1 \otimes q^2$ and $p_1 = \sum p^1 \otimes p^2$ are the elements defined in (19).

- The comultiplication and counit are given by the formulas
  \[
  \Delta_{B^\ast}(\varphi) = \sum_{i,j=1}^n \langle \varphi, [(g^1 \cdot j)(-1)g^2 \cdot b](g^1 \cdot j)(0) \rangle b \otimes f^i b
  \]
  \[
  \varepsilon_{B^\ast}(\varphi) = \varphi(1_B)
  \]
  for any $\varphi \in B^\ast$, where $f^{-1} = \sum g^1 \otimes g^2$ was defined in (16), $(\cdot b)_{i=1}^{\infty}$ is a basis of $B$ and $(\cdot b^\ast)_{i=1}^{\infty}$ the corresponding dual basis of $B^\ast$.

- The antipode is given by
  \[
  S_{B^\ast} = S^\ast, \text{ i.e. } S_{B^\ast}(\varphi) = \varphi \circ S,
  \]
  for all $\varphi \in B^\ast$.

**Proposition 5.4.** Let $B \in \mathcal{H}^d$ a Hopf algebra. Then $B^\ast$ is a right $B$-Hopf module, with structure:
\[
\langle \varphi \cdot b, b' \rangle = \sum \langle \varphi, [(U^1 \cdot b)(-1)U^2 \cdot b'] S((U^1 \cdot b)(0)) \rangle,
\]

\[
\langle \varphi, b, b' \rangle
\]

\[
\sum \langle \varphi, [(U^1 \cdot b)(-1)U^2 \cdot b'] S((U^1 \cdot b)(0)) \rangle,
\]

\[
\sum \langle \varphi, [(U^1 \cdot b)(-1)U^2 \cdot b'] S((U^1 \cdot b)(0)) \rangle,
\]

\[
\sum \langle \varphi, [(U^1 \cdot b)(-1)U^2 \cdot b'] S((U^1 \cdot b)(0)) \rangle,
\]
\[ \rho_{B^*}(\varphi) = \sum_{i=1}^{n} (S(\bar{\rho}^i) \cdot b_{(-1)} \cdot [b \ast (\bar{\rho}^i \cdot \varphi)] \otimes (S(\bar{\rho}^i) \cdot b)_0), \]

for all \( \varphi \in B^* \), \( b, b' \in B \), where

\[ U = \sum U^1 \otimes U^2 := \sum g^1 S(q^2) \otimes g^2 S(q^1), \]

\[ p_L = \sum \bar{\rho}^1 \otimes \bar{\rho}^2, \quad q_R = \sum q^1 \otimes q^2 \text{ and } f^{-1} = \sum g^1 \otimes g^2 \text{ are the elements defined by (21), (19) and (16), and } \{b\}_{i=1}^T \text{ is a basis of } B \text{ with corresponding dual basis } \{b^*_i\}_{i=1}^T. \]

Moreover,

\[ B^{\ast \text{co}B} = \{ A \in B^* | \sum (\bar{\rho}^i \cdot \varphi) \ast (\bar{\rho}^2 \cdot A) = \varphi(1_B)A \text{ for all } \varphi \in B^* \}. \]

**Proof.** If \( B \) is a Hopf algebra in a braided rigid monoidal category \( \mathcal{C} \), then \( B^* \) is a right Hopf \( B \)-module, as follows.

- the right \( B \)-module structure \( \triangleright : B^* \otimes B \rightarrow B \) on \( B^* \) is the composition

\[ \begin{array}{c}
\begin{array}{c}
B^* \otimes B \\
\xrightarrow{\text{id}_{B^*} \otimes \text{coev}_B} \\
\xrightarrow{a_{B^* \otimes B,B}^{-1}} \\
\xrightarrow{a_{B^* \otimes B,B} \otimes \text{id}_{B^*}} \\
\xrightarrow{\text{ev}_{B^*} \otimes \text{id}_{B^*}} \\
\xrightarrow{r_{B^*}} \\
1 \otimes B^* \\
\xrightarrow{\text{id}_{B^*} \otimes 1} \\
B^* \\
\xrightarrow{\text{coev}_B \otimes \text{id}_{B^*}} \\
\xrightarrow{\text{id}_{B^*} \otimes m_{B^*}} \\
B \otimes (B^* \otimes B^*) \\
\end{array}
\end{array} \]

- the right \( B \)-comodule structure \( \bar{\rho}_{B^*} : B^* \rightarrow B^* \otimes B \) on \( B^* \) is the composition

\[ \begin{array}{c}
\begin{array}{c}
B^* \\
\xrightarrow{r_{B^*}} \\
\xrightarrow{a_{B^* \otimes B,B}^{-1}} \\
\xrightarrow{a_{B^* \otimes B,B} \otimes \text{id}_{B^*}} \\
\xrightarrow{\text{coev}_B \otimes \text{id}_{B^*}} \\
\xrightarrow{\text{id}_{B^*} \otimes m_{B^*}} \\
1 \otimes B^* \\
\xrightarrow{\text{id}_{B^*} \otimes 1} \\
B^* \otimes B. \\
\end{array}
\end{array} \]

Let \( \gamma = \sum \gamma^1 \otimes \gamma^2 \) and \( f^{-1} = \sum g^1 \otimes g^2 \) be the elements defined in (14) and (16).

By (98), we have, for all \( \varphi \in B^* \) and \( b, b' \in B \):

\[ \langle \varphi \leftarrow b, b' \rangle = \sum \langle \varphi, S(X^1 p_1^1) \alpha \cdot \left[ (S(X^3 p_2^2 \cdot S(b))_{(-1)} X^3 p_2^2 \cdot b') (X^3 p_2^2 \cdot S(b))_{(0)} \right] \rangle \]

(67, 45) \[ = \sum \langle \varphi, (S(X^1 p_1^1) \alpha_1 X^2 p_1^2 \cdot S(b))_{(-1)} S(X^1 p_1^1) \alpha_2 X^3 p_2^2 \cdot b' \rangle \]

(67, 45) \[ (S(X^1 p_1^1) \alpha_1 X^2 p_1^2 \cdot S(b))_{(0)} \]
The additional structure is the following. Finally, by (99) we have which is just (95). (96) follows easily by (99), the details are left to the reader. Finally, by (99) we have

\[ \Lambda \in B^{*\text{co}(B)} \iff \varrho_{B,\Lambda}^{\text{co}}(\Lambda) = \Lambda \otimes 1_B \]
\[ \iff c_{B,B'} \circ \varrho_{B,\Lambda}^{\text{co}}(\Lambda) = c_{B',B}^{\text{co}}(\Lambda \otimes 1_B) \]
\[ \iff \sum_{i=1}^n S(\tilde{p}^i) \cdot b \otimes i^b \ast (\tilde{p}^2 \cdot \Lambda) = 1_B \otimes \Lambda \]
\[ \iff \sum (\tilde{p}^1 \cdot \varphi) \ast (\tilde{p}^2 \cdot \Lambda) = \varphi(1_B)\Lambda, \text{ for all } \varphi \in B^*. \]

We define the space of left integrals by \( I_l(B^*) = B^{*\text{co}(B)} \). From the Fundamental Theorem for Hopf modules, we then obtain.

**Corollary 5.5.** Let \( H \) be a quasi-Hopf algebra and \( B \) a finite dimensional Hopf algebra in \( H \text{-}\text{YD} \). Then \( I_l(B^*) \otimes B \simeq B^* \) as right \( B \)-Hopf modules. In particular, \( \text{dim}_k(I_l(B^*)) = 1 \).

Now, let \( H \) be a quasi-Hopf algebra and \( H_0 \) the \( H \)-module algebra described in Section 4. If \((H, R)\) is quasitriangular, then \( H_0 \) is a Hopf algebra in \( H \text{-}\text{YD} \), see [5]. The additional structure is the following.

\[ \lambda_{H_0}(h) = \sum R^2 \otimes R^1 \triangleright h, \]  
\[ \Delta(h) = \sum h_\perp \otimes h_\perp \]  
\[ = \sum x^1 X^1 h_1 g^1 S(x^2 R^2 Y^3 X_2^3) \otimes x^2 R^1 \triangleright g^1 X^2 h_2 g^2 S(y^2 X_1^3), \]  
\[ \varepsilon(h) = \varepsilon(h), \]  
\[ \varepsilon(h) = \sum X^1 R^2 p^2 S(q^1 (X^2 R^1 p^1 \triangleright h) S(q^2 X^3), \] for all \( h \in H \), where \( R = \sum R^2 \otimes R^2 \) and \( f^{-1} = \sum g^1 \otimes g^2 \), \( p_R = \sum p^1 \otimes p^2 \) and \( q_R = \sum q^1 \otimes q^2 \) are the elements defined by (16), (19) and (20). By the above arguments, if \( H \) is a finite dimensional Hopf algebra, then \( H_0^* \) is also a Hopf algebra.
in \( H \otimes YD \), with structure

\[
\langle \varphi \ast \Psi \rangle (h) = \sum \langle \varphi, f^2 R^2 \triangleright h_2 \rangle \langle \Psi, f^4 R^1 \triangleright h_1 \rangle S(Y^3 x^3),
\]

\[
\langle \Psi, f^4 R^1 \triangleright X^2 x_2 h_2 g^2 S(X^3 x^3) \rangle,
\]

\[
\sum \langle \varphi, f^2 \triangleright Y^2 R^2 X^1 x_1 h_1 g^1 S(Y^3 x^3) \rangle \langle \Psi, f^4 R^1 \triangleright h_1 \rangle S(Y^3 x^3),
\]

\[
= \sum \langle \varphi, f^2 \triangleright Y^2 R^2 X^1 x_1 h_1 g^1 S(Y^3 x^3) \rangle \langle \Psi, f^4 R^1 \triangleright h_1 \rangle S(Y^3 x^3),
\]

\[
1_{H^*_0} = \varepsilon,
\]

\[
\Delta H^*_0 (\varphi) = \sum_{i,j=1}^n \langle \varphi, (R^2 g^2 \triangleright e_2)(R^1 g^1 \triangleright e_1) \rangle^i \otimes \langle \varphi, (R^2 g^2 \triangleright e_2)(R^1 g^1 \triangleright e_1) \rangle^j e,
\]

\[
\varepsilon H^*_0 (\varphi) = \varphi(\beta),
\]

\[
S H^*_0 (\varphi) = v \circ S,
\]

for all \( h \in H \) and \( \varphi \in H^* \), where \( R^{-1} = \sum R^1 \otimes R^2, \{i e_1 \}_{i=1}^{\tilde{n}} \) is a basis of \( H \) and \( \{i e_1 \}_{i=1}^{\tilde{n}} \) the corresponding dual basis of \( H^* \). In this particular case we have

\[
\mathcal{I}_i(H^*_0) = \{ \Lambda \in H^* \mid \sum \Lambda(S(\tilde{p}^3) f^3 R^1 \triangleright h_1) S(\tilde{p}^1) f^3 R^2 \triangleright h_2 = \Lambda(\beta) \}, \text{ for all } h \in H \}.
\]

REFERENCES


