BRAIDINGS ON THE CATEGORY OF BIMODULES, AZUMAYA ALGEBRAS AND EPIMORPHISMS OF RINGS

A. L. AGORE, S. CAENEPEEL, AND G. MILITARU

Abstract. Let A be an algebra over a commutative ring k. We investigate braidings on the category of A-bimodules. Braidings are in bijective correspondence to R-matrices, these are elements in $A \otimes A \otimes A$ satisfying certain axioms. If A is commutative, then there exists a braiding if and only if $k \to A$ is an epimorphism in the category of rings, and then the R-matrix is trivial. We show that all braidings are symmetries. If the invariants functor $G = (-)^A : {}_A\mathcal{M}_A \to \mathcal{M}_k$ is separable, then it is fully faithfull, and we have a symmetry on the category of bimodules. This (almost) classifies the symmetries. We provide several examples. Our computations lead to new examples of solutions of the quantum Yang-Baxter equation.

Introduction

Motivated by the study of the quantum Yang-Baxter equation, Drinfeld [8] introduced the concept of quasitriangular Hopf algebra, also named strict quantum group. One of the fundamental results is the relation between quasitriangular structures on a Hopf algebra and braided monoidal categories, namely, there exists a bijective correspondence between the set of braidings on the representation category of H and the set of quasitriangular structures on H. A concise treatment of this material is presented in [11, Sec. 10.4].

In this paper, we consider the following question: given an algebra A over a commutative ring k, when is the monoidal category of A-bimodules braided? Proceeding as in [11, Sec. 10.4], we obtain that braidings on ${}_A\mathcal{M}_A$ are in bijective correspondence to a kind of generalized R-matrices, these are elements in the threefold tensor product $A \otimes A \otimes A$, satisfying a list of technical equations, see Theorem 2.1. This list of equations can be reduced to two equations, a centralizing condition and a normality condition, see Theorem 2.4. Moreover, all braidings on the category of bimodules are symmetries. If an R-matrix exists, then we call (A^e, R) quasitriangular, the terminology will be explained in Remark 2.2.

The invariants functor $G = (-)^A : {}_A\mathcal{M}_A \to \mathcal{M}_k$ has a left adjoint $F = A \otimes -$. We prove that G is a separable functor [12, 13] if and only if G is fully faithful and this property implies quasitriangularity of (A^e, R). Actually, if A is free as a k-module, then these

---

2010 *Mathematics Subject Classification.* 16T10, 16T05, 16S40.

Key words and phrases. epimorphism of rings, braided category, R-matrix, Azumaya algebra, separable functor, quantum Yang-Baxter equation.

A.L. Agore is research fellow “aspirant” of FWO-Vlaanderen; the first and the second author are supported by research project G.0117.10 “Equivariant Brauer groups and Galois deformations” from FWO-Vlaanderen.

properties are equivalent, see Theorem 2.6, and in this case the braiding on the category of A-bimodules is unique. Azumaya algebras were introduced in [1] under the name centrale separable algebras; a more restrictive class was considered earlier by Azumaya in [2]. Azumaya algebras are the proper generalization of central simple algebras to commutative rings. The Brauer group consists of the set of Morita equivalence classes of Azumaya algebras. There exists a large literature on Azumaya algebras and the Brauer group, see for example the reference list in [4]. A is an Azumaya algebra if and only if G is an equivalence of categories, and then G is separable. Therefore the category of bimodules over an Azumaya algebra is braided monoidal; in the case where A is a matrix algebra or a quaternion algebra, we have an explicit formula for the R-matrix, see Example 2.8 and Example 2.9. For A commutative, we have a complete classification: A^e has a quasitriangular structure if and only if the ring morphism $k \to A$ is an epimorphism in the category of rings, see Proposition 2.3. In Theorem 2.12 we show that a quasitriangular structure on A^e gives rise to a new solution of the quantum Yang-Baxter equation.

1. Preliminary results

1.1. **Azumaya algebras.** Let k be a commutative ring and A a k-algebra. Unadorned $\otimes$ means $\otimes_k$. ${}_A\mathcal{M}_A$ is the k-linear category of A-bimodules. It is well-known that we have a pair of adjoint functors (F, G) between the category of k-modules $\mathcal{M}_k$ and the category of A-bimodules ${}_A\mathcal{M}_A$. For a k-module N, $F(N) = A \otimes N$, with A-bimodule structure $a(b \otimes n)c = abc \otimes n$, for all a, b, $c \in A$ and $n \in N$. For an A-bimodule M, $G(M) = M^A = \{m \in M \mid am = ma, \forall a \in A\} \cong {}_A\mathrm{Hom}_A(A, M)$. The unit η and the counit ε of the adjoint pair (F, G) are given by the formulas

$$\begin{array}{rlcr} \eta_N: & N \to (A \otimes N)^A & ; & \eta_N(n) = 1 \otimes n; \\ \varepsilon_M: & A \otimes M^A \to M & ; & \varepsilon_M(a \otimes m) = am = ma \end{array}$$

for all $n \in N$, $a \in A$ and $m \in M^A$. Recall that A is an Azumaya algebra if A is faithfully projective as a k-module, that is, A is finitely generated, projective and faithful, and the algebra map

$$F: \ A^e = A \otimes A^{\mathrm{op}} \to \mathrm{End}_k(A), \ \ F(a \otimes b)(x) = axb \tag{1}$$

is an isomorphism. Azumaya algebras can be characterized in several ways; perhaps the most natural characterization is the following: A is an Azumaya algebra if and only if the adjoint pair (F, G) is a pair of inverse equivalences, see [10, Theorem III.5.1]. Another characterization is that A is central and separable as a k-algebra.

1.2. **Separable functors.** Recall from [12] that a covariant functor $F: \mathcal{C} \to \mathcal{D}$ is called separable if the natural transformation

$$\mathcal{F}: \ \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \ ; \ \mathcal{F}_{C,C'}(f) = F(f)$$

splits, that is, there is a natural transformation

$$\mathcal{P}: \ \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \to \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$$

such that $\mathcal{P}\circ\mathcal{F}$ is the identity natural transformation. Rafael's Theorem [13] states that the left adjoint F in an adjoint pair of functors (F,G) is separable if and only if the unit of the adjunction $\eta : 1_{\mathcal{C}} \to GF$ splits; the right adjoint G is separable if and only if the counit $\varepsilon : FG \to 1_{\mathcal{D}}$ cosplits, that is, there exists a natural transformation $\zeta : 1_{\mathcal{D}} \to FG$ such that $\varepsilon\circ\zeta$ is the identity natural transformation. A detailed study of separable functors can be found in [5].

1.3. **Braided monoidal categories.** A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, called the tensor product, an object $I \in \mathcal{C}$ called the unit object, and natural isomorphisms $a : \otimes\circ(\otimes\times\mathcal{C}) \to \otimes\circ(\mathcal{C}\times\otimes)$ (the associativity constraint), $l : \otimes\circ(I\times\mathcal{C}) \to \mathcal{C}$ (the left unit constraint) and $r : \otimes\circ(\mathcal{C}\times I) \to \mathcal{C}$ (the right unit constraint). a, l and r have to satisfy certain coherence conditions, we refer to [9, XI.2] for a detailed discussion. $\mathcal{C}$ is called strict if a, l and r are the identities on $\mathcal{C}$. McLane's coherence Theorem asserts that every monoidal category is monoidal equivalent to a strict one, see [9, XI.5]. The categories that we will consider are - technically spoken - not strict, but they can be threated as if they were strict.
Let $\tau : \mathcal{C}\times\mathcal{C} \to \mathcal{C}\times\mathcal{C}$ be the flip functor. A prebraiding on $\mathcal{C}$ is a natural transformation $c : \otimes \to \otimes\circ\tau$ satisfying the following equations, for all $U, V, W \in \mathcal{C}$:

$$c_{U,V\otimes W} = (V\otimes c_{U,W})\circ(c_{U,V}\otimes W) \; ; \; c_{U\otimes V,W} = (c_{U,W}\otimes V)\circ(U\otimes c_{V,W}).$$

c is called a braiding if it is a natural isomorphism. c is called a symmetry if $c^{-1}_{U,V} = c_{V,U}$, for all $U, V \in \mathcal{C}$. We refer to [9, XIII.1] for more detail.

2. Braidings on the category of bimodules

Let A be an algebra over a commutative ring k. $A^{(n)}$ will be a shorter notation for the n-fold tensor product $A\otimes\cdots\otimes A$. An element $R \in A^{(3)}$ will be denoted by $R = R^1\otimes R^2\otimes R^3$, where summation is implicitly understood. $({}_A\mathcal{M}_A, -\otimes_A -, A)$ is a monoidal category. Our aim is to investigate braidings on ${}_A\mathcal{M}_A$.

Theorem 2.1. *There is a bijective correspondence between braidings c on ${}_A\mathcal{M}_A$ and invertible elements $R = R^1\otimes R^2\otimes R^3 \in A^{(3)}$ (summation implicitly understood) satisfying the following conditions, for all $a \in A$:*

$$R^1\otimes R^2\otimes aR^3 = R^1a\otimes R^2\otimes R^3 \tag{2}$$

$$aR^1\otimes R^2\otimes R^3 = R^1\otimes R^2a\otimes R^3 \tag{3}$$

$$R^1\otimes aR^2\otimes R^3 = R^1\otimes R^2\otimes R^3a \tag{4}$$

$$R^1\otimes R^2\otimes 1\otimes R^3 = r^1R^1\otimes r^2\otimes r^3R^2\otimes R^3 \tag{5}$$

$$R^1\otimes 1\otimes R^2\otimes R^3 = R^1\otimes R^2r^1\otimes r^2\otimes R^3r^3 \tag{6}$$

where $r = r^1\otimes r^2\otimes r^3 = R$. We call R the R-matrix corresponding to the braiding c, and we say that $(A^e = A\otimes A^{\mathrm{op}}, R)$ is quasitriangular, see Remark 2.2 for an explication of this terminology. In the case where the braiding c is a symmetry, we call $(A^e = A\otimes A^{\mathrm{op}}, R)$ triangular. The braiding c corresponding to R is given by the formula

$$c_{M,N} : M\otimes_A N \to N\otimes_A M, \quad c_{M,N}(m\otimes_A n) = R^1nR^2\otimes_A mR^3. \tag{7}$$

for all M, $M \in {}_A\mathcal{M}_A$, $m \in M$ *and* $n \in N$.

Proof. $A^{(2)}$ is an A-bimodule via the usual actions $a(x \otimes y)b = ax \otimes yb$, for all a, b, x, $y \in A$. Let $c : {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \to {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A$ be a braiding on ${}_A\mathcal{M}_A$. For each $M, N \in {}_A\mathcal{M}_A$, we have an A-bimodule isomorphism $c_{M,N} : M \otimes_A N \to N \otimes_A M$, that is natural in M and N. Now consider

$$c_{A^{(2)},A^{(2)}} : A^{(3)} \cong A^{(2)} \otimes_A A^{(2)} \to A^{(3)} \cong A^{(2)} \otimes_A A^{(2)},$$

and let $R = c_{A^{(2)},A^{(2)}}(1 \otimes 1 \otimes 1)$. c is completely determined by R. For $M, N \in {}_A\mathcal{M}_A$, $m \in M$ and $n \in N$, we consider the A-bimodule maps $f_m : A^{(2)} \to M$ and $g_n : A^{(2)} \to N$ given by the formulas $f_m(a \otimes b) = amb$ and $g_n(a \otimes b) = anb$. From the naturality of c, it follows that the diagram

$$\begin{tikzcd}
A^{(3)} \cong A^{(2)} \otimes_A A^{(2)} \arrow[r, "c_{A^{(2)},A^{(2)}}"] \arrow[d, "f_m \otimes_A g_n"'] & A^{(3)} \cong A^{(2)} \otimes_A A^{(2)} \arrow[d, "g_n \otimes_A f_m"] \\
M \otimes_A N \arrow[r, "c_{M,N}"] & N \otimes_A M
\end{tikzcd}$$

commutes. (7) now follows after we evaluate the diagram at $1 \otimes 1 \otimes 1$.
Obviously $c_{M,N}(ma \otimes_A n) = c_{M,N}(m \otimes_A an)$. Furthermore, $c_{M,N}(am \otimes_A n) = ac_{M,N}(a \otimes_A n)$ and $c_{M,N}(m \otimes_A na) = c_{M,N}(a \otimes_A n)a$ since $c_{M,N}$ is a bimodule map. If we write these three formulas down in the case where $M = N = A^{(2)}$, and $m = n = 1 \otimes 1$, then we obtain (2-4). c satisfies the two triangle equalities

$$\begin{aligned}
c_{M \otimes_A N,P} &= (c_{M,P} \otimes_A N) \circ (M \otimes_A c_{N,P});\\
c_{M,N \otimes_A P} &= (N \otimes_A c_{M,P}) \circ (c_{M,N} \otimes_A P).
\end{aligned}$$

The first equality is equivalent to

$$R^1pR^2 \otimes_A m \otimes_A nR^3 = r^1R^1pR^2r^2 \otimes_A mr^3 \otimes_A nR^3,$$

for all $m \in M$, $n \in N$ and $p \in P$. If we take $M = N = P = A^{(2)}$ and $m = n = p = 1 \otimes 1$, then we find that $R^1 \otimes R^2 \otimes 1 \otimes R^3 = r^1R^1 \otimes R^2r^2 \otimes r^3 \otimes R^3$. Applying (4), we find that (5) holds. In a similar way, the second triangle equality implies (6).
We can apply the same arguments to the inverse braiding c^{-1}. This gives $S = S^1 \otimes S^2 \otimes S^3 = c^{-1}_{A^{(2)},A^{(2)}}(1 \otimes 1 \otimes 1)$. Then we have that

$$m \otimes_A n = (c^{-1}_{N,M} \circ c_{M,N})(m \otimes_A n) = c^{-1}_{N,M}(R^1nR^2 \otimes_A mR^3) = S^1mR^3S^2 \otimes_A R^1nR^2S^3.$$

Now take $m = n = 1 \otimes 1 \in A^{(2)}$. Then we find

$$\begin{aligned}
1 \otimes 1 \otimes 1 &= S^1 \otimes R^3S^2R^1 \otimes R^2S^3 \overset{(3)}{=} R^1S^1 \otimes R^3S^2 \otimes R^2S^3\\
&\overset{(4)}{=} R^1S^1 \otimes S^2 \otimes R^2S^3R^3 \overset{(2)}{=} R^1S^1R^2 \otimes S^2 \otimes S^3R^3 \overset{(3)}{=} S^1R^2 \otimes S^2R^1 \otimes S^3R^3.
\end{aligned}$$

In a similar way, we have that $R^1S^2 \otimes R^2S^1 \otimes R^3S^3 = 1 \otimes 1 \otimes 1$, and it follows that $S^2 \otimes S^1 \otimes S^3$ is the inverse of $R^1 \otimes R^2 \otimes R^3$.
Conversely, assume that $R \in A^{(3)}$ is invertible and satisfies (2-6). Then we define c using (7). Straightforward computations show that c is a braiding on ${}_A\mathcal{M}_A$. $\square$

Remarks 2.2. 1. It is well-known that A^e is an A-bialgebroid, and A-bimodules are precisely the left A^e-modules. The notion of quasitriangular bialgebroid is developed in [7]. Then (A^e, R) is quasitriangular in the sense of Theorem 2.1 if and only if (A^e, R) is a quasitriangular bialgebroid in the sense of [7], justifying our terminology. [7, Prop. 3.13], see also [3, Theorem 3.16], states that the category of modules over a quasitriangular bialgebroid is braided; one implication in Theorem 2.1 follows from this. In order to avoid technicalities from the theory of bialgebroids, we preferred to give a consistent and elementary proof of Theorem 2.1.
2. (5-6) can be rewritten as $R^{124} = R^{123}R^{134}$ and $R^{134} = R^{124}R^{234}$ in the algebra $A^{(4)}$.

Proposition 2.3. *Let A be a k-algebra. Then:*

1. *If a monomial $x \otimes y \otimes z$ is an R-matrix, then it is equal to $1 \otimes 1 \otimes 1$.*
2. *$1 \otimes 1 \otimes 1$ is an R-matrix if and only if $u_A : k \to A$ is an epimorphism of rings.*
3. *If A is commutative, then (A^e, R) is quasitriangular if and only if $R = 1 \otimes 1 \otimes 1$ and $u_A : k \to A$ is an epimorphism in the category of rings.*

Proof. 1. Let $R = x \otimes y \otimes z$ be an R-matrix. From (5-6), it follows that

$$x \otimes 1 \otimes y \otimes z = x \otimes yx \otimes y \otimes z^2 \text{ and } x \otimes y \otimes 1 \otimes z = x^2 \otimes y \otimes zy \otimes z.$$

Since R is invertible, this implies that

$$1 \otimes 1 \otimes 1 \otimes 1 = 1 \otimes yx \otimes 1 \otimes z \text{ and } 1 \otimes 1 \otimes 1 \otimes 1 = x \otimes 1 \otimes zy \otimes 1,$$

and, multiplying tensor factors, we find that $1 \otimes 1 = yx \otimes z$ and $1 \otimes 1 = x \otimes zy$. It then follows that $yxz = xzy = 1$, hence y is invertible with $y^{-1} = xz$. Finally

$$x \otimes y \otimes z = x \otimes y^2 y^{-1} \otimes z \overset{(4)}{=} x \otimes yy^{-1} \otimes zy = 1 \otimes 1 \otimes 1.$$

2. If $R = 1 \otimes 1 \otimes 1$, then the three centralizing conditions (2-4) are equivalent to $a \otimes 1 = 1 \otimes a$, for all $a \in A$, which is equivalent to $u_A : k \to A$ being an epimorphism of rings, see [14].
3. Assume that (A^e, R) is quasitriangular.

$$\begin{aligned} R^1 \otimes R^2 \otimes 1 \otimes R^3 &\overset{(5)}{=} r^1R^1 \otimes r^2 \otimes r^3R^2 \otimes R^3 \\ &\overset{(4)}{=} r^1R^1 \otimes R^2r^2 \otimes r^3 \otimes R^3 = \sum R^1r^1 \otimes R^2r^2 \otimes r^3 \otimes R^3. \end{aligned}$$

At the third step, we used the fact that A is commutative. From the fact that R is invertible, it follows that $R^1 \otimes R^2 \otimes 1 \otimes R^3 = 1 \otimes 1 \otimes 1 \otimes 1$ and $R = 1 \otimes 1 \otimes 1$. The rest of the proof follows from 2. □

Let c be a braiding on ${}_A\mathcal{M}_A$, and R the corresponding R-matrix. Then (A^e, R) is triangular, that is, c is a symmetry, if and only if $S = R$, this means that

$$R^2r^1 \otimes R^1r^2 \otimes R^3r^3 = 1 \otimes 1 \otimes 1, \tag{8}$$

that is, $R^{-1} = R^2 \otimes R^1 \otimes R^3$.

Theorem 2.4. *Let A be a k-algebra. Every braiding on ${}_A\mathcal{M}_A$ is a symmetry. There is a bijective correspondence between the symmetries on ${}_A\mathcal{M}_A$, and elements $R \in A^{(3)}$ satisfying (4) and the normalizing condition*

$$R^1R^2 \otimes R^3 = R^2 \otimes R^3R^1 = 1 \otimes 1. \tag{9}$$

In this situation, R is invariant under cyclic permutation of the tensor factors,

$$R = R^2 \otimes R^3 \otimes R^1 = R^3 \otimes R^1 \otimes R^2, \tag{10}$$

and we have the additional normalizing condition

$$R^1 \otimes R^2R^3 = 1 \otimes 1. \tag{11}$$

Proof. For a braiding c consider the corresponding R-matrix R as in Theorem 2.1. We have seen in Theorem 2.1 that R is invertible and satisfies (2-6). Multiplying the second and the third tensor factor in (6), we find that $R = R^1 \otimes R^2r^1r^2 \otimes R^3r^3 = R(1 \otimes r^1r^2 \otimes r^3)$. From the fact that R is invertible, it follows that $1 \otimes 1 \otimes 1 = 1 \otimes r^1r^2 \otimes r^3$, and the first normalizing condition of (9) follows after we multiply the first two tensor factors. On the other hand, if we apply the flip map on the last two positions in (5) we obtain that $R^1 \otimes R^2 \otimes R^3 \otimes 1 = r^1R^1 \otimes r^2 \otimes R^3 \otimes r^3R^2$. Multiplying the last two position we obtain:

$$R = r^1R^1 \otimes r^2 \otimes R^3r^3R^2 \overset{(2)}{=} r^1R^3R^1 \otimes r^2 \otimes r^3R^2 = R(R^3R^1 \otimes 1 \otimes R^2)$$

As R is invertible it follows that $R^3R^1 \otimes R^2 = 1 \otimes 1$.

Conversely, assume that R satisfies (4) and (9). We will show that R is an R-matrix satisfying (8). Applying Theorem 2.1 we have a braiding c corresponding to R, and it follows from the observations preceding Theorem 2.4 that it is a symmetry. First we show that R is invariant under cyclic permutation of the tensor factors.

$$\begin{aligned} R^3 \otimes R^1 \otimes R^2 &\overset{(9)}{=} R^3r^1r^2 \otimes r^3R^1 \otimes R^2 \overset{(4)}{=} R^3r^2 \otimes r^3R^1 \otimes r^1R^2 \\ &\overset{(4)}{=} r^2 \otimes r^3R^3R^1 \otimes r^1R^2 \overset{(9)}{=} r^2 \otimes r^3 \otimes r^1. \end{aligned}$$

This implies immediately that the central conditions (2-3) are also satisfied. Next we show that (5-6) are satisfied.

$$\begin{aligned} r^1R^1 \otimes r^2 \otimes r^3R^2 \otimes R^3 &\overset{(3)}{=} R^1 \otimes r^2 \otimes r^3R^2r^1 \otimes R^3 \\ &\overset{(4)}{=} R^1 \otimes R^2r^2 \otimes r^3r^1 \otimes R^3 \overset{(9)}{=} R^1 \otimes R^2 \otimes 1 \otimes R^3; \\ R^1 \otimes R^2r^1 \otimes r^2 \otimes R^3r^3 &\overset{(4)}{=} R^1 \otimes r^3R^2r^1 \otimes r^2 \otimes R^3 \\ &\overset{(4)}{=} R^1 \otimes r^3r^1 \otimes R^2r^2 \otimes R^3 \overset{(9)}{=} R^1 \otimes 1 \otimes R^2 \otimes R^3. \end{aligned}$$

Finally, we prove that (8) holds

$$\begin{aligned} R^1r^2 \otimes R^2r^1 \otimes R^3r^3 &\overset{(3)}{=} R^1r^2R^2 \otimes r^1 \otimes R^3r^3 \overset{(4)}{=} R^1R^2 \otimes r^1 \otimes R^3r^2r^3 \\ &\overset{(9)}{=} 1 \otimes r^1 \otimes r^2r^3 \overset{(10)}{=} 1 \otimes r^3 \otimes r^1r^2 \overset{(9)}{=} 1 \otimes 1 \otimes 1. \end{aligned}$$

□

Before we state our next main result Theorem 2.6, we need a technical Lemma. If $M \in {}_A\mathcal{M}_A$, then $A \otimes M$ is a $k \otimes A$-bimodule, and we can consider

$$(A \otimes M)^{k \otimes A} = \{\sum_i a_i \otimes m_i \in A \otimes M \mid \sum_i a_i \otimes am_i = \sum_i a_i \otimes m_ia, \text{ for all } a \in A\}.$$

If $M = A^{(2)}$, then $(A \otimes A^{(2)})^{k \otimes A}$ is the set of elements $R \in A^{(3)}$ satisfying (4). We have a map $\alpha_M : A \otimes M^A \to (A \otimes M)^A$, $\alpha_M(a \otimes m) = a \otimes m$.

Lemma 2.5. *Let M be an A-bimodule. The map α_M is injective if A is flat as a k-module, and bijective if A is free as a k-module.*

Proof. If A is flat, then $A\otimes M^A\to A\otimes M$ is injective, and then α_M is also injective. Assume that A is free as a k-module, and let $\{e_j\mid j\in I\}$ be a free basis of A. Assume that $x=\sum_i a_i\otimes m_i\in (A\otimes M)^{k\otimes A}$. For all i, we can write $a_i=\sum_{j\in I}\alpha_i^j e_j$, for some $\alpha_i^j\in k$. Then $x=\sum_{j\in I}e_j\otimes\big(\sum_i\alpha_i^j m_i\big)$. Now

$$x=\sum_{j\in I}e_j\otimes\big(\sum_i\alpha_i^j a m_i\big)=\sum_i a_i\otimes a m_i=\sum_i a_i\otimes m_i a=\sum_{j\in I}e_j\otimes\big(\sum_i\alpha_i^j m_i a\big),$$

hence $\sum_i\alpha_i^j a m_i=\sum_i\alpha_i^j m_i a$, for all $j\in I$, and $\sum_i\alpha_i^j m_i\in M^A$. We conclude that $x=\sum_{j\in I}e_j\otimes\big(\sum_i\alpha_i^j m_i\big)\in\operatorname{Im}\alpha_M$, and this shows that α_M is surjective. □

Theorem 2.6. *Let A be a k-algebra A, and consider the conditions:*

1. *(F,G) is a pair of inverse equivalences, that is, A is an Azumaya algebra;*
2. *The functor $G=(-)^A:\ {}_A\mathcal{M}_A\to\mathcal{M}_k$ is fully faithful;*
3. *the functor $G=(-)^A:\ {}_A\mathcal{M}_A\to\mathcal{M}_k$ is separable;*
4. *there exists $R=R^1\otimes R^2\otimes R^3\in A\otimes(A\otimes A)^A$ such that $R^1R^2\otimes R^3=1\otimes 1$;*
5. *there exists a unique $R=R^1\otimes R^2\otimes R^3\in A\otimes(A\otimes A)^A$ such that $R^1R^2\otimes R^3=1\otimes 1$;*
6. *there exists a braiding on ${}_A\mathcal{M}_A$, that is, there exists $R\in A^{(3)}$ such that (A^e,R) is quasitriangular.*

Then $(1)\Rightarrow(2)\Leftrightarrow(3)\Leftrightarrow(4)\Leftrightarrow(5)\Rightarrow(6)$. If A is central, then $(2)\Rightarrow(1)$. If A is free as a k-module, then $(6)\Rightarrow(5)$, and in this case the braiding on ${}_A\mathcal{M}_A$ is unique. If k is a field, and A is finite dimensional, then $(6)\Rightarrow(1)$, and all six assertions are equivalent.

Proof. $\underline{(1)\Rightarrow(2)}$, $(2)\Rightarrow(3)$ and $(5)\Rightarrow(4)$ are trivial.
$\underline{(3)\Rightarrow(4)}$. If G is separable, then we have a natural transformation $\zeta:\ 1\Rightarrow FG$ such that $\varepsilon_M\circ\zeta_M=M$, for all $M\in{}_A\mathcal{M}_A$. Now let $R=\zeta_{A^{(2)}}(1\otimes 1)=R^1\otimes R^2\otimes R^3\in FG(A^{(2)})=A\otimes(A\otimes A)^A$. Then $1\otimes 1=(\varepsilon_{A^{(2)}}\circ\zeta_{A^{(2)}})(1\otimes 1)=R^1R^2\otimes R^3$.
The natural transformation ζ is completely determined by R. For an A-bimodule M and $m\in M$, we define f_m as in the proof of Theorem 2.1. From the naturality of ζ, it follows that the diagram

$$\begin{tikzcd} A^{(2)} \arrow[r, "\zeta_{A^{(2)}}"] \arrow[d, "f_m"'] & A\otimes(A\otimes A)^A \arrow[d, "A\otimes(f_m)^A"] \\ M \arrow[r, "\zeta_M"] & A\otimes M^A \end{tikzcd}$$

commutes. Evaluating the diagram at $1\otimes 1$, we find that

$$\zeta_M(m)=R^1\otimes R^2 m R^3. \tag{12}$$

$\underline{(4)\Rightarrow(6)}$. Write $R=\sum_i a_i\otimes b_i$, with $a_i\in A$ and $b_i\in(A\otimes A)^A$. Then $R^2\otimes R^3R^1=\sum_i b_i a_i=\sum_i a_i b_i=R^1R^2\otimes R^3=1\otimes 1$, hence $\alpha_{A^{(2)}}(R)\in(A\otimes A^{(2)})^{k\otimes A}$ satisfies (4), and it follows from Theorem 2.4 that $\alpha_{A^{(2)}}(R)$ determines a braiding on ${}_A\mathcal{M}_A$. It also follows from Theorem 2.4 that (3) and (11) are satisfied.
$\underline{(4)\Rightarrow(2)}$. Given $R\in A\otimes(A\otimes A)^A$ satisfying $R^1R^2\otimes R^3=1\otimes 1$, we define ζ using (12). It follows immediately that $(\varepsilon_M\circ\zeta_M)(m)=\varepsilon(R^1\otimes R^2mR^3)=R^1R^2mR^3=m$.

We have seen in the proof of 4) ⇒ 6) that (3) and (11) are satisfied. For $a_i \in A$ and $m_i \in M^A$, we then compute

$$\begin{aligned}(\zeta_M \circ \varepsilon_M)(\sum_i a_i \otimes m_i) &= \sum_i R^1 \otimes R^2 a_i m_i R^3 \overset{(3)}{=} \sum_i a_i R^1 \otimes R^2 m_i R^3\\ &= \sum_i a_i R^1 \otimes R^2 R^3 m_i \overset{(11)}{=} \sum_i a_i \otimes m_i.\end{aligned}$$

This shows that ε is a natural transformation with inverse ζ, and G is fully faithful.
(2) ⇒ (5). We have already seen that 2) implies 4), and this shows that R exists. If G is fully faithful, then ε_M is invertible, for all $M \in {}_A\mathcal{M}_A$. If $R \in A \otimes (A \otimes A)^A$ satisfies $R^1R^2 \otimes R^3 = 1 \otimes 1$, then $\varepsilon_{A\otimes A}(R) = 1 \otimes 1$, hence $R = \varepsilon^{-1}_{A\otimes A}(1 \otimes 1)$.
(6) ⇒ (4). From (5), it follows that there exists $R \in (A \otimes A^{(2)})^{k\otimes A}$ such that $R^1R^2 \otimes R^3 = 1 \otimes 1$, see Theorem 2.4. $\alpha_{A^{(2)}}$ is bijective, see Lemma 2.5, hence $\alpha^{-1}_{A^{(2)}}(R) \in A \otimes (A \otimes A)^A$ satisfies (3). The uniqueness of R follows from (4).
(4) ⇒ (1). Assume that A is central. From (4), it follows that $\varepsilon_{A\otimes A} : A \otimes (A \otimes A)^A \to A \otimes A$ is surjective, and then it follows from [1, Theorem 3.1] that A is separable over $Z(A) = k$. Thus A is central separable, and therefore Azumaya.
(6) ⇒ (1). If k is a field, then A is free, so (6) implies (5), and, a fortiori, (2). Then $\varepsilon_A : A \otimes A^A \to A$ is an isomorphism of A-bimodules, and therefore also of vector spaces. A count of dimensions shows that $\dim_k(Z(A)) = \dim_k(A^A) = 1$, so that $Z(A) = k1_A$, and A is central, and then (1) follows from (2). □

For any k-algebra A, the functor $F : \mathcal{M}_k \to {}_A\mathcal{M}_A$ is strong monoidal. Indeed, for any $N, N' \in \mathcal{M}_k$, we have natural isomorphisms $\varphi_0 : F(k) = A \otimes k \to A$ and

$$\varphi_{N,N'} : F(N) \otimes_A F(N') = (A \otimes N) \otimes_A (A \otimes N') \to F(N \otimes N') = A \otimes N \otimes N'$$

satisfying all the necessary axioms, see [9].

Proposition 2.7. *If $G = (-)^A : {}_A\mathcal{M}_A \to \mathcal{M}_k$ is separable, then the symmetry on ${}_A\mathcal{M}_A$ is such that F preserves the symmetry.*

Proof. We have to show that the following diagram commutes

$$\begin{tikzcd}
(A \otimes N) \otimes_A (A \otimes N') \arrow[r, "\varphi_{N,N'}"] \arrow[d, "c_{A\otimes N, A\otimes N'}"'] & A \otimes N \otimes N' \arrow[d, "A\otimes\tau_{N,N'}"] \\
(A \otimes N') \otimes_A (A \otimes N) \arrow[r, "\varphi_{N',N}"] & A \otimes N' \otimes N
\end{tikzcd}$$

Here $\tau_{N,N'} : N \otimes N' \to N' \otimes N$ is the usual switch map. For $a, b \in A$, $n \in N$ and $n' \in N$, we compute

$$\begin{aligned}(\varphi_{N',N} \circ c_{A\otimes N, A\otimes N'})((a \otimes n) \otimes_A (b \otimes n')) &\overset{(7)}{=} \varphi_{N',N}\big((R^1bR^2 \otimes n') \otimes_A aR^3 \otimes n\big)\\ &= R^1bR^2aR^3 \otimes n' \otimes n \overset{(2)}{=} R^1R^2abR^3 \otimes n' \otimes n\\ &\overset{(9)}{=} ab \otimes n' \otimes n = ab \otimes \tau_{N,N'}(n \otimes n')\\ &= ((A \otimes \tau_{N,N'}) \circ \varphi_{N,N'})((a \otimes n) \otimes_A (b \otimes n')).\end{aligned}$$

□

If A is an Azumaya algebra, then it follows from Theorem 2.6 that we have a symmetry on the category of A-bimodules ${}_A\mathcal{M}_A$. In Examples 2.8 and 2.9, we give explicit formulas for R in the case where A is a matrix ring or a quaternion algebra; in both cases A is free, so that the R-matrix is unique.

Example 2.8. Let $A = M_n(k)$ be a matrix algebra. Then

$$R = \sum_{i,j,k=1}^{n} e_{ij} \otimes e_{ki} \otimes e_{jk},$$

where e_{ij} is the elementary matrix with 1 in the (i, j)-position and 0 elsewhere. Indeed, For all indices i, j, p, q, we have

$$e_{pq}(\sum_{k=1}^{n} e_{ki} \otimes e_{jk}) = e_{pi} \otimes e_{jq} = (\sum_{k=1}^{n} e_{ki} \otimes e_{jk})e_{pq},$$

hence $\sum_{k=1}^{n} e_{ki} \otimes e_{jk} \in (A \otimes A)^A$ and $R = \sum_{i,j=1}^{n} e_{ij} \otimes (\sum_{k=1}^{n} e_{ki} \otimes e_{jk}) \in A \otimes (A \otimes A)^A$. Finally

$$\sum_{i,j,k=1}^{n} e_{ij}e_{ki} \otimes e_{jk} = \sum_{i,j=1^n} e_{ii} \otimes e_{jj} = 1 \otimes 1.$$

Example 2.9. Let K be a commutative ring, such that 2 is invertible in K, and take two invertible elements $a, b \in K$. The generalized quaternion algebra $A = {}^aK^b$ is the free K-module with basis $\{1, i, j, k\}$ and multiplication defined by

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji = k.$$

It is well-known that A is an Azumaya algebra. The corresponding R-matrix is

$$\begin{aligned} R = {} & \frac{1}{4}(1 \otimes 1 \otimes 1) + \frac{1}{4a}(1 \otimes i \otimes i + i \otimes 1 \otimes i + i \otimes i \otimes 1) \\ + {} & \frac{1}{4b}(1 \otimes j \otimes j + j \otimes 1 \otimes j + j \otimes j \otimes 1) - \frac{1}{4ab}(1 \otimes k \otimes k + k \otimes 1 \otimes k + k \otimes k \otimes 1) \\ + {} & \frac{1}{4ab}(i \otimes j \otimes k + j \otimes k \otimes i + k \otimes i \otimes j) - \frac{1}{4ab}(j \otimes i \otimes k + k \otimes j \otimes i + i \otimes k \otimes j). \end{aligned}$$

It is easy to show that R satisfies (4) and (9). Indeed,

$$\begin{aligned} R^1R^2 \otimes R^3 = {} & \frac{1}{4}(1 \otimes 1) + \frac{1}{4a}(i \otimes i + i \otimes i + a \otimes 1) \\ + {} & \frac{1}{4b}(j \otimes j + j \otimes j + b \otimes 1) - \frac{1}{4ab}(k \otimes k + k \otimes k - ab \otimes 1) \\ + {} & \frac{1}{4ab}(k \otimes k - bi \otimes i - aj \otimes j) + \frac{1}{4ab}(k \otimes k - bi \otimes i - aj \otimes j) = 1 \otimes 1, \end{aligned}$$

proving the first normalization from (9); the second one follows in a similar manner. An elementary computation shows that $R^1 \otimes xR^2 \otimes R^3 = R^1 \otimes R^2 \otimes R^3x$, for $x = i, j, k$, and this proves (4).

Example 2.10. If A is an Azumaya algebra, then there exists an R such that (A^e, R) is quasitriangular. The converse is not true, it suffices to consider $\mathbb{Q}$ as a $\mathbb{Z}$-algebra. Since $\mathbb{Z} \subset \mathbb{Q}$ is an epimorphism of rings, it follows from Proposition 2.3 that $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}, 1 \otimes 1 \otimes 1)$ is quasitriangular; it is obvious that $\mathbb{Q}$ is not a $\mathbb{Z}$-Azumaya algebra.

Example 2.11. Let A and B be k-algebras, and assume that (A^e, R) and (B^e, S) are quasitriangular. It is straightforward to show that $((A \otimes B)^e, T)$, with $T = R^1 \otimes S^1 \otimes R^2 \otimes S^2 \otimes R^3 \otimes S^3 \in (A \otimes B)^{(3)}$, is quasitriangular.

We end this paper with a new class of solutions of the quantum Yang-Baxter equation, coming from R-matrices on algebras.

Theorem 2.12. *Let (A^e, R) be quasitriangular $V \in {}_A\mathcal{M}_A$. Then the map*

$$\Omega : V \otimes V \to V \otimes V, \qquad \Omega(v \otimes w) = R^1\, w R^2 \otimes R^3\, v$$

is a solution of the quantum Yang-Baxter equation $\Omega^{12}\,\Omega^{13}\,\Omega^{23} = \Omega^{23}\,\Omega^{13}\,\Omega^{12}$ in $\mathrm{End}(V^{(3)})$.

Proof. For all v, w, $t \in V$ we have:

$$\begin{aligned}
\Omega^{12}\,\Omega^{13}\,\Omega^{23}(v \otimes w \otimes t) &= \Omega^{12}\,\Omega^{13}(v \otimes R^1\, t\, R^2 \otimes R^3\, w)\\
&= \Omega^{12}(r^1\, R^3\, w\, r^2 \otimes R^1\, t\, R^2 \otimes r^3\, v) = S^1\, R^1\, t\, R^2\, S^2 \otimes S^3\, r^1\, R^3\, w\, r^2 \otimes r^3\, v\\
&\overset{(3)}{=} R^1\, t\, R^2\, S^1\, S^2 \otimes S^3\, r^1\, R^3\, w\, r^2 \otimes r^3\, v \overset{(9)}{=} R^1\, t\, R^2 \otimes r^1\, R^3\, w\, r^2 \otimes r^3\, v\\
&\overset{(2)}{=} R^1\, t\, R^2 \otimes r^1\, w\, r^2 \otimes R^3\, r^3\, v;\\
\Omega^{23}\,\Omega^{13}\,\Omega^{12}(v \otimes w \otimes t) &= \Omega^{23}\,\Omega^{13}(R^1\, w\, R^2 \otimes R^3\, v \otimes t)\\
&= \Omega^{23}(r^1\, t\, r^2 \otimes R^3\, v \otimes r^3\, R^1\, w\, R)^2) = r^1\, t\, r^2 \otimes S^1\, r^3\, R^1\, w\, R^2\, S^2 \otimes S^3\, R^3\, v\\
&\overset{(2)}{=} r^1\, t\, r^2 \otimes S^1\, R^1\, w\, R^2\, S^2 \otimes r^3\, S^3\, R^3\, v \overset{(3)}{=} r^1\, t\, r^2 \otimes R^1\, w\, R^2\, S^1\, S^2 \otimes r^3\, S^3\, R^3\, v\\
&\overset{(9)}{=} r^1\, t\, r^2 \otimes R^1\, w\, R^2 \otimes r^3\, R^3\, v,
\end{aligned}$$

where $R = r = S$. It follows that $\Omega^{12}\,\Omega^{13}\,\Omega^{23} = \Omega^{23}\,\Omega^{13}\,\Omega^{12}$. □

References

[1] M. Auslander, O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1960), 367–409.

[2] G. Azumaya, On maximally central algebras, *Nagoya Math. J.* **2** (1951), 119–150.

[3] G. Böhm, Hopf algebroids, *Handbook Algebra* **6** (2009), 173–235.

[4] S. Caenepeel, Brauer groups, Hopf algebras and Galois theory, *K-Monographs in Mathematics* **4**, Kluwer Academic Publishers, Dordrecht, 1998.

[5] S. Caenepeel, G. Militaru and S. Zhu, Frobenius and separable functors for generalized module categories and nonlinear equations, *Lecture Notes in Math.* **1787**, Springer Verlag, Berlin, 2002.

[6] M. Cipolla, Discesa fedelmente piatta dei moduli, *Rend. Circ. Mat. Palermo (2)* **25** (1976), 43–46.

[7] J. Donin, A. Mudrov, Quantum groupoids and dynamical categories, *J. Algebra* **296** (2006), 348–384.

[8] V. Drinfeld, Quantum groups, *Proc. Int. Cong. Math. Berkeley* **1** (1986), 789–820.

[9] C. Kassel, "Quantum Groups", *Graduate Texts Math.* **155**, Springer Verlag, Berlin, 1995.

[10] M. A. Knus and M. Ojanguren, "Théorie de la descente et algèbres d'Azumaya", *Lecture Notes in Math.* **389**, Springer Verlag, Berlin, 1974.

[11] S. Montgomery, "Hopf algebras and their actions on rings", American Mathematical Society, Providence, 1993.

[12] C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen, Separable functors applied to graded rings, *J. Algebra* **123** (1989), 397–413.
[13] M. D. Rafael, Separable functors revisited, *Comm. in Algebra* **18** (1990), 1445–1459.
[14] L. Silver, Noncommutative localization and applications, *J. Algebra* **7** (1967), 44–76.

Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

E-mail address: `ana.agore@vub.ac.be and ana.agore@gmail.com`

Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

E-mail address: `scaenepe@vub.ac.be`

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

E-mail address: `gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com`