NONCOMMUTATIVE COHERENT STATES FOR QUANTUM GROUPS

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Abstract

Generalized coherent states (GCS) of Perelomov type are extended to quantum groups by making use of universal $\mathcal{T}$-matrix. Taking $SU_q(2)$ and $SU_q(1,1)$ as examples, properties of the GCS are studied in some detail. Well-known properties of GCS for corresponding Lie groups are lifted up to the quantum group setting. It is also shown that the GCS's naturally provide complex description of noncommutative homogeneous spaces of quantum groups.

Introduction

Quantum groups are highly nontrivial examples of noncommutative and noncocommutative Hopf algebra. Their importance has been recognized in various fields in both mathematics and physics such as integrable models, knot theory, basic hypergeometric functions, noncommutative geometry and so on. From the viewpoint of noncommutative geometry, quantum groups are regarded as a noncommutative extension of Lie groups, that is, they may be regarded as transformation matrices in noncommutative space [13]. This enables us to develop theories of physics on noncommutative space in a way similar to commutative counterparts. Many works have been done along this line inspired by the pioneering works of Manin [13], Woronowicz [24],Wess and Zumino [23].
One of the quantum mechanical objects having group theoretical nature is the generalized coherent state (GCS) introduced in [18, 4]. It is quite useful object in analysis of wide variety of quantum systems ranging from simple system like time dependent oscillator to quantum filed theory [19, 10, 27]. Furthermore, the GCS has nice mathematical features. It provides Kähler potential for the manifold on which the GCS is defined. Representations of Lie groups are constructed by using the GCS. Based on the aforementioned observations, one may anticipate that quantum group extension of GCS would be useful mathematical tool for quantum physics and open a way to noncommutative Kähler geometry. As a first step, we study GCS for simple quantum groups and associated noncommutative spaces in the present work. Among numerous publications on "coherent states" for quantum group, the work by Jurčo and Šťovíček is the only one adequate for our purpose [9], since full Hopf algebra structure of the quantum groups and their dual is reflected in their definition of coherent states. In [9], a general theory for quantum group coherent states is provided, however, no examples are discussed. We here

study explicit cases for simple quantum groups $SU_q(2)$ and $SU_q(1,1)$ by adopting a slightly different approach to the quantum group coherent state. In our approach, quantum analogue of the relation between Lie group and Lie algebra is fully used. In order to illustrate our approach, we review the GCS for Lie group $SU(2)$ briefly in the next section. Our setting of $SU_q(2)$ and $SU_q(1,1)$ is summarized in §2 as well as the universal $\mathcal{T}$-matrix that is a key ingredient of our construction of the GCS. The GCS for $SU_q(2)$ is introduced and its properties are investigated in §3 and §4. We present our main results without proof. Derivation or proof of the results are found in [3]. We discuss the GCS for $SU_q(1,1)$ in §5. §6 is concluding remarks.

1 Generalized coherent state for $SU(2)$

The GCS can be defined for arbitrary Lie groups [19, 27]. The definition is based on the observation that the boson coherent state $|z\rangle$ is obtained by application of an element of the Heisenberg group on the vacuum state:

$$|z\rangle = \exp(za^\dagger - \bar{z}a)\,|0\rangle\,, \quad z \in \mathbb{C} \tag{1.1}$$

where $a^\dagger, a$ are boson creation and annihilation operators. The exponential function in (1.1) gives an element of Heisenberg group. For any Lie groups, one can adopt this construction as a definition of GCS, that is, the GCS is a state obtained by applying group elements on a certain state belonging to the representation space of the group. For instance, coherent state for $SU(2)$ is defined by angular momentum operators as follows:

$$|\xi\rangle = \exp(\xi J_+ - \bar{\xi} J_-)\,|j\ -j\rangle\,, \quad \xi \in \mathbb{C} \tag{1.2}$$

We do not distinguish two GCS differing from each other only by a phase factor. Therefore the GCS (1.2) is regarded as a function on 2-sphere, *i.e.,* homogeneous space $SU(2)/U(1) \approx S^2$. Indeed, the expectation values of angular momentum operators with respect to the GCS give a complex description of 2-sphere:

$$\langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2 = j^2. \tag{1.3}$$

Geometry of the homogeneous spaces is fully described by the GCS [17]. Let $z_i, \bar{z}_i$ be complex coordinates of the homogeneous space and $ds^2 = \omega_{ij}\, dz^i d\bar{z}^j$ be a metric on it. Then the invariant 2-form ω is determined by the GCS:

$$\omega = \omega_{ij} dz_i \wedge d\bar{z}_j = \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j, \qquad F = -\ln |N|^2, \tag{1.4}$$

where N is given by $\langle j\ -j\,|\,\xi\rangle$ for $SU(2)$. Furthermore, with this invariant 2-form ω one can define Poisson bracket on the homogeneous space by

$$\{f, g\} = i\omega^{ij} \left(\frac{\partial f}{\partial z^i} \frac{\partial g}{\partial \bar{z}^j} - \frac{\partial f}{\partial \bar{z}^i} \frac{\partial g}{\partial z^j} \right). \tag{1.5}$$

In this way, GCS provides a natural framework for studying geometry of homogeneous spaces of Lie groups and physics in such spaces.
Another important feature of the GCS is resolution of unity:

$$\int |\xi\rangle\, \langle\xi|\, d\mu = 1, \tag{1.6}$$

where $d\mu$ is a invariant measure on the homogeneous space. When applications of GCS to quantum systems are concerned, this property plays the most fundamental role.

2 Universal $\mathcal{T}$-matrix and real form for quantum $SL(2,\mathbb{C})$

The well-known quantum algebra $\mathcal{U} = U_q[sl(2,\mathbb{C})]$ endowed with a Hopf $*$-structure is generated by three elements $J_\pm,\ J_0$ subject to the relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_0], \tag{2.1}$$

where the q-number is defined as usual:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{2.2}$$

The coproduct Δ, the counit ε and the antipode S maps are given below

$$\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \qquad \Delta(J_\pm) = J_\pm \otimes q^{J_0} + q^{-J_0} \otimes J_\pm, \tag{2.3}$$

$$\varepsilon(J_0) = \varepsilon(J_\pm) = 0, \tag{2.4}$$

$$S(J_0) = -J_0, \qquad S(J_\pm) = -q^{\pm 1} J_\pm. \tag{2.5}$$

A set of monomials $E_{k\ell m} = J_+^k J_0^\ell J_-^m$, $(k,\ell,m \in \mathbb{Z}_{\geq 0})$ provide the basis of the universal enveloping algebra $\mathcal{U}$.
The algebra dual to $\mathcal{U}$ is the quantum group $\mathcal{A} = SL_q(2,\mathbb{C})$. One can determine the basis elements of the algebra $\mathcal{A}$ and their Hopf structure *à la* Fronsdal and Galindo [8] where they started from two parameter deformation of $SL(2)$, and reached its dual algebra. Reversing their construction, we derive the basis vectors $e^{k\ell m}$ of the quantum group $\mathcal{A}$ by employing their dual relations with the known basis set $E_{k\ell m}$ of $\mathcal{U}$:

$$\langle\, e^{k\ell m},\, E_{k'\ell' m'} \,\rangle = \delta^k_{k'}\, \delta^\ell_{\ell'}\, \delta^m_{m'}. \tag{2.6}$$

Denoting the generators of $\mathcal{A}$ as

$$e^{100} = x, \qquad e^{010} = z, \qquad e^{001} = y,$$

we determine the algebraic relations among the generators:

$$[x,y] = 0, \qquad [x,z] = 2\ln q\, x, \qquad [y,z] = 2\ln q\, y. \tag{2.7}$$

These are one parameter reduction of the relations in [8]. The full set of the dual basis $e^{k\ell m}$ are derived inductively:

$$e^{k\ell m} = \frac{x^k}{[k]!} \frac{(z-(k-m)\ln q)^\ell}{\ell!} \frac{y^m}{[m]!}. \tag{2.8}$$

It follows that the universal $\mathcal{T}$-matrix, defined by $\mathcal{T} = \sum_{k\ell m} e^{k\ell m} \otimes E_{k\ell m}$, has the following closed form:

$$\mathcal{T} = \left(\sum_{k=0}^{\infty} \frac{(x \otimes J_+ q^{-J_0})^k}{(k)_{q^{-2}}!} \right) e^{z \otimes J_0} \left(\sum_{m=0}^{\infty} \frac{(y \otimes q^{J_0} J_-)^m}{(m)_{q^2}!} \right), \tag{2.9}$$

where

$$(n)_q = \frac{1-q^n}{1-q}. \tag{2.10}$$

We emphasize that the classical $q \to 1$ limit of the universal $\mathcal{T}$-matrix (2.9) yields the usual exponential mapping from the Lie algebra $sl(2,\mathbb{C})$ to the Lie group $SL(2,\mathbb{C})$. The universal $\mathcal{T}$-matrix (2.9) reproduces the standard matrix expression of $\mathcal{A}$ at the fundamental representation (π) of $\mathcal{U}$:

$$\pi(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \pi(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \pi(J_0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{2.11}$$

Denoting the 2×2 matrix by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv (id \otimes \pi)(\mathcal{T}) = \begin{pmatrix} e^{z/2} + xe^{-z/2}y & q^{1/2}xe^{-z/2} \\ q^{-1/2}e^{-z/2}y & e^{-z/2} \end{pmatrix}, \tag{2.12}$$

one verifies that the elements a, b, c, d satisfy the familiar defining relations of the quantum group $\mathcal{A}$.

We now turn to real form of $\mathcal{U}$ and $\mathcal{A}$. In the following part of this paper we assume that $-1 < q < 1$ $(q \neq 0)$. Real forms of $\mathcal{U}$ are defined by the following $*$-involution on the generators of $\mathcal{U}$:

$$J_+^* = \pm J_-, \qquad J_-^* = \pm J_+, \qquad J_0^* = J_0 \tag{2.13}$$

Plus (minus) sign defines the real form $U_q[su(2)]$ $(U_q[su(1,1)])$ [14]. The $*$-involution on the elements of $\mathcal{A}$

$$a^* = d, \qquad b^* = \mp q^{-1}c, \qquad c^* = \mp qb, \qquad d^* = a, \tag{2.14}$$

defines the quantum groups $SU_q(2)$ and $SU_q(1,1)$[24, 14]. The $*$-involution on the generators of $\mathcal{A}$ can be read off from (2.12) and (2.14):

$$x^* = \mp q^{-1/2}ca^{-1}, \qquad e^{z^*/2} = a^{-1}, \qquad y^* = \mp q^{1/2}a^{-1}b, \tag{2.15}$$

and it is found to maintain a close kinship to the antipode:

$$S(x) = \pm y^*, \qquad S(y) = \pm x^*, \qquad S(z) = z^*. \tag{2.16}$$

Introducing the element

$$\zeta = -qxe^{-z}y = -qbc, \tag{2.17}$$

the $*$-map (2.15) is rewritten as

$$x^* = \mp\frac{1}{1-\zeta}ye^{-z}, \qquad e^{z^*/2} = \frac{1}{1-\zeta}e^{-z/2}, \qquad y^* = \mp\frac{1}{1-\zeta}e^{-z}x. \tag{2.18}$$

One immediately notices that the element ζ is real: $\zeta^* = \zeta$. It is observed that x, y, z and their $*$-involutions do not commute. For instance, x and x^* satisfy the relations:

$$xx^* = \frac{q^{-2}x^*x}{1 \pm (q-q^{-1})x^*x}, \qquad x^*x = \frac{q^2xx^*}{1 \mp q^2(q-q^{-1})xx^*}. \tag{2.19}$$

The coproduct of $\mathcal{A}$ can be regarded as a left/right coaction of $\mathcal{A}$ on $\mathcal{A}$ itself. This leads a left and a right coaction of $\mathcal{A}$ on x and y, respectively. Denoting these coactions by $\varphi(x)$, $\varphi(y)$ we have

$$\begin{aligned} \varphi(x) &= q^{-1/2}\Delta(bd^{-1}) \\ &= (a \otimes x + b \otimes q^{-1/2})(q^{1/2}c \otimes x + d \otimes 1)^{-1}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \varphi(y) &= q^{1/2}\Delta(d^{-1}c) \\ &= (q^{-1/2}y \otimes b + 1 \otimes d)^{-1}(y \otimes a + q^{1/2} \otimes c). \end{aligned} \tag{2.21}$$

It is easy to verify that (2.19) is covariant under the coaction (2.20).

3 Coherent states for quantum $SU(2)$

The universal $\mathcal{T}$-matrix (2.9) was constructed as a quantum analogue of an exponential mapping. The GCS for $SU_q(2)$ is defined as a quantum analogue of the construction of § 1. Namely, we consider a representation of $U_q[su(2)]$ and take a fiducial vector from the representation space. The coherent state is the state obtained by transformation of the fiducial vector by the universal $\mathcal{T}$-matrix. Let us take spin j representation of $U_q[su(2)]$:

$$\begin{aligned} J_{\pm}\,|jm\rangle &= \sqrt{[j\mp m][j\pm m+1]}\,|j\,m\pm 1\rangle\,, \\ J_0\,|jm\rangle &= m\,|jm\rangle\,, \end{aligned} \tag{3.1}$$

where j is a non-negative integer or a positive half-integer. This is a unitary representation of $U_q[su(2)]$. The GCS for $SU_q(2)$ is defined by

$$|x,z\rangle = \mathcal{T}\,|j\ -j\rangle\,. \tag{3.2}$$

Repeated use of (3.1) gives us the explicit form of the GCS:

$$|x,z\rangle = \sum_{n=0}^{2j} q^{nj} \begin{bmatrix} 2j \\ n \end{bmatrix}_q^{1/2} x^n e^{-jz}\,|j\ -j+n\rangle \tag{3.3}$$

$$= e^{-jz}\sum_{n=0}^{2j} q^{-nj} \begin{bmatrix} 2j \\ n \end{bmatrix}_q^{1/2} x^n\,|j\ -j+n\rangle\,, \tag{3.4}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}. \tag{3.5}$$

By definition, the GCS (3.2) has unit norm. Proof by direct computation is also easy [3].
One of the most important properties of the GCS is ”resolution of unity”. This is essential for applications of coherent states to physical problems, path integrals, representation theory, and so on. Despite the noncommutativity of $SU_q(2)$, the coherent state (3.3) satisfies the resolution of unity with respect to an invariant integration over quantum groups. Invariant integration over the quantum group $SU_q(2)$ is developed in [24, 25, 16]. Let $\mathcal{G}$ be an arbitrary quantum group. A linear functional $H : \mathcal{G} \longrightarrow \mathbb{C}$ is said to be *normalized bi-invariant integral* if

1. $H[\,1_{\mathcal{G}}\,] = 1$,
2. for any $f \in \mathcal{G}$

$$(H\otimes id)[\,\Delta(f)\,] = (id\otimes H)[\,\Delta(f)\,] = H[\,f\,]. \tag{3.6}$$

Let $\mathcal{V}$ be an algebra dual to $\mathcal{G}$. Writing the coproduct of $f\in\mathcal{G}$ as $\Delta(f) = \sum_k f_k\otimes f^k$, the left and the right action of $Z\in\mathcal{V}$ on f are defined by

$$Z\triangleright f = \sum_k f_k\langle Z, f^k\rangle, \qquad f\triangleleft Z = \sum_k \langle Z, f_k\rangle f^k. \tag{3.7}$$

Then one can prove the left and the right invariance of the integral H :

$$H[\,Z\triangleright f\,] = H[\,f\triangleleft Z\,] = \varepsilon(Z)H[\,f\,]. \tag{3.8}$$

Proposition 3.1. *With respect to the invariant integration of $SU_q(2)$, the GCS for $SU_q(2)$ satisfies the resolution of unity relation:*

$$(2j+1)_{q^2}H[\ |x,z\rangle\langle x,z|\ ] = 1. \tag{3.9}$$

The coherent state for the quantum group $SU_q(2)$ allows for easy generalizations of some properties whose classical counterparts are well-known [19]. The discussions presented here in conjunction with the results obtained in the succeeding sections show that many key characteristics of the $SU(2)$ coherent state can be lifted up to the quantum group setting.

Proposition 3.2. *The GCS for $SU_q(2)$ enjoys the following properties.*

1. *The overlap of two coherent states is defined as an object in $SU_q(2)^{\otimes 2}$ [9]. Let us introduce two independent copies of the coherent state distinguished by the subscripts: $|x_a, z_a\rangle$, $a = 1,2$.*

$$\langle x_1, z_1\,|x_2, z_2\rangle = (e^{-jz_1^*}\otimes e^{-jz_2})\sum_{n=0}^{2j}\begin{bmatrix} 2j \\ n \end{bmatrix}_q (x_1^*\otimes x_2)^n. \tag{3.10}$$

   In (3.10) the exponential terms contained in the parenthesis commute with the factored sum.

2. *There exists an operator which annihilates the coherent state:*

$$(J_- + (1+q^{2j})x\,[J_0] - q^{2j}x^2J_+)\,|x,z\rangle = 0. \tag{3.11}$$

3. *The coherent state is an eigenvector of the operator Γ defined by*

$$\Gamma = (1-(q^j+q^{-j})\zeta)\,q^{J_0}\,[J_0] - (1-q^j\zeta)\,x\,q^{J_0}J_+ + q^{-j-1}e^{-z}y\,q^{J_0}J_-. \tag{3.12}$$

   The eigenvalue relation reads as

$$\Gamma\,|x,z\rangle = -q^{-j}\,[j]\,|x,z\rangle\,. \tag{3.13}$$

4 Coherent states and noncommutative 2-sphere

In this section, we investigate a connection of the GCS for $SU_q(2)$ to noncommutative 2-sphere of Podleś, refereed to as q-sphere [20]. Considering the embedding of q-sphere into $SU_q(2)$:

$$x_{-1} = \sqrt{1+q^2}\,ab, \qquad x_0 = 1+(q+q^{-1})\,bc \qquad x_1 = \sqrt{1+q^{-2}}\,dc, \tag{4.1}$$

where $x_{\pm 1}$ and x_0 are the generators of q-sphere, it turns out that the q-sphere is identified with the homogeneous space $SU_q(2)/U(1)$[7, 11]. Let us consider expectation values of some specific elements in $U_q[su(2)]$ with respect to the GCS for $SU_q(2)$:

$$\begin{aligned} X_+ &\equiv \langle x,z|\,J_+q^{-J_0}\,|x,z\rangle = [2j]\,(1-\zeta)\,x^*, \\ X_- &\equiv \langle x,z|\,q^{-J_0}J_-\,|x,z\rangle = [2j]\,x\,(1-\zeta), \\ X_0 &\equiv \langle x,z|\,q^{-J_0}\,[J_0]\,|x,z\rangle = q^{-2}\,[2j]\,\zeta - q^j\,[j]\,. \end{aligned} \tag{4.2}$$

These expectation values, after suitable scaling given by

$$
\begin{aligned}
x_1 &= -q\frac{\sqrt{[2]}}{[2j]}X_+ = -q\sqrt{[2]}(1-\zeta)x^*,\\
x_0 &= 1-q\frac{[2]}{[2j]}(X_0+q^j\,[j]) = 1-q^{-1}\,[2]\,\zeta,\\
x_{-1} &= \frac{\sqrt{[2]}}{[2j]}X_- = \sqrt{[2]}x(1-\zeta),
\end{aligned} \tag{4.3}
$$

satisfy the defining relations of the q-sphere, while maintaining the following $*$-involution map:

$$x_1^* = -qx_{-1}, \qquad x_0^* = x_0, \qquad x_{-1}^* = -q^{-1}x_1. \tag{4.4}$$

The relation

$$\zeta = \frac{qx^*x}{1+qx^*x} = \frac{q^3xx^*}{1+qxx^*} \tag{4.5}$$

ensures that the coordinate x_k are functions of only x and x^*. Thus the coherent state gives a natural complex description of the q-sphere. We emphasize that we did not introduce any additional assumption to obtain the above complexification. It is a direct consequence of the definitions of $U_q[su(2)]$ and its dual together with the finite dimensional representations of $U_q[su(2)]$. We remark that complex description of the q-sphere has been previously considered in [21, 6]. The commutation relation between x and x^* derived therein, however, differs from the one given in this work.

In order to develop Kähler geometry on q-sphere, the first step may be differential calculus in our complex description, namely, differential calculus in coordinates x, x^*. Such a differential calculus is considered in [6] based on different noncommutative coordinates from ours. Here we should develop the differential calculus for our noncommutative coordinates. The embedding of q-sphere into $SU_q(2)$ allows us to infer the differential structures on the q-sphere from the well-known [24, 26] covariant differential calculi on $SU_q(2)$. We use the so-called left-covariant 3D calculus [24] on $SU_q(2)$ because of the reasons advanced in [6]. The obtained differential calculus is listed below. The relation between coordinates and their differentials:

$$
\begin{aligned}
&x\,dx = q^2dx\,x, \qquad x^*dx^* = q^{-2}dx^*\,x^*,\\
&dx\,x^* = q^{-2}f_-(x^*x)x^*dx, \quad dx^*\,x = q^2xf_+(x^*x)dx^*,
\end{aligned} \tag{4.6}
$$

where

$$f_\pm(x^*x) = \frac{1-\zeta}{1-q^{\pm4}\zeta} = \frac{1}{1+(1-q^{\pm4})\,qx^*x}. \tag{4.7}$$

The nilpotency of the complex differentials follow:

$$(dx)^2 = (dx^*)^2 = 0. \tag{4.8}$$

Determination of the commutation relation between dx and dx^* requires a lengthy computation. Obtained relation reads:

$$dx\,dx^* = -\frac{f_-(x^*x)}{q^2}\frac{1-q^2\omega x^*x}{1+\omega x^*x}\,dx^*\,dx, \tag{4.9}$$

where $\omega = q - q^{-1}$. The differential calculus on q-sphere allows us to compute exterior derivative of the coordinates expressed in terms of dx and dx^*. For instance, the following relations may be established:

$$
\begin{aligned}
dX_+ &= [2j]\,\frac{1-q^2\omega t}{(1+qt)^2}\,(dx^* - q^{-1}(x^*)^2 dx),\\
dX_- &= \frac{[2j]}{f_-(x^*x)}\,\frac{1+\omega t}{(1+qt)^2}\,(dx - q^3 x^2 dx^*),\\
dX_0 &= [2j]\,\frac{1+\omega t}{(1+qt)^2}\,(qx\,dx^* + q^{-1}x^*\,dx).
\end{aligned}
\tag{4.10}
$$

If we define Kähler potential on q-sphere by

$$
F(x,x^*) = -\ln \langle j-j\,|\,\Psi\rangle^* \langle j-j\,|\,\Psi\rangle\,, \tag{4.11}
$$

then it follows:

$$
F(x,x^*) = 2j\ln(1+qx^*x) - \sum_{k=0}^{2j-1} \ln(1+(1-q^{-2(k+1)})qx^*x). \tag{4.12}
$$

Even though x and x^* are noncommuting, the first term has the same form as in the classical case. The other terms reflect noncommutative nature of the q-sphere and vanish in the classical limit. Note that the argument of the logarithm in (4.11) is the normalization factor of the coherent state. In principle, one can compute Kähler 2-form using (4.12) and our differential calculus. This will be presented in a future work.

5 Coherent states for quantum $SU(1,1)$

One can construct the GCS for "noncompact" quantum group $SU_q(1,1)$ in a way similar to $SU_q(2)$. We use the basis of $U_q[su(1,1)]$ familiar in physics:

$$
K_+ = J_+, \qquad K_0 = J_0, \qquad K_- = -J_-. \tag{5.1}
$$

The $*$-involution for this basis has simpler form: $K_\pm^* = K_\mp$, $K_0^* = K_0$. Unitary irreducible representations of $U_q[su(1,1)]$ have been classified in [15]. There are some inequivalent representations and each representation may give different GCS for $SU_q(1,1)$. We here consider the most often used representation used in physics, *i.e.*, positive discrete series:

$$
\begin{aligned}
&K_0\,|\kappa\mu\rangle = \mu\,|\kappa\mu\rangle\,,\\
&K_\pm\,|\kappa\mu\rangle = \sqrt{[\mu\pm\kappa][\mu\mp\kappa\pm 1]}\,|\kappa\,\mu\pm 1\rangle\,,\\
&C\,|\kappa\mu\rangle = [\kappa][\kappa-1]\,|\kappa\mu\rangle\,,
\end{aligned}
\tag{5.2}
$$

where $\mu = \kappa, \kappa+1, \kappa+2, \cdots$, and κ is a positive integer or positive half-integer. The GCS for $SU_q(1,1)$ is defined by

$$
\begin{aligned}
|x,z\rangle &= \mathcal{T}\,|\kappa\kappa\rangle\\
&= \sum_{n=0}^{\infty} q^{-\kappa n} \left[\begin{array}{c} 2\kappa-1+n \\ n \end{array}\right]_q^{1/2} x^n e^{\kappa z}\,|\kappa\,\kappa+n\rangle\,.
\end{aligned}
\tag{5.3}
$$

The GCS (5.3) has unit norm.

Proposition 5.1. *The GCS for $SU_q(1,1)$ enjoys the following properties.*

1. *The overlap of two coherent states is given by*

$$\langle x_1, z_1 \,|\, x_2, z_2\rangle = (e^{\kappa z_1^*} \otimes e^{\kappa z_2}) \sum_{n=0}^{\infty} \left[\begin{array}{c} 2\kappa - 1 + n \\ n \end{array} \right]_q (x_1^* \otimes x_2)^n. \tag{5.4}$$

2. *There exists an operator which annihilates the coherent state:*

$$(K_- - (1+q^{-2\kappa})x\,[K_0] + q^{-2\kappa}x^2 K_+)\,|x,z\rangle = 0. \tag{5.5}$$

3. *The coherent state is an eigenvector of the operator Γ defined by*

$$\Gamma = (1+(q^{\kappa}+q^{-\kappa})\zeta)\,q^{K_0}\,[K_0] - (1+q^{-\kappa}\zeta)\,xq^{K_0}K_+ + q^{\kappa-1}e^{-z}y\,q^{K_0}K_-, \tag{5.6}$$

   Eigenvalue relation reads

$$\Gamma\,|x,z\rangle = q^{\kappa}\,[\kappa]\,|x,z\rangle\,. \tag{5.7}$$

The homogeneous space $SU_q(1,1)/U(1)$ is a q-analogue of hyperboloid. It turns out that the GCS (5.3) naturally gives a complex description of the q-analogue of hyperboloid. To see this, let us consider expectation values of some specific elements of $U_q[su(1,1)]$ with respect to the GCS (5.3):

$$X_+ = \langle x,z|\,K_+q^{-K_0}\,|x,z\rangle\,, \qquad X_0 = \langle x,z|\,q^{-K_0}\,[K_0]\,|x,z\rangle\,, \qquad X_- = \langle x,z|\,q^{-K_0}K_-\,|x,z\rangle\,.$$

These expectation values, after suitable scaling given by

$$\begin{aligned} x_1 &= \frac{q\sqrt{[2]}}{[2\kappa]}X_+ = q\sqrt{[2]}(1-\zeta)x^*, \\ x_0 &= 1 + \frac{q\,[2]}{[2\kappa]}\,(X_0 - q^{-\kappa}\,[\kappa]) = 1 - q^{-1}\,[2]\,\zeta, \\ x_{-1} &= \frac{\sqrt{[2]}}{[2\kappa]}X_- = \sqrt{[2]}\,x(1-\zeta). \end{aligned} \tag{5.8}$$

satisfy the commutation relations of q-hyperboloid with the following $*$-involution:

$$x_1^* = qx_{-1}, \qquad x_0^* = x_0, \qquad x_{-1}^* = q^{-1}x_1. \tag{5.9}$$

The relation

$$\zeta = \frac{-qx^*x}{1-qx^*x} = \frac{-q^3xx^*}{1-qxx^*}. \tag{5.10}$$

ensures that the coordinate x_k are functions of only x and x^*. We remark that the results given in this section are quantum group extension of the known results for $SU(1,1)$ GCS [19]. The resolution of unity for the $SU_q(1,1)$ GCS is a nontrivial question because the homogeneous space is noncompact. Complex differential geometry on this noncommutative homogeneous space is also an open problem. They will be discussed in future.

6 Concluding remarks

We have investigated the GCS for $SU_q(2)$ and $SU_q(1,1)$. It has been shown that they allow for easy generalizations of some properties whose classical counterparts are well-known. Besides the properties presented in this paper, the GCS for $SU_q(2)$ has the following properties similar to the $SU(2)$ GCS [3]. (i) $U_q[su(2)]$ is represented in the coherent state basis, that is, the generators of $U_q[su(2)]$ can be expressed as operators acting on the polynomial in x, (ii) by taking the limit of $j \to \infty$, the GCS (3.3) is reduced to the one for quantum Heisenberg group introduced in [5], (iii) by employing the parity operator method used for $SU(2)$ GCS [22], entanglement can be introduced to the tensor product of two $SU_q(2)$ GCS. These observations imply that the GCS's for quantum groups considered in this paper may be an appropriate extension of the GCS for Lie groups. It is therefore worth investigating the GCS for higher rank quantum groups. It is also an interesting problem to study the case for quantum supergroups. Using the GCS for quantum supergroups, one may able to study Kähler geometry on noncommutative superspace. The simplest example is the quantum supergroup $OSp_q(1/2)$ [12]. The universal $\mathcal{T}$-matrix for $OSp_q(1/2)$ has been obtained [1] and the finite dimensional representations of $OSp_q(1/2)$ are well studied [12, 28, 1, 2]. We are ready to study GCS for $OSp_q(1/2)$. This work is in progress.

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