A THREE-DIMENSIONAL TIME-DEPENDENT NUMERICAL MODEL FOR POLAR ICE SHEETS: SOME BASIC TESTING WITH A STABLE AND EFFICIENT FINITE DIFFERENCE SCHEME

Philippe HUYBRECHTS

GEOGRAFISCH INSTITUUT - VRIJE UNIVERSITEIT BRUSSEL
pleinlaan 2 1050 brussel belgium
A THREE-DIMENSIONAL TIME-DEPENDENT NUMERICAL MODEL FOR POLAR ICE SHEETS: SOME BASIC TESTING WITH A STABLE AND EFFICIENT FINITE DIFFERENCE SCHEME

Philippe HUYBRECHTS

Geografisch Instituut, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium

Report 86-1

October 1986
Abstract

In this report a detailed description is given of a three-dimensional time-dependent numerical model of cold ice sheet flow, designed for climatic change experiments. This model computes the fully coupled velocity and temperature fields (coupled by means of the Arrhenius equation, relating the flow parameter to temperature), as well as the geographical distribution of mass.

In order to solve the stress equilibrium equations, approximations have been made according to the 'shallow ice approximation' (variations in the longitudinal directions are small, as compared to the vertical) and the 'hydrostatic assumption' (implying zero normal stress deviator components). The time-dependent energy balance equation accounts for vertical heat conduction, advection and heat generation by internal deformation.

The equations of motion and temperature are then integrated on a 3-D grid (vertically scaled to local ice thickness and with high resolution in the lower layers, where the shear concentrates), employing a finite-difference scheme based upon the alternating-direction-implicit method.

Some basic test runs, involving a schematic East Antarctic flowline, proved the scheme to be stable, relatively efficient and seems to produce very realistic solutions. Satisfactorily results for the temperature solution are obtained with an 'optimal' vertical resolution of 10 layers, gradually thinning towards the base and with a lowermost grid spacing of 2% of the local ice thickness.

In its present form, the model serves as a 'core' for a general polar ice sheet model we are developing for the Antarctic Ice Sheet. For this purpose, the model will be coupled in the near future with an ice shelf model, in order to allow for changes in the grounded ice domain due to environmental changes.
1. Introduction

2. Model Description
   2.1 Ice Deformation
   2.2 Heat Transfer
   2.3 Bedrock Adjustment

3. Numerical procedure of solution
   3.1 Coordinate Transformation
   3.2 Finite-difference scheme

4. Some basic testing

5. Final remarks

References
1. Introduction

In polar ice sheets, temperature and flow are strongly related: temperature determines to a large extent the viscosity of ice whereas the velocity field contributes in the heat equation through the advective and frictional terms. Physically, the ice temperature-velocity coupling may, under suitable circumstances, lead to runaway situations, in which a large amount of continental ice is discharged into the ocean in a relatively short time (the so-called creep instability). From a climatological point of view, on the other hand, temperature-depth profiles provide evidence of the history of our climate, once we understand how ice sheets move and vary under changing conditions. Interpretation of isotopic profiles derived from deep ice cores also depends on the flow and past behaviour of ice sheets. The interaction of the physical processes involved is generally so complex that numerical computer models are needed to solve the coupled thermomechanical equations. Such models also possess the potential to identify particle trajectories and construct age-depth profiles in response to climatic conditions over the ice sheet.

Historically, the first papers on the temperature distribution in ice sheets did not deal with the interaction of velocity and temperature, but were attempts to give a theoretical explanation for observed temperature profiles (e.g. Robin, 1955; Weertman, 1968; Philberth and Federer, 1971). In these calculations simple one-dimensional (vertical) steady-state models were involved. Due to the neglect of horizontal advection these models could only be applied in regions close to the ice divide. The moving-column model, based on a vertical model, but taking into account in a crude way the effect of horizontal advection, has subsequently been used to investigate two-dimensional (vertical plane) temperature distributions, see for example Budd et al. (1971) where this model is used extensively under steady-state conditions to assess the "Derived characteristics of the Antarctic Ice Sheet". This approach has been used widely by the Australian group since, also in more dynamic situations (e.g. Budd et al., 1976; Young, 1981; Budd et al., 1984). Critical to the applicability of the moving-column model is the vertical shear in the horizontal velocity as the assumption that the column remains vertical is not realistic far from an ice divide. Nevertheless, when ice motion is almost entirely by basal sliding, or when all velocity shear is concentrated in the lowest layers, the moving-column model works well. The model is also restricted to a stationary flow pattern since ice thickness is not a prognostic variable but instead used as input to derive the vertical mean horizontal velocity along a flowline.

To date, the most ambitious model is that of Jenssen (1977). It is a three-dimensional model incorporating the mutual interaction between ice flow and its thermodynamics. The approach taken for the flow part is quite similar to the three-dimensional (isothermal) model of Mahaffy (1976) to study the time-dependent behaviour of an arbitrarily shaped ice mass. Jenssen introduced a scaled vertical coordinate, transformed the relevant continuity and thermodynamic equation (prognostic
equations for ice thickness and temperature) and tried to solve the system numerically. In applying the scheme to the Greenland Ice Sheet, however, numerical instabilities occurred, forcing the calculations to be interrupted after 1000 years of integration. Nevertheless, this model appears to be the first that dealt with the ice flow-temperature coupling in a truly dynamic fashion.

In the last few years new attempts were made to solve the coupled thermomechanical equations for "shallow ice flow" [for a more rigorous definition of shallow ice flow, see Hutter (1983)]. Yakowitz et al. (1985) have presented numerical solutions for the steady-state two- (vertical plane) dimensional case based on a treatment given by Morland (1984). However, it appears that numerical problems are still encountered, leading to spurious solutions for the vertical velocity profiles.

We recently developed a three-dimensional, fully time-dependent ice sheet model including a treatment of the (mechanical) bedrock response. Providing the ice sheet deforms internally by shear strain the model computes the fully coupled velocity and temperature field as well as the geographical distribution of mass. The inputs of the model are the mass-balance, temperature-dependent flow-law coefficients, the thermal parameters, surface temperature and the basal temperature gradient. We introduced a finite-difference scheme based upon the alternating-direction-implicit method to solve the system numerically. This scheme appears to be stable, relatively efficient and seems to produce very realistic solutions. In this report we give an elaborate description of the numerical model and illustrate its possibilities while presenting the results of some basic test runs involving a very schematic polar continental ice sheet.

2. Model description

2.1 Ice Deformation

Defining a right-handed Cartesian coordinate system \(x, y, z\), the \(x-y\) plane being parallel to the geoid and the \(z\)-axis vertically upwards \((z = 0 \text{ at sea level})\), ice thickness will be denoted by \(H(x, y)\) and bedrock elevation by \(h(x, y)\). Figure 1 shows the model geometry. Units are in the common SI-system, except time which will mostly be expressed in years.

Fundamental to the derivation of the equations of motion are the mechanical equations expressing balance between body forces (gravity in this case) and surface forces (stresses) acting on an element of ice. Since accelerations in an ice sheet can be neglected Newton's second law for a continuum then reads:
\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho g
\]

with \( \tau_{ij} \) the stress tensor components \([\text{Nm}^{-2}]\), \(g\) acceleration of gravity \([9.81 \text{ ms}^{-1}]\), and \(\rho\) ice density \([910 \text{ kgm}^{-3}]\), assumed to be constant.

As a complete solution to the stress equilibrium equations (1) - (3) is not known additional assumptions appropriate to large ice sheets are made as follows. Bedrock and surface slopes are supposed to be sufficiently small so that normal stress deviators can be neglected in effect stating that hydrostatic equilibrium prevails everywhere (or: longitudinal stress equals hydrostatic stress). This implies that the longitudinal strain-rate components are negligible compared to the shear strain-rate components and \( \frac{\partial \tau_{xx}}{\partial z} \approx \frac{\partial \tau_{xz}}{\partial x}, \frac{\partial \tau_{yz}}{\partial y}. \) The validity of this assumption for small slopes has been discussed by Nye (1969). Although breaking down at the edge (large slopes and therefore substantial non-zero longitudinal stress deviators) and at the centre (longitudinal strain-rates prevailing) the inclusion of longitudinal stress deviators hardly causes a modification of the profile of ice sheets larger than about 30 km (Weertman, 1961). Neglecting longitudinal strain-rates also limits the use of the model to cases where basal sliding is relatively unimportant and requires slopes to be smoothed over a distance an order of magnitude greater than ice thickness to circumvent problems associated with small scale bedrock irregularities (Budd, 1970). This condition will generally be met when resolving the equations on a computational grid with typical spacing in the 20-100 km
range. With these limitations in mind, however, the above presupposition has shown general applicability in large-scale ice sheet modelling (e.g. Mahaffy(1976), Birchfield(1977), Oerlemans(1981), Pollard(1983)).

So:
\[
\tau_{xx} = \tau_{yy} = \tau_{zz} = 0 \tag{4}
\]
\[
\tau_{xx} = \tau_{yy} = \tau_{zz} \tag{5}
\]

with \( \tau_{ij} \) the normal components of the deviatoric stress tensor \( \tau_{ij} \), defined as the stress tensor \( \tau_{ij} \) minus the hydrostatic component:

\[
\tau_{ij} = \tau_{ij} - \frac{\delta_{ij} \tau_{kk}}{3} \quad \text{i,j,k = x,y,z} \tag{6}
\]

where \( \delta_{ij} \) is the Kronecker delta.

With the further remark that the ice sheet does not experience shear at its sides (\( \tau_{xy} = 0 \)), the ice sheet will be assumed to deform by shear under its own weight in planes parallel to the geoid with \( \tau_{xz}, \tau_{yz} \) the only non-zero stress-deviator components.

The equations of motion (1) - (3) then reduce to:

\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{7}
\]
\[
\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{8}
\]
\[
\frac{\partial \tau_{zz}}{\partial z} = \rho g \tag{9}
\]

Integrating (9) from the surface \( z = H + h \) to a height \( z \) neglecting atmospheric pressure (which does not influence ice deformation) yields an expression for the normal stress components:

\[
\tau_{zz} (z) = -\rho g(H + h - z) = \tau_{xx} (z) = \tau_{yy} (z) \tag{10}
\]
from which follow, after integrating (7), (8) from the surface (which can support no shear) to a height \( z \) and substituting (10) equations for shear stress:

\[
\tau_{xz}(z) = \tau_{xz}(z) = -\rho g(H + h - z) \frac{\partial(H + h)}{\partial x}
\]

(11)

\[
\tau_{yx}(z) = \tau_{yz}(z) = -\rho g(H + h - z) \frac{\partial(H + h)}{\partial y}
\]

(12)

It should be noted at this point that a minor contribution due to the local bedrock and surface slopes has been neglected. Since these slopes are generally small (of the order of \( 10^{-3} \)), no serious error is introduced. This point should however be kept in mind when choosing boundary conditions for the vertical velocity component.

Now the relevant stress equations have been derived we can proceed to a derivation of the velocity field by means of an expression relating deformation to stress. The flow law governing the creep of polycrystalline ice can be written (Glen, 1955; Paterson, 1981) as:

\[
\dot{\varepsilon}_{ij} = A(T^*) \tau_*^{n-1} \tau_{ij}
\]

(13)

In this expression \( \dot{\varepsilon}_{ij} \) \([a^{-1}]\) are the strain rate components related to velocity gradients by definition, \( \tau_* \) the effective stress defined in terms of all the stress deviator components so that it is independent of the coordinate system, \( n \) the flow law exponent and \( A [Pa^n a^{-1}] \) the flow-law coefficient. The values of \( n \) and \( A \) probably introduce a main uncertainty in the model as their values are not exactly known. For a discussion, see e.g. Journal of Glaciology, 1978, no. 85 on the physics and chemistry of ice; Paterson, 1981, chap. 3. In this study we will follow the recommendations of Paterson and adopt a mean value of \( n = 3 \). \( A \) depends on such factors as ice temperature, crystal size and orientation, impurity content, meltwater and possibly other factors. Here only the temperature dependence of \( A \) will be taken into account. This is an approximation since information on how variations in ice properties affect flow is lacking. Laboratory experiments suggest that \( A(T) \) can be expressed by the Arrhenius relationship:

\[
A(T^*) = m \cdot a \exp \left\{ \frac{-Q}{RT^*} \right\}
\]

(14)

where \( a \) is independent of temperature, \( R \) is the gas constant \([8.314 Jmol^{-1} K^{-1}]\), \( Q \) the activation
energy for creep, and $T^*$ absolute temperature corrected for the dependence of the melting point on pressure ($T^* = T + 8.7 \times 10^{-4} (H + h - z)$, with $T$ measured in K). $m$ is a tuning parameter in order to slightly adjust the height-width ratio. With the following values for $a$ and $Q$:

\[
\begin{align*}
T^* < 263.15 \text{ K} & \quad a = 1.14 \times 10^{-5} \text{ Pa}^{-3} \text{ year}^{-1} & Q = 60 \text{ kJmol}^{-1} \\
T^* \geq 263.15 \text{ K} & \quad a = 5.47 \times 10^{-10} \text{ Pa}^{-3} \text{ year}^{-1} & Q = 139 \text{ kJmol}^{-1}
\end{align*}
\]

$A(T^*)$ lies within the bounds as put forward by Paterson and Budd (1982). The higher value of $Q$ for $T^* \geq -10$ appears to be connected with enhanced creep due to the presence of liquid water at grain boundaries. Figure 2 shows the adopted temperature dependence of the flow law coefficient. In practice, however, $A$ (controlling the height-to-width ratio) will serve a 'tuning' purpose. While nevertheless retaining the general temperature dependence as in (14) in this way the effects of crystal fabric and impurity content (that soften the ice) or basal sliding (if not explicitly modelled) can be implicitly dealt with. As mentioned by Paterson and Budd (1982) these factors may alter $A$ by up to an order of magnitude. Comparison of computed velocity profiles with observations (that are sparse) will in practice have to shed more light as to the proper choice of $A$ and $n$.

![Fig. 2: Temperature dependence of the flow law coefficient $A$ as used in the model](image)

\[\text{Fig. 2: Temperature dependence of the flow law coefficient } A \text{ as used in the model}\]

\[\text{: Paterson and Budd (1982)}\]

\[\text{---: Adopted relation (Eq. 14)}\]
Regarding the assumptions, (13) reduces to its two-component form as:

\[
\dot{\varepsilon}_{xz} = \frac{1}{2} \left[ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] = A \tau_{xz}^{n-1} \tau_{xz} \tag{17}
\]

\[
\dot{\varepsilon}_{yz} = \frac{1}{2} \left[ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] = A \tau_{yz}^{n-1} \tau_{yz} \tag{18}
\]

\[
\tau_* = \left[ \tau_{xz}^2 + \tau_{yz}^2 \right]^{1/2} \tag{19}
\]

where \( u, v, w \) are the \( x, y, z \) components of the three-dimensional velocity vector \( \mathbf{V} \) [m/s]. Integrating (17),(18) with respect to \( z \) and assuming that \( \partial w/\partial x \ll \partial u/\partial z \); \( \partial w/\partial y \ll \partial v/\partial z \) immediately yields an expression for the horizontal velocity vector \( \mathbf{v} \):

\[
\mathbf{v}(z) - \mathbf{v}(h) = -2(\rho g)^n \left[ \nabla (H+h) \cdot \nabla (H+h) \right]^{n-1/2} \nabla (H+h) \int_h^z A(H+h-z)^n \, dz \tag{20}
\]

\( \mathbf{v}(h) \), the two-dimensional basal-sliding velocity, enters here as a boundary condition; the expression \( \mathbf{v}(z) - \mathbf{v}(h) \) expresses the deformational part of the horizontal velocity. Since \( A \) depends on temperature (and therefore on position) equation (20) now has to be evaluated numerically, given the temperature distribution. Integrating (20) once more from the base to the surface will lead to an expression for the depth averaged horizontal velocity \( \mathbf{v} \):

\[
\mathbf{v}_H = -2(\rho g)^n \left[ \nabla (H+h) \cdot \nabla (H+h) \right]^{n-1/2} \nabla (H+h) \int_h^z \int_h^z A(H+h-z)^n \, dz \, dz \tag{21}
\]

Substituting (18) in the vertically integrated continuity equation:

\[
\frac{\partial H}{\partial t} = - \nabla \cdot (\mathbf{v}H) + M - S \tag{22}
\]

together with the condition that \( H \geq 0 \) then yields an expression setting out the time-dependent evolution of the ice sheet. The two-dimensional \( \nabla \)-operator is understood here as being parallel to the
x-y plane. \( M \) is the mass balance (accumulation minus surface melting) and \( S \) the basal melting rate [both expressed in \( \text{m a}^{-1} \) ice depth]. The flux divergence term in (22) can also be written as

\[-\nabla \cdot D \nabla (H + h)\]. Here \( D \) may be interpreted as a nonlinear diffusion coefficient depending strongly on ice thickness and surface slope given by the scalar component of (21). Since generally any sliding theory contains basal shear stress and therefore \( \nabla (H + h) \) basal sliding can easily be incorporated in \( D \).

To complete the description of the velocity field one more equation is needed for the vertical velocity component \( w \). Conservation of mass for an incompressible material reads:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{or} \quad \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 0
\]  

(23)

yielding immediately:

\[
w(z) - w(h) = -\int_{h}^{z} \nabla \cdot \mathbf{v}(z) \, dz
\]  

(24)

Note that an additional vertical velocity component due to firn compaction in the upper layers is ignored here, since ice density is kept constant.

The kinematic boundary conditions at the upper and lower ice surfaces read:

\[
w(h) = \frac{\partial h}{\partial t} + \mathbf{v}(h) \cdot \nabla h
\]  

(25)

\[
w(H + h) = \frac{\partial (H + h)}{\partial t} + \mathbf{v}(H + h) \cdot \nabla (H + h) - M + S
\]  

(26)

Equations (20) and (24) now fully describe the three-dimensional velocity field \( \mathbf{v} \), with the only limitation that \( \mathbf{v} \) is not allowed to change direction with depth, its direction being solely determined by the local surface slope. Finally, to close the set of equations specifying ice deformation the temperature distribution within the ice sheet needs to be known simultaneously in order to adjust the flow-law parameter \( A \). This leads us on to the next section.
2.2 Heat Transfer

Taking the same coordinate system fixed in space as in fig. 1 the general thermodynamic equation governing the transfer of heat in a continuum (Paterson, 1981, p.199-200, Oerlemans and Van der Veen, 1984, p.78) reads:

\[ \frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \nabla^2 T - \nabla \cdot \nabla T + \frac{\Phi}{\rho c_p} \]  \hspace{1cm} (27)

Here $T$ is absolute temperature [K], $t$ time [s], $k$ thermal conductivity $[6.62 \times 10^7 \text{ Jm}^{-1}\text{K}^{-1}\text{a}^{-1}]$, $\rho$ ice density $[910 \text{ kgm}^{-3}]$, $c_p$ specific heat capacity $[2009 \text{ Jkg}^{-1}\text{K}^{-1}]$, $V$ three-dimensional ice velocity $[\text{m}^2\text{a}^{-1}]$ and $\Phi$ internal frictional heating $[\text{Jm}^{-3}\text{a}^{-1}]$ due to deformation. Thus in the general case heat is transferred both by diffusion and advection and generated internally by deformational friction. Simplifications made in (27) include constant density, thermal conductivity and heat capacity (depending on density and temperature) and the omission of melting and refreezing processes in the density-variable firn layer (which is of minor importance on the scales considered). Furthermore, horizontal conduction in an ice sheet can be disregarded as temperature gradients in the horizontal directions are usually small compared to the vertical gradient, thus $\nabla^2 T$ can safely be replaced by $\partial^2 T/\partial z^2$.

The internal heating rate per unit volume can then be expressed as:

\[ \Phi = \sum_{ij} \dot{\varepsilon}_{ij} \tau_{ij} \] \hspace{1cm} (28)

Assuming that the deformational heating due to longitudinal strain-rates is small compared with that due to the horizontal shear strain-rates (which is certainly true near the base where heating is greatest, Paterson, 1981, p.200) leads to:

\[ \Phi = 2 \dot{\varepsilon}_{xz} \tau_{xz} + 2 \dot{\varepsilon}_{yz} \tau_{yz} = -\rho g(H + h - z) \frac{\partial^2 v}{\partial z^2} \cdot \nabla (H + h) \] \hspace{1cm} (29)

according to (11), (12) and the assumptions made in (17), (18).

Boundary conditions are chosen as follows. At the surface, temperature is set equal to the mean
annual air-temperature at that altitude and location. At the base, the ice sheet gains heat both from
sliding friction and the geothermal heat flux. Neglecting heat interaction with the underlying bed these
ccontributions can most easily be incorporated in the basal temperature gradient:

\[
\left\{ \frac{\partial T}{\partial z} \right\}_b = \gamma - \frac{\tau_b \cdot v(h)}{k},
\]

where \(\gamma\) [Km\(^{-1}\)] is the prescribed geothermal gradient (= \(G/k\), \(G\) being the geothermal heat flux [taken here as \(-1.32 \times 10^6\) Jm\(^{-2}\)a\(^{-1}\) or \(-4.2 \times 10^{-2}\) Wm\(^{-2}\)], \(\tau_b\) two-dimensional basal shear stress and \(v(h)\) basal sliding velocity.

Phase changes at the base are incorporated in the model by keeping the basal temperature at the
pressure melting point whenever it is reached or basal melt water is present (in case a relation is added
that governs melt water flow). The basal melt rate \(S\) [ma\(^{-1}\)] (positive when melting, negative when
freezing) can then be calculated as follows:

\[
S = \frac{k}{\rho L} \left[ \left\{ \frac{\partial T}{\partial z} \right\}_c - \left\{ \frac{\partial T}{\partial z} \right\}_b \right]
\]

where \((\partial T/\partial z)_c\) is the basal temperature gradient after correction for the pressure melting point,
\((\partial T/\partial z)_b\) the basal temperature gradient as given in (30) and \(L\) the specific latent heat of fusion
[\(3.35 \times 10^5\) Jkg\(^{-1}\)]. The pressure melting point is given (e.g. Paterson, 1981, p. 193) by:

\[
T_{pmp} = T_0 - \beta (H + h - z)
\]

with \(T_0 = 273.15\) K, the triple-point of water, \(H\) ice thickness and \(\beta = 8.7 \times 10^{-4}\) Km\(^{-1}\) of ice.

This concludes the general description of the coupled temperature-ice flow model.

2.3. Bedrock Adjustment

Large ice sheets exert considerable pressures on the underlying bed. Since the relaxation time of
isostatic adjustment is comparable to the reaction of the ice sheet to changing environmental conditions the model also contains a description of the bedrock response. At present consider an elastic lithosphere resting on a viscous astenosphere. The deflection $w$ [m] of the lithosphere then follows from local hydrostatic equilibrium:

$$w = -\frac{\rho}{\rho_m} H$$

(33)

where $\rho$ is ice density [910 kg m$^{-3}$], $\rho_m$ mantle density [3000 kg m$^{-3}$] and $H$ ice thickness.

The response of the underlying viscous substratum leads to a diffusion equation (e.g. Oerlemans and Van der Veen, p.119) for bedrock elevation $h$:

$$\frac{\partial h}{\partial t} = D_a \nabla^2 (h - h_0 + w)$$

(34)

In this expression, $h_0$ is the undisturbed bedrock topography and $D_a$ [$10^8$ m$^2$ a$^{-1}$] a diffusion coefficient. Using equation (34) implies also that the characteristic time scale for bedrock sinking depends on the size of the load. With a typical length scale of 1500 km this leads to a relaxation time for bedrock adjustment $T = L^2 / D_a = 22500$ years.

3. Numerical procedure of solution

3.1. Coordinate Transformation

Integrating the thermodynamic equation in the computational stage on a grid fixed in space is rather inconvenient, as the upper and lower ice boundaries will generally not coincide with grid-points. Instead of decreasing the grid-size (implying larger computing times) or devising some or other interpolation scheme these problems may quite readily be overcome by introducing a new vertical coordinate $\xi$, scaled to ice thickness. This approach was originally proposed by Jenssen (1977) in analogy with the s-coordinate system in numerical weather forecasting.

The stretched dimensionless vertical coordinate is defined by:
\[ \zeta = \frac{H + h - z}{H} \]  

such that \( \zeta = 0 \) at the upper surface and \( \zeta = 1 \) at the base for all values of \( x, y \) and \( t \). The derivative of a variable \( f \) in the \( (x,y,z,t) \)-system (subscript \( z \) denotes a derivative in the \( x,y,z \)-system; \( \zeta \) stands for the new \( x,y,\zeta \)-system) is then transformed (e.g. Haltiner, 1971, p.194) as:

\[
\begin{align*}
\frac{\partial f}{\partial r} \bigg|_z &= \frac{\partial f}{\partial r} \zeta + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial r} \\
\frac{\partial f}{\partial z} \bigg|_z &= \frac{\partial \zeta}{\partial z} \frac{\partial f}{\partial \zeta}
\end{align*}
\]

leading to:

\[
\begin{align*}
\frac{\partial f}{\partial x} \bigg|_z &= \frac{\partial f}{\partial x} \zeta + \frac{1}{H} \frac{\partial f}{\partial \zeta} \left[ \frac{\partial (H+h)}{\partial x} - \zeta \frac{\partial H}{\partial x} \right] \\
\frac{\partial f}{\partial y} \bigg|_z &= \frac{\partial f}{\partial y} \zeta + \frac{1}{H} \frac{\partial f}{\partial \zeta} \left[ \frac{\partial (H+h)}{\partial y} - \zeta \frac{\partial H}{\partial y} \right] \\
\frac{\partial f}{\partial t} \bigg|_z &= \frac{\partial f}{\partial t} \zeta + \frac{1}{H} \frac{\partial f}{\partial \zeta} \left[ \frac{\partial (H+h)}{\partial t} - \zeta \frac{\partial H}{\partial t} \right] \\
\frac{\partial f}{\partial z} &= \frac{1}{H} \frac{\partial f}{\partial \zeta} \\
\frac{\partial^2 f}{\partial z^2} &= \frac{1}{H^2} \frac{\partial^2 f}{\partial \zeta^2}
\end{align*}
\]

In the now adopted layer approach the three-dimensional time-dependent \( V \)- and \( T \)-fields will be computed at levels of constant \( \zeta \) (the layer interfaces). The relevant equations (20), (21), (24) and (27) then become:
\[ \tilde{v}(Q) - \tilde{v}(1) = 2(\rho g H)^n H \left[ \nabla (H+h) \cdot \nabla (H+h) \right]^{n-1} \frac{H}{2} \nabla (H+h) \int_1^n A(T^*) \zeta^d d\zeta \] (43)

\[ \tilde{v}H = H \int_0^1 \tilde{v}(\zeta) d\zeta + \tilde{v}(1)H \] (44)

\[ w(\zeta) = \int_1^\zeta \left[ H \left\{ \frac{\partial u}{\partial x} \zeta + \frac{\partial v}{\partial y} \zeta \right\} + \frac{\partial v^*}{\partial \zeta} \cdot \left\{ \nabla (H+h) - \zeta \nabla H \right\} \right] d\zeta + w(1) \] (45)

\[ \frac{\partial T}{\partial t} = \frac{k}{\rho c_p H^2} \frac{\partial^2 T}{\partial \zeta^2} - u \frac{\partial T}{\partial x} \zeta - v \frac{\partial T}{\partial y} \zeta \]

\[ - \frac{1}{H} \frac{\partial}{\partial \zeta} \left[ \frac{\partial (H+h)}{\partial t} + \tilde{v} \cdot \nabla (H+h) - \zeta \left\{ \frac{\partial H}{\partial t} + \tilde{v} \cdot \nabla H \right\} - w \right] \]

\[ + \frac{g}{c_p} \zeta \frac{\partial v^*}{\partial \zeta} \cdot \nabla (H+h) \] (46)

with lower boundary condition:

\[ \left\{ \frac{\partial T}{\partial \zeta} \right\}_b = \frac{H}{k} \left[ G + \tilde{t}_b \cdot \tilde{v}(1) \right] \] (47)

and a similar transformation for (31).

It should be noted here that the velocity components \( u, v, w \) and the two-dimensional \( \nabla \)-operator are taken in the \( x, y, z \)-system.

### 3.2 Finite difference scheme

The equation describing ice sheet evolution gives rise to a nonlinear parabolic differential equation and may be interpreted as a diffusion equation for ice thickness \( H \):
\[
\frac{\partial H}{\partial t} = -\nabla \cdot (D \nabla (H + h)) + M - S = -\nabla \cdot (vH) + M - S
\]  

(48)

where in the layer approach the "vertically integrated" diffusion coefficient \(D\) now contains a deformational part evaluated at layers of constant \(\zeta\) \((D^1)\) and a contribution from basal sliding \((D^8)\):

\[
D = H \int_0^1 D^1(\zeta) \, d\zeta + D^8
\]  

(49)

As defined here \(D\) is a negative quantity equal to the scalar component of \((44)\).

Although in glaciology there appears to be at present a surge of interest in finite element modelling techniques (e.g. Hooke et al., 1979; Nixon et al., 1985), we have opted for a finite difference approach to solve \((48)\). This is mainly given in by the fact that the finite-element method does not seem to be suited for the type of studies we are envisaging. Apart from other complications the finite element method appears to be unable to deal with changing boundaries (Hutter, 1983, p.463) and does not seem to be as efficient in terms of computer time, especially when one wants to integrate a 3-D model for Antarctica, say, 100,000 years forward in time.

Having replaced the derivatives by finite differences the resulting equations are usually quite readily solved with an explicit integration scheme. However, such a scheme has the important drawback that in order to preserve stability time steps necessarily have to be taken small. On the other hand, a fully implicit scheme would be highly unpractical as it would require the solution of a set of nonlinear equations in two space dimensions at the advanced time level. Alternatively, an alternating-direction-implicit (ADI) scheme proved to perform very satisfactorily. This is a two-step method involving the solution of sets of equations along lines parallel to the \(x\)- and \(y\)-axes at the first and second step respectively. In order to avoid the complication of having to solve a set of nonlinear equations the scheme was adapted by evaluating the diffusion coefficient at the old time step. Although not unconditionally stable, partly due to strong nonlinear coupling through the source terms, it allows time steps to be chosen an order of magnitude larger than an explicit scheme, without a noticeable loss of accuracy (steady state solutions corresponding up to 5 figures). Solving the resulting tridiagonal linear systems with an equivalent to Gaussian elimination even turned out to take up less processing time than an explicit scheme with the same time step, leading to substantial savings of computer resources.

Consider a rectangular domain in \(x,y\)-space. Let \(i,j,k\) then denote the index numbers of a grid point with \(N_x, N_y, N\zeta\) their total numbers along the respective axes and \(t\) the time. \(D^1_{i,j,k,t}\) are then
integrated numerically from (see also fig.3):

\[
D^1_{i,j,k,t} = 2(\rho g)^n H^{n+1}_{i,j,t} \left[ G_x^2 + G_y^2 \right]^{\frac{n-1}{2}} A(T^*_{i,j,k+1,t}) + A(T^*_{i,j,k,t}) \cdot \frac{\left[ \xi_{k+1} + \xi_{k} \right]^n}{2} \\
\cdot \left[ \xi_{k} - \xi_{k-1} \right] + D^1_{i,j,k+1,t} \quad k = N\xi - 1, 1
\]

(50)

with

\[
G_x = \frac{(H+h)_{i+1,j,t} - (H+h)_{i,j,t}}{2\Delta x}
\]

(51)

\[
G_y = \frac{(H+h)_{i,j+1,t} - (H+h)_{i,j-1,t}}{2\Delta y}
\]

(52)

and boundary condition \( D^1_{i,j,N\xi,t} = 0 \), from which follow values for \( D_{i,j,t} \):

\[
D_{i,j,t} = H_{i,j,t} \sum_{k=2}^{N\xi} \left\{ \frac{D^1_{i,j,k,t} + D^1_{i,j,k-1,t}}{2} \cdot \frac{\xi_{k} - \xi_{k-1}}{2} \right\} + D^0_{i,j,t}
\]

(53)

---

fig.3: Grid representation for the calculation of \( D^1_{i,j,k,t} \).
The ADI-scheme is then constructed with the Peaceman-Rachford formula (Mitchell and Griffiths, 1980, p.60) employing a staggered grid in space, in effect calculating mass fluxes in between grid points with a mean diffusivity. Smoothing in this way helps to keep the integration stable and has the additional advantage that it enables the ice sheet to expand to regions where the mass balance is negative. This technique is widely used in the ice sheet models of Oerlemans and Van der Veen (1984):

\[ H_{i,j,t+1/2} + \frac{\Delta t}{2\Delta x} \cdot \left[ F_{i+1/2,j,t+1/2} - F_{i-1/2,j,t+1/2} \right] = \]

\[ H_{i,j,t} - \frac{\Delta t}{2\Delta y} \cdot \left[ F_{i,j+1/2,t} - F_{i,j-1/2,t} \right] + \frac{\Delta t}{2} \cdot \left[ M_{i,j,t} - S_{i,j,t} \right] \]

\[ i = 2, Nx-1 \quad \text{for} \quad j = 2, Ny - 1 \quad (54) \]

and similar sets of equations along rows of constant \( i \):

\[ fig.4 : \text{Grid used to perform the first x-ADI step along rows of constant } j. \]

Thick lines refer to the implicit direction

\[ fig.5 : \text{Grid used to perform the second y-ADI step along rows of constant } i. \]

Thick lines refer to the implicit direction
with mass fluxes \( F \) written in (54) as:

\[
F_{i+1/2,j,t+1/2} = \left[ \frac{H_{i+1/2,j,t+1/2} + h_{i+1,j,t,t} - H_{i,j,t+1/2} - h_{i,j,t}}{\Delta x} \right] \cdot \left[ \frac{D_{i+1,j,t} + D_{i+1,j,t}}{2} \right]
\]  

(56)

and analogues for \( F_{i-1/2,j,t+1/2}, F_{i,j+1/2,t}, F_{i,j-1/2,t} \), and in (55):

\[
F_{i,j+1/2,t+1} = \left[ \frac{H_{i,j+1/2,t+1} + h_{i,j+1,t+1/2} - H_{i,j,t+1} - h_{i,j,t+1/2}}{\Delta y} \right] \cdot \left[ \frac{D_{i,j,t+1/2} + D_{i,j+1,t+1/2}}{2} \right]
\]  

(57)

and analogues.

\[\text{fig.6: Representation of the staggered grid used to calculate mass fluxes in between grid points}\]
Rearranging the terms in (54) leads to a tridiagonal set of equations for row $j$ of the type:

$$-\alpha_{i,j,t} H_{i-1,j,t+1/2} + \beta_{i,j,t} H_{i,j,t+1/2} - \gamma_{i,j,t} H_{i+1,j,t+1/2} = \delta_{i,j,t} \quad i = 2, N_x-1$$  \hspace{1cm} (58)

with boundary conditions $H_{1,j} = H_{N_x,j}$, usually set as zero ice thickness (edge) or alternatively zero thickness gradient (divide). The time dependent elements of the coefficient matrix $\alpha, \beta, \gamma$ for the first $x$-ADI step and $\delta$ (coefficients for the second step are analogues) then read ($i = 2, N_x -1$):

$$\alpha_{i,j,t} = -\frac{\Delta t}{2(\Delta x)^2} \left[ \frac{D_{i-1,j,t} + D_{i,j,t}}{2} \right]$$  \hspace{1cm} (59)

$$\beta_{i,j,t} = 1 - \frac{\Delta t}{2(\Delta x)^2} \left[ \frac{D_{i-1,j,t} + 2D_{i,j,t} + D_{i+1,j,t}}{2} \right]$$  \hspace{1cm} (60)

$$\gamma_{i,j,t} = -\frac{\Delta t}{2(\Delta x)^2} \left[ \frac{D_{i,j,t} + D_{i+1,j,t}}{2} \right]$$  \hspace{1cm} (61)

$$\delta_{i,j,t} = H_{i,j,t} - \frac{\Delta t}{2\Delta y} \left[ F_{i,j+1/2,t} - F_{i,j-1/2,t} \right] + \frac{\Delta t}{2} \left[ M_{i,j,t} - S_{i,j,t} \right]$$

$$+ \gamma_{i,j,t} \cdot h_{i+1,j,t} \cdot (\beta_{i,j,t} - 1) \cdot h_{i,j,t} + \alpha_{i,j,t} \cdot h_{i-1,j,t}$$  \hspace{1cm} (62)

With the additional conditions (that are fulfilled) that:

$$\alpha_{i,j,t}, \beta_{i,j,t}, \gamma_{i,j,t} > 0 \quad \text{and} \quad \beta_{i,j,t} \geq \alpha_{i,j,t} + \gamma_{i,j,t}$$  \hspace{1cm} (63)

a highly efficient method is available for solving the tridiagonal system as follows (Mitchell and Griffiths, 1980, p.32):
The new ice-thicknesses at the intermediate time step are then found from backsubstitution:

\[
\begin{align*}
H_{i,j,t+1/2} &= w_{i-1,j,t} H_{i,j,t} + g_{i,j,t} \\
H_{i,j,t+1/2} &= w_{i,j,t} H_{i+1,j,t+1/2} + g_{i,j,t}
\end{align*}
\]

and condition \( H_{i,j,t+1/2} \geq 0 \)

This procedure involving the solution of \( Ny - 2 \) tridiagonal sets of \( Nx \) equations with \( Nx - 2 \) unknowns along rows of constant \( j \) is then repeated along rows of constant \( i \) to complete the integration forward in time. Temperature and bedrock elevation are updated every half time step. It is to be observed that the ADI-coefficients are time dependent and need continuous re-evaluation. As presented here the scheme does not allow for variable grid sizes in the horizontal direction as well as non-rectangular domains. As shown in Mitchell and Griffiths (1980), however, the scheme contains enough flexibility to be amended in the above fashion.

Realizing that mass fluxes are only known in between grid points, velocity components are also calculated in between grid points and subsequently interpolated onto grid points as follows:

\[
\begin{align*}
\text{Realizing that mass fluxes are only known in between grid points, velocity components are also calculated in between grid points and subsequently interpolated onto grid points as follows:}
\end{align*}
\]

\[
\begin{align*}
u_{i+1/2,j,k,t} &= 2 \left[ H_{i,j,t} D_{i,j,k,t}^1 + H_{i+1,j,t} D_{i+1,j,k,t}^1 + D_{i,j,t}^x + D_{i+1,j,t}^x \right] \\
&= \left[ \frac{(H+h)_{i+1,j,t} - (H+h)_{i,j,t}}{\Delta x} \right] \\
&= \left[ \frac{(H+h)_{i,j,t} + H_{i+1,j,t}}{\Delta x} \right]
\end{align*}
\]
and a similar procedure for $v_{i,j,k,t}$ using $v_{i,j+1/2,k,t}$ and $v_{i,j-1/2,k,t}$.

Finally, the vertical velocity components at layer $k$ follow from a straightforward numerical integration of (45) however keeping in mind that horizontal velocities are evaluated in between grid points:

$$w_{i,j,k,t} = \left\{ \begin{array}{l}
\left[ \frac{H_{i,j,t} + H_{i,j,t} + H_{i,j+1,t}}{4} \right] \left[ \frac{u_{i+1/2,j,k,t} + u_{i+1/2,j,k+1,t} - u_{i-1/2,j,k,t} - u_{i-1/2,j,k+1,t}}{2\Delta x} \right] \\
+ \left[ \frac{H_{i,j-1,t} + H_{i,j,t} + H_{i,j+1,t}}{4} \right] \left[ \frac{v_{i,j+1/2,k,t} + v_{i,j+1/2,k+1,t} - v_{i,j-1/2,k,t} - v_{i,j-1/2,k+1,t}}{2\Delta y} \right] \\
+ \left[ \frac{u_{i,j,k,t} - u_{i,j,k+1,t}}{\zeta_k - \zeta_{k+1}} \right] \left[ \frac{(1 - \zeta_k) H_{i+1,j,t} + h_{i+1,j,t} - (1 - \zeta_{k}) H_{i-1,j,t} - h_{i-1,j,t}}{2\Delta x} \right] \\
+ \left[ \frac{v_{i,j,k,t} - v_{i,j,k+1,t}}{\zeta_k - \zeta_{k+1}} \right] \left[ \frac{(1 - \zeta_k) H_{i,j+1,t} + h_{i,j+1,t} - (1 - \zeta_{k}) H_{i,j-1,t} - h_{i,j-1,t}}{2\Delta y} \right] \\
\end{array} \right\}$$

$$k = N\zeta - 1, 1 \quad (68)$$

Comparison of the calculated $w$ at the surface with the upper kinematic boundary condition (26) revealed that the scheme conserves mass very well.

For reasons of efficiency, accuracy and stability properties an ADI-scheme was chosen as well to compute the 2-D bedrock response. Since (34) is an ordinary parabolic differential equation it is a straightforward matter to write down the corresponding equations:

$$\alpha = \gamma = \frac{D_a \cdot \Delta t}{2 (\Delta x)^2} \quad (69)$$

$$\beta = 1 + \frac{D_a \cdot \Delta t}{(\Delta x)^2} \quad (70)$$
In this case the ADI-coefficients $\alpha, \beta, \gamma$ are independent of time and only need evaluation once. At the grid edge, boundary conditions follow from isostatic equilibrium.

With all necessary variables every half time step at hand we can now proceed to the numerical solution of the thermodynamic equation. A three-step ADI-technique looks cumbersome in this case as it would require a procedure to follow changing boundaries in order to avoid heat 'leaking through' the ice sheet edge. Realizing that the critical time step for stability is set by the smallest grid spacing, which is along $\zeta$, it seems only natural to make the terms involving $\zeta$-derivatives implicit (i.e. diffusion, vertical advection and the various correction terms). Calculations pointed out that this approach influenced accuracy only minimal, with resulting temperatures corresponding to $10^{-2}$K in comparison with a fully explicit scheme. Even with time steps up to 500-1000 years, the calculations turned out to be fully stable. However, a note concerning the horizontal advective terms is in order here, as replacing the derivatives by central differences turned out to generate oscillations in the solution. This problem is usually circumvented in diffusion-convection equations with a high Peclet number (i.e. a constant proportional to the ratio of advective velocity and diffusivity) by introducing an artificial horizontal diffusion process. (e.g. Mitchell and Griffiths, 1980, p.224; Gladwell and Wait, 1979, chap. 11). We use an 'upstream' difference scheme that can be shown to introduce an artificial horizontal diffusivity equal to $u \Delta x / 2$, with $u$ velocity and $\Delta x$ the grid spacing. Besides stabilizing the integration, this influences results only marginally, as in most cases the associated artificial heat transfer turns out to be an order of magnitude smaller than the horizontal heat advection. Additionally, this approach ensures calculations to be pursued right till the edge as the horizontal velocity vector points outwards. Again, the resulting tridiagonal set of $N\zeta$ equations in $N\zeta - 2$ unknowns is most easily solved by Gaussian elimination. Boundary conditions follow from the mean annual surface temperature and the basal temperature gradient respectively, while initially ice temperatures are set equal to the corresponding air temperatures at that altitude. The resulting set of

$$
\delta_{i,j,t} = h_{i,j,t} + \frac{D_a \Delta t}{2(\Delta x)} \left[ \frac{\rho}{\rho_m} \left( H_{i+1,j,t+1/2} - 2H_{i,j,t+1/2} + H_{i-1,j,t+1/2} \right) - h_{i+1,j}^0 + 2h_{i,j}^0 - h_{i-1,j}^0 \right] 
$$

$$
+ \frac{D_a \Delta t}{2(\Delta y)^2} \left[ h_{i,j+1,t+1/2} - 2h_{i,j,t} + h_{i,j-1,t} + \frac{\rho}{\rho_m} \left( H_{i,j+1,t+1/2} - 2H_{i,j,t+1/2} + H_{i,j-1,t+1/2} \right) 
$$

$$
- h_{i,j+1}^0 + 2h_{i,j}^0 - h_{i,j-1}^0 \right] \quad i = 2, N_x - 1
$$

(71)
equations reads:

- \( \alpha_{i,j,k,t} T_{i,j,k-1,t+1} + \beta_{i,j,k,t} T_{i,j,k,t+1} - \gamma_{i,j,k,t} T_{i,j,k+1,t+1} = \delta_{i,j,k,t} \)

\( k = 2, N\zeta - 1 \)

with:

\[
\alpha_{i,j,k,t} = \frac{2k \Delta t}{\rho c_p H_{i,j}^2 (\zeta_{k+1} - \zeta_k)(\zeta_k - \zeta_{k-1})} + \left[ \frac{\Delta t}{H_{i,j} (\zeta_{k+1} - \zeta_k)} \right] \cdot
\]

\[
\begin{aligned}
&\left[ u_{i,j,k} \cdot \frac{(1-\zeta_k)H_{i+1,j} + h_{i+1,j} - (1-\zeta_k)H_{i-1,j} - h_{i-1,j}}{2\Delta x} + v_{i,j,k} \cdot \frac{(1-\zeta_k)H_{i,j+1} + h_{i,j+1} - (1-\zeta_k)H_{i,j-1} - h_{i,j-1}}{2\Delta y} \\
&+ \frac{(1-\zeta_k)H_{i,j,t+1} + h_{i,j,t+1} - (1-\zeta_k)H_{i,j,t} - h_{i,j,t}}{\Delta t} \right] \\
\end{aligned}
\]

\[
\beta_{i,j,k,t} = 1 + \frac{2k \Delta t}{\rho c_p H_{i,j}^2 (\zeta_{k+1} - \zeta_k)(\zeta_k - \zeta_{k-1})} \]

\[
\gamma_{i,j,k,t} = \frac{2k \Delta t}{\rho c_p H_{i,j}^2 (\zeta_{k+1} - \zeta_k)(\zeta_k - \zeta_{k-1})} - \left[ \frac{\Delta t}{H_{i,j} (\zeta_{k+1} - \zeta_k)} \right] \cdot
\]

\[
\begin{aligned}
&\left[ u_{i,j,k} \cdot \frac{(1-\zeta_k)H_{i+1,j} + h_{i+1,j} - (1-\zeta_k)H_{i-1,j} - h_{i-1,j}}{2\Delta x} + v_{i,j,k} \cdot \frac{(1-\zeta_k)H_{i,j+1} + h_{i,j+1} - (1-\zeta_k)H_{i,j-1} - h_{i,j-1}}{2\Delta y} \\
&+ \frac{(1-\zeta_k)H_{i,j,t+1} + h_{i,j,t+1} - (1-\zeta_k)H_{i,j,t} - h_{i,j,t}}{\Delta t} \right] \\
\end{aligned}
\]

\[
\delta_{i,j,k,t} = T_{i,j,k,t} + \left[ \frac{u_{i,j,k+1} - u_{i,j,k-1}}{\zeta_{k+1} - \zeta_k} \cdot \frac{H_{i+1,j} + h_{i+1,j} - H_{i-1,j} - h_{i-1,j}}{2\Delta x} + \right. \\
\left. \frac{v_{i,j,k+1} - v_{i,j,k-1}}{\zeta_{k+1} - \zeta_k} \cdot \frac{H_{i,j+1} + h_{i,j+1} - H_{i,j-1} - h_{i,j-1}}{2\Delta y} \right] \cdot g \frac{\zeta_k \Delta t}{c_p} - \Delta t [ A_x + A_y ]
\]
with

\[ A_x = u_{i,j,k} \cdot \frac{T_{i,j,k} - T_{i-1,j,k}}{\Delta x} \]

if \( v_{i,j,k} > 0 \)

\[ A_x = u_{i,j,k} \cdot \frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta x} \]

if \( v_{i,j,k} < 0 \) (76)

and analogues.

Although the conditions mentioned in (63) are not met in all cases, intensive testing showed no sign of spurious behaviour.

4. Some basic testing

In order to illustrate the basic behaviour of the model, a series of steady state experiments was set up. They mainly aim at presenting plots of basic variables and at assessing a "suitable" vertical resolution. To keep things simple complete symmetry in the x and y directions was put forward so only a two-dimensional cross-section of a half ice sheet was actually computed. For this purpose we considered a "clean" flowline with a smooth bedrock profile extending from the ice divide at \( x = 0 \) to a fixed edge position at \( x = 1500 \) km. At present, features at the ice-bedrock interface such as basal sliding and basal water flow and at the edge (coupled ice shelf and the possibility of moving boundaries), were isolated from the experiments.

![Figure 7: Full model steady state ice thickness profile. Labelled columns denote locations of vertical profiles; AA' = divide; BB' = 500 km; CC' = 1000 km; DD' = edge. This model run has 10 unevenly distributed layers in the vertical, \( \Delta x = 50 \) km, \( \Delta t = 50 \) a.](image)

Having in mind a schematic East Antarctic flowline, mass-balance [ma\(^{-1}\)] and mean annual surface
The temperature [°C] were parameterized by the following linear relations:

\[
M = \max \left[ 0.05; -2 \times 10^{-4}(H+h) + 0.65 \right] \text{ m ice/year} \tag{77}
\]
\[
T_s = -0.012(H+h) - 15 \text{ °C} \tag{78}
\]

such that at a surface elevation of \((H + h) = 3000 \text{ m}\), \(M = 0.05 \text{ m a}^{-1}\), \(T_s = -51 \text{ °C}\) and at sea level \(M = 0.65 \text{ m a}^{-1}\), \(T_s = -15 \text{ °C}\) respectively.

Initially, computations started with zero ice thickness and bedrock elevation. A steady state was usually reached after 150000 years of integration. With \(\Delta x = 50 \text{ km}\), \(\Delta t = 50 \text{ years}\) and 10 layers in the vertical this required, including graphical output, about 450 CP seconds on a CDC-Cyber 750 computer.

**fig.8**: Surface mass balance in the steady state ice sheet corresponding to fig.7

**fig.9**: Vertical mean velocity in the steady state ice sheet corresponding to fig.7
Figs 7-15 show steady state plots of the various variables involved in the "standard" model run. In order to obtain a realistic height-to-width ratio it proved necessary, however, to multiply the a's in (15), (16) by a factor 20. This might, apart from crystal fabric effects and the uncertainty in n, be the consequence of the neglect of basal sliding forcing the bulk of the movement to result from deformation only. Obviously, this is not always the case in real ice sheets, as the ice at the edge is often channelled in outlet glaciers or ice streams with a considerable amount of the motion most probably due to basal sliding. The overall profile as depicted in fig. 7 resembles quite realistically an ideal continental ice sheet cross-section. The main effect of the inclusion of the ice flow-temperature feedback appears to be a less pronounced convex surface profile as compared to an isothermal model run. As a result of an as yet improper treatment of the edge (fixed boundary and the neglect of normal stress deviators and an ice shelf) fig. 9 shows the expected asymptotic increase of the vertical mean velocity towards the edge. Requiring continuity it would actually attain infinity at the edge itself.

As far as can be judged from sparse observations (Robin, 1983) the model produces very realistic temperature profiles (figs. 10 and 15) including features as the reversed temperature gradient towards the edge, a slight upward convex T-profile in the basal layers due to layer heating and the overall importance of friction and consequently warm basal ice towards the outer edge. As the possibility of a temperate basal layer and basal sliding frictional heating are not included yet, the basal melting rate has an upper limit corresponding to the geothermal heat flux (fig. 12).
**Fig. 12**: Basal melting rates corresponding to fig. 11

**Fig. 13**: Vertical profiles of the $x$-velocity component corresponding to the columns in fig. 7 (a,b,c,d).

For comparison the $x$-velocity distribution is added of a steady state ice sheet without temperature dependence of $A$ (fig. 13e).
The x-velocity profiles resemble almost block-flow, which is still more pronounced towards the edge (fig.13). Non-zero x-velocities at the divide (fig.13a) are due to the smoothing procedure in the numerical scheme. At x = 1500 km (fig.13d) over 50% of the velocity shear appears to be concentrated in the lowermost 5% of ice. This is a direct consequence of the temperature dependence of the creep behaviour of ice. In the lowermost 500m or so, shear strain rates easily differ by a factor 10 due to temperature, while the corresponding factor due to stress differences is typically around 2. In the isothermal case (with no temperature-velocity coupling) the share of the total velocity gradient in the lower 5% reduces to about 18% (fig.13e). In this case velocity is according to the derivation in (20) an (n+1)-order function of depth, n being the flow-law exponent. As observations are almost absent it is difficult to assess the reality of the computed profiles but a velocity-depth profile from Dye 3 in Greenland (Gundestrup and Hansen, 1984) seems to point to a more gradual velocity decrease with depth. On the other hand, the Dye 3 borehole shows a marked discontinuity in the deformation characteristics at the Wisconsin/Holocene transition, with the Wisconsin ice deforming more easily, probably due to dissolved impurities. Also, n = 3 might not be the appropriate value, a point taken up recently by Doake and Wolff (1985) who postulate a linear flow law with n = 1 for polar ice. The longitudinal strain rate $\dot{e}_{xx}$ as a function of z thus appears to be approximately constant in the upper 80 or 90% or so of the ice sheet. Consequently, according to the incompressibility condition $\dot{e}_{xx} = -\dot{e}_{zz} = -\partial w / \partial z$, it is not surprising to find the z-velocity component profiles to be almost linear (fig.14e). Positive w's towards the edge are the upward velocity component due to the sloping bedrock. Fig.14e shows the model run corresponding to fig.13e (no temperature-velocity coupling) and fig.14f results from an experiment without bedrock adjustment and thus zero basal w-boundary condition.
The vertical resolution appears to be very critical with regard to the temperature solution, while the influence on the overall ice thickness distribution remains rather limited (the vertical resolution affects the accuracy of the numerical integration of the vertically integrated diffusivity $D_{i,j,k}$). Increasing the vertical resolution, more specifically in the lower layers, tends to produce a colder base towards the interior and a warmer base in the outer part (fig.16). This is because the bulk of the velocity shear is concentrated in the basal layers and a crude resolution there leads to a loss of frictional heating and horizontal advection respectively. Fig.16d is the 'standard model run' with a lowermost grid spacing of $\Delta \zeta = 0.02$. Adding one more basal layer at $\zeta = 0.99$ does improve the basal temperature solution (fig.16f) but starts to put too heavy demands on the numerical scheme as the minimum thickness to start temperature calculations has to be adapted likewise ($\Delta z = 5$ m is set as a lower limit). To make sure that no frictional heating or horizontal advection is lost the lower but one layer interface should be located in a place where $\partial^2 v / \partial z^2$ becomes negligible (fig.18). Obviously, to catch the essential features of the $x$-velocity profile with depth more layers should be added in the lower part of the ice sheet (fig.18e). On the other hand, the longitudinal resolution appears to be less critical to the temperature solution (fig.19), although the choice of $\Delta x$ will of course be mainly given in by the topographical resolution one desires. In the case of $\Delta x = 100$ km the maximum allowable time step for stability is, the limiting factor being the ice thickness continuity integration, at least twice the value considered ($\Delta t = 100$ a). Since the temperature and velocity calculations are uncoupled during one time step, the stability criterion should not necessarily be fully exploited in order to preserve sufficient transient detail. For reasons given before, the longitudinal resolution has a marked influence on the velocity at the edge (fig.20).
fig. 15: Steady state temperature profiles corresponding to the columns in fig.7 and fig.10
fig. 16: Steady state basal temperatures in experiments where the vertical resolution is altered. (a): 5 evenly distributed layers with $\Delta \zeta = 0.2$; (b): 10 evenly distributed layers with $\Delta \zeta = 0.1$; (c): 20 evenly distributed layers with $\Delta \zeta = 0.05$; (d): 10 unevenly distributed layers with $\zeta = 0; 0.15; 0.30; 0.45; 0.60; 0.75; 0.83; 0.90; 0.95; 0.98; 1.00$; this version corresponds to fig. 7 (e): an extra layer $\zeta = 0.99$ is added at the bottom.

fig. 17: Temperature profiles at the edge corresponding to the experiments described in fig. 16 (a,b,c,d,e).
Figure 18: X-velocity component profiles at the edge corresponding to the experiments described in Figure 16 (a,b,c,d,e).
Fig. 19: Steady state basal temperatures in experiments where the horizontal resolution is altered; (a): $\Delta x = 25$ km, $\Delta t = 10$ a; (b): $\Delta x = 100$ km, $\Delta t = 100$ a; (c): $\Delta x = 50$ km, $\Delta t = 50$ a; this version corresponds to fig. 7.

Fig. 20: Steady state vertical mean velocity corresponding to the experiments in fig. 19 (a,b,c).

Finally, fig. 21 shows several model runs in which one or more terms in the heat transfer equation are "switched off". They basically show what one expects (see e.g. Oerlemans and Van der Veen (1984),...
chapter 5).

\[ x - 1000 \text{ km} \]
\[ z = 1000 \text{ km} \]
\[ x = 1000 \text{ km} \]

\( T_{\text{Temperature}} \) in °C

(a) only downward diffusion is dealt with;
(b) downward advection is added;
(c) horizontal advection is added;
(d) friction is dealt with;
(e) only diffusion and friction;
(f) model run with downward advection, diffusion and friction.

fig. 21: Steady state temperature profiles for \( x = 1000 \text{ km} \) in a model run without temperature dependence of \( A \) (corresponding to fig. 13e; fig. 14e).

5. Final Remarks

In this report a three-dimensional numerical model for studying the time-dependent behaviour of cold ice masses was developed. It differs mainly from previous models that it computes the mutually interactive temperature and velocity fields and that no a priori assumptions are made regarding their profiles or a steady state. The model is especially designed for use in regions where horizontal shear
is the dominant flow mechanism, i.e. in grounded parts of continental ice sheets. However, it contains enough degrees of freedom and produces relevant boundary conditions to take into account features like basal sliding. As such the model serves as a 'core' towards the development of a general polar ice sheet.

As far as can be judged from available data regarding temperature profiles the model produces very realistic solutions. About ten layers in the vertical concentrated towards the base of the ice sheet appear to be sufficient to capture their essential characteristics. The main restriction regarding the velocity with depth distribution appears to be a firm knowledge with respect to the creep behaviour of ice.

Employing the alternating-direction implicit method to solve the coupled equations numerically leads to considerable savings of computer resources with respect to a straightforward explicit integration scheme. The full three-dimensional model can be expected to require on CDC Cyber 750 computer approximately $6N_xN_yN_z$ memory positions and $50N_xN_yN_z/\Delta t$ central processor seconds for a 100000 year integration (with $N_x$, $N_y$, $N_z$ the number of gridpoints along their respective axes and $\Delta t$ the time step in years). Thus, tentatively representing the whole of Antarctica by a 5500 x 5500 km box, with $\Delta x = \Delta y = 100$ km, $\Delta t = 100$ years and ten layers in the vertical brings a 200000 year integration within the limits of the aforementioned machine. However, this still means that for a reasonable resolution of, say, 50 km, the model should be implemented on a vector computer, for example on a CRAY, see table below.

<table>
<thead>
<tr>
<th>grid spacing</th>
<th>time step</th>
<th>memory positions</th>
<th>CP seconds for 100000 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 km</td>
<td>200 years</td>
<td>$= 0.2 \times 10^6$</td>
<td>$= 8500$ CP sec</td>
</tr>
<tr>
<td>50 km</td>
<td>50 years</td>
<td>$= 0.8 \times 10^6$</td>
<td>$= 135000$ CP sec</td>
</tr>
<tr>
<td>20 km</td>
<td>8 years</td>
<td>$= 5.0 \times 10^6$</td>
<td>too much</td>
</tr>
</tbody>
</table>

*table 1: estimated memory positions and calculation time on a CDC Cyber 750 computer for a 100000-year integration on Antarctica, represented by a 5500 x 5500 km box*

At present studies are under way at assessing more thoroughly the potential importance of the temperature-velocity feedback with respect to changing climatic conditions. Preliminary calculations seem to suggest that the mechanism of creep instability s.s. is too weak to give rise to surge-like
behaviour, in contrast with previous theoretical studies on the subject. Obviously, a good deal of
ground still needs to be covered before the model can be expected to be applied to the whole Antarctic
Ice Sheet. Problems involved not only relate to computational procedures (faster and more efficient
numerical techniques) but also to the underlying physics (key problems like features at the
ice-bedrock interface, grounding-line movement) and more complete observational data.

Acknowledgement
Philippe Huybrechts is supported by the Belgian National Fund for Scientific Research (N.F.W.O.)
and in part sponsored by the Belgian Ministry of Science Policy under contract ANTAR /04.
References

Budd W.F. ; Jenssen D. and Radok U. (1971) : Derived physical characteristics of the Antarctic Ice Sheet, ANARE Interim Report, Series A(TIV), Glaciology, Publ.120, 178p + maps
Budd W.F. ; Young N. and Austin C.R. (1976) : Measured and computed temperature distributions in the Law Dome Ice Cap, Antarctica, J. Glaciology 16, 99-110
Hooke R. Le B. et al. (1979) : Calculations of velocity and temperature in a polar glacier using the finite-element method, J. Glaciology 24, 131-146
Jenssen D. (1977) : A three-dimensional polar ice-sheet model, J. Glaciology 18, 373-389
Nixon W.A. et al. (1985) : Applications and limitations of finite element modeling to glaciers: a case study, J. Geophys. Res. 90, 11303-11311
Philibert K. and Federer B. (1971) : On the temperature profile and age profile in the central part of cold ice sheets, *J. Glaciology* 10, 3-14
Robin G. de Q. (1955) : Ice movement and temperature distribution in glaciers and ice sheets, *J. Glaciology* 2, 523-532
Weertman J. (1968) : Comparison between measured and theoretical temperature profiles of the Camp Century, Greenland, borehole, *J. Geophys. Res.* 73, 2691-2700
Young N.W. (1981) : Responses of ice sheets to environmental changes, *IAHS Publ.* 131, 331-360