Slow traveling waves on a finite interval for Burgers’ type equations *

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Abstract

In this paper we study for small positive \( \varepsilon \) the slow motion of the solution for evolution equations of Burgers’ type with small diffusion,

\[
    u_t = \varepsilon u_{xx} + F(u)_x, \quad u(x,0) = u_0(x), \quad u(\pm 1,t) = u_\pm, \tag{*}
\]

on the bounded spatial domain \([-1,1]\); \( F \) is a smooth non-positive function having only a finite number of zeros (at least two) between \( u_- \) and \( u_+ \), all of finite order. The initial and boundary value problem \( (*) \) has a unique asymptotically stable equilibrium solution that attracts all solutions starting with continuous initial data \( u_0 \). On an interval \([-1 - c_0\varepsilon, 1 + c_0\varepsilon]\), \( c_0 > 0 \) the differential equation has slow speed traveling wave solutions generated by profiles that satisfy the boundary conditions of \( (*) \). During a long but finite time interval, such traveling waves suitably describe the slow long term behaviour of the solution of evolution problem \( (*) \). Their speed characterizes the local velocity of the slow motion with algebraic precision (w.r.t. \( \varepsilon \)) in general, and with exponential precision, if \( F \) has only two zeros of first order located at \( u_- \) and \( u_+ \). A solution that starts near a traveling wave, moves in a small neighborhood of such a traveling wave during a long time interval \((0,T)\). If \( F \) has zeros of order higher than 1, the equilibrium and the traveling wave are multi-shock solutions of \( (*) \). This situation differs strongly from the case where \( F \) has only a 1st order zero at both \( u_- \) studied by the authors in a previous paper. In this paper we consider multi-shock solutions of \( (*) \). Moreover, we improve some results of the previous paper, allowing a larger ball of initial data.

1 Introduction

In this paper we study the slow motion of internal layers in the solution of an equation of Burgers’ type (1D advection-diffusion) with small diffusion. This type of problems has been the subject of much recent interest. We highlight some ideas and results and give a few references to other papers. Slow motion is studied for reaction-diffusion (or Allan-Cahn or scalar Ginzburg-Landau) equations among others in [4], [9], [10] and for Cahn-Hilliard equations in [8], [1]. For convection-diffusion problems it is among others studied by formal methods in [22], [17] and more rigorously in [12] and, using non-standard analysis, in [6]. Connected to “slow motion” is the rather vague concept of “metastability”. Generally, a solution is called metastable, when its motion and change of form can be noticed on very long time intervals only, such that the authors in [18] remark, that “only the most persistent won’t be tempted to abort any computation prematurely, since the solution seems to be stationary”. The length of such a “very long time interval” is expressed as a function of the small positive parameter \( \varepsilon \); it is polynomially \( O(\varepsilon^{-\alpha}) \) or exponentially \( O(e^{\alpha/\varepsilon}) \) large (for some \( \alpha > 0 \), and depends heavily on the problem under consideration. The mechanism that generates

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such behaviour generally is some type of “resonance” due to one (or several) small eigenvalues of the linearized operator. Typically the non-linear equation has a one-parameter family of approximate solutions and the derivative w.r.t. this parameter provides an approximate eigenfunction with a small eigenvalue for the Frechet-derivative of the non-linear equation. From the point of view of the calculation of asymptotic approximations, we often see that standard methods like “matched asymptotic expansions” break down, because of the difference in scale between polynomially and exponentially small terms, e.g. see [9], [22], [17], and in related linear problems [11].

Although the phenomena look analogous, the analysis in the three types of equations, mentioned above, is quite different. In the case of a reaction-diffusion equation with a bistable nonlinearity like \( u_t = \varepsilon^2 u_{xx} + u^3 - u \), a slow solution hops between the “stable” states 0 and 1 and the interfaces between those states move with exponentially slow velocity, see Carr & Pego [9]. The profile of the interfaces hardly depends on its location and its derivative is an approximate eigenfunction with an exponentially small eigenvalue, see [3]. In [23] and [24], Ward formally derives formulae for the (exponentially small) velocity of the interfaces by projection onto this space of approximate eigenfunctions (otherwise stated, using formally Fredholm’s alternative). The analysis of the small eigenvalues of the linearization around a profile is relatively easy in this case, because the linearized operator is not far from being selfadjoint. In the case of the fourth order Cahn-Hilliard equation, eigenvalues of the linearization around a traveling wave solution (which has an exponentially slow speed), we are able to characterize metastability as a contraction transversal to the travelling wave. A major problem is that the linearized operator (around a traveling wave) is far from being selfadjoint for small \( \varepsilon \) and that the transformation to make it selfadjoint uses a weight function that is exponentially small outside an \( O(\sqrt{\varepsilon}) \) neighbourhood of the transition layer location. In this paper we improve this metastability result and we extend our analysis to the case, where \( F \) is non-convex, where multiple transition layers may occur and where the slow velocity is of algebraic order \( O(\varepsilon^n) \) only.

On the half strip \((-1, 1) \times \mathbb{R}_+\) we consider the Initial & Boundary Value (IBV) problem,

\[
(\text{IBV}) \quad \begin{cases} 
  u_t = \varepsilon u_{xx} + F(u)_x, & \text{if } (x, t) \in (-1, 1) \times \mathbb{R}_+ \\
  u(x, 0) = u_0(x), & \text{if } x \in (-1, 1) \\
  u(\pm 1, t) = u_0(\pm 1) = u_\pm, & \text{if } t \in \mathbb{R}_+ 
\end{cases}
\]

with continuous initial condition \( u_0(x) \) and compatible constant boundary values \( u(1, t) = u_0(1) = u_+ \) and \( u(-1, t) = u_0(-1) = u_- \); \( \varepsilon \) is a small positive parameter and \( F \) is a smooth function. Since the (IBV)-problem is not changed, if a constant is added to \( F \), we may assume that \( F \) is non-positive on the closed interval \([u_-, u_+]\) and has a maximum zero there. Also, since the classical solution of (IBV) is bounded we may assume that \( F \) has compact support. In [12] we have analyzed the case where \( F(u) \) has simple zeros at \( u = u_\pm \) and is strictly negative in between; we have established
traveling shock waves with exponentially small velocity and a unique stable equilibrium solution, which has a shock layer somewhere in the interior of the interval.

In this paper we significantly improve some of the results in [12] (theorems 3.9, 3.11 and 5.10, 5.12) and we analyse the situation, where \( F \) has several zeros between \( u_\text{-} \) and \( u_\text{+} \). In that case we may have several internal transition layers, which are only of algebraic order as pointed out by O’Malley [20] and Bohé [5]. Let us make formulation easy and suppose that \( F \) has several zeros between \( u_\text{-} \) and \( u_\text{+} \). In that case we may have several internal transition layers, which are only of algebraic order as pointed out by O’Malley [20] and Bohé [5].

The main hypothesis on \( F \):

\[ F \in C^\infty(\mathbb{R}) \] has a finite number of zeros on the interval \([u_\text{-}, u_\text{+}]\), but at least two; all zeros are of finite order.

The set of all zeros is denoted by \( \{t_1 < t_2 < \cdots < t_\ell\} \) and the subset of zeros of maximal order \( p \) is denoted by \( \{s_1 < s_2 < \cdots < s_n\} \). The Taylor coefficients at those zeros are denoted by \( \zeta_j \) and \( \xi_j \) respectively,

\[
F(s) = -\zeta_j \left| s - t_j \right|^\nu (1 + O(s - t_j)), \quad 1 \leq j \leq \ell, \quad \text{if } u_\text{-} \leq s \leq u_\text{+}, \quad (1.4)
\]

where \( \nu := \max\{r_j | j = 1 \cdots \ell\} \) and \( q := \max\{r_j < p | j = 1 \cdots \ell\} \). We shall show, that all zeros of lower order can be ignored, except possibly either the smallest outer zero \( t_1 < s_1 \) or the largest outer zero \( t_\ell > s_n \), such that we can simplify notation and denote its order by \( r \). Since the class of problems (IBV) is invariant w.r.t. linear change of variables \( u \mapsto au + \beta, \alpha > 0 \), we always can ensure that \( u_\text{-} = -1, a := u_\text{+} > 0 \) and that the zero level is between \( s_1 \) and the next zero of \( F \) (or between \( t_1 \) and \( s_1 \) if \( n = 1 \)). Thus, if \( n \geq 2 \), we always suppose \( s_1 < 0 \). This will simplify several formulae in section 2.

**Example 1:** In fig. 1 we have plotted the equilibrium solution \( u \) in a typical example, where \( F(u) = -\cos^2(2\pi u) \). The graph of the function \( F \) at the left is inverted to make clear the correspondence between the zeros (maxima) of \( F \) and the flat regions of \( u \).
Figure 2: 3D plot of the evolution of the solution $u(x,t)$ of problem (IBV) with nonlinearity $F(u) := -3(u + 1) (x + .25)^2 (x - 1)^2$ and initial profile $u_0(x) = x$, integrated numerically using an explicit monotone scheme.

Figure 3: The solution $u$ of fig.2 as a function of $x$ at $t = 0$, $t = 0.17$, $t = 2.3$ and $t = 19$. The graph of the function $u \rightarrow F(u)$ in the center is inverted to make clear the correspondence between the zeros (maxima) of $F$ and the flat regions of $u$. The graph shows two slow (internal) traveling waves; the right one clearly moves (much) slower than the left one.

**Example 2:** In fig. 2 and fig. 3 we show the evolution of the solution in the case where $\varepsilon := 0.05$ and where the non-linear function $F(u) := -3(u + 1) (x + .25)^2 (x - 1)^2$ has three zeros (maxima), two of order 2 and the third of order 1. The (approximate) solution has been integrated numerically.
using an explicit monotone scheme. It displays nicely the behaviour, derived in section 2.4. After a fast initial phase, flat zones develop at the levels $-1, -\frac{1}{4}$ and $+1$ and transition layers in between. Those transition layers move slowly to the right and left boundaries; due to the fact that $-1$ is a zero of $F$ of first order, i.e. of lower than both other zeros, the corresponding jump from $-1$ to $-\frac{1}{4}$ moves faster than the other one.

**RESULTS:** In section 2 we study the structure of the stationary solution and the traveling wave profiles that satisfy the boundary conditions (1.3). There is a unique stationary solution, which is monotone. We show existence of an one parameter family of monotone traveling wave profiles, characterized by flat regions, corresponding to zeros of $F$ of maximal order, by transitions layers between those zones and by possible boundary layers at both endpoints, as pictured in fig. 1. We find the exact location of the internal transition layers and compute their number. If all zeros of lower order are between $s_1$ and $s_n$ then the equilibrium and the traveling wave profile both have $n - 1$ internal layers. If $n = 1$ then the equilibrium has only two boundary layers and the traveling wave profile has one internal layer and two boundary layers. Finally, if $n \geq 2$ and there are lower order zeros outside the interval $[s_1, s_n]$ then the equilibrium has $n - 1$ internal layers, while the traveling wave profile may have $n$ internal layers.

In section 3 we use the maximum principle as in [12] to investigate the evolution of the solution of (IBV), starting at a traveling wave profile. In Corollary 3.2 we prove estimates in both maximum norm and weighted Sobolev norm w.r.t. $x \in [-1, 1]$, showing that the solution stays in a small neighbourhood of the traveling wave profile during a long time interval $T_\varepsilon = O(\varepsilon^{-\frac{1}{p-1}})$ if $p \geq 2$. The case $p = 1$ was considered in [12], Corollary 5.4.

In section 4 we study the stability of the equilibrium solution. The global stability (cf. Theorem 4.8) is already proved in [12]. Here our concern is the local (asymptotic) stability. As in [12] we use linearization around the equilibrium and contraction methods. To this end first we estimate the eigenvalues of the corresponding linear operator. The case $p \geq 2$ (cf. Theorem 4.6) is easy and the efforts are devoted to the case $p = 1$ (cf. Theorem 4.7). Our goal is to improve significantly the result of Theorem 3.9 in [12], allowing a larger class of initial data. Namely, the $H^1$-ball of very small radius $\rho = e^{-c/\varepsilon}$, $c > 0$, is replaced by a larger ball of radius $\rho = \sqrt{\varepsilon}$. (see lemma 4.5) The method relies on a further development of the geometric approach used in [12] to prove lemmas 3.5 and 3.7. This approach is an usual one in the stability theory for dynamical systems [14]. We use that the gap between the smallest eigenvalue and the next one is of order unity (what is true only in the case $p = 1$) and prove that in the $H^1$-ball of radius $\rho = \sqrt{\varepsilon}$ of the initial data, there is a submanifold $Y_\varepsilon$ of codimension one, orthogonal to the ground eigenfunction at zero, that is fast decaying in the sense that the trajectory starting from $Y_\varepsilon$ is approaching the equilibrium with a rate of convergence governed by the next eigenvalue (of order one). The main idea is first to replace the true eigenfunction by a one parameter family of approximate eigenfunctions, that give rise to trajectories, easily controlled by the maximum principle. The second step consists of decomposing the initial data into two transversal parts: one is generated by an approximate eigenfunction and the other is taken from a corresponding fast decaying submanifold. (cf.formula (4.55)).

In section 5 we generalize the results from section 3 about metastability of the slow motion (Theorems 5.4–5.5), taking the initial data from a small neighbourhood of a traveling wave profile. In the case $p = 1$ (Theorem 5.5) we use the same geometric approach as in the previous section. The main difference is that the traveling wave can approximate the solution only during a finite time interval. According to the above decomposition of the initial data, the trajectory is decomposed into two parts. The first part is very small as $\varepsilon \to 0$, while the second part is drifting exponentially slowly during an exponentially long time interval, thus exhibiting a metastable behaviour.

**WEIGHTED NORMS.** For the study of convergence we use the standard $L^2(-1,1)$-norm $\| \cdot \|$ and
the $\varepsilon$-dependent Sobolev norm $\| \cdot \|_1$ defined by
\[
\| u \|^2 := \varepsilon^2 \| u' \|^2 + \| u \|^2 = \int_{-1}^{1} \left\{ \varepsilon^2 |u'(x)|^2 + |u(x)|^2 \right\} \, dx.
\] (1.5)
Moreover we shall consider weighted Sobolev norms $\| \cdot \|_h$ for given weight functions $h(x)^2$,
\[
\| u \|_h := \| u h \|_1, \quad \text{or} \quad \| u \|^2_h = \int_{-1}^{1} \left\{ \varepsilon^2 |(u(x) h(x))'|^2 + |u(x) h(x)|^2 \right\} \, dx.
\] (1.6)
Two norms $\| \cdot \|_{h_1}$ and $\| \cdot \|_{h_2}$ or two weight functions $h_1$ or $h_2$ are said to be equivalent if positive constants $c_1$ and $c_2$ exist such that
\[
c_1 \| u \|_{h_1} \leq \| u \|_{h_2} \leq c_2 \| u \|_{h_1} \quad \text{for all} \quad u \quad \text{or} \quad c_1 h_1(x) \leq h_2(x) \leq c_2 h_1(x);
\]
notation: $\| \cdot \|_{h_1} \asymp \| \cdot \|_{h_2}$ or $h_1 \asymp h_2$ respectively.

2 Traveling wave profiles

In this section we study the structure of the traveling wave profiles. Formally, we can view them as profiles of solutions of the differential equation (1.1) on the whole line of the form,
\[
u(x; t) = \varphi(x - Vt),
\]
which travel with constant velocity $V$ and are normalized by the condition $\varphi(x_0) = 0$. Because of this normalization we have assumed $s_1 < 0 < t_\ell \leq a$. Since we are interested only in their behaviour between $u_- = -1$ and $u_+ = a$, we impose on them the boundary conditions (1.3). So we define the traveling wave profile $\Phi(x; x_0)$ as the solution of the equation on the interval $[-1, 1]$,
\[
\varepsilon \Phi'' + (F(\Phi))' + V \Phi' = 0,
\] (2.1)
that satisfies the boundary conditions and the normalization condition
\[
\Phi(-1; x_0) = -1 \quad \Phi(1; x_0) = a \quad \text{and} \quad \Phi(x_0; x_0) = 0.
\] (2.2)
Alternatively, $\Phi$ may be viewed as a sub- or supersolution to the stationary solution of (IBV), depending on the sign of $V$. $\Phi$ is a monotone function, given implicitly by the relation
\[
x = x_0 + \varepsilon \int_0^{\Phi(x; x_0)} \frac{ds}{g(s)}, \quad g(s) := C - Vs - F(s),
\] (2.3)
provided we can find constants $C(\varepsilon, x_0)$ and $V(\varepsilon, x_0)$ such that $g(s) > 0$ for all $s \in [-1, a]$ and all $0 < \varepsilon \leq \varepsilon_0$. These constants satisfy the equations
\[
\int_0^{a} \frac{ds}{C - Vs - F(s)} = \frac{1 - x_0}{\varepsilon}, \quad \text{(a)}
\]
\[
\int_{-1}^{0} \frac{ds}{C - Vs - F(s)} = \frac{1 + x_0}{\varepsilon}, \quad \text{(b)}
\] (2.4)
Since $F$ has zeros $s_1 < 0 < t_\ell$ and since the integrand in both equations must be positive, both $C - Vs_1$ and $C - Vt_\ell$ must be positive. Hence
\[
C > 0 \quad \text{and} \quad C/s_1 < V < C/t_\ell.
\] (2.5)
2.1 Basic estimates

For the analysis of the asymptotics of equations of type (2.4) we need the behaviour of the integral $I(C,V)$ for small $C$ and $V$,

$$I(C,V) := \int_0^\delta \frac{ds}{C - Vs - F(s)},$$

(2.6)

where the smooth function $F$ has a zero of (integral) order $p \geq 2$ at $s = 0$ and $F^{(p)}(0) = -p! \xi$. If $p = 1$, this integral is of the order $O(\log C)$ as is easily seen.

**Lemma 2.1** Let $F: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function satisfying

$$F(s) = -\xi s^p - g(s) s^{p+1} \quad \text{with} \quad 0 < m \leq \min\{\xi, \xi + sg(s)\} \quad \text{and} \quad |g(s)| \leq M \quad (s \geq 0)$$

for some integer $p \geq 2$ and positive constants $m$ and $M$. If $C$ and $V > 0$ are real variables, satisfying $0 < C \leq 1$ and

$$(p - 1) m^{-\frac{1}{p+1}} V^{\frac{2}{p+1}} \leq C$$

(2.7)

then the integral (2.6) satisfies the estimate

$$\left| I(C,V) - \gamma_p \xi^{-1/p} C^{-1+1/p} \right| \leq K_1 V C^{-2 + \frac{2}{p}} + \begin{cases} K_2 (1 + |\log C|) & \text{if } p = 2, \\ K_2 C^{-1+2/p} & \text{if } p > 2, \end{cases}$$

(2.8)

for some positive constants $K_1$ and $K_2$ and with $\gamma_p := \int_0^\infty \frac{ds}{1 + s^p}$.

**Proof:** By Hölder’s inequality we can estimate

$$Vs \leq \frac{m s^p}{p} + \frac{p - 1}{p} m^{-\frac{1}{p+1}} V^{\frac{2}{p+1}} \leq \frac{1}{p} (\xi s^p + C).$$

Hence, $C - V s - F(s) \geq (1 - \frac{1}{p})(C + m s^p)$ and

$$\left| \int_0^\delta \frac{ds}{C - Vs - F(s)} - \int_0^\delta \frac{ds}{C + \xi s^p} \right| \leq \frac{p}{p - 1} \int_0^\delta \frac{V s + s^{p+1} |g(s)|}{(C + m s^p)^2} ds$$

$$\leq \frac{p}{p - 1} \left\{ m^{-\frac{2}{p}} V C^{-2 + \frac{2}{p}} \int_0^\delta \frac{(\frac{\pi}{2})^{1/p}}{(1 + t^p)^2} + M m^{-\frac{p+2}{p}} C^{-1+2/p} \int_0^\delta \frac{(\frac{\pi}{2})^{1/p}}{(1 + t^p)^2} \right\}. $$

Finally, we have

$$\gamma_p = \int_0^\infty \frac{ds}{1 + s^p} \quad \text{and} \quad 0 \leq \gamma_p C^{-1+1/p} \xi^{-1/p} - \int_0^\delta \frac{ds}{C + \xi s^p} = \int_\delta^\infty \frac{ds}{C + \xi s^p} \leq \frac{1}{\xi} \frac{\delta^{1-p}}{p - 1}.$$

This implies (2.8).

**Remark 2.2** The integral above can be expressed as an incomplete Beta function, see [19] section 9.2.5,

$$\int_0^\delta \frac{ds}{1 + s^p} = \frac{1}{p} B \left( \frac{1}{p}, 1 - \frac{1}{p}, \frac{\delta^p}{1 + \delta^p} \right) \quad \text{such that} \quad \gamma_p = \int_0^\infty \frac{ds}{1 + s^p} = \frac{\pi}{p \sin \frac{\pi}{p}}. $$

(2.9)
2.2 The equilibrium solution

In the special case \( V = 0 \) we obtain from (2.1)–(2.2) the stationary or equilibrium solution of the problem (IBV). We shall denote it by \( \Phi_\varepsilon \). Its zero \( x_\varepsilon \) with \( \Phi_\varepsilon(x_\varepsilon) = 0 \) we shall call the equilibrium point. This definition is somewhat arbitrary, any constant could have been taken instead, but this one matches the choice of the normalization point. The corresponding constant \( C \), to be denoted by \( C_\varepsilon(\varepsilon) \) in the sequel, is the unique positive solution of the equation

\[
I(\varepsilon) := \int_{-1}^{a} \frac{ds}{c - F(s)} = \frac{2}{\varepsilon}.
\]

(2.10)

It is unique, because the maximum of \( F \) on \([-1, a]\) is zero, and because the integral is a monotonically decreasing function of \( \varepsilon \) with range \((0, \infty)\) for \( c > 0 \). Clearly, \( C_\varepsilon(\varepsilon) \) tends to zero as \( \varepsilon \to 0 \). This implies that \( \Phi_\varepsilon \) and \( x_\varepsilon \) are uniquely defined by

\[
x_\varepsilon = -1 + \varepsilon \int_{-1}^{0} \frac{ds}{C_\varepsilon(\varepsilon) - F(s)} \quad \text{and} \quad x = -1 + \varepsilon \int_{-1}^{\Phi_\varepsilon(x)} \frac{ds}{C_\varepsilon(\varepsilon) - F(s)}.
\]

(2.11)

Positivity of the integrand implies that \( \Phi_\varepsilon \) is strictly increasing. The asymptotics of \( C_\varepsilon \) and \( \Phi_\varepsilon \) for small \( \varepsilon \) is governed by the behaviour of \( F \) in the neighbourhood of its zeros of highest order. For the case \( p = 1 \) we refer to [12, eq. (2.24)], and for the case \( p \geq 2 \) we prove the following result:

**PROPOSITION 2.3** (The structure of the equilibrium)

Let \( \xi_j \) be the Taylor coefficients of \( F \) at \( s_j \) as given in (1.4), let \( \gamma_p \) as in (2.9) and let \( \delta_j \) and \( R_\varepsilon \) be defined by

\[
d_j := \begin{cases} 2 & \text{if } -1 < s_j < a, \\ 1 & \text{if } s_j = -1 \text{ or } s_j = a, \end{cases} \quad \text{and} \quad R_\varepsilon = \begin{cases} \varepsilon \log \varepsilon & \text{if } p = 2, \\ \varepsilon^{p-1} & \text{if } p > 2, \quad q \leq \frac{1}{2}p, \\ \varepsilon^{p-q} & \text{if } p > 2, \quad q > \frac{1}{2}p. \end{cases}
\]

(2.12)

The constant \( C_\varepsilon(\varepsilon) \) has the asymptotics as \( \varepsilon \to 0 \),

\[
C_\varepsilon(\varepsilon) = \left\{ \frac{\varepsilon \gamma_p}{2} \sum_{i=1}^{n} \delta_i \xi_i^{-1/p} \right\}^{p-1} (1 + O(R_\varepsilon)),
\]

(2.13)

and the equilibrium \( \Phi_\varepsilon \) has \( n - 1 \) internal transition layers of width \( O(R_\varepsilon) \) at the points

\[
x_i := -1 + \frac{\sum_{j=1}^{i} \delta_j \xi_j^{-1/p}}{\sum_{j=1}^{n} \delta_j \xi_j^{-1/p}}, \quad 1 \leq i \leq n - 1,
\]

(2.14)

where it switches from a value below \( s_i + \gamma \) to a value above \( s_{i+1} - \gamma \) for all small \( \gamma > 0 \). In addition, if \( s_1 > -1 \) then there is a boundary layer at the boundary point \( x = -1 \), where \( \Phi_\varepsilon \) jumps from \( -1 \) to \( s_1 \). Analogously, if \( s_n < a \), there is a boundary layer at the point \( x = 1 \), where \( \Phi_\varepsilon \) jumps from \( s_n \) to \( a \).

**PROOF.** To prove eq. (2.13) we apply lemma 2.1 to the integral \( I(\varepsilon) \) in (2.10). The main contributions to the integral come from neighbourhoods of the zeros of \( F \). According to (2.8) a zero of order \( r \geq 2 \) contributes \( \gamma_r \xi_r^{-1/r} c^{-1+1/r} \) if it is at the boundary and twice this amount if it is internal. A zero of order 1 contributes a term of order \( O(\log \varepsilon) \). Apparently for small \( \varepsilon \) the dominant terms are generated by zeros of order \( p \) and terms from lower order zeros are of strictly lower order (for \( c \to 0 \)). Hence,

\[
I(\varepsilon) = \sum_{j=1}^{n} \delta_j \gamma_p \xi_j^{-1/p} \frac{1-\varepsilon p}{\varepsilon^{p-1}} (1 + O(R(c))), \quad R(c) := \begin{cases} c^{1/2} \log c & \text{if } p = 2, \\ c^{1/p} & \text{if } p > 2 \text{ and } q \leq \frac{1}{2}p, \\ c^{1/q - 1/p} & \text{if } p > 2 \text{ and } q > \frac{1}{2}p. \end{cases}
\]

(2.15)
2.3 Existence and uniqueness of the traveling wave profiles

Using the asymptotics of $I(c)$ and its relation (2.10) to $2/\varepsilon$, we conclude first that $C_\varepsilon(\varepsilon) \asymp O(\varepsilon^{\frac{2}{1-p}})$. Inserting this in the error term of (2.15) we find from (2.10) the asymptotics of eq. (2.13).

To prove the relations (2.14) we define for any given level $b \in (-1, a)$ the corresponding point $x_\varepsilon(b)$ such that $\Phi_\varepsilon(x_\varepsilon(b)) = b$, i.e.

$$x_\varepsilon(b) := -1 + \varepsilon \int_{-1}^{b} \frac{ds}{C_\varepsilon(\varepsilon) - F(s)} = 1 - \varepsilon \int_{b}^{1} \frac{ds}{C_\varepsilon(\varepsilon) - F(s)}.$$

(2.16)

Using the same procedure as above and the asymptotics (2.13) of $C_\varepsilon$, and taking into account other possible zeros of $F$, we derive

$$x_\varepsilon(b) = \begin{cases} 
O(\varepsilon) & \text{if } -1 < b < t_1, \\
-1 + O(\varepsilon \log \varepsilon) & \text{if } t_1 = -1 \leq b < t_2 = s_1 \text{ and the order of } t_1 \text{ is } 1, \\
O(R_\varepsilon) & \text{if } t_1 \leq b < t_2 = s_1 \text{ and the order of } t_1 \text{ is } q > 1, \\
x_i + O(R_\varepsilon) & \text{if } s_i < b < s_{i+1} \text{ and } 1 \leq i \leq n - 1, \\
O(R_\varepsilon) & \text{if } t_{i-1} = s_n < b < t_i \text{ and the order of } t_i \text{ is } q > 1, \\
1 + O(\varepsilon \log \varepsilon) & \text{if } t_{i-1} = s_n < b < t_i = a \text{ and the order of } t_i \text{ is } 1, \\
O(\varepsilon) & \text{if } t_i < b < a.
\end{cases}$$

(2.17)

This implies the asymptotic behaviour of $\Phi_\varepsilon$. Finally, note that

$$x_\varepsilon(s_i) = \frac{1}{2} (x_{i-1} + x_i)(1 + O(R_\varepsilon)), \quad 1 \leq i \leq n,$$

(2.18)

if we (for this formula alone) use the notation $x_0 := -1$ and $x_n := 1$.

**Remark 2.4 (Location of the equilibrium point)** As a consequence of (2.17) we find the asymptotics of the equilibrium point $x_\varepsilon := x_\varepsilon(0) = x_1(1 + O(R_\varepsilon))$. If $n = 1$ (and $s_1 < 0 < t_1$), this equilibrium point is located at the boundary, $x_\varepsilon := x_\varepsilon(0) = 1 + O(R_\varepsilon)$.

2.3 Existence and uniqueness of the traveling wave profiles

Our analysis is based on the equations (2.4) for $C$ and $V$. First we prove existence of a unique solution $(C(\varepsilon, x_0), V(\varepsilon, x_0))$ of (2.4) and monotonicity w.r.t. $x_0$. Next we derive a rough estimate $C \asymp \varepsilon^{\frac{2}{1-p}}$ and $V = O(C)$. Finally, we derive the leading order asymptotics of $C$ and $V$ and with it the asymptotics of $\Phi$.

**Proposition 2.5 (Existence and uniqueness of the traveling wave profile)**

For every fixed $x_0 \in (-1, 1)$ and all $0 < \varepsilon \leq \varepsilon_0$ equation (2.4) has a unique solution $C(\varepsilon, x_0), V(\varepsilon, x_0)$ satisfying $C > 0$ and $(x_\varepsilon - x_0)V > 0$. Hence the boundary value problem (2.1)–(2.2) has a unique solution $x \mapsto \Phi(x; x_0)$ which is strictly increasing.

**Proof.** We only have to prove that the system (2.4) has a unique solution $(C, V)$. We consider the case $x_0 < x_\varepsilon$ implying $V > 0$; the case $x_0 > x_\varepsilon$ is analogous. First we solve $v(\varepsilon, x_0)$ from (2.4a) for fixed $c$ and then substitute this in (2.4b). We rewrite (2.4a) as $H(c, v; x_0) = \varepsilon$, where

$$H(c, v; y) := \frac{(1 - y)}{\int_{0}^{a} \frac{ds}{c - vs - F(s)}} \quad \text{for} \quad c > 0 \quad \text{and} \quad 0 \leq v \leq v_0(c) := \min_{t_1 \leq s \leq a} \frac{c - F(s)}{s}.$$

Because $v_0(c) \leq c/t_1$ by definition, the numerator in the integrand is strictly positive for all $s \in [0, a]$ if $v < v_0$ and it has a zero if $v = v_0$. Hence, $H$ is well-defined and smooth in its domain of definition and satisfies $H(c, v_0(c); x_0) = 0$. For the same reason as in eq. (2.10), the equation $H(c, 0; x_0) = \varepsilon$ has a unique solution $\overline{c}(\varepsilon, x_0) > 0$. By definition (2.11) the special case $H(c, v; x_\varepsilon) = \varepsilon$ is solved
by \( \tau(\varepsilon, x_0) = C_\varepsilon(\varepsilon) \) and \( v = 0 \). Since \( \partial_y \tau(\varepsilon, y) = -\partial_y H/\partial_c H > 0 \) and \( x_0 < x_\varepsilon \), we conclude \( \tau(\varepsilon, x_0) < C_\varepsilon(\varepsilon) \).

Since \( \partial_c H > 0 \) and \( H(\tau; 0; x_0) = \varepsilon \), we have \( H(c; 0; x_0) > \varepsilon \) if \( c > \tau(\varepsilon, x_0) \). Moreover, \( H(c, v; x_0) \searrow 0 \) as \( v \to v_0(c) \). Hence, the equation \( H(c, v; x_0) = \varepsilon \) has a solution \( v(c; \varepsilon, x_0) \geq 0 \) for each \( c \geq \tau(\varepsilon, x_0) \). This solution is unique in the interval \( 0 < v < v_0(c) \) because \( \partial_c H < 0 \).

Differentiating \( H(c, v(c; \varepsilon, x_0); x_0) = \varepsilon \) implicitly to \( c \) we find \( \partial_c v(c; \varepsilon, x_0) > 0 \).

This solution we insert in (2.4b), resulting in the equation for the constant \( c \)

\[
\eta(c; \varepsilon, x_0) = \frac{1 + x_0}{\varepsilon}
\]

where \( \eta(c; \varepsilon, x_0) := \int_{-1}^{0} \frac{ds}{c - v(c; \varepsilon, x_0)s - F(s)} \) and \( c > \tau \).

Because \( \tau(\varepsilon, x_0) < C_\varepsilon(\varepsilon) \) and \( v(\tau(\varepsilon, x_0); \varepsilon, x_0) = 0 \) we have

\[
\eta(\tau; \varepsilon, x_0) = \int_{-1}^{0} \frac{ds}{\tau - F(s)} \geq \int_{-1}^{0} \frac{ds}{C_\varepsilon(\varepsilon) - F(s)} = \eta(C_\varepsilon(\varepsilon); \varepsilon, x_0) = \frac{1 + x_\varepsilon}{\varepsilon} > \frac{1 + x_0}{\varepsilon}.
\]

Since \( \eta(c; \varepsilon, x_0) \searrow 0 \) as \( c \to \infty \) and \( \partial_c \eta < 0 \), it follows that there is a unique function \( C(\varepsilon, x_0) \) such that

\[
\eta(C(\varepsilon, x_0); \varepsilon, x_0) = \frac{1 + x_0}{\varepsilon} \quad \text{and} \quad C(\varepsilon, x_0) > \tau.
\]

Thus system (2.4) has a unique solution \( C(\varepsilon, x_0) > 0 \) and \( V(\varepsilon, x_0) := v(C(\varepsilon, x_0); \varepsilon, x_0) > 0 \).

**Remark 2.6 (Monotonicity of \( \Phi, V \) w.r.t. \( x_0 \)).**

The traveling wave profile \( \Phi(x; x_0) \) and its velocity \( V(\varepsilon, x_0) \) are monotone functions of \( x_0 \),

\[
\frac{\partial \Phi}{\partial x_0} \leq 0, \quad \frac{\partial V}{\partial x_0} < 0.
\]

(2.19)

This can be proved by the same arguments as in the proof of proposition 2.2 in [12].

**Lemma 2.7** If \( \ell \geq 2, n \geq 1 \) and \( p \geq 2 \) (i.e. \( F \) has at least two zeros \( s_1 < 0 < t_\ell \) on \([-1, a] \) and at least one among them has the order \( p \geq 2 \)), then

\[
C(\varepsilon, x_0) \sim \varepsilon^{\frac{p}{p-1}} \quad \text{and} \quad V(\varepsilon, x_0) = O(\varepsilon^{\frac{p}{p-1}}) \quad (\varepsilon \to 0).
\]

(2.20)

**Proof.** Under the assumption \( x_0 \leq x_\varepsilon \), and hence \( 0 \leq V \leq C/t_\ell \), we can estimate the second integral of (2.4a) from above and below by

\[
\frac{1 + x_0}{\varepsilon} = \int_{-1}^{0} \frac{ds}{C - V s - F(s)}
\]

\[
\left\{ \begin{array}{ll}
\leq \int_{-1}^{0} \frac{ds}{C - F(s)} = \delta_1 \gamma_p \xi^{-1} \varepsilon^{\frac{1-p}{p}} C^{\frac{1}{p}} + O(\log C) & \text{if } p = 2, \\
O(C^{\frac{2}{p}} + C^{\frac{1-p}{p}}) & \text{if } p > 2.
\end{array} \right.
\]

(2.21)

Clearly, this implies (2.20). If \( x_0 \geq x_\varepsilon \), and hence \( C/s_1 \leq V < 0 \), we can do the same with (2.4a) on \([0, a]\), provided \( n \geq 2 \), i.e. provided the interval contains a zero of order \( p \). If \( n = 1 \) such a zero is absent; however, the equilibrium point then is in the right-hand boundary layer and we need not consider the case \( x_0 > x_\varepsilon \).
2.4 The structure of the traveling wave profile

We derive the structure of the traveling wave profile from the leading order asymptotics of the constants $C$ and $V$ that satisfy eq. (2.4). Guided by the knowledge of the asymptotic order of $C$ and $V$ by lemma 2.7, we do in a first step the local asymptotic analysis around the zeros of $F$ of the integrals involved. In a second step we consider the full expansion of those integrals and derive the leading orders of $C$ and $V$ together with an error estimate. In a third step we use this knowledge to characterize the structure of $\Phi$. In order to avoid a lengthy statement of all resulting formulae in advance, we shall first derive all formulae and afterwards summarize the results in a theorem.

The structure of $\Phi$ depends strongly on the locations of the zeros of $F$ of maximal order on the interval $[u_-, u_+]$ and possibly on the rightmost (if $V > 0$) or leftmost (if $V < 0$) zero of $F$, even if this is of lower order. However, it does not depend on all other zeros of lower order. The reason, that a zero $t_j$ of lower order does not contribute to the leading order asymptotics of $C$ and $V$, is the following. If $V > 0$ and $j < \ell$, eq. (2.5) implies $C - V t_j = C - V t_{\ell} + V(t_{\ell} - t_j) \geq V(t_{\ell} - t_j)$ so that

$$V < \frac{C - V t_{\ell}}{t_{\ell} - t_j} \quad \text{and} \quad C - V t_j \geq \begin{cases} C(t_{\ell} - t_j)/t_{\ell} & \text{if } t_j > 0, \\ C & \text{if } t_j < 0. \end{cases}$$

Hence lemma 2.1 applies to the restriction of the integral in (2.4) to a neighbourhood of $t_j$; if this zero has order $r$, this results by (2.20) for some suitable positive $\alpha$ in the estimate

$$\int_{[t_j - \alpha, t_j + \alpha] \cap [-1, 1]} \frac{ds}{C - V s - F(s)} = O((C - V t_j)^{-1 + \frac{\alpha}{p}}) = O(\varepsilon^{1 + \frac{\alpha}{p} - \frac{r - \alpha}{p - r}}).$$

If $r < p$, this is of strictly lower order than the r.h.s. in (2.4) is. If $r = p$ (and $t_j = s_i$ with $i < n$) this integral is of the order $O(1/\varepsilon)$; more accurately, it satisfies

$$\int_{[s_i - \alpha, s_i + \alpha] \cap [-1, 1]} \frac{ds}{C - V s - F(s)} = \delta \gamma_p \zeta_i^{-\frac{1}{2}}(C - V s_i)^{-1 + \frac{1}{p}} + \begin{cases} O(|\log C|) & \text{if } p = 2, \\ O(C^{-1/2/p}) & \text{if } p > 2, \end{cases}$$

(2.24)

Likewise, only the lower order zero $t_1 < s_1$ (if present) may have a non-negligible contribution in the asymptotic expansion of (2.4) if $V < 0$.

For an estimate, analogous to (2.24), in a neighbourhood of the outer zeros $t_1$ and $t_\ell$ we need an inequality of type (2.7) between $|V|$ and $C - V t_1$ and $C - V t_\ell$ respectively. Assume $V > 0$ and let $t_\ell$ be a zero of order $r$. If $t_\ell < a$ (and hence $r \geq 2$), lemma 2.1 implies existence of a positive constant $k_3$ such that

$$\frac{2}{\varepsilon} \int_{[t_\ell - \alpha, t_\ell + \alpha] \cap [-1, 1]} \frac{ds}{C - V t_\ell - V(s - t_\ell) - F(s)} \geq \int_{[t_\ell - \alpha, t_\ell + \alpha] \cap [-1, 1]} \frac{ds}{C - V t_\ell - F(s)} \geq k_3(C - V t_\ell)^{-1 + \frac{1}{p}},$$

(2.25)

implying the lower bound

$$C - V t_\ell \geq \left(\frac{2}{\varepsilon} k_3\right)^{\frac{1}{p - 1}} = k_\alpha \varepsilon^{\frac{1}{p - 1}}.$$
where \( d_\ell := C - V t_\ell \). If \( t_\ell = a \) and \( r = 1 \), we have for \( d_\ell \to 0 \)
\[
\int_{a - \alpha}^{a} \frac{ds}{C - V s - F(s)} = \int_{0}^{a} \frac{dt}{C - V a + V t + \zeta t + O(t^2)} = - \zeta \log(d_\ell)(1 + O(V)) + O(1).
\] (2.27)

If \( V < 0 \) we find analogous expressions for the contribution of a neighbourhood of \( t_1 \) to the integrals in (2.4).

Together, the equations (2.23–2.26–2.27) enable us to make an asymptotic expansion of the equations (2.4), by which \( C \) and \( V \) are defined. We shall show, that functions \( c(x_o) \) and \( v(x_o) \) (independent of \( \varepsilon \)) exist such that
\[
C(\varepsilon, x_o) = c(x_o)\varepsilon^\frac{\rho}{p-1}(1 + O(R_1)) \quad \text{and} \quad V(\varepsilon, x_o) = v(x_o)\varepsilon^\frac{\rho}{p-1}(1 + O(R_2)),
\] (2.28)
and that the error terms effectively are of the order \( O(R_\varepsilon) \) and we discuss the uniformity of this error w.r.t. \( x_o \). We shall consider only the case \( x_o < x_\varepsilon \) implying \( V > 0 \); the other case, \( x_o > x_\varepsilon \), is completely analogous. So we may ignore in the sequel all zeros of lower order, except \( t_\ell \). It is convenient to consider consecutively three cases:

i. Both outer zeros \( t_1 = s_1 \) and \( s_0 = t_0 \) are of maximal order \( p \geq 2 \); then necessarily \( n \geq 2 \).

ii. There is only one zero of maximal order \( p \geq 2 \) and at least one zero of lower order, i.e. \( n = 1 \).

We shall consider the case \( s_1 < 0 \) and \( t_0 > 0 \), the latter having order \( r < p \); the case \( s_1 > 0 \) and \( t_0 < 0 \) is analogous.

iii. There are at least two zeros of maximal order \( p \geq 2 \) and at least one outer zero of lower order; without loss of generality we may assume \( s_1 < 0 < s_n < t_\ell \), the latter having order \( r < p \).

In case (i) expansion of (2.4) yields the equations
\[
\delta_1 \gamma_1 \xi_1^{\frac{1}{p}} (C(\varepsilon, x_o) - s_1 V(\varepsilon, x_o))^{-1 + \frac{1}{p}} (1 + O(R_\varepsilon)) = \frac{1 + x_o}{\varepsilon}, \quad (a)
\]
\[
\sum_{i=2}^{n} \delta_1 \gamma_i \xi_i^{\frac{1}{p}} (C(\varepsilon, x_o) - s_i V(\varepsilon, x_o))^{-1 + \frac{1}{p}} (1 + O(R_\varepsilon)) = \frac{1 - x_o}{\varepsilon}. \quad (b)
\] (2.29)

Rescaling \((C, V) = \varepsilon^\frac{p}{p-1}(\tilde{C}, \tilde{V})\) and using (2.20), we find
\[
\delta_1 \gamma_1 \xi_1^{\frac{1}{p}} (\tilde{C}(\varepsilon, x_o) - s_1 \tilde{V}(\varepsilon, x_o))^{-1 + \frac{1}{p}} = 1 + x_0 + O(R_\varepsilon), \quad (a)
\]
\[
\sum_{i=2}^{n} \delta_1 \gamma_i \xi_i^{\frac{1}{p}} (\tilde{C}(\varepsilon, x_o) - s_i \tilde{V}(\varepsilon, x_o))^{-1 + \frac{1}{p}} = 1 - x_0 + O(R_\varepsilon). \quad (b)
\] (2.30)

Because the Jacobian of this system is bounded away from zero by (2.25) if \( \varepsilon \leq \varepsilon_0 \), this implies
\[
(C, V) = \varepsilon^\frac{p}{p-1} (\tilde{C}, \tilde{V}) = \varepsilon^\frac{p}{p-1} (c(x_o), v(x_o))(1 + O(R_\varepsilon)), \quad (3.1)
\]
where \((c(x_o), v(x_o))\) is the solution of the system
\[
\delta_1 \gamma_1 \xi_1^{\frac{1}{p}} (c(x_o) - s_1 v(x_o))^{-1 + \frac{1}{p}} = 1 + x_o, \quad (a)
\]
\[
\sum_{i=2}^{n} \delta_1 \gamma_i \xi_i^{\frac{1}{p}} (c(x_o) - s_i v(x_o))^{-1 + \frac{1}{p}} = 1 - x_o. \quad (b)
\] (2.32)

To find the switch points of the traveling wave, we argue as in the proof of Proposition 2.3. We define for every given level \( b \in (-1, a) \) the corresponding point \( X_\varepsilon(b) \) such that \( \Phi(X_\varepsilon(b); x_o) = b \), i.e.
\[
X_\varepsilon(b) := -1 + \varepsilon \int_{-1}^{b} C(\varepsilon, x_o) - V(\varepsilon, x_o) s - F(s) ds.
\] (2.33)
In the same way as in formula (2.17) we obtain the switch points of the traveling wave, \( X_1 = x_o < X_2 < ... < X_{n-1} \), defined by

\[
X_i := -1 + \gamma_p \sum_{j=1}^{i} \delta_j \xi_j^{-\frac{1}{p}} (c(x_o) - s_j v(x_o))^{-1 + \frac{1}{p}}, \quad 1 \leq i \leq n - 1
\]  

(2.34)

At those points the traveling wave has transition layers; within a region of width \( R_\varepsilon \) around \( X_i \) its value jumps from approximately \( s_i \) to \( s_{i+1} \).

**In case (ii)** expansion of (2.4) yields the equations

\[
\begin{align*}
\delta_1 \gamma_p \xi_1^{-\frac{1}{p}} (C(\varepsilon, x_o) - s_1 V(\varepsilon, x_o))^{-1 + \frac{1}{p}} (1 + O(\tilde{R}_\varepsilon)) &= \frac{1 + x_o}{\varepsilon}, \\
\delta_r \xi_i^{-\frac{1}{p}} (C(\varepsilon, x_o) - t_i V(\varepsilon, x_o))^{-1 + \frac{1}{p}} (1 + E) &= \frac{1 - x_o}{\varepsilon} (1 + O(\tilde{R}_\varepsilon)), \quad \text{if } r \geq 2, \\
-\frac{1}{\xi_i} \log(C(\varepsilon, x_o) - t_i V(\varepsilon, x_o)) + E &= \frac{1 - x_o}{\varepsilon} (1 + O(\tilde{R}_\varepsilon)), \quad \text{if } r = 1,
\end{align*}
\]  

(2.35)

where \( E \) stands for the error terms in either (2.26) or (2.27) and where \( \tilde{R}_\varepsilon \) is defined as \( R_\varepsilon \) in (2.12), except that \( q \) is taken as the maximum order of all zeros of \( F \) of lower order, except \( t_\ell \), because this zero is treated separately. \( E \) captures the errors in the asymptotic expansion of the part of the integral (2.5a) in the neighbourhood of \( t_\ell \), whereas the term \( \tilde{R}_\varepsilon \) in (2.35) captures the errors on the remaining part of the interval (including the other zeros of lower order, if present). The second equation (2.35b) implies that \( C - V t_\ell \) is of the order \( O(\varepsilon^{-r-1}) \) if \( r \geq 2 \) and is exponentially small if \( r = 1 \), which is of strictly smaller order than \( C \) and \( V \) are. Moreover, the error term \( E \) is of the order \( O(\varepsilon^{-r-1}) \) if \( r \geq 2 \) and of order unity if \( r = 1 \). Hence,

\[
C(\varepsilon, x_o) - t_\ell V(\varepsilon, x_o) = \begin{cases} (\frac{\varepsilon \delta_r \gamma_p}{1 - x_o})^{-\frac{1}{r+1}} (1 + O(R_\varepsilon)), & \text{if } r \geq 2, \\ O(\exp(-\xi_\ell (1 - x_o) / \varepsilon)), & \text{if } r = 1. \end{cases}
\]  

(2.36)

This forces equality \( c = t_\ell v \), such that \( c \) and \( v \) can be calculated explicitly from (2.35a),

\[
c(x_o) = t_\ell v(x_o) = \frac{t_\ell \xi_\ell^{-\frac{1}{p}}}{1 - s_1} \left( \frac{\delta_1 \gamma_p}{1 + x_o} \right)^{\frac{p}{p+1}}.
\]  

(2.37)

The error terms in \( C \) and \( V \) are both of the form \( (1 + O(R_\varepsilon)) \).

Analogously, if \( t_1 < 0 < s_1 \) we obtain

\[
c(x_o) = t_1 v(x_o) = \frac{t_1 \xi_1^{-\frac{1}{p}}}{1 - s_1} \left( \frac{\delta_1 \gamma_p}{1 - x_o} \right)^{\frac{p}{p+1}}.
\]  

(2.38)

Expanding (2.33) likewise, we see that in this case there is only one switch point \( X_1 := x_o \), where \( \Phi \) jumps from the value \( s_1 \) to \( t_\ell \) (approximately).

**In case (iii)** expansion of (2.4) with \( r \geq 2 \) yields a combination of (2.29) and (2.35),

\[
\begin{align*}
\delta_1 \gamma_p s_1^{-\frac{1}{p}} (C - s_1 V)^{-1 + \frac{1}{p}} (1 + O(\tilde{R}_\varepsilon)) &= \frac{1 + x_o}{\varepsilon}, \\
\delta_\ell \gamma_r \xi_\ell^{-\frac{1}{p}} (C - t_\ell V)^{-1 + \frac{1}{p}} (1 + E) + \sum_{i=2}^{n} \delta_i \gamma_p \xi_i^{-\frac{1}{p}} (C - s_i V)^{-1 + \frac{1}{p}} (1 + O(\tilde{R}_\varepsilon)) &= \frac{1 - x_o}{\varepsilon},
\end{align*}
\]  

(2.39)

where \( \tilde{R}_\varepsilon \) is defined as \( R_\varepsilon \) in (2.12), except that \( q \) is taken as the maximum order of all zeros of \( F \) of lower order, except \( t_\ell \), because this zero is treated separately. An analogous expression is
obtained for \( r = 1 \). Inserting formally the ansatz (2.28) under the assumption \( c - t_\ell v > 0 \) (it is non-negative anyway), we find that the contribution from \( t_\ell \) in the second equation is subdominant and disappears like the contributions of all other lower order zeros. Hence \( c \) and \( v \) satisfy system (2.32). Inserting the inequality \( c - t_\ell v > 0 \) in (2.32a) and in the sum of both equations we obtain the inequalities

\[
\delta_1 \gamma_{p} \xi_1^{-1/2} (t_\ell v - s_1 v)^{-1 + 1/p} < 1 + x_0, \quad \text{and} \quad \sum_{i=1}^{n} \delta_1 \gamma_{p} \xi_i^{-1/2} (t_\ell v - s_i v)^{-1 + 1/p} < 2.
\]

We can eliminate \( v \) in the first inequality by the second and obtain:

\[
x_o > Y_\ell := -1 + \frac{2 \delta_1 \gamma_{1}^{-1/p} (t_\ell - s_1)^{-1 + 1/p}}{\sum_{i=1}^{n} \delta_1 \gamma_{p} \xi_i^{-1/2} (t_\ell - s_i)^{-1 + 1/p}}.
\]

We remark that the inequality (2.40) is equivalent to the condition \( c - t_\ell v > 0 \). Apparently this inequality (2.40) is a necessary condition, under which the solution of (2.32) provides the leading order of the asymptotics of the solution of (2.39). Conversely, if \( (c(x_o), v(x_o)) \) is the solution of (2.32) (which is known to exist from case (ii)) and if \( Y_\ell < x_o < x_\ell \), then the inequality \( c - t_\ell v > 0 \) is satisfied and we show that this solution indeed provides the leading order of \((C,V)\) as follows. Substituting (2.28) in (2.39a), we find that the error terms \( R_1 \) and \( R_2 \) satisfy

\[
c R_1 - s_1 v R_2 = O(\tilde{R}_\ell).
\]

Since \( c \) is strictly positive by lemma 2.7, we can use this equation to eliminate \( R_1 \) from (2.39b), in which we have done the same substitution; expansion leads to the second equation (2.32b) for \( c \) and \( v \) and to the estimate for the remainder \( R_2 \)

\[
R_2 = O \left( R_\ell + \varepsilon^{\frac{p-1}{p-1}} (c(x_o) - t_\ell v(x_o))^{-\frac{1}{p-1}} \right),
\]

in which we explicitly have conserved the error due to the presence of \( t_\ell \). Although it is bounded by \( O(\tilde{R}_\ell) \), this term may dominate the error if the coefficient \( c(x_o) - t_\ell v(x_o) \) is small. This shows that the solution of (2.32) indeed is the leading order of \((C,V)\) if \( Y_\ell < x_o < x_\ell \).

Although the formulae are somewhat different, this conclusion remains the same, if the order of \( t_\ell \) is \( r = 1 \).

If \( x_o < Y_\ell \), the solution of (2.32) gives values for \( c \) and \( v \) that do not satisfy the condition \( c - t_\ell v > 0 \), and thus it cannot provide the leading order for the solution of (2.39). This means, that \( C - t_\ell V \) must be of smaller order than \( C \). This imposes equality of the leading orders, \( c(x_o) - t_\ell v(x_o) = 0 \). Thus \( c \) and \( v \) can be computed from (2.39a) as in case (ii) and are given by (2.37). On the other hand, \( C - V t_\ell \) cannot be too small, since it is bounded from below by \( k_0 \varepsilon^{\frac{1}{p-1}} \) as shown in (2.25). In order to show that it is exactly of this order, we substitute in (2.39) (with \( r \geq 2 \)) the ansatz

\[
C(\varepsilon, x_o) - t_\ell V(\varepsilon, x_o) = \varepsilon^{1/r - 1} (d(x_o) + R_1(\varepsilon)) \quad \text{and} \quad V(\varepsilon, x_o) = \varepsilon^{1/p - 1} (v(x_o) + R_2(\varepsilon)),
\]

where both \( R_1 \) and \( R_2 \) are \( o(1) \) for \( \varepsilon \to 0 \). From the first equation and the sum of both we obtain

\[
(t_\ell - s_1) + R_2 + \varepsilon^{1/r - 1} d(x_o) + R_1 = O(\tilde{R}_\ell)
\]

\[
\varepsilon \delta_1 \gamma_{c_\ell}^{-1/2} (C - t_\ell V)^{1/r - 1} (1 + E) = \delta_1 \gamma_{c_\ell}^{-1/2} (d(x_o) + R_1)^{1/r - 1} (1 + E) =
\]

\[
= 2 - \sum_{i=1}^{n} \delta_1 \gamma_{p} \xi_i^{-1/2} \left( (t_\ell - s_i)(v + R_2) + \varepsilon^{1/r - 1} d(x_o) + R_1 \right)^{1/p} + O(\tilde{R}_\ell) =
\]

\[
= \frac{Y_\ell - x_o}{1 + Y_\ell} + O(R_2) + O(\varepsilon^{1/r - 1} + \varepsilon^{1/p - 1}) + O(\tilde{R}_\ell).
\]
This shows:

\[ d(x_o) = \left( \frac{\delta \gamma_t (1 + Y_t)}{\xi_1^{1/\tau} Y_t - x_o} \right)^{1/\tau}, \quad R_1(\varepsilon) = O(R \varepsilon) \quad \text{and} \quad R_2(\varepsilon) = O(R \varepsilon). \]  

(2.44)

We remark, that \( d(x_o) \) tends to infinity, when \( x_o \rightarrow Y_t \) from below, showing again the non-uniformity of the asymptotic expansion, if \( x_o \) is in the neighbourhood of the exceptional point \( Y_t \) (or \( Y_1 \)). If \( r = 1, t_\ell = 1 \) and the term due \( t_\ell \) in (2.39b) is replaced by \(-\zeta_t \log(C - V) + E\) as in (2.35b). The ansatz \( C - V = O(\varepsilon^{-n/\gamma}) \) for some \( \alpha > 0 \) (to be determined) leads to the equation

\[ -\frac{1}{\zeta_t} \log(C - V) + E = \frac{1}{\varepsilon} \left\{ \frac{Y_t}{1 + Y_t} + O(R_2) + O(\tilde{R}_e) + O(\varepsilon^{-1/\gamma}) \right\}, \]

implying

\[ C(\varepsilon, x_o) - V(\varepsilon, x_o) = O \left( \text{exp} - \frac{\zeta_t}{\varepsilon} \frac{Y_t - x_o}{1 + Y_t} \right) \]  

(2.45)

Finally, we derive from the expansion of the integral (2.33), that the traveling wave \( \Phi \) in case (iii) has the same \( n - 1 \) switch points (2.34) as it has in case (i), if \( Y_t < x_o < x_e \). If \( x_o < Y_t \), then the \( n^{th} \) point \( X_n \) in (2.34) is smaller than 1 and gives an additional switch point. In a neighbourhood of this point \( \Phi \) jumps from (approximately) \( s_n \) to \( t_\ell \). Because we can eliminate \( c \) and \( v \) from this expression by (2.37), we find explicitly

\[ X_i := -1 + (1 + x_o) \sum_{j=1}^{i} \frac{\delta j}{\delta 1} \left( \frac{\xi_j}{\xi_1} \right)^{-\frac{1}{\gamma}} \left( \frac{t_\ell - s_j}{t_\ell - s_1} \right)^{-\frac{1}{\gamma}}, \quad 1 \leq i \leq n. \]  

(2.46)

If \( t_\ell < s_1 \) with \( n \geq 2 \) we find analogously in the case \( x_o > x_e \) (\( V < 0 \)), that \( c \) and \( v \) satisfy the equations

\[ c(x_o) = t_1 v(x_o) \quad \text{and} \quad \gamma_p \sum_{i=2}^{n} \delta_i \xi_i^{-1/p}(c(x_o) - s_i v(x_o))^{1/p - 1} = 1 - x_o \]  

(2.47)

and that the critical point is given by

\[ Y_1 := -1 + \frac{2\delta t_1 \xi_1^{-1/p}(s_1 - t_1)^{-\frac{1}{1+p}}}{\sum_{i=1}^{n} \delta_i \xi_i^{-1/p}(s_i - t_1)^{-\frac{1}{1+p}}}. \]  

(2.48)

If \( x_o < x_o < Y_1 \), then \( \Phi \) has \( n - 1 \) jumping points, given by (2.34). Otherwise, if \( x_o > Y_1 \), then an additional transition layer is located at \( X_o \) between \(-1 \) and \( x_o \). We find the explicit expression:

\[ X_i := 1 - (1 - x_o) \sum_{j=i+1}^{n} \frac{\delta_j \xi_j^{-1/p}(s_j - t_1)^{-\frac{1}{1+p}}}{\sum_{j=2}^{n} \delta_j \xi_j^{-1/p}(s_j - t_1)^{-\frac{1}{1+p}}}, \quad 0 \leq i \leq n - 1. \]  

(2.49)

**Remark 2.8** The critical points \( Y_1 \) and \( Y_t \) are defined in all cases by (2.48) and (2.40) respectively. If \( t_\ell = s_n \) (\( n \geq 2 \)) formula (2.40) implies \( Y_t = -1 \); it implies \( Y_t = 1 \) if \( n = 1 \). Likewise eq. (2.48) implies \( Y_1 = 1 \) if \( t_1 = s_1 \) (\( n > 1 \)) and \( Y_1 = -1 \) if \( n = 1 \) (and \( t_1 < 0 < s_1 \)).

**Proposition 2.9** (The structure of the traveling wave profile)

Let the traveling wave profile \( \Phi \) be the solution of (2.1–2.2) and let \( \delta > 0 \) be a (small) constant. \( \Phi \) is completely determined by the solution \((C, V)\) of (2.4), which for \( \varepsilon \rightarrow 0 \) satisfy the asymptotics

\[ (C(\varepsilon, x_o), V(\varepsilon, x_o)) = (c(x_o), v(x_o)) \varepsilon^{\frac{p}{\gamma}} (1 + O(R \varepsilon)), \]  

(2.50)

uniformly for all \( x_o \in [-1 + \delta, Y_1 - \delta] \cup [Y_1 + \delta, Y_1 - \delta] \cup [Y_1 + \delta, 1 - \delta] \). If \( Y_t < x_o < Y_1 \), the leading order term \((c, v)\) is solution of eq. (2.32) and \( \Phi \) has \( n - 1 \) internal
transition layers at the points $X_i$ ($i = 1 \cdots n-1$), defined in (2.34).

If $x_0 < Y_1$ the leading order term $(c, v)$ is given by (2.37) and it is given by (2.38) for $n = 1$ and
by (2.47) for $n > 1$ if $x_0 > Y_1$. In those cases $\Phi$ has $n$ internal switch points given in (2.46) when
$x_0 < Y_1$ and in (2.49) if $x_0 > Y_1$.

Proof: The only point in the analysis above we did not consider explicitly is the uniformity w.r.t. $x_0$. However, this is obvious from the formulae. ■

Above we have only shown that the value of $\Phi$ at a switch point jumps from the neighbourhood
of one zero of $F$ to the next one and that it is approximately constant in between. We easily can
compute the order of approximation at such a flat zone as $\varepsilon \to 0$.

**Corollary 2.10** If $p \geq 2$, $\Phi'$ is positive and a positive constant $k_1$ exists such that,

$$\Phi'(x; x_0) \geq k_1 \frac{1}{\varepsilon^{p-1}} \text{ uniformly for all } \left\{ \begin{array}{ll}
\varepsilon \in [-1, 1] & \text{if } Y_1 - \delta < x_0 < Y_1 + \delta, \\
x \in [-1, X_n - \delta] & \text{if } x_0 < Y_1 - \delta, \\
x \in [X_n + \delta, 1] & \text{if } Y_1 - \delta < x_0,
\end{array} \right. (2.51)$$

$$\Phi'(x; x_0) \geq k_1 \frac{1}{\varepsilon^{p-1}} \text{ uniformly for all } \left\{ \begin{array}{ll}
x \in [X_n - \delta, 1] & \text{if } x_0 < Y_1 + \delta, \\
x \in [-1, X_n + \delta] & \text{if } x_0 > Y_1 - \delta \text{ and } r \geq 2.
\end{array} \right. (2.52)$$

Proof: For any $b \in [-1, a]$ we have by (2.3)

$$\varepsilon \Phi'(x; x_0) = C(\varepsilon, x_0) - V(\varepsilon, x_0) b - F(b) = g(b),$$

where $\Phi(X_i(b); x_0) = b$. It is sufficient to estimate $g(b)$ only in the neighbourhoods of the zeros
$t_i$, $1 \leq i \leq \ell$ of $F$, where it has its minima. As in the proof of lemma 2.1 we find in the
neighbourhood of $t_j$ the lower bound $g(s) \geq (1 - 1/p)(C - t_j V + 2 \zeta_j (s - t_j)^r)$. The lower bound
(2.51) is a consequence of (2.24) and (2.52) follows from (2.25). ■

**Corollary 2.11** Let $X_\varepsilon$ be as in (2.33). In the flat zone at the level $s_i$ (with $1 \leq i \leq n$) the interval

$$[y_i^-, y_i^+] = [X_\varepsilon(s_i - \alpha \varepsilon^{p-1}), X_\varepsilon(s_i + \alpha \varepsilon^{p-1})] \cap [-1, 1]$$

is of non-vanishing length, i.e. $y_i^+ - y_i^- \geq c_1(\alpha) > 0$, for any (fixed) $\alpha > 0$. The estimates are valid uniformly for $0 < \varepsilon \leq \varepsilon_0$ and $x_0 \in [-1 - \delta, Y_1 - \delta] \cup [Y_1 + \delta, Y_1 - \delta] \cup [Y_1 + \delta, 1 - \delta]$ and

$x \in [y_i^-, y_i^+]$:

$$|\Phi(x; x_0) - s_i| \leq \varepsilon^{\frac{1}{p-1}} \text{ and } \Phi'(x; x_0) = O(\varepsilon^{\frac{1}{p-1}}). (2.54)$$

If $Y_1 < x_0 < Y_1$ and $\beta > 0$ we can choose $\alpha$ so large, that the length of the complement of the union of those flat zones is smaller than $\beta$ (provided $\varepsilon \leq \varepsilon_\beta$).

Proof. From the definition and the previous theorem we find a constant $c_1 > 0$ such that

$$y_i^+ - y_i^- = \varepsilon \int_{s_i - \alpha \varepsilon^{p-1}}^{s_i + \alpha \varepsilon^{p-1}} \frac{d\tau}{C - V s - F(s)} = \int_{-\alpha}^{\alpha} \frac{d\tau}{c - vs_i + \xi_\tau \tau^p (1 + O(R_\tau))} \geq c_1,$$  

provided $-1 < s_i < a$; if $s_i = -1$ or $s_i = a$ is a boundary point we skip the irrelevant parts of the integrals. The estimate of $\Phi$ follows by monotonocity and the estimate of $\Phi'$ follows from the differential equation $\varepsilon \Phi' = C - V \Phi - F(\Phi)$ using (2.20) and (1.4). Finally, its is clear that the sum of the integrals (2.55) tends to the sum of both equations in (2.32) if $\alpha$ tends to $\infty$, leaving an arbitrary small remainder for sufficiently large $\alpha$. ■

**Remark 2.12** In the cases $x_0 < Y_1$ and $x_0 > Y_1$ analogous estimates hold in neighbourhoods of $t_1$ and $t_1$ respectively, if $\varepsilon^{\frac{1}{p-1}}$ is replaced by $\varepsilon^{\frac{1}{p-1}}$ for $r \geq 2$ and by an exponentially small order for $r = 1$. Moreover, if this interval is added to the union above, we again can get an arbitrarily small rest.
In section 4 we shall linearize equation (1.1) around a traveling wave profile and use the derivative (4.4) of this profile in the weight function $h$,

$$h^{-2}(x; x_0) := \varepsilon \Phi'(x; x_0) e^{V_{x_0}}.$$  

**Proposition 2.13** (estimate of the weight)  
If $p \geq 2$ the weight satisfies for some constant $c > 0$ the $L^2(-1, 1)$-norm estimate

$$\|h(\cdot; x_0)\| \leq \varepsilon^{-\frac{p}{2(1-p)}}$$  

if $Y_1 + \delta \leq x_0 \leq Y_1 - \delta$

$$\|h(\cdot; x_0)\| \leq \varepsilon^{-\frac{p}{r}}$$  

if $x_0 > Y_1 - \delta$ or $x_0 < Y_1 + \delta$ and $r \geq 2$

uniformly w.r.t. $x_0$ for fixed small $\delta > 0$ and $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0 > 0$ depends on $\delta$.

**Proof.** This is an easy consequence of corollary 2.10.  

**2.5 Supplement concerning the case $p = 1$.**

In [12] we have studied this case extensively. For an enhancement of some of those results, we need more information about the traveling wave profile $\Phi(x; x_0)$ in this case. As in the case $p > 1$ it is the solution of (2.1–2.2), in which we can assume $\alpha = 1$ without loss of generality, and where $F$ has exactly two zeros, $F(\pm 1) = 0$. The profile can be described in terms of a function $\psi$,

$$\Phi(x; x_0) = \psi\left(\frac{x - x_0}{\varepsilon}, C(\varepsilon, x_0), V(\varepsilon, x_0)\right),$$  

where $\psi$ is defined on the whole line and satisfies the initial value problem

$$\psi' = C(\varepsilon, x_0) - V(\varepsilon, x_0)\psi - F(\psi), \quad \psi(0, C, V) = 0.$$  

The constants $C(\varepsilon, x_0)$, $V(\varepsilon, x_0)$ are derived from the integrals (2.4) and are exponentially small (see Proposition 2.1 in [12]),

$$C(\varepsilon, x_0), \quad V(\varepsilon, x_0) = O(R_{\varepsilon}(\delta)),$$  

uniformly w.r.t. $x_0 \in [-1 + \delta, 1 - \delta]$, $\delta > 0$, $0 < \varepsilon \leq \varepsilon_0$, where

$$R_{\varepsilon}(\delta) := \exp(-F(a)\varepsilon) + \exp(F(1)\varepsilon).$$  

Note also that (2.58–2.59) imply

$$\varepsilon \Phi'(\pm 1; x_0) = O(R_{\varepsilon}(\delta)).$$  

The boundary conditions (2.4) can be written as

$$\psi\left(-\frac{1 + x_0}{\varepsilon}, C(\varepsilon, x_0), V(\varepsilon, x_0)\right) = -1, \quad \psi\left(\frac{1 - x_0}{\varepsilon}, C(\varepsilon, x_0), V(\varepsilon, x_0)\right) = 1.$$  

Using the fact that the Jacobian determinant of this system w.r.t. $(C, V)$ is bounded away from zero for $\varepsilon \in (0, \varepsilon_0)$ (see eq. (2.17) in [12]), we derive the estimates

$$\frac{\partial C}{\partial x_0}, \frac{\partial V}{\partial x_0} = O(R_{\varepsilon}(\delta)),$$  

uniformly w.r.t. $x_0 \in [-1 + \delta, 1 - \delta]$, $\delta > 0$, $0 < \varepsilon \leq \varepsilon_0$. For the same reason (see eq. (2.16) in [12]), we have the uniform estimates

$$\psi_c, \psi_v, \psi'_c, \psi'_v = O(1),$$  

where $\psi_c := \frac{\partial \psi}{\partial x}$, $\psi'_c := \frac{\partial^2 \psi(\eta, c, x)}{\partial \eta \partial c}$ and analogously for $\psi_v, \psi'_v$. Finally, we find for the corresponding weight function $\tilde{h}$ the estimates

$$\|\tilde{h}(x; x_0)\| \approx \sqrt{\varepsilon/R_{\varepsilon}(\delta)},$$  

$$\frac{\varepsilon}{\tilde{h}} \frac{\partial \tilde{h}}{\partial x} = O(1),$$  

uniformly w.r.t. $x_0 \in [-1 + \delta, 1 - \delta]$, $0 < \varepsilon \leq \varepsilon_0$. 


3 Evolution starting at a travelling wave profile

A particular class of solutions of (IBV) consists of solutions that start at a travelling wave profile and keep more or less their original form, while converging to the equilibrium. For any \(x_0 \in [-1 + \delta, 1 - \delta]\), \(\delta > 0\) we consider the solution \(u(x, t; x_0)\) of the problem (IBV) which satisfies the special initial condition

\[
u(x, 0; x_0) = \Phi(x; x_0) .
\]

(3.1)

As we are comparing such a solution to the traveling wave \(t \mapsto \Phi(x - Vt; x_0)\) for small \(Vt\), we have to ascertain existence of a continuation of \(\Phi\) to a larger interval \([-1 - c_0\varepsilon, 1 + c_0\varepsilon]\) for some positive constant \(c_0\). With \(C\) and \(V\) given, the traveling wave profile \(\Phi\) is in neighbourhoods of \(\pm 1\) solution of the initial value problems

\[
w' = \frac{1}{\varepsilon}(C - Vw - F(w)), \quad w(1) = a \text{ and } w(-1) = -1 \text{ respectively.}
\]

Since the right-hand side is a smooth function of \(w\) with a Lipschitz constant of order \(O(1/\varepsilon)\), Picard’s existence theorem ensures existence of the solution \(\Phi\) on intervals \([-1 - c_0\varepsilon, 1 + c_0\varepsilon]\) and \([-1 - c_0\varepsilon, 1 + c_0\varepsilon]\) respectively for some \((\varepsilon\text{-independent})\) positive constant \(c_0\), thus extending it to a smooth function on \([-1 - c_0\varepsilon, 1 + c_0\varepsilon]\).

Using the maximum principle we conclude as in [12] Corollary 5.1, that this solution of (1.1–1.3–3.1) is squeezed between \(\Phi(x; x_0)\) and \(\Phi(x - Vt; x_0)\), provided \(|Vt| \leq c_0\varepsilon\). For \(V > 0\) (i.e. \(x_0 < x_e\)) we have

\[
\Phi(x; x_0) \geq u(x, t; x_0) \geq \Phi(x - Vt; x_0),
\]

(3.2)

and

\[
\Phi(x; x_0) \geq u(x, t; x_0) \geq \Phi_e(x).
\]

(3.3)

For \(V < 0\) the inequalities are reversed. Since \(u\) is squeezed between \(\Phi(\cdot; x_0)\) and \(\Phi_e(\cdot)\), which share the same boundary conditions at \(x = \pm 1\), the derivative \(u_x\) satisfies at the boundaries the inequalities (see eq. (5.5) in [12]) if

\[
\Phi'(1; x_0) \leq u_x(1, t; x_0) \leq \Phi'_e(1), \quad \Phi'_e(-1) \leq u_x(-1, t; x_0) \leq \Phi'(-1; x_0),
\]

(3.4)

for \(V > 0\), and the reverse for \(V < 0\). In particular, using \(\varepsilon \Phi' = C - V \Phi - F(\Phi)\) and \(\varepsilon \Phi'_e = C(\varepsilon, x_e) - F(\Phi_e)\) this implies

\[
\varepsilon u_x(-1, t; x_0) = F(-1) + O(R_\varepsilon) \quad \text{and} \quad \varepsilon u_x(-1, t; x_0) = F'(a) + O(R_\varepsilon),
\]

(3.5)

where \(R_\varepsilon\) is defined by (2.60) for \(p = 1\) and by (2.12) for \(p > 1\).

If there are no boundary layers, i.e. if \(F(-1) = F(a) = 0\), then the derivatives of \(\Phi\) at the boundaries \(\pm 1\) are small, and we can use the monotonicity of \(u\) w.r.t. \(t\), \(u_t V < 0\), see Corollary 5.2 in [12], and the equation \(u_t = (u_x + F(u))_x\), to derive the estimate

\[
\varepsilon u_x(1, t; x_0) \leq \varepsilon u_x(x, t; x_0) + F(u) \leq \varepsilon u_x(-1, t; x_0) \quad \text{if} \quad V > 0, \; F(-1) = F'(a) = 0 .
\]

(3.6)

and the reverse inequality if \(V < 0\). This implies that \(u_x\) is positive. Moreover, in the same way as in lemma 5.3 in [12] we infer from the monotonicity of \(u\) w.r.t. \(u_t V < 0\),

**Lemma 3.1 (pointestimate)**

Let \(F(-1) = F(a) = 0\). Then there is a constant \(c > 0\), depending only on \(F\), such that

\[
\varepsilon |u_x(x, t; x_0) - \Phi'(x; x_0)| \leq c |u(x, t; x_0) - \Phi(x; x_0)| + (1 + a)|V|
\]

(3.7)

and

\[
\varepsilon |u_x(x, t; x_e) - \Phi'_e(x)| \leq c |u(x, t; x_0) - \Phi_e(x)| + \varepsilon \max_{\pm}|\Phi'_e(\pm 1) - \Phi'(\pm 1; x_0)|
\]

(3.8)

uniformly w.r.t. \(x_0 \in [-1 + \delta, 1 - \delta]\).
Now we can formulate our first result about metastability. It concerns only the special solutions starting at a traveling wave profile. Namely, the solution stays in a small neighborhood of the traveling wave profile and has almost the same form during a long time interval (cf. Corollary 5.4 in [12]).

**Corollary 3.2** Let \( x_o \neq x_e \) and \( F(-1) = F(a) = 0 \). If \( |V| t \leq c_o \varepsilon \) then the solution of (1.1)–(1.3)–(3.1) satisfies the pointwise estimate

\[
\varepsilon|u_x(x, t; x_o) - \Phi'(x; x_o)| + |u(x, t; x_o) - \Phi(x; x_o)| \leq c|V|t\Phi'(x; x_o) + (1 + a)|V|, 
\]

and the estimate in the weighted Sobolev norm,

\[
\|u(x, t; x_o) - \Phi(x; x_o)\|_h \leq c|V|t/\sqrt{\varepsilon} + c|V| ||h||. 
\]

### 4 Stability of the equilibrium solution

In this section we prove results about stability of the equilibrium solution, using positivity of the smallest eigenvalue of the linearization and contraction methods (see for example [15]). We consider both the case where both zeros of \( F \) have order \( p = 1 \) as in [12] and the case where at least one zero of \( F \) has order \( p > 1 \). The main difference between those cases is, that the gap between the bottom eigenvalue and the next one is of order unity if \( p = 1 \), whereas the eigenvalues accumulate at 0 for \( \varepsilon \to 0 \) if \( p > 1 \).

We start our analysis by the linearization around a traveling wave profile \( \Phi(x; x_o) \) (as in [12], section 3.a) and we derive some estimates for eigenvalues and for Sobolev norms. Next we consider contraction around the equilibrium solution (for general \( p \)). Finally we show that in the case \( p = 1 \) we can obtain (much) stronger convergence results than we had in [12].

#### 4.1 Linearization around a traveling wave profile

Let

\[
v(x, t) = u(x, t) - \Phi(x; x_o). \tag{4.1}
\]

Then (cf. (3.2)[12])

\[
v_t = \varepsilon v_{xx} + F'(\Phi) v_x + \Phi' F''(\Phi) v - V \Phi' + g_1(v),
\]

\[
v(x, 0) = u_o(x) - \Phi(x; x_o), \text{ and } v(\pm1, t) = 0, 
\]

\[
g_1(v) := v^2 (v_x + \Phi') g_2(v) + v v_x F''(\Phi), \quad g_2(v) := \int_0^1 (1 - s) F''(\Phi + sv) \, ds. 
\]

The linear part of the spatial operator is a selfadjoint operator acting in a weighted \( L^2(-1, 1) \) where (the square of) the weight \( h \) is given by the formula:

\[
h^2(x; x_o) := D(\varepsilon, x_o) \exp \left( \frac{1}{\varepsilon} \int_{-1}^x F'(\Phi(y; x_o)) \, dy \right). \tag{4.3}
\]

The constant of integration \( D(\varepsilon, x_o) \) is fixed by \( D(\varepsilon, x_o)^{-1} := \Phi'(-1; x_o) \). Using the differential equation \( \Phi'' + F'(\Phi) \Phi' + V \Phi = 0 \) we find an alternative expression for the weight,

\[
h^{-2}(x; x_o) = \varepsilon \Phi'(x; x_o) e^{V \frac{x+1}{\varepsilon}}. \tag{4.4}
\]
The exponential factor in (4.4) is essentially equal to 1 because $V/\varepsilon = O(\varepsilon^{-\frac{1}{2}})$ (or exponentially small if $p = 1$). For convenience we prefer to work in an unweighted $L^2$ with a stretched time variable. So we transform the functions by the square root of the weight, 

$$w(x,t/\varepsilon) := v(x,t) h(x;x_0);$$

the equation in the transformed variable $w$ then reads 

$$w_t + Aw = r(w) + g, \quad w(x,0) = w_0(x), \quad w(\pm 1,t) = 0,$$

where $A := -\varepsilon^2 \partial_x^2 + q$ and (cf. (3.6)[12])

$$q(x;x_o) = \frac{1}{4} [F'(\Phi(x;x_o))]^2 - \frac{1}{4} F''(\Phi(x;x_o)) \varepsilon \Phi'(x;x_o),$$

$$w_0(x) = (u_0(x) - \Phi(x;x_o)) h(x;x_o),$$

$$g(x) = -V[\varepsilon \Phi'(x;x_o)]^{1/2} \exp(-V \frac{\varepsilon^{1/2}}{2\varepsilon}),$$

$$r(w) = g_2(h^{-1}w) \left\{ \varepsilon w^2 w_x h^{-2} - \frac{1}{2} w^2 F'(\Phi)_h^{-2} + \varepsilon w^2 h^{-1} \Phi' \right\} + h^{-1} F''(\Phi) \varepsilon w w_x - \frac{1}{2} w^2 F''(\Phi).$$

In an argument we sometimes have to switch from the problem (4.2) “in $v$-coordinates” to its formulation in “$w$-coordinates” and vice versa. Moreover, when using those functions $g_1, g_2, g, q, r,$ and $w_0$, defined above, we should always keep in mind, that they implicitly depend on $\varepsilon$ and $x_o$ via $\Phi$. Due to its special form the $L^2$-norm of the inhomogeneous term $g$ satisfies

$$\|g\|^2 = \varepsilon(1 + a)V(1 + O(R_\varepsilon)).$$

Using the embedding estimate $\|u\|_{L^2} \leq \sqrt{\frac{\varepsilon}{\pi}} \|u\|_1$, for all $u \in H^1_\varepsilon(-1,1)$, we can bound the nonlinear term $r$ by (see (3.9) and (3.10) in [12]),

$$\|r(w)\| < \frac{a_1}{\sqrt{\varepsilon}} \|w\|_1 + \frac{a_1}{\varepsilon} \|w\|_1^3$$

and analogously the difference by

$$\|r(v) - r(w)\| \leq \frac{a_2}{\sqrt{\varepsilon}} \left( \|v\|_1 + \|w\|_1 + \frac{1}{\varepsilon}(\|v\|_1^3 + \|w\|_1^3) \right) \|v - w\|_1$$

for some positive constants $a_1, a_2$, depending only on $F$. These estimates are uniform for all $x_o \in [-1 + \delta, 1 - \delta]$ given a fixed $\delta > 0$. Another form of the last inequality is useful in estimates of variations of $r$ around a function $z$. We define

$$r_z(y) := r(y + z) - r(z).$$

Because $r(w)$ is linear w.r.t. the derivative $w_x$, it can be written in the form,

$$r_z(y) = y r_1(y,z,z_x) + \varepsilon y x r_2(y,z),$$

and the functions $r_1$ and $r_2$ can be bounded as follows:

$$\|r_1(y,z,z_x)\| \leq a_3(\|y\|_1 + \|z\|_1 + \frac{1}{\varepsilon}(\|y\|_1^3 + \|z\|_1^3)),$$

$$\|r_2(y,z)\|_{L^\infty} \leq \frac{a_4}{\sqrt{\varepsilon}}(\|y\|_1 + \|z\|_1 + \frac{1}{\varepsilon}(\|y\|_1^3 + \|z\|_1^3)).$$
where the norm in (4.13) is $L^2(-1,1)$ (cf. (3.11),[12]). Analogously, we derive estimates for the differences of $r_1$ and $r_2$
\[
\|r_1(y,z,z_x) - r_1(y_0,z,z_x)\| \leq \leq \hat{a}_3 \|y-y_0\|_1 \left(1 + \|y\|_1^2 + \|y_0\|_1^2 + \|z\|_1^2 + \frac{\|y\|_1^3 + \|y_0\|_1^3 + \|z\|_1^3}{\varepsilon}\right) \tag{4.15}
\]
\[
\|r_2(y,z) - r_2(y_0,z)\|_{L^\infty} \leq \leq \frac{\hat{a}_4 \|y-y_0\|_1}{\sqrt{\varepsilon}} \left(1 + \|y\|_1^2 + \|y_0\|_1^2 + \|z\|_1^2 + \frac{\|y\|_1^3 + \|y_0\|_1^3 + \|z\|_1^3}{\varepsilon}\right). \tag{4.16}
\]

The operator $A$, defined in (4.6) is a selfadjoint operator in $L^2(-1,1)$ on the domain

$$
\mathcal{D}(A) := H^1_0(-1,1) \cap H^2(-1,1).
$$

We can consider the (symmetrized) equation (4.6) for the variation $w$ around the (fixed) profile $\Phi(\cdot,x_0)$ as a Cauchy problem in $C^1([0,T],L^2(-1,1))$, which is equivalent to the integral equation

$$
w = Gw, \quad \text{where} \quad Gw(\cdot,t) := e^{-tA}w_0 + \int_0^t e^{-A(t-s)}(r(w(\cdot,s)) + g)ds. \tag{4.17}
$$

Due to the special form of the potential $q$, the quadratic form $v \mapsto (Av,v)$ satisfies the relation

$$
(Av,v) = \int_{-1}^{1} |v'(x) + \frac{1}{2}F'(\Phi(x;x_0))v(x)|^2 dx, \quad \text{for} \quad v \in \mathcal{D}(A), \tag{4.18}
$$

implying that $A$ is a positive operator. For each $\varepsilon > 0$ the operator $A$ has only simple isolated eigenvalues $0 < \lambda_0(\varepsilon) < \lambda_1(\varepsilon) < \cdots$. We denote by $\{\omega_j(x)\}$ the corresponding orthonormal eigenfunctions, which form a complete set in $L^2(-1,1)$. We have the equivalence (cf. (3.19) in [12])

$$
\|v\|_1 \approx \|A^{1/2}v\| + \|v\|, \quad v \in H^1_0(-1,1) \cap \mathcal{D}(A), \tag{4.19}
$$

and the following estimates (cf. (3.20), (3.21) and (3.22) in [12]):

$$
\|A^{\alpha}e^{-tA}v\| \leq ce^{-t\lambda_0/2}\|v\|, \quad t > 0, \quad v \in L^2(-1,1), \tag{4.20}
$$

$$
\|e^{-tA}v\|_1 \leq (t^{-1/2}e^{-t\lambda_0/2} + \sqrt{q_0 + 1}e^{-\lambda_0 t})\|v\| \quad t > 0, \quad v \in L^2(-1,1), \tag{4.21}
$$

and

$$
\|e^{-tA}v\|_1 \leq \sqrt{2(q_0 + 1)e^{-\lambda_0 t}}\|v\|_1, \quad t > 0, \quad v \in H^1(-1,1), \tag{4.22}
$$

where $q_0 := \max(x,\varepsilon)$ is a constant dependent on $F$ only.

In the case $p = 1$ ($F$ has two zeros of order 1), we have shown that the bottom eigenvalue $\lambda_0(\varepsilon)$ of $A(0)$ is exponentially small and that the next smallest eigenvalue is larger than some positive constant not depending on $\varepsilon$. More precisely, we have the equivalence

$$
\lambda_0(\varepsilon) \asymp h^{-2}(1;x_0) + h^{-2}(-1;x_0), \quad \text{if} \quad p = 1, \tag{4.23}
$$

uniformly w.r.t. $x_0 \in [-1 + \delta,1 - \delta]$, $\delta > 0$, and $\varepsilon \in (0,\varepsilon_0)$, cf. [12], lemma 3.1 and eq. (3.39).

If $p \geq 2$ ($F$ has at least one zero of order $p \geq 2$), the behaviour of the eigenvalues is completely different in that they all tend to zero for $\varepsilon \to 0$. A constant $c_1$ (not depending on $\varepsilon$) exists, such that

$$
\lambda_j(\varepsilon) \leq c_1\varepsilon^2(j + 1)^2, \quad j \geq 0, \tag{4.24}
$$

uniformly w.r.t. $j$, for all $\varepsilon \in (0,\varepsilon_0)$ and $x_0 \in [-1 + \delta,1 - \delta]$, $|x_0 - Y_i| \leq \delta > 0$ ($i = 1,k$). To prove this we consider the operator $B := -\varepsilon^2 D_y^2 + q$ on the interval $[y_-,y_+]$, defined in (2.53), with Dirichlet boundary conditions. According to Corollary (2.11), $\Phi(x;x_0)$ is approximately constant and equal to $s_1 + \varepsilon(\varepsilon^{-2})$ on this interval $[y_-,y_+]$. Since $s_1$ is a zero of order $p - 1$ of $F'$, we find $F'(\Phi) = O(\varepsilon)$ and $F''(\Phi) = O(\varepsilon^{p-2})$, hence $q = O(\varepsilon^2)$ uniformly w.r.t. $x \in [y_-,y_+]$. Using the minimax characterization of the eigenvalues (see [21] or [13]), we obtain the estimate $\lambda_j(\varepsilon) \leq \lambda_j(B)$, where $\lambda_j(B)$ is the $j$-th eigenvalue of $B$. Hence $\lambda_j(\varepsilon) \leq c_1\varepsilon^2(j + 1)^2$ for some $c_1 > 0$. 

4.1 Linearization around a traveling wave profile

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4.2 Contraction around the equilibrium solution

If we consider in (4.1) the variation around the equilibrium solution $\Phi_\varepsilon$, the velocity $V$ is zero and the inhomogeneous term disappears. So we obtain the homogeneous (non-linear) equation

$$w_t + Aw = r(w), \quad w(x, 0) = w_0(x). \quad (4.25)$$

Using the (strict) positivity of the bottom eigenvalue as in [15], theorem 5.1.1, it is easily seen that the equilibrium $w = 0$ is asymptotically stable:

**Lemma 4.1** (asymptotic stability) There exist positive constants $c_0$ and $c_1$ depending on $F$ only, such that for all functions $w_0 \in H^1_0(-1,1)$ satisfying

$$\|w_0\|_1 \leq c_1 \varepsilon, \quad 0 < \varepsilon < c_0 \lambda_0(\varepsilon)^{1/2},$$

the solution of (4.25) exists and satisfies

$$\|w(\cdot, t)\|_1 \leq \varepsilon e^{-\lambda_0(\varepsilon)t/2}, \quad \text{for all} \quad t > 0. \quad (4.26)$$

The radius of contraction in this lemma is bounded by the smallest eigenvalue, that is exponentially small in the case $p = 1$ and at most $O(\varepsilon^2)$ for $p \geq 2$. Since there is only one equilibrium, the solution of (IBV) has to converge to it eventually for every initial condition. However, this lemma gives an upper bound for the rate of convergence only in the end, if the solution is already very near to equilibrium.

In the case $p = 1$ we have enlarged this domain of attraction (see lemma 3.5 in [12]) to a ball of radius $O(\sqrt{\varepsilon})$ in a subspace of codimension 1 (roughly) orthogonal to the bottom eigenfunction, using the fact that all other eigenvalues of $A$ remain bounded away from zero. In [12] lemma 3.7, we have shown convergence (whose rate is dominated by the second eigenvalue) if the initial function $w_0$ has the special form $w_0 = \omega + k(\omega, \sigma)\omega_0 + \sigma\omega_0$, where $\omega$ is an arbitrary element in the ball $\|\omega\|_1 \leq c_1 \varepsilon$, $0 < \varepsilon < c_0 \sqrt{\varepsilon}$, orthogonal to the bottom eigenfunction $\omega_0$, and the parameter $\sigma$ is very small, $|\sigma| \leq c_2 \lambda_0 \varepsilon$. Apparently, the component in the direction of $\omega_0$ in an initial condition causes a problem. In order to improve the result and to show convergence from the whole ball of radius $\varepsilon < c_0 \sqrt{\varepsilon}$, we consider the particular problem of evolution of solutions starting approximately at $\omega_0$, or better, at an approximation of the form

$$z_{0, \gamma}(x) := [\Phi(x; x_0 + \gamma \varepsilon) - \Phi(x; x_0)]h(x; x_0), \quad |\gamma| \leq \gamma_0. \quad (4.27)$$

The reason for this choice is that we have a good control over the solution starting at a travelling wave profile (cf. section 3). Moreover, the difference of two neighbouring traveling wave profiles automatically satisfies the boundary conditions $z_{0, \gamma}(\pm 1) = 0$, such that we need not bother about (tiny) boundary layer corrections.

We compare $z_{0, \gamma}$ to the true eigenfunction and define $E_1$ as the orthogonal projection onto the span of the bottom eigenfunction $\omega_0$ and $E_2$ as its orthogonal complement, i.e. $E_1 v = (v, \omega_0) \omega_0$ and $E_2 = Id. - E_1$.

**Lemma 4.2** In the case $p = 1$, the function $z_{0, \gamma}$ is an approximate eigenfunction satisfying

$$\mu(\gamma) := (z_{0, \gamma}, \omega_0) = -\gamma \sqrt{\varepsilon} \left(1 + O(\gamma) + O(\sqrt{R_\varepsilon(\delta)/\varepsilon})\right) \quad \text{and} \quad \|E_2 z_{0, \gamma}\| \leq c \gamma \mu(\gamma) \quad (4.28)$$

$$\|E_j z_{0, \gamma}\| \leq c|\gamma|\sqrt{\varepsilon}, \quad (j = 1, 2), \quad (4.29)$$

uniformly w.r.t. $x_0 \in [-1 + \delta, 1 - \delta], \delta > 0, |\gamma| \leq \gamma_0, 0 < \varepsilon \leq \varepsilon_0$, provided $\gamma_0$ and $\varepsilon_0$ are small enough.
\textbf{4.2 Contraction around the equilibrium solution}

**Proof.** Using the notations (2.57) we can write
\[ h^{-1}z_{0,\gamma} = \psi\left(\frac{x-x_0-\gamma\varepsilon}{\varepsilon}, C_0, V_0\right) - \psi\left(\frac{x-x_0}{\varepsilon}, C_0, V_0\right) + \psi\left(\frac{x-x_0-\gamma\varepsilon}{\varepsilon}, C_\gamma, V_\gamma\right) - \psi\left(\frac{x-x_0}{\varepsilon}, C_0, V_0\right), \]
where \( C_\gamma := C(\varepsilon, x_0 + \gamma\varepsilon) \) and \( V_\gamma := V(\varepsilon, x_0 + \gamma\varepsilon) \). Hence by Taylor's formula,
\[ h^{-1}z_{0,\gamma} = -\gamma\psi'(\frac{x-x_0-\gamma\varepsilon}{\varepsilon}, C_0, V_0) + O(\gamma^2)|\psi''| + (C_\gamma - C_0)\psi_c + (V_\gamma - V_0)\psi. \]
(4.30)

By the differential equation (2.58) we have \(|\psi''| \leq c\psi'\). On the other hand corollary 2.5 [12] says that
\[ \psi'(\eta + u, C_0, V_0) \approx \psi'(\eta, C_0, V_0), \]
uniformly w.r.t. \( \eta, x_0, u, \varepsilon \) if \(|\eta| \leq \eta_0/\varepsilon, x_0 \in [-1+\delta, 1-\delta], \delta > 0, \eta_0 > 0 \|u\| \leq u_0, 0 < \varepsilon \leq \varepsilon_0 \).
Thus \( \psi'' \) from (4.30) has a bound
\[ |\psi''| \leq c\psi'(\frac{x-x_0}{\varepsilon}, C_0, V_0). \]
(4.32)

Using also the estimates (2.62) we get from (4.30),
\[ h^{-1}z_{0,\gamma}(x) = (-\gamma + O(\gamma^2))\varepsilon \Phi'(x; x_0) + |\gamma|O(R_\varepsilon(\delta)), \]
(4.33)
uniformly w.r.t. \( x_0 \in [-1 + \delta, 1 - \delta], |\gamma| \leq \gamma_0, 0 < \varepsilon \leq \varepsilon_0 \). Using (4.4) and the bound \( V(\varepsilon, x_0) = O(R_\varepsilon(\delta)) \) (see eq. (2.7) in [12]), we can rewrite (4.33) in the form
\[ z_{0,\gamma}(x) = (-\gamma + O(\gamma^2))h^{-1}(x; x_0) + |\gamma|h(x; x_0)O(R_\varepsilon(\delta)/\varepsilon). \]
(4.34)

Now we notice that \( h^{-1}(x; x_0) \) corresponds to an approximation of the eigenfunction (see [12], section 3.b, where we used it in the derivation of the asymptotics of \( \lambda_\varepsilon(\varepsilon) \)). In particular (cf. (3.37) and (3.39) in [12]) we have
\[ (h^{-1}(x; x_0), \omega_0(x)) = \sqrt{2\varepsilon} + O(\sqrt{\varepsilon}R_\varepsilon(\delta)). \]
(4.35)

Now (4.28) follows from (4.34), (4.35) and (2.64). Furthermore, (4.34) implies
\[ \|z_{0,\gamma}\| = |\gamma|\sqrt{\varepsilon}(1 + O(\gamma + \sqrt{\varepsilon})), \quad |\gamma| \leq \gamma_0. \]
(4.36)

Since \( \|\omega_0\| \leq c_1 \) we get (4.29) for \( j = 1 \) from (4.36). It remains to prove
\[ \|z_{0,\gamma}\| \leq c|\gamma|\sqrt{\varepsilon}, \quad |\gamma| \leq \gamma_0. \]
(4.37)

Because of (4.36) we only have to prove
\[ \|\varepsilon z'_{0,\gamma}\| \leq c|\gamma|\sqrt{\varepsilon}, \]
(4.38)
where \( z'_{0,\gamma} := \frac{\partial}{\partial x} z_{0,\gamma}(x) \). Since
\[ \varepsilon z'_{0,\gamma} = h \left[ \psi'(\frac{x-x_0-\gamma\varepsilon}{\varepsilon}, C_\gamma, V_\gamma) - \psi'(\frac{x-x_0}{\varepsilon}, C_0, V_0) \right] + \frac{\varepsilon h'}{h} z_{0,\gamma}, \]
we can use Taylor's formula as above and the estimates (4.32–4.36–2.63–2.65) to obtain (4.38).

**Remark 4.3** If we apply the estimates of lemma 4.2 to the difference
\[ [\Phi(x; x_0 + \gamma\varepsilon) - \Phi(x; x_0 + \tilde{\gamma}\varepsilon)] h(x; x_0), \]
we see that this difference is an approximate eigenfunction too, and that the inner product with the true bottom eigenfunction \( \omega_0 \) yields the estimate
\[ \mu(\gamma) - \mu(\tilde{\gamma}) = - (\gamma - \tilde{\gamma}) \sqrt{2\varepsilon} \left( 1 + O(|\gamma| + |\tilde{\gamma}| + \sqrt{R_\varepsilon(\delta)}) \right) \]
(4.39)
The next step in our approach (if $p = 1$) is to estimate the solution of the evolution problem (4.6) starting at the approximate eigenfunction $z_{0, \gamma}$ in the particular case, where $x_o = x_e$,

$$\partial_t z_\gamma + A z_\gamma = r(z_\gamma), \quad z_\gamma(x, 0) = z_{0, \gamma}(x). \quad (4.40)$$

**Lemma 4.4.** If $x_o = x_e$, the solution of (4.40) satisfies the estimate

$$\|z_\gamma(\cdot, t)\|_1 \leq c|\gamma|\sqrt{\varepsilon}, \quad |\gamma| \leq \gamma_0 \quad \text{and} \quad 0 < \varepsilon \leq \varepsilon_0,$$

uniformly for all $t \geq 0$ and all $\gamma$ and $\varepsilon$, provided $\gamma_0, \varepsilon_0$ are small enough.

**Proof.** By (4.1) and (4.5) $z_\gamma$ is related to the solution $u(\cdot, \cdot; x_o)$ of (1.1–1.3–3.1),

$$z_\gamma(x, t/\varepsilon) = [u(x, t; x_o + \gamma \varepsilon) - \Phi(x; x_o)]h(x; x_o). \quad (4.42)$$

Since $x_o = x_e$, eq. (3.2) implies $|z_\gamma(x, t/\varepsilon)| \leq |z_{0, \gamma}(x)|$; using (3.36), we obtain from it the $L_2$-estimate

$$\|z_\gamma(\cdot, t)\| \leq c|\gamma|\sqrt{\varepsilon}, \quad |\gamma| \leq \gamma_0. \quad (4.43)$$

In order to estimate the norm of the derivative we differentiate (4.42)

$$\frac{\partial}{\partial x} z_\gamma(x, t/\varepsilon) = \left[u_x(x, t; x_e + \gamma \varepsilon) - \Phi'(x) \right] h(x; x_e) + z_\gamma(x, t/\varepsilon) F'(\Phi(x))/2\varepsilon,$$

The second term in it is bounded by (4.43) and for the first term we use (3.8) together with the estimate on the boundary values of $\Phi'$:

$$\varepsilon|\Phi'(\pm 1) - \Phi'(\pm 1; x_e + \gamma \varepsilon)| \leq c|\gamma|R(\delta), \quad |\gamma| \leq \gamma_0.$$

This implies

$$\|\varepsilon\frac{\partial}{\partial x} z_\gamma(\cdot, t)\| \leq c|\gamma|\sqrt{\varepsilon}, \quad \text{for all} \quad |\gamma| \leq \gamma_0, \quad \text{and} \quad 0 < \varepsilon \leq \varepsilon_0,$$

provided $\gamma_0$ and $\varepsilon_0$ are sufficiently small. Thus the lemma is proved.

Next we split the solution $w$ of the equation (4.25) in two parts: $w = y_\gamma + z_\gamma$ for $t \geq 0$, where $\gamma$ is a free constant, to be determined later on. Then $y_\gamma$ satisfies the equation

$$\partial_t y_\gamma + Ay_\gamma = r_\gamma(y_\gamma) \quad \text{with} \quad y_\gamma(\cdot, 0) = w_o - z_{0, \gamma}, \quad (4.44)$$

where

$$r_\gamma(y_\gamma) := r(y_\gamma + z_\gamma) - r(z_\gamma). \quad (4.45)$$

Using the same technique as in [14] and [15] and in [12] lemma 3.5, we prove that the equation (4.44) has a fast decaying solution for every (sufficiently small) $y_\gamma(\cdot, 0)$ in a submanifold of codimension one. Such a solution satisfies the bound

$$\|y_\gamma(\cdot, t)\|_1 \leq c_1 e^{-\lambda_1 t/2}, \quad t > 0, \quad |\gamma| \leq \gamma_0, \quad 0 < \varepsilon \leq \varepsilon_0. \quad (4.46)$$

for some constants $c_1, \gamma_0$, depending only on $F$, where $0 < c_1 < c_3\sqrt{\varepsilon}$. Moreover, we show that this submanifold is transversal to $\{z_{0, \gamma} \mid \gamma \in \mathbb{R}\}$ in a ball around the origin, such that each $w_o$ in this ball has a unique decomposition $y_\gamma(\cdot, 0) + z_{0, \gamma}$, its orthogonal complement.

Equivalent to (4.44) is the integral equation

$$y_\gamma = e^{-At_{\gamma}} y_{\gamma}(\cdot, 0) + \int_0^t e^{A(t-s)} r_\gamma(y_\gamma(\cdot, t)) \, dt. \quad (4.47)$$

Assume (4.47) has a solution $y_\gamma$ with initial value $y_\gamma(\cdot, 0) = \omega + \kappa(\omega, \gamma) \omega_o$ and $\omega \in \mathbb{R}(E_2)$ and assume that $y_\gamma$ is in the cone

$$S_\delta := \{\omega \in C([0, \infty); H^1_\alpha(-1, 1)) \mid \|\omega(\cdot, t)\|_1 \leq \delta e^{-\lambda_1 t/2} \}. \quad (4.48)$$
for some positive $\varrho$. The projection of (4.47) onto the span of $\omega_0$ is

$$e^{\lambda_0 t} E_1 y_\gamma = \kappa(\omega, \gamma) \omega_0 + \int_0^t e^{\lambda_0 s} E_1 r_\gamma(y_\gamma(\cdot, t)) \, dt.$$  \hfill (4.49)

The estimates (4.12–4.13–4.14) imply the inequality

$$\| r_\gamma(y_\gamma(\cdot, t)) \| \leq \| y_\gamma \|_{L_\infty} \| r_1 \| + \| \partial_t y_\gamma \| \| r_2 \|_{L_\infty} \leq \frac{c}{\sqrt{\varepsilon}} \| y_\gamma \|_1 \left( \| y_\gamma \|_1 + \| z_\gamma \|_1 + \| y_\gamma \|_1^2 + \| z_\gamma \|_1^2 \right).$$  \hfill (4.50)

Since $y_\gamma \in S_{\varrho}$, the integral in (4.49) converges for $t \to \infty$ to a finite value and the left-hand side vanishes, such that

$$\kappa(\omega, \gamma) \omega_0 = - \int_0^\infty e^{\lambda_0 s} E_1 r_\gamma(y_\gamma(\cdot, t)) \, dt.$$  \hfill (4.51)

Clearly, $\kappa$ is a smooth function of $\gamma$. To this expression we apply (4.15–4.16) to derive the asymptotic estimates of $\kappa$ for small $\gamma$ and $\varrho$,

$$| \kappa(\omega, \gamma) | = \varrho O(\gamma_1 + \gamma_0) \quad \text{and} \quad | \kappa(\omega, \gamma) - \kappa(\omega, 0) | = | \gamma | \sqrt{\varepsilon} O(\gamma_1 + \gamma_0),$$  \hfill (4.52)

provided $| \gamma | \leq \gamma_0$ and $\varrho \leq \gamma_1 \sqrt{\varepsilon}$. As a consequence, the derivative of $\kappa$ w.r.t. $\gamma$ at $\gamma = 0$ can be made sufficiently small by the choice of $\gamma_0$ and $\gamma_1$.

The projection $E_2$ applied to (4.48) yields the integral equation (see (3.45) in [12]):

$$y_\gamma = \mathcal{G} y_\gamma \quad \text{where} \quad \mathcal{G} \text{ is the integral operator}$$

$$\mathcal{G} y_\gamma(\cdot, t) := e^{-t A_2} \omega + \int_0^t e^{-(t-s) A_2} E_2 r_\gamma(y_\gamma(\cdot, s)) \, ds - \int_t^\infty e^{-(t-s) \lambda_0} E_1 r_\gamma(y_\gamma(\cdot, s)) \, ds,$$  \hfill (4.53)

and where $A_2 = A E_2$. From the estimates (4.12–4.16) we can find positive constants $\varepsilon_0$, $\gamma_0$ and $\gamma_1$ such that $\mathcal{G}$ maps $S_{\varrho}$ into itself and is a contraction, uniformly for all

$$(\varepsilon, \gamma, \varrho) \in (0, \varepsilon_0) \times [-\gamma_0, \gamma_0] \times (0, \gamma_1 \sqrt{\varepsilon}).$$  \hfill (4.54)

This implies that (4.44) has a unique solution $w(\cdot, t) = y_\gamma(\cdot, t) + z_\gamma(\cdot, t)$ that satisfies the initial condition

$$w(\cdot, 0) = \omega + \kappa(\gamma, \omega) \omega_0 + z_{0, \gamma}$$  \hfill (4.55)

for every $\gamma \in [-\gamma_0, \gamma_0]$, $\varepsilon \in (0, \varepsilon_0)$ and $\omega \in \mathcal{R}(E_2)$ with $\| \omega \|_1 \leq \gamma_1 \sqrt{\varepsilon}$. It remains to show, that every $w_0$ in a ball of radius $O(\sqrt{\varepsilon})$ around zero admits this representation. Projecting this equation by $E_1$ and $E_2$ we find

$$(w_0, \omega_0) = \kappa(\gamma, \omega) + (z_{0, \gamma}, \omega_0) = \kappa(\gamma, \omega) + \mu(\gamma) \quad \text{and} \quad E_2 w_0 = \omega + E_2 z_{0, \gamma}$$

Since the second equation yields an $\omega \in \mathcal{R}(E_2)$ for every $\gamma$, it suffices to find a $\gamma$ that solves the equation

$$\mu(\gamma) = (w_0, \omega_0) - \kappa(\gamma, E_2 w_0 - z_{0, \gamma})$$  \hfill (4.56)

As a consequence of (4.52), the derivative of the function $\gamma \mapsto \kappa(\omega, \gamma)$ is strictly smaller than $\sqrt{2 \varepsilon}$ uniformly w.r.t. $\varepsilon$ and $\omega$ provided $\varepsilon_0$, $\gamma_0$ and $\varrho_1$ are small enough, whereas the derivative of $\mu$ is equal to $\sqrt{2 \varepsilon}(1 + O(1))$, according to (4.39). Hence, equation (4.56) has a unique solution for all $w_0 \in H^1_0(\gamma, \gamma_0)$ satisfying $\| w_0 \|_1 \leq \gamma_1 \sqrt{\varepsilon}$, provided $\varrho_1$ is small enough.

Thus we have proved the following lemma:
Lemma 4.5 There exist constants \( c_0, c_1 \) and \( \varepsilon_0 \), depending only on \( F \), such that if

\[
\|w_0\|_1 \leq \varrho, \quad w_0 \in H^1_{\alpha}(-1,1), \quad 0 < \varrho < c_0 \sqrt{\varepsilon},
\]

then the solution of (4.25) satisfies

\[
\|w(\cdot,t)\|_1 \leq c_1 \varrho, \quad t > 0, \quad 0 < \varepsilon \leq \varepsilon_0.
\]

Moreover, there exist constants \( c_2, c_3, c_4 \), such that

\[
\|w(\cdot,t) - z_\gamma(\cdot,t)\|_1 \leq c_3 \varrho e^{-\lambda_1 t/2},
\]

where \( z_\gamma \) is the solution of the problem (4.40) and

\[
\|z_\gamma(\cdot,t)\|_1 \leq c_4 |\gamma|\sqrt{\varepsilon}.
\]

Now we can return to \( e \)-coordinates and state our results about local stability of the equilibrium solution. As a consequence of lemmas 4.1 and 4.5, taking into account the equivalence, \( \|u\|_h \asymp \|uh\|_1 \) (see proposition 2.6 in [12]), we have

Theorem 4.6 (asymptotic stability for all \( p \geq 1 \))

There exist positive constants \( k_0 \) and \( k_1 \), depending only on \( F \) and \( \varepsilon_0 > 0 \), such that if \( u \) is the solution of the problem (IBV) and

\[
\|u_0 - \Phi_e\|_{h_e} \leq k_1 \varrho_1, \quad 0 < \varrho_1 < k_0 \lambda_0(\varepsilon) \sqrt{\varepsilon},
\]

where \( h_e(x) := h(x; x_e) \) defined in (4.4), then

\[
\|u(\cdot,t) - \Phi_e\|_{h_e} \leq \varrho_1 e^{-\lambda_1 t/2\varepsilon}, \quad \text{for all } 0 < t < \infty, \quad 0 < \varepsilon < \varepsilon_0.
\]

Theorem 4.7 (enhanced asymptotic stability for \( p = 1 \))

There exist constants \( k_0, k_1 \) and \( \varepsilon_0 \), depending only on \( F \), such that if \( \|u_0 - \Phi_e\|_{h_e} \leq \varrho \) with \( 0 < \varrho < k_0 \sqrt{\varepsilon} \) then

\[
\|u(\cdot,t) - \Phi_e\|_{h_e} \leq k_1 \varrho, \quad 0 < t < \infty, \quad 0 < \varepsilon < \varepsilon_0.
\]

Moreover, there exist constants \( k_2, k_3, k_4 \) and \( \gamma = \gamma(\varepsilon, \varrho) \), \( |\gamma|\sqrt{\varepsilon} \leq k_2 \varrho \), such that

\[
\|u(\cdot,t) - u_\gamma(\cdot,t)\|_{h_e} \leq k_3 \varrho e^{-\lambda_1 t/2\varepsilon},
\]

where \( u_\gamma \) is the solution of the problem (1.1)–(1.3) with initial data \( u_\gamma(x,0) = \Phi(x; x_e + \gamma \varepsilon) \). In addition,

\[
\|u_\gamma(\cdot,t) - \Phi_e\|_{h_e} \leq k_4 |\gamma|\sqrt{\varepsilon} \leq k_2 k_4 \varrho.
\]

Finally, we have the following result about global stability of the equilibrium (see theorem 4.9 in [12]).

Theorem 4.8 (global stability of the equilibrium)

The solution of the problem (IBV) with any continuous initial data \( u_0 \) is attracted by the equilibrium \( \Phi_e \), i.e.

\[
u(x,t) \to \Phi_e(x) \quad \text{and} \quad u_x(x,t) \to \Phi'_e(x) \quad \text{as} \quad t \to \infty,
\]

uniformly for \( x \in [-1,1] \) (and uniformly w.r.t. \( u_0 \) in a bounded set in \( C[-1,1] \)).

For the proof see theorem 4.9 in [12].
5 Metastability of the slow motion

In this section we explain the behaviour of the solution when it is still far away from the equilibrium state. We consider only the case where the initial data is near a traveling wave profile and prove that the solution moves in a small neighbourhood of the profile with slow speed during a long time interval \((0, T_\varepsilon)\). So we study (in \(v\)-coordinates) the inhomogeneous equation (4.6) or the equivalent integral equation (4.17) for small initial value \(w_0\).

5.1 Contraction around a traveling wave profile

Using contraction methods, see theorem 5.1.1 in [15] and lemma 5.6 in [12], we easily prove, that a solution of (IBV) that starts near to a traveling wave profile, stays in its vicinity on a long time scale. A phenomenon dubbed “metastability”.

**Lemma 5.1** There exist positive constants \(c_1\) and \(c_2\), depending on \(F\) only, such that for all \(w_0 \in H^1_0(-1, 1)\) satisfying
\[
\|w_0\|_1 \leq c_1 \sqrt{\varepsilon |V(\varepsilon, x_0)|}
\] (5.1)
the solution of (4.6) satisfies
\[
\|w(\cdot, t)\|_1 \leq \sqrt{\varepsilon |V(\varepsilon, x_0)|}, \quad \text{for all} \quad 0 \leq t \leq T := \frac{c_2}{\sqrt{|V(\varepsilon, x_0)|}}. \tag{5.2}
\]

**Proof:** Define the ball \(B_\sigma := \{w \in C^1([0, T], H^1_0(-1, 1)) \mid w(\cdot, 0) = w_0 \& \|w\|_1 \leq \sigma\}\) for given \(T\) and \(\sigma\). Because of (4.8–4.9–4.10), the integral operator \(\mathcal{G}\) maps \(B_\sigma\) into itself for suitable values of \(c_1\), \(c_2\) and \(\sigma = \sqrt{\varepsilon |V(\varepsilon, x_0)|}\) and is a contraction. This implies, that a solution starting at a distance of order \(O(\sigma)\), stays at this distance during a time span of order \(O(1/\sqrt{|V(\varepsilon, x_0)|})\). \(\blacksquare\)

For the case \(p = 1\) we can improve this result in the sense that we can allow a larger class of initial data, replacing the very small ball of radius \(O(\sqrt{\varepsilon})\) by a larger one of size \(O(1/\sqrt{\varepsilon})\), analogously to lemma 4.5. As in section 4.2 we consider first the particular solution \(z_\gamma\), that starts at the approximate bottom eigenfunction \(z_{0\gamma}\), and consider variations around it afterwards. So \(z_\gamma\) is the solution of the problem
\[
\partial_t z_\gamma + A z_\gamma = r(z_\gamma) + g, \quad z_\gamma(x, 0) = z_{0\gamma}(x). \tag{5.3}
\]
The main difference with the case of lemma 4.5 is that the solution \(z_{0\gamma}(\cdot, t)\) (starting at 0) is not identically zero due to the inhomogeneous term \(g\); so we cannot have a result like lemma 4.4 eq. (4.41). Instead we compare \(z_\gamma\) and \(z_0\) and we show that they remain near to each other on a long time scale.

**Lemma 5.2** There exist constants \(c_1, c_2, \varepsilon_0, \gamma_0\), depending only on \(F\), such that if \(x_0 \in [-1 + \delta, 1 - \delta]\), \(\delta > 0\) then
\[
\|z_\gamma(\cdot, t) - z_{0\gamma}(\cdot, t)\|_1 \leq c_1 |\gamma| \sqrt{\varepsilon}, \quad |\gamma| \leq \gamma_0, \quad 0 < \varepsilon \leq \varepsilon_0, \tag{5.4}
\]
for all \(0 < t < T\), where \(T := c_2/\sqrt{\varepsilon_0(\delta)}\).

**Proof.** For small values of \(\gamma\) we use a contraction argument, as in lemma 5.1. The validity of this method has such a small \(\gamma\)-range because it uses rather rough estimates of the non-linear term. For larger values of \(\gamma\) we use estimate (3.8), which gives a bound for the derivative in terms in the solution itself plus some exponentially small error, that is negligible only if the functions are sufficiently large.

By definition (4.42) we have the relation
\[
z_\gamma(x, t/\varepsilon) - z_{0\gamma}(x, t/\varepsilon) = [u(x, t; x_0 + \gamma \varepsilon) - u(x, t; x_0)]h(x, x_0) \tag{5.5}
\]
By (3.4–3.5–3.6) the difference in $v$-coordinates satisfies for some $c > 0$ the pointwise estimate

$$
\varepsilon|u_x(x, t; x_0 + \gamma\varepsilon) - u_x(x, t; x_0)| \leq c|u(x, t; x_0 + \gamma\varepsilon) - u(x, t; x_0)| + cR_\varepsilon(\delta). \tag{5.6}
$$

Using (2.64) we transform this into the norm-estimate in $w$-coordinates

$$
\|\varepsilon \partial_x [z_\gamma(\cdot, t) - z_0(\cdot, t)]\| \leq c \|z_\gamma(\cdot, t) - z_0(\cdot, t)\| + c\sqrt{R_\varepsilon(\delta)}. \tag{5.7}
$$

Hence, for an estimate in $\| \cdot \|_*$-norm an $L^2$-estimate of the difference suffices provided $\gamma$ is not too small. To this aim we use a little trick and write the difference as an integral of the derivative w.r.t. $\gamma$. So we consider the derivative $u'(x, t; \gamma) := \partial_\gamma (\gamma \mapsto u(x, t; x_0 + \varepsilon\gamma))$ of $u$. For given $u = u(x, t; x_0 + \varepsilon\gamma)$ it satisfies the linear problem

$$
u'_t = \varepsilon u'_{xx} + F'(u) u_x' + F''(u) u_x'' + F''(u) u_x', \quad u'(\pm 1, t) = 0, \quad u'(0, t; x_0 + \gamma\varepsilon) = \frac{\partial}{\partial \gamma} \Phi(x; x_0 + \gamma\varepsilon). \tag{5.8}
$$

To estimate $u'$ we symmetrize as before in eq. (4.2) the equation (5.8) by the choice $\tilde{w}(x, t/\varepsilon; \gamma) := u'_0(x, t; x_0 + \gamma\varepsilon) h(x, t; x_0 + \gamma\varepsilon)$, using the (time-dependent) weight $h$,

$$
\tilde{h}^2(x, t; x_0 + \gamma\varepsilon) := D(\varepsilon, x_0 + \gamma\varepsilon) \exp \left( \frac{1}{\varepsilon} \int_{-1}^{x} F'(u(y, t; x_0 + \gamma\varepsilon)) dy \right).
$$

This gives the linear initial value problem for the unknown function $\tilde{w}$,

$$
\partial_t \tilde{w} + \tilde{A}(t) \tilde{w} = 0, \quad \tilde{w}(x, 0; \gamma) = h(x; x_0 + \gamma\varepsilon) \frac{\partial}{\partial \gamma} \Phi(x; x_0 + \gamma\varepsilon) \tag{5.9}
$$

where the potential $\tilde{q}$ is given by

$$
\tilde{q}(x, t; x_0 + \gamma\varepsilon) := \frac{1}{4} \left[ F'(u(x, t; x_0 + \gamma\varepsilon)) \right]^2 - \frac{1}{2} \varepsilon F''(u(x, t; x_0 + \gamma\varepsilon)) u_x(x, t; x_0 + \gamma\varepsilon). \tag{5.10}
$$

This potential has the same form as $q$ has in eq. (4.7) implying a formula like (4.18), so that this operator $\tilde{A}(t)$ is positive too. Multiplying (5.9) by $\tilde{w}$ and integrating we obtain the estimate

$$
\|\tilde{w}(\cdot, t; \gamma)\|^2 - \|\tilde{w}(\cdot, 0; \gamma)\|^2 = -2 \int_{0}^{t} (\tilde{A}(s) \tilde{w}(\cdot, s; \gamma), \tilde{w}(\cdot, s; \gamma)) ds,
$$

where $(\cdot, \cdot)$ is the usual inner product in $L^2(-1, 1)$. Hence $\|\tilde{w}\|$ is non-increasing,

$$
\|\tilde{w}(\cdot, t; \gamma)\| \leq \|\tilde{w}_0\|. \tag{5.11}
$$

Using the same method as in the proof of (4.29) we obtain (since $\|hu\|_1 \asymp \|u\|_h$),

$$
\|\tilde{w}(\cdot, 0; \gamma)\| \leq c\sqrt{\varepsilon}, \quad \text{uniformly w.r.t. } |\gamma| \leq \gamma_0, \ 0 < \varepsilon \leq \varepsilon_0. \tag{5.12}
$$

This function $\tilde{w}$ is related to the difference (5.5),

$$
z_\gamma(x, t/\varepsilon) - z_0(x, t/\varepsilon) = \int_{0}^{\gamma} \tilde{w}(\cdot, t; \gamma) h(x; x_0 + \gamma\varepsilon) d\gamma. \tag{5.13}
$$

We estimate the quotient of both weight functions using corollary 3.2,

$$
\frac{h(x; x_0)^2}{\tilde{h}^2(x, t; x_0 + \gamma\varepsilon)^2} = \frac{D(\varepsilon, x_0)^2}{D(\varepsilon, x_0 + \gamma\varepsilon)^2} \exp \left( \frac{1}{\varepsilon} \int_{-1}^{x} F'(\Phi(y; x_0)) - F'(u(y, t; x_0 + \gamma\varepsilon)) dy \right) \leq \frac{\Phi(-1, x_0 + \gamma\varepsilon)}{\Phi(-1, x_0)} \exp \left( \frac{K}{\varepsilon} \int_{-1}^{x} \Phi(y; x_0) - u(y, t; x_0 + \gamma\varepsilon) dy \right) \leq M \exp(cK|V|^{1/2}).
$$
where $K$ is the Lipschitz constant of $F$ and $M$ a bound for the quotient of the $D$’s. Since the inverse quotient satisfies the same bound, we have the equation

$$h(x, t; x_0 + \gamma \varepsilon) \simeq h(x; x_0) \quad \text{if} \quad t R_\varepsilon(\delta) \leq \varepsilon \quad \text{and} \quad |x| \leq 1,$$

and we can estimate the $L^2$-norm of the difference (5.13) by $\|\gamma w(\cdot, 0; \gamma)\|$ if $t R_\varepsilon(\delta) \leq \varepsilon$. Thus (5.12) and (5.13) give

$$\|z_\gamma(\cdot, t) - z_0(\cdot, t)\| \leq c |\gamma| \sqrt{\varepsilon} \quad \text{if} \quad t R_\varepsilon(\delta) \leq \varepsilon,$$

uniformly w.r.t. $|\gamma| \leq \gamma_0, \ 0 < \varepsilon \leq \varepsilon_0$ and $x_0 \in [-1 + \delta, 1 - \delta]$. Hence, from (5.7) and (5.14) we find a constant $c$ so that

$$\|z_\gamma(\cdot, t) - z_0(\cdot, t)\|_1 \leq c |\gamma| \sqrt{\varepsilon} + c \sqrt{\varepsilon R_\varepsilon(\delta)} \quad \text{if} \quad t R_\varepsilon(\delta) < \varepsilon,$$

(5.15)

For a given a positive constant $C_1$, we can find the constant $c_1$ (depending on $c$) in (5.4), such that it is true for all $|\gamma| \geq C_1 \sqrt{R_\varepsilon(\delta)}$.

To prove (5.4) for the case $|\gamma| \leq C_1 \sqrt{R_\varepsilon(\delta)}$ we use the contraction method (see lemma 5.6 in [12]). We start with the estimate

$$\|z_\gamma(\cdot, t)\|_1 \leq c (\sqrt{\varepsilon R_\varepsilon(\delta)} + |\gamma| \sqrt{\varepsilon}) \quad \text{if} \quad 0 < t < T_1, \ \text{uniformly w.r.t.} \quad 0 < \varepsilon \leq \varepsilon_0.$$ (5.16)

which is a consequence of (3.9), because of the identity

$$z_\gamma(x, t) = [u(x, t; x_0 + \gamma \varepsilon) - \Phi(x; x_0 + \gamma \varepsilon)]h(x; x_0) + z_0(\gamma)(x).$$

From (5.3) it follows that $w_\gamma := z_\gamma - z_0$ satisfies the homogeneous equation

$$\partial_t w_\gamma + Aw_\gamma = r(w_\gamma + z_0) - r(z_0), \ w_\gamma(x, 0) = z_0(\gamma)(x),$$

(5.17)

where according to (4.37)

$$\|z_0(\gamma)\|_1 \leq c |\gamma| \sqrt{\varepsilon} \leq C_2 \sqrt{\varepsilon R_\varepsilon(\delta)} \quad \text{if} \quad |\gamma| \leq C_1 \sqrt{R_\varepsilon(\delta)}.$$ (5.18)

To estimate $\|w_\gamma(\cdot, t)\|_1$ we consider the corresponding integral equation

$$w(\cdot, t) = Gw, \quad Gw := e^{-At}z_0(\gamma) + \int_0^t e^{-A(t-s)}r(w + z_0) - r(z_0)) \, ds.$$ (5.19)

It is sufficient to show that $G$ is a contraction in the ball

$$B_{\varrho} := \{w \in C^1([0, T], H^1_0((-1, 1))) \mid w(\cdot, 0) = z_0(\gamma) \& \|w\|_1 \leq \varrho\}$$

where $\varrho := C_3 \sqrt{\varepsilon R_\varepsilon(\delta)}$ and where $C_3 > 0$ will be chosen below. Using (5.16–4.10–4.20–4.21–4.22) we get the estimates

$$\|Gw(\cdot, t)\|_1 \leq \frac{C_4}{C_3} \varrho + C_5 \varrho^2 T / \sqrt{\varepsilon}, \quad 0 < t < T,$$

$$\sup_{0 < t < T} \|Gv(\cdot, t) - Gw(\cdot, t)\|_1 \leq C_6 \varrho \frac{T}{\sqrt{\varepsilon}} \sup_{0 < t < T} \|v(\cdot, t) - w(\cdot, t)\|_1.$$ (5.19)

Choosing $C_3$ and $T := c_2 / \sqrt{R_\varepsilon(\delta)}$ appropriately, we see that $G$ is a contraction in the ball $B_{\varrho}$ for all $0 < t < T$. Hence

$$\|z_\gamma(\cdot, t) - z_0(\cdot, t)\|_1 \leq c_1 |\gamma| \sqrt{\varepsilon}, \quad 0 < t < T,$$

if $|\gamma| \leq C_1 \sqrt{R_\varepsilon(\delta)}$. Together with (5.15) this proves the lemma.
LEMMA 5.3 (case $p = 1$) There exist constants $c_0, c_1, c_2, \varepsilon_0$, depending only on $F$, such that if $x_o \in [-1 + \delta, 1 - \delta], \delta > 0$, and

$$\|w_o\|_1 \leq \varrho, \quad 0 < \varrho < c_0 \sqrt{\varepsilon},$$

(5.20)

then

$$\|w(\cdot, t)\|_1 \leq c_1 \varrho, \quad \text{for all } 0 < t < T, \quad \text{and } 0 < \varepsilon < \varepsilon_0,$$

(5.21)

where $T := c_2/\sqrt{R_\varepsilon(\delta)}$, $R_\varepsilon(\delta)$ being given by (2.60). Moreover, there exist constants $c_3, c_4, c_5$ and $\gamma = \gamma(\varepsilon, \varrho)$ with $|\gamma| \sqrt{\varepsilon} \leq c_3 \varrho$, such that

$$\|w(\cdot, t) - z_\gamma(\cdot, t)\|_1 \leq c_4 \varrho e^{-\lambda_1 t/2}, \quad \text{for all } 0 < t < T, \quad \text{and } 0 < \varepsilon < \varepsilon_0,$$

(5.22)

and (5.16) implies

$$\|\tilde{z}_\gamma(\cdot, t)\|_1 \leq c \sqrt{\varepsilon} \sqrt{\varepsilon} + |\gamma| \sqrt{\varepsilon}, \quad \text{uniformly w.r.t. } |\gamma| \leq \gamma_0, \quad 0 < \varepsilon \leq \varepsilon_0, \quad \text{for all } t > 0,$$

(5.23)

and (5.16) implies

$$\|\tilde{z}_\gamma(\cdot, t)\|_1 \leq c \sqrt{\varepsilon} \sqrt{\varepsilon} + |\gamma| \sqrt{\varepsilon}, \quad \text{uniformly w.r.t. } |\gamma| \leq \gamma_0, \quad 0 < \varepsilon \leq \varepsilon_0, \quad \text{for all } t > 0.$$
References