Permutation Group Theory

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CHAPTER 1

Introduction

This course is a manifestation of the authors tastes and his attempt to explain some of the trade in finite permutation group theory. Those of a wiser standing could tell you the history of permutation group theory, or indeed group theory itself, which could provide motivation and context for some of the material in this course; the author may have to opt out of such a challenge as such an exercise would exceed the page limit of these notes. However, it can be trusted that group theory is an important part of mathematics; the embodiment and abstraction of symmetry, that mysteriously pervades the far corners of pure and applied mathematics. One of the aims of this course is to provide a foundation in group actions to a level where the practitioner can communicate with a variety of researchers.

Another aim of this course is to see a structure theorem similar to that of the celebrated O’Nan-Scott Theorem. Due to the biases of the author, the structure of primitive groups for which the O’Nan-Scott Theorem entails, will be seen through the world of a larger class of permutation groups; the so-called innately transitive groups. The main reason being that the fundamental theory does not change all that much and the actions of minimal normal subgroups are brought to the fore.

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\(^1\)When this was written, it was intended as notes for a Summer Course at RMIT in January 2006.
CHAPTER 2

Group actions

2.1. The basic definitions

In most books on group actions, or permutation groups, they begin with two definitions of a group action. The main reason for this, is that most group theorists actually use both definitions when they are thinking. One is more abstract than the other, which group theorists would call a “permutation representation”, whilst the concrete version is used when there is an apparent and readily visible “action” of a group. In any case, a group action is just that; a group “acting” upon a set.

**Definition 2.1.1 (...with cumbersome notation).** Let $G$ be a group and let $\Omega$ be a set. Then a group action is a function $\mu$ from $\Omega \times G$ to $\Omega$ such that the following hold:

- (i) $\mu(\mu(\omega, g), h) = \mu(\omega, gh)$, for all $\omega \in \Omega$ and $g, h \in G$;
- (ii) $\mu(\omega, 1) = \omega$ for any $\omega \in \Omega$.

Often, we will not write $\mu$ explicitly and use the notation (made popular by H. Wielandt)

$$\omega^g$$

to denote the image of $(\omega, g)$ under $\mu$. This exponential notation has some pros and cons. One of the detriments of writing the action this way is that the group elements appear to the right of the set elements, whereas we are accustomed to functions appearing on the left of set elements. However, one benefit of it is that it really looks like the group is acting on the set, and it has the mnemonic property of obeying power laws.

**Definition 2.1.2 (...without cumbersome notation).** Let $G$ be a group and let $\Omega$ be a set. Then a group action is a function of the form $\omega \mapsto \omega^g$ (from $\Omega \times G$ to $\Omega$) such that the following hold:

- (i) $(\omega^g)^h = \omega^{gh}$, for all $\omega \in \Omega$ and $g, h \in G$;
- (ii) $\omega^1 = \omega$ for any $\omega \in \Omega$.

From now on, we will not write $\mu$ explicitly and use the notation (made popular by H. Wielandt)

$$\omega^g$$

to denote the image of $(\omega, g)$ under $\mu$. This exponential notation has some pros and cons. One of the detriments of writing the action this way is that the group elements appear to the right of the set elements, whereas we are accustomed to functions appearing on the left of set elements. However, one benefit of it is that it really looks like the group is acting on the set, and it has the mnemonic property of obeying power laws.

**Example 2.1.3.**

(i) The symmetric group on $n$ letters acts naturally on the set $\{1, 2, \ldots, n\}$. For example, if $n = 6$, we have

- $1^{(123)} = 2$,
- $3^{(12)(35)} = 5$, and
- $6^{(126)(65)(351)} = 3$.

(ii) The general linear group $\text{GL}_d(\mathbb{R})$ acts naturally on the vector space $\mathbb{R}^d$. Specifically, if $M$ is a $d \times d$ invertible matrix over the integers, and $(x_1, x_2, \ldots, x_d)$ is a $d$-tuple of reals, then

$$(x_1, x_2, \ldots, x_d)^M = (x_1, x_2, \ldots, x_d)M.$$ 

(iii) A group $G$ acts naturally on itself by right multiplication:

$$u^g := ug$$

for all $u, g \in G$. Note that the action by left multiplication is not necessarily a group action as we may have $(gh)u \neq h(gu)$.

(iv) A group $G$ acts naturally on itself by conjugation:

$$u^g := g^{-1}ug$$

for all $u, g \in G$. 


2.1. THE BASIC DEFINITIONS

(v) There always exists a trivial action of a group $G$ on a set $\Omega$:

$$\omega^g := \omega$$

for all $\omega \in \Omega$ and $g \in G$.

(vi) The symmetries of an $n$-gon, the so-called Dihedral group $D_{2n}$, acts naturally on the $n$-gon. We will see later that we can identify $D_{2n}$ with a subgroup of $S_n$.

The abstract definition, which is sometimes called a “permutation representation”, is useful when describing the image and kernel of a group action. It also provides an exercise for a group actions course! Basically, a group action is nothing more than seeing your abstract group as a group of permutations. Recall that $\text{Sym}(\Omega)$ is the set of all bijections of $\Omega$, which are called permutations by another name, with the binary operation of function composition rendering this set a group.

**Definition 2.1.4 (Permutation Representation).** Let $G$ be a group and let $\Omega$ be a set. Then a permutation representation of $G$ on $\Omega$ is a homomorphism from $G$ into $\text{Sym}(\Omega)$.

We see now that group actions and permutation representations are really just the same things.

**Lemma 2.1.5.** Let $G$ be a group and let $\Omega$ be a set.

(i) If $\mu : \Omega \times G \to \Omega$ is a group action, then the map $\psi : G \to \text{Sym}(\Omega)$ defined by

$$\psi(g)(\omega) = \mu(\omega, g^{-1})$$

for all $g \in G$ and $\omega \in \Omega$, is a permutation representation of $G$ on $\text{Sym}(\Omega)$.

(ii) If $\psi : G \to \text{Sym}(\Omega)$ is a permutation representation, then the following defines a group action of $G$ on $\Omega$:

$$\omega^g := \psi(g^{-1})(\omega).$$

**Proof.** Exercise. $\square$

Note that the inverses appearing in the above lemma are due to our preference for writing group elements on the right of set elements whilst retaining the traditional notation of functions on the left.

**Example 2.1.6.** Let $\text{Aut}(N)$ be the group of automorphisms of a group $N$. Then $\text{Aut}(N)$ acts naturally on $N$: $n^\tau := \tau(n)$ for all $\tau \in \text{Aut}(N)$ and $n \in N$.

Suppose we have a group $H$ acting on $N$ whose permutation representation $H \to \text{Sym}(N)$ is such that its image is contained in $\text{Aut}(N)$. Then we say that $H$ is acting by automorphisms on $N$. In this case, we can define the semidirect product

$$N \rtimes H$$

with respect to this action, where the binary operation is defined by

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 n_2^{h_1^{-1}}, h_1 h_2)$$

for all $n_1, n_2 \in N$ and $h_1, h_2 \in H$. (Note: the inverse in the definition of the binary operation is necessary to ensure that associativity holds.) If the action of $H$ on $N$ is trivial, that is $n^h = n$ for all $n \in N$ and $h \in H$, then $N \rtimes H$ is just the direct product of $N$ and $H$.

Note that the group $S_3$ can act on $\{1, \ldots, n\}$ for any choice of $n > 3$ by acting naturally on the elements $\{1, 2, 3\}$ and fixing all the rest. Hence such identification of redundancy is important in the study of group actions, as is the ubiquitous notion of “kernel” that pervades algebra.

**Definition 2.1.7 (Kernel/Faithful).** The kernel of a group action is the kernel of the permutation representation arising from it. Alternatively, the kernel of a group $G$ acting on a set $\Omega$ is the set

$$G_\Omega := \{g \in G : \omega^g = \omega, \text{ for all } \omega \in \Omega\}.$$

If the kernel of the action of a group $G$ on a set $\Omega$ consists only of the identity element, then we say that $G$ acts faithfully on $\Omega$.

**Example 2.1.8.** The full symmetric group $\text{Sym}(\Omega)$ of a set $\Omega$ acts faithfully. In fact, any subgroup of a group acting faithfully is also faithful. A permutation group is a subgroup of some $\text{Sym}(\Omega)$. Hence every permutation group is faithful. We will see later that the converse is also true!
CHAPTER 2. GROUP ACTIONS

2.2. ORBITS AND STABILISERS

EXERCISES 2.1.9.

(i) Prove Lemma 2.1.5.

(ii) Let $V$ be a vector space, and show that $\text{GL}(V)$ (the group of invertible linear maps on $V$) acts naturally on the set of 1-dimensional subspaces of $V$. What is the kernel of this action?

(iii) Prove that if $\alpha^g = \beta$ and $h = g^{-1}$, then $\beta^h = \alpha$.

(iv) Suppose we have group actions $\mu_1 : \Omega \times G_1 \to \Omega_1$ and $\mu_2 : \Omega_2 \times G_2 \to \Omega_2$, and let $G = G_1 \times G_2$.

(a) Let $\Omega := \Omega_1 \cup \Omega_2$ (disjoint union). Show that the map $\mu_1 + \mu_2 : \Omega \times G \to \Omega$ defined by

$$(\mu_1 + \mu_2)(\omega, (g_1, g_2)) = \begin{cases} \mu_1(\omega, g_1) & \text{if } \omega \in \Omega_1 \\ \mu_2(\omega, g_2) & \text{if } \omega \in \Omega_2, \end{cases}$$

is a group action. This is called the disjoint sum of $\mu_1$ and $\mu_2$.

(b) Let $\Omega := \Omega_1 \times \Omega_2$ (Cartesian product). Show that the map $\mu_1 \times \mu_2 : \Omega \times G \to \Omega$ defined by

$$(\mu_1 \times \mu_2)((\omega_1, \omega_2), (g_1, g_2)) = (\mu_1(\omega_1, g_1), \mu_2(\omega_2, g_2))$$

is a group action. This is called the product of $\mu_1$ and $\mu_2$.

(v) Let $S^1$ be the unit circle in the complex plane. Show that the following defines a group action of $\mathbb{R}$ on $S^1$: $(e^{it})^r = e^{itr}$ for all $x \in [0, 2\pi)$ and $r \in \mathbb{R}$. What is the kernel of this action?

(vi) Let $G$ be a group, and define the centre of $G$ to be the set of elements of $G$ which commute with every element of $G$. That is, $Z(G) = \{g \in G : (\forall x \in G) xg = gx\}$. Using a group action, show that $Z(G)$ is a normal subgroup of $G$.

(vii) Let $G$ be a group, and define the norm of $G$ to be the set of elements of $G$ which commute with every subgroup of $G$. That is, $N(G) = \{g \in G : (\forall H \leq G) Hg = gH\}$. Using a group action, show that $N(G)$ is a normal subgroup of $G$.

(viii) Find an example of a group for which $Z(G) \neq N(G)$.

2.2. Orbits and stabilisers

The two most important concepts in the theory of group actions are that of an orbit and the stabiliser. An orbit can be thought of as what you get when you spin an element under the action of $G$.

**Definition 2.2.1** (Orbit). Let $G$ be a group acting on a set $\Omega$, and let $\omega$ be an element of $\Omega$. Then the orbit of $\omega$ under $G$ is the subset of $\Omega$ defined by

$$\omega^G := \{\omega^g : g \in G\}.$$

So for example, the orbit of 1 under the subgroup $\langle (123), (23)(45) \rangle$ of $S_5$, is $\{1, 2, 3\}$. We will consider two types of subgroups which arise from stabilising a set or completely fixing a set.

**Definition 2.2.2** (Stabilisers, set-wise and point-wise). Let $G$ be a group acting on a set $\Omega$, and let $\Sigma$ be a subset of $\Omega$. Then the set-wise stabiliser of $\Sigma$ in $G$ is the subgroup

$$G_{\Sigma} = \{g \in G : \sigma^g \in \Sigma, \text{ for all } \sigma \in \Sigma\}.$$

The point-wise stabiliser of $\Sigma$ in $G$ is the subgroup

$$G_{(\Sigma)} = \{g \in G : \sigma^g = \sigma, \text{ for all } \sigma \in \Sigma\}.$$

If $\Sigma = \{\sigma\}$, then we write $G_{\sigma}$ for $G_{\Sigma}$.

We will give some example of stabilisers as well as display some properties that follow from the definitions of orbit and stabiliser.

**Example 2.2.3.**

(i) Let $G$ be a group acting on a set $\Omega$. Then we have the following properties of the orbits of $G$ on $\Omega$:

(a) If $\omega \in \Omega$, then $\omega \in \omega^G$. Therefore, the orbits of $G$ are nonempty and cover all of $\Omega$.

(b) If $\alpha^G \neq \beta^G$, then $\alpha^G \cap \beta^G = \emptyset$. Therefore, the orbits of $G$ are disjoint.

Therefore, the orbits of $G$ form a partition of $\Omega$.

(ii) If $\alpha, \beta \in \Omega$, then $G_{\alpha} \cap G_{\beta} = G_{(\{\alpha, \beta\})}$. We will usually abbreviate by writing $G_{\alpha, \beta}$ for this group.
2.2. ORBITS AND STABILISERS

(iii) The kernel of the action of \( G \) on \( \Omega \) fits with our notation. That is, the kernel of the action is the pointwise stabiliser of the whole set! So we have

\[
G_{(\Omega)} = \bigcap_{\omega \in \Omega} G_{\omega}.
\]

**Definition 2.2.4 (Transitivity).** Let \( G \) be a group acting on a set \( \Omega \). If \( G \) has just one orbit of \( \Omega \), namely the whole set itself, then we say that \( G \) is transitive on \( \Omega \). Alternatively, \( G \) is transitive on \( \Omega \) if for any pair of points \( \omega_1, \omega_2 \in \Omega \), there exists an element \( g \in G \) such that

\[
\omega_1^g = \omega_2.
\]

Transitive actions turn up a lot in group theory. For example, the right coset action on a subgroup is one such action.

**Example 2.2.5.** Let \( G \) be a group and let \( H \) be a subgroup of \( G \). Then we can define a group action of \( G \) on the right cosets of \( H \) by

\[
(Hu)g = Hug
\]

for all right cosets \( Hu \) and group elements \( g \in G \). Moreover, \( G \) acts transitively on the set \( G/H \) of right cosets of \( H \) in \( G \). The stabiliser in \( G \) of the trivial coset \( H \) is the core of \( H \) in \( G \), which is the normal subgroup of \( G \) defined by

\[
\text{Core}_G(H) = \bigcap_{g \in G} g^{-1}Hg.
\]

**Theorem 2.2.6 (The Orbit-Stabiliser Theorem).**

Let \( G \) be a group acting on a set \( \Omega \) and let \( \omega \in \Omega \). Then:

(i) There is a bijective correspondence between the elements of \( \omega^G \) and the set of right cosets \( G/G_{\omega} \).

(ii) If \( \Omega \) is finite, then \( |G| = |\omega^G| \cdot |G_{\omega}| \).

**Proof.** The bijection is simply the map

\[
G_{\omega^g} \mapsto \omega^g.
\]

It is an exercise to prove that this is indeed an injective and surjective map. \( \square \)

**Definition 2.2.7 (Semiregular and Regular).** Let \( G \) be a group acting on set \( \Omega \). If the only element of \( G \) which fixes some element of \( \Omega \) is the identity, then the action of \( G \) is semiregular. If \( G \) is transitive and semiregular on \( \Omega \), then we say that \( G \) is regular on \( \Omega \).

As an example, consider the cyclic group \( \mathbb{Z}_n \) acting on itself by right translation:

\[ x^g := x + g \]

for all \( x, g \in \mathbb{Z}_n \). Then no element, except the zero of \( \mathbb{Z}_n \), can fix any element of \( \mathbb{Z}_n \). Hence \( \mathbb{Z}_n \) is semiregular in this action. Moreover, \( \mathbb{Z}_n \) is transitive and hence regular.

Let \( G \) be a group acting on a set \( \Omega \). Recall that the orbits of \( G \) on \( \Omega \) form a partition of \( \Omega \). So there is a decomposition of \( \Omega \) which is left invariant under the action of \( G \). So in some sense, one can think of a transitive action as one which does not fix a decomposition of a set into a disjoint union of proper nonempty sets. Similarly, we could ask when a group action preserves a different structure on the set. One of the first to come to mind is a weakening of the decomposition of the kind just described, where the group is allowed to permute the parts of the partition (and not just fix them in place). That is, \( G \) preserves a block-system.

**Definition 2.2.8 (Block).** Let \( G \) be a group acting on a set \( \Omega \) and let \( \Sigma \) be a nonempty subset of \( \Omega \). If for all elements \( g \in G \) we have

\[
\Sigma^g = \Sigma \text{ or } \Sigma^g \cap \Sigma = \emptyset
\]

then we say that \( \Sigma \) is a block for \( G \). If \( G \) is transitive, then an orbit of a block is called a block-system for \( G \).

A block-system of a group \( G \) acting on a set \( \Omega \), is a partition which is left invariant by the action of \( G \). That is, for each part \( P \) of the partition and element \( g \in G \), we have that \( P^g \) is a part of the partition. Hence a block-system is a \( G \)-invariant partition. As was the case for transitive groups, we have a name for group actions which do not stabilise a (nontrivial) block-system.
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2.2. ORBITS AND STABILISERS

Definition 2.2.9 (Primitivity). Let $G$ be a group acting transitively on a set $\Omega$. If the only $G$-invariant partitions of $\Omega$ are trivial, that is, $\{\Omega\}$ and $\{\{\omega\} : \omega \in \Omega\}$, then we say that the action of $G$ is primitive.

So a transitive group action is primitive if it has no nontrivial blocks.

Example 2.2.10.
(i) The full symmetric group $\text{Sym}(\Omega)$ acting on a set $\Omega$ is primitive.
(ii) The symmetry group of the cube is imprimitive (not primitive) as the four antipodal pairs of vertices of the cube form a block-system.

Theorem 2.2.11. Let $G$ be a group acting transitively on a set $\Omega$. Then there is a bijection between the collection of subgroups of $G$ containing a point stabiliser $G_\alpha$, and the blocks of $G$ containing $\alpha$.

Proof. This bijection is simply $J \mapsto \alpha J$ defined on all subgroups $J$ of $G$ which contain $G_\alpha$. □

So we have the following corollary.

Corollary 2.2.12. Provided $\Omega$ has more than one element, $G$ is primitive on $\Omega$ if and only if a point stabiliser $G_\alpha$ is a maximal subgroup of $G$.

Proof. Exercise. □

Exercises 2.2.13. Let $G$ be a group acting on a set $\Omega$.

(i) Suppose $G$ is transitive and faithful, let $N$ be a normal subgroup of $G$, and let $\omega \in \Omega$. Show that the following are equivalent:
   - $N$ is transitive;
   - $G = NG_\omega$;
   - $G = G_\omega N$.

(ii) Prove that if $G$ is transitive, faithful, and abelian, then it is also regular.

(iii) Show that if $G$ acts faithfully and transitively, and the size of $\Omega$ is a prime number, then $G$ is primitive.

(iv) Suppose $G$ is transitive and let $B_1$ and $B_2$ be two blocks for $G$. Show that if $B_1$ and $B_2$ are not disjoint, then $B_1 \cap B_2$ is a block for $G$.

(v) Let $H$ be a subgroup of $G$. Show that $\text{Core}_G(H)$ is the largest normal subgroup of $G$ contained in $H$.

(vi) An equivalence relation $\sim$ on $\Omega$ is a congruence (w.r.t $G$) if whenever $\alpha \sim \beta$, we have $\alpha^g \sim \beta^g$ for any $g \in G$. Show that if $G$ is transitive and $\sim$ is a congruence, then the equivalence classes of $\sim$ form a block-system for $G$. Conversely, show that a block-system for $G$ gives rise to a congruence on $\Omega$.

(vii) Suppose $G$ is transitive and let $B$ be a block for $G$. Show that the set-wise stabiliser $G_B$ acts transitively on $B$.

(viii) Suppose $G$ is transitive. Show that if $\alpha$ and $\beta$ are two points of $\Omega$, then $G_\alpha$ and $G_\beta$ are conjugate in $G$.

(ix) We say that $G$ is quasiprimitive if every nontrivial normal subgroup of $G$ is transitive.
   (a) Prove that every primitive group is quasiprimitive.
   (b) A subgroup $H$ of $G$ is quasimaximal if for every normal subgroup $N$ of $G$, either $N \leq H$ or $G = NH$.
      Show that $G$ is quasiprimitive if and only if $G$ is transitive and has a quasimaximal point stabiliser.
   (c) A partition of $\Omega$ is $G$-normal if it is the set of orbits of some normal subgroup of $G$. Show that $G$ is quasiprimitive if and only if $G$ is transitive and every $G$-normal partition of $\Omega$ is trivial.
2.3. Morphisms

In many areas of mathematics, there is usually a way to distinguish between objects. For example, in group theory the objects are considered the same if they are isomorphic. However in permutation group theory, we would like a more discriminating equivalence relation than group isomorphism. For example, the symmetric group on 3 letters may act naturally on 3 elements, but may also act on \{4, \ldots, n\} by fixing every element of \{1, 2, 3\}. These actions are very different (only one of them is transitive), but the group in question is the same.

Derek Robinson [14, pp. 32] calls the equivalence relation below, a “similarity”, whereas Neumann, Stoy, and Thompson [12, pp. 32] and Peter Cameron [6, pp. 3] call the same concept an “equivalence” and “G-space isomorphism” respectively. We adopt a similar standard to that of Dixon and Mortimer [7, pp. 17].

DEFINITION 2.3.1 (Permutational Isomorphism). Let \( G \) be a group acting on a set \( \Omega \) and let \( H \) be a group acting on a set \( \Delta \). Then \( G \) on \( \Omega \) is permutationally isomorphic to \( H \) on \( \Delta \) if there is an isomorphism \( \theta : G \to H \) and a bijection \( \mu : \Omega \to \Delta \) such that for all \( g \in G \) and \( \omega \in \Omega \), we have \( \mu(\omega^g) = \mu(\omega)^\theta(g) \). We call the pair \((\theta, \mu)\) a permutational isomorphism.

\[
\begin{array}{ccc}
\Omega & \xrightarrow{g} & \Omega \\
\downarrow{\mu} & & \downarrow{\mu} \\
\Delta & \xrightarrow{\theta(g)} & \Delta
\end{array}
\]

Figure 1. A permutational isomorphism from \( G \) on \( \Omega \) to \( H \) on \( \Delta \) is an isomorphism \( \theta : G \to H \) together with a bijection \( \mu : \Omega \to \Delta \) such that the above diagram commutes (for each \( g \in G \)).

With this definition we can state what we always knew to be true – a faithful group action is just the same as a permutational group!

LEMA 2.3.2. If \( G \) acts faithfully on \( \Omega \), then \( G \) is permutationally isomorphic to a permutation group on \( \Omega \).

PROOF. The permutation representation \( \psi : G \to \text{Sym}(\Omega) \) induced by the action of \( G \) on \( \Omega \) is injective. Hence \( \psi \), together with the identity map on \( \Omega \), form a permutational isomorphism from \( G \) on \( \Omega \) to \( \psi(G) \) on \( \Omega \). \( \square \)

We also define the weaker notion of permutational transformation.

DEFINITION 2.3.3 (Permutational Transformation). Let \( G \) be a group acting on a set \( \Omega \) and let \( H \) be a group acting on a set \( \Delta \). Then \((\theta, \mu)\) is a permutational transformation from \( G \) on \( \Omega \) to \( H \) on \( \Delta \) if \( \theta : G \to H \) is a group homomorphism and \( \mu : \Omega \to \Delta \) is a function such that for all \( g \in G \) and \( \omega \in \Omega \), we have \( \mu(\omega^g) = \mu(\omega)^\theta(g) \).

So a permutational isomorphism is a permutational transformation \((\theta, \mu)\) such that \( \theta \) and \( \mu \) are each bijections.

Example 2.3.4. Let \( G \) be a group acting on a set \( \Omega \).

(a) Suppose \( G \) is not faithful on \( \Omega \), and let \( E \) be the kernel of its action. Let \( \theta : G \to G/E \) be the canonical projection homomorphism and let \( \mu \) be the identity map on \( \Omega \). Then \((\theta, \mu)\) is a permutational transformation to \( G/E \) with action on \( \Omega \) defined by \( \theta^E(g) := \theta(g) \) for all \( \omega \in \Omega \) and \( Eg \in G/E \). We call \( G/E \) the permutation group induced by \( G \) on \( \Omega \). Sometimes we will write \( G^\Omega \) for this permutation group.

(b) Let \( \mathcal{B} \) be a \( G \)-invariant partition of \( \Omega \). Then there is an induced action of \( G \) on \( \mathcal{B} \) given by \( B^g := \{b^g : b \in B\} \) for all \( B \in \mathcal{B} \) and \( g \in G \). This is called a quotient action of \( G \). We can reframe this concept in terms of permutational transformations. Let \( E \) be the kernel of the action of \( G \) on \( \mathcal{B} \), and consider the faithful action of \( G/E \) on \( \mathcal{B} \) given by \( B^g = B^f \) for all \( B \in \mathcal{B} \) and \( Eg \in G/E \). Let \( \theta : G \to G/E \) be the canonical

\[1\] Dixon and Mortimer use the term permutation isomorphic, but in this course, we prefer to use the adjective form permutationally isomorphic.
projection homomorphism and let \( \mu : \Omega \rightarrow \mathcal{B} \) be the map which selects for each element of \( \Omega \), the unique part of \( \mathcal{B} \) it belongs to. Then \((\theta, \mu)\) is a permutational transformation from the action of \( G \) on \( \Omega \) to the quotient action of \( G/E \) on \( \mathcal{B} \).

(c) There is also a converse to the last example. Let \((\theta, \mu)\) be a permutational transformation with domain \( G \) acting on \( \Omega \). Recall that if \( f \) is a function with domain \( \Omega \), then a fibre of \( f \) is a preimage of a point in the image of \( f \). The fibres of \( \mu \) form a \( G \)-invariant partition \( \mathcal{B} \) of \( \Omega \), and it turns out that the permutation group induced by the action of \( G \) on \( \mathcal{B} \) is permutationally isomorphic to the action of \( \theta(G) \) on \( \mu(\Omega) \).

Note that in the definition of a permutational transformation, if \( G \) acts faithfully on \( \Omega \), it may not be true that \( \theta(G) \) acts faithfully on \( \mu(\Omega) \).

**Definition 2.3.6** (Faithful Permutational Transformation). We say that \((\theta, \mu)\) from \( G \) on \( \Omega \) to \( H \) on \( \Delta \) is a faithful permutational transformation if

\[
\{ g \in G : \mu(\omega^g) = \mu(\omega), \text{ for all } \omega \in \Omega \}
\]

that is, \( \theta(G) \) acts faithfully on \( \mu(\Omega) \).

If \( \omega \in \Omega \), then we denote the \( f \)-preimage of \( f(\omega) \) by \([\omega]_f\). We have the following elementary result for permutational transformations.

**Theorem 2.3.7** ("The First Isomorphism Theorem" for Group Actions). Let \( G \) be a group acting on a set \( \Omega \) and let \( H \) be a group acting on a set \( \Gamma \), and suppose that \((\theta, \mu)\) is a permutational transformation from \( G \) on \( \Omega \) to \( H \) on \( \Gamma \). Then there is an action of \( G/\ker \theta \) on the fibres of \( \mu \) defined by

\[
[\omega]_{[\ker \theta]} := [\omega^g]_{\mu},
\]

for all \( g \in G \) and \( \omega \in \Omega \), and this action is permutationally isomorphic to the action of \( \theta(G) \) on \( \mu(\Omega) \).

**Proof.** Exercise.

**Corollary 2.3.8.** Let \( G \) be a group acting on a set \( \Omega \) and let \( H \) be a group acting on a set \( \Gamma \), and suppose that \((\theta, \mu)\) is a faithful permutational transformation from \( G \) on \( \Omega \) to \( H \) on \( \Gamma \). Then we have the following:

(a) If \( G \) is transitive on \( \Omega \), then \( \theta(G) \) is transitive on \( \mu(\Omega) \).

(b) If \( \theta(G) \) is semiregular, then \( G/\ker \theta \) is semiregular on \( \Omega \) (in its natural action).

(c) If \( G \) is faithful and primitive on \( \Omega \) and \( \theta \) is nontrivial, then \( \theta(G)^{\mu(\Omega)} \) is permutationally isomorphic to \( G^{\Omega} \).

The following lemma provides another way to view permutational isomorphism between two transitive permutation groups.

**Lemma 2.3.9.** If two permutation groups \( G \) and \( H \) are both transitive, then they are permutationally isomorphic if and only if there is an isomorphism \( \theta : G \rightarrow H \) such that \( \theta \) maps a point stabiliser of \( G \) onto some point stabiliser of \( H \).

**Proof.** Exercise.

Let \( G \) have two transitive actions on a finite set \( \Omega \), and let \( G_a \) and \( G_b \) be point stabilisers for these respective actions. Note that these actions are permutationally isomorphic if there is an automorphism \( \tau \) of \( G \) such that \( \tau(G_a) = G_b \). Some texts require that \( \tau \) be an inner automorphism, that is, these two actions of \( G \) are permutationally isomorphic if and only if \( G_a \) and \( G_b \) are conjugate. In this course, we will adopt the weaker definition as given by Definition 2.3.1. Now \( S_6 \) has two conjugacy classes of subgroups that are isomorphic to \( S_5 \), so for some authors, \( S_6 \) has two primitive actions on 6 points. However, the union of these conjugacy classes forms an orbit under the automorphism group of \( S_6 \), and so we will regard \( S_6 \) as having one primitive action on 6 points up to permutational isomorphism.
2.3. MORPHISMS

CHAPTER 2. GROUP ACTIONS

THEOREM 2.3.10. Let $G$ be a group acting transitively on $\Omega$. Then for all $\alpha \in \Omega$, the action of $G$ on $\Omega$ is permutationally isomorphic to the right coset action of $G$ on $G/G_\alpha$.

PROOF. Let $\alpha \in \Omega$. Note that the stabiliser of the trivial coset $G_\alpha$ in the right coset action of $G$, is $G_\alpha$. So since $G$ is transitive on $\Omega$ and transitive on $G/G_\alpha$, then the respective actions are permutationally isomorphic by Lemma 2.3.9. □

Primitive groups play a crucial role in the study of maximal subgroups of abstract finite groups. By the correspondence theorem, if $N$ is a normal subgroup of $G$, then under the natural projection, there is a one-to-one correspondence between the maximal subgroups of $G$ containing $N$, and the maximal subgroups of $G/N$. Let $M_N$ be the set of core-free maximal subgroups of $G/N$. It is not difficult to prove that we can identify the set of all maximal subgroups of $G$ with the disjoint union $\bigcup_{N \trianglelefteq G} M_N$. So one may analyse maximal subgroups of finite groups via this reduction to the study of core-free maximal subgroups of finite groups, which is no different than analysing finite primitive groups as the following shows.

We say that $H$ is core-free in $G$ if $\text{Core}_G(H) = 1$. We have the following very important and well-known result.

THEOREM 2.3.11. Let $G$ be a finite group. Then $G$ has a non-trivial primitive permutation representation if and only if it has a core-free maximal subgroup.

PROOF. Suppose $G$ is a finite primitive permutation group on a set $\Omega$ and let $\alpha \in \Omega$. Then by Corollary 2.2.12, $G_\alpha$ is a maximal subgroup of $G$. Since $G$ is transitive and faithful, we have $\text{Core}_G(G_\alpha) = \cap \{ G_\alpha^g : g \in G \} = \cap \{ G_\alpha^\omega : \omega \in \Omega \} = 1$. Therefore $G$ has a core-free maximal subgroup. Conversely, suppose $G$ has a core-free maximal subgroup $H$. By Lemma 2.3.10, $G$ acts transitively and faithfully on the right cosets of $H$ in the right coset action. Since the stabiliser of the trivial coset in this action is $H$, it follows from Corollary 2.2.12 that $G$ is primitive. □

EXERCISES 2.3.12.

(i) Prove that “permutational isomorphism” is an equivalence relation on the set of finite permutation groups.

(ii) Prove Lemma 2.3.9.

(iii) In this question, we focus on centralisers, which will have an important part in a later section.

(a) Prove the following:

LEMA 2.3.13. Let $G$ and $G'$ be groups with normal subgroups $K$ and $K'$ respectively. Let $\theta$ be an epimorphism from $G$ onto $G'$ which restricts to an isomorphism of $K$ onto $K'$. Then $C_{G'}(K') = \theta(C_G(K))$.

(b) Prove the following:

LEMA 2.3.14. Let $\Omega$ be a finite set. Then

(i) A subgroup $J$ of $\text{Sym}(\Omega)$ is semiregular if and only if $C_{\text{Sym}(\Omega)}(J)$ is transitive.

(ii) Any group centralising a transitive subgroup of $\text{Sym}(\Omega)$ is semiregular.
CHAPTER 2. GROUP ACTIONS

2.4. MULTIPLE TRANSITIVITY

2.4. Multiple Transitivity

Given a group $G$ acting on a set $\Omega$, there is a natural action of $G$ on the $n$-fold Cartesian product of $\Omega$ defined by

$$(\omega_1, \omega_2, \ldots, \omega_n)^g := (\omega_1^g, \omega_2^g, \ldots, \omega_n^g)$$

for all $\omega_1, \omega_2, \ldots, \omega_n \in \Omega$ and $g \in G$. Now for each orbit $\omega^G$ of $G$ on $\Omega$, there is a corresponding orbit $\left((\omega, \omega, \ldots, \omega)^G\right)$ of $G$ on $\Omega^n$. These orbits will be considered non-interesting and so we concentrate on the orbits of $G$ on $n$-tuples of distinct elements.

DEFINITION 2.4.1 (Multiple Transitivity). Let $G$ be a group acting on a set $\Omega$ and let $n$ be a positive integer. If $n \geq 2$, we say that $G$ is $n$-transitive if $G$ is transitive on all $n$-tuples of distinct elements of $\Omega$. As a convention, $G$ is 1-transitive if it is transitive on $\Omega$.

LEMMA 2.4.2. Let $G$ be a group acting transitively on a finite set $\Omega$, let $\omega \in \Omega$, and let $k > 1$. Then $G$ is $k$-transitive if and only if $G_{\omega}$ is $(k - 1)$-transitive on $\Omega \setminus \{\omega\}$.

PROOF. Exercise. \(\square\)

Example 2.4.3.

(i) The symmetric group $S_n$ is $n$-transitive.

(ii) The alternating group $A_n$ is $(n - 2)$-transitive.

(iii) Let $V$ be a $d$-dimensional vector space over a field of order $q$. Then the affine group $AGL(V)$ acts 2-transitively on the nonzero vectors of $V$. If $q = 2$, we have that $AGL(V)$ is 3-transitive.

(iv) Let $PGL_d(q)$ be the projective general linear group; i.e., the permutation group induced by the action of $GL_d(q)$ on the one-dimensional subspaces of the $d$-dimensional vector space $V_d(q)$ over $GF(q)$. Then $PGL_d(q)$ acts 2-transitively on the one-dimensional subspaces of $V_d(q)$.

(v) The simple groups $M_{12}$ and $M_{24}$ act 5-transitively in their natural actions (on the Steiner systems $S(5, 6, 12)$ and $S(5, 8, 24)$ respectively).

LEMMA 2.4.4. Let $G$ be a group acting 2-transitively on a finite set $\Omega$. Then $G$ is primitive.

PROOF. Exercise. \(\square\)

THEOREM 2.4.5 (W. Burnside). Let $G$ be a group acting 2-transitively on a finite set $\Omega$, and let $M$ be a minimal normal subgroup of $G$. Then one of the following occurs:

(i) $M$ is elementary abelian and can be identified with a $d$-dimensional vector space $V$ over $GF(p)$. Moreover, $G$ is permutationally isomorphic to a subgroup of $AGL(V)$;

(ii) $M$ is a nonabelian simple group acting primitively, and there is an embedding of $G$ in $\text{Aut}(M)$.

It was shown by using the Classification of Finite Simple Groups that the only 6-transitive finite permutation groups are symmetric and alternating groups (see [5]). To this day, there is still no proof that does not use the CFSG.
CHAPTER 3

Normal Subgroups of Transitive Groups

3.1. Minimal Normal Subgroups

If $G$ is a group, then a normal subgroup $M$ of $G$ is a minimal normal subgroup if the only normal subgroups of $G$ contained in $M$ are the trivial subgroup and $M$ itself. One of the themes of this course is the study of finite permutation groups which have a transitive minimal normal subgroup. So it would be useful to record some properties of minimal normal subgroups. For example, any minimal normal subgroup $M$ of a group $G$ is characteristically simple, since every characteristic subgroup of $M$ is a normal subgroup of $G$.

**Definition 3.1.1 (Characteristic Subgroups).** Let $G$ be a group and let $H$ be a subgroup of $G$. We say that $H$ is a characteristic subgroup of $G$ if it is Aut($G$)-invariant; that is, $H^\tau = H$ for all $\tau \in \text{Aut}(G)$. If there are no proper nontrivial subgroups of $H$ that are characteristic subgroups of $G$, then we say that $H$ is characteristically simple.

**Example 3.1.2.**
(i) An abelian group for which every element has order at most a prime $p$ (i.e., has exponent $p$), is called an elementary abelian $p$-group. Every elementary abelian $p$-group is characteristically simple.
(ii) Let $T$ be a nonabelian simple group, and consider the direct product $M = T^n$. Then $M$ is characteristically simple.

**Exercises 3.1.3.**
(i) Prove that the centre of a group $G$ is a characteristic subgroup of $G$.
(ii) The derived subgroup $G'$ of a group $G$ is the smallest normal subgroup of $G$ which leaves an abelian quotient (i.e., $G' = \cap\{N \trianglelefteq G : G/N \text{ is abelian}\}$). Show that $G'$ is a characteristic subgroup of $G$.
(iii) Prove that a characteristic subgroup of $G$ is normal in $G$.
(iv) Prove that if $H$ is characteristic in $K$ and $K$ is normal in $G$, then $H$ is normal in $G$.
(v) Show that a minimal normal subgroup of a group is characteristically simple.

We now prove that a characteristically simple group is a direct product of isomorphic simple groups, a classical result which can be found in Robinson’s book [14, 3.3.15].

**Lemma 3.1.4.** A finite group is characteristically simple if and only if it is a direct product of isomorphic simple groups.

**Proof.** We have one direction from the examples above (which we have not proved, but is not difficult to do so). Let $G$ be a characteristically simple group and let $M$ be a minimal normal subgroup of $G$. Then the group generated by all images $M^\tau$, where $\tau \in \text{Aut}(G)$, is clearly a characteristic subgroup of $G$ and so must equal $G$. Now suppose that $H$ is a normal subgroup of $G$ which is a direct product of some of the $M^\tau$, and suppose furthermore, that it is maximal with respect to this property. If $H$ is a proper subgroup of $G$, then there is an automorphism $\tau$ such that $M^\tau$ is not contained in $H$. Now $M^\tau \cap H$ is a normal subgroup of $G$ contained in $M^\tau$. So since $M^\tau$ is a minimal normal subgroup, we must have that $M^\tau \cap H = 1$. However, this implies that $H$ is properly contained in $M^\tau \times H$, which contradicts maximality of $H$. Thus $G = H$. Now a nontrivial normal subgroup of $M$ is also normal in $G$ (as $G$ is a direct product of $M$ and some images of $M$ under Aut($G$)). By minimality of $M$, it follows that $M$ is simple.

**Exercises 3.1.5.** Let $G$ be a group. Prove that
(i) Any pair of distinct minimal normal subgroups of $G$ intersect trivially.
(ii) Any pair of distinct minimal normal subgroups of $G$ centralise each other.
3.2. Regular Normal Subgroups

In the study of groups, it is a common technique to represent a group as a linear group or permutation group. In this section, we will be concerned with the representation of a group $G$ as a group of automorphisms of one of its normal subgroups. Let $M$ be a normal subgroup of $G$. Then $G$ acts naturally by automorphisms on $M$ (by conjugation). The kernel of this action is the centraliser $C_G(M)$. Indeed, we could take $M$ to be $G$, in which case the permutation group induced by this action is isomorphic to $G/Z(G)$. Let $g \in G$ and let $\tau_g : G \to G$ be defined by $\tau_g(x) = g^{-1}xg$ for all $x \in G$. The map $\tau_g$ is called an inner automorphism of $G$, and the set of all $\tau_g$, denoted $\text{Inn}(G)$, is called the inner automorphism group of $G$. So $\text{Inn}(G)$ is isomorphic to $G/Z(G)$.

A group is said to be almost simple if it has a nonabelian simple unique minimal normal subgroup. If a group $G$ has a normal subgroup $K$ such that $C_G(K) = 1$ (for example, if $G$ is almost simple with minimal normal subgroup $K$), then $G$ can be embedded in $\text{Aut}(K)$ such that its image contains $\text{Inn}(K)$.

**Definition 3.2.1** (Right Regular and Left Regular Representations). Let $g \in G$ and let the permutations $\rho_g$ and $\lambda_g$ of $G$ be defined by

$$\rho_g(x) = gx, \quad \lambda_g(x) = g^{-1}x$$

for $x \in G$. The right regular representation of $G$ is the subgroup $G_R$ of $\text{Sym}(G)$ given by $\{\rho_g : g \in G\}$. Similarly, the left regular representation of $G$ is $G_L := \{\lambda_h : h \in G\}$.

As we will see later, the holomorph of a group arises when we have a regular action.

**Definition 3.2.2** (Holomorph). Let $G$ be a group and consider the natural action of $\text{Aut}(G)$ on $G \times \text{Aut}(G)$. Then the semidirect product $G_R \rtimes \text{Aut}(G)$ given by this action is called the holomorph of $G$, which we denote by $\text{Hol}(G)$.

**Exercises 3.2.3.** Let $G$ be a group.

(a) Verify that $\text{Hol}(G)$ is a subgroup of $\text{Sym}(G)$.

(b) Show that $G_L = C_{\text{Sym}(G)}(G_R)$ and $G_R = C_{\text{Sym}(G)}(G_L)$.

(c) Prove that $\text{Hol}(G) = N_{\text{Sym}(G)}(G_R)$.

Let $G$ be a finite permutation group acting on a set $\Omega$, let $M$ be a normal subgroup of $G$, and suppose that $M$ acts regularly on $\Omega$. So the action of $M$ on $\Omega$ is permutationally isomorphic to the action of $M$ on the cosets of the trivial group. This action is in turn permutationally isomorphic to the action of $M$ on itself by right multiplication. The next lemma shows that $G$ on $\Omega$ is permutationally isomorphic to a subgroup of $\text{Hol}(M)$ acting naturally on $M$.

**Lemma 3.2.4.** Let $M$ be a permutation group acting regularly on a finite set $\Omega$. Then there is a permutational isomorphism $(\theta, \mu)$ with $\theta : N_{\text{Sym}(\Omega)}(M) \to \text{Hol}(M)$ and $\mu : \Omega \to M$ such that

(i) $\theta(M) = M_R$.

(ii) $\theta(C_{\text{Sym}(\Omega)}(M)) = M_L$, and

(iii) if $M$ has trivial centre, then $\theta(C_{\text{Sym}(\Omega)}(M) \rtimes M) = M_R \rtimes \text{Inn}(M)$.

**Proof.** Since $M$ acts regularly on $\Omega$, for a fixed $\alpha \in \Omega$, the map $\alpha^g \mapsto g$ is a bijection from $\Omega$ to $M$. Let $\mu$ be this map and let $\theta : N_{\text{Sym}(\Omega)}(M) \to \text{Hol}(M)$ be the map where, for all $\tau \in N_{\text{Sym}(\Omega)}(M)$, we have $\theta(\tau) = \mu^{-1} \circ \tau \circ \mu$. The rest we leave as an exercise.

As promised in the introduction, we will demonstrate that we can describe the structure of innately transitive groups (which are defined later) by the actions of their minimal normal subgroups. The following lemma is crucial in achieving this goal.

**Lemma 3.2.5.** Every finite permutation group $G$ on $\Omega$ has at most two distinct transitive minimal normal subgroups. Moreover, if $M_1$ and $M_2$ are distinct transitive minimal normal subgroups of $G$, then

(i) $G$ is primitive,

(ii) $C_G(M_1) = M_2$ and $C_G(M_2) = M_1$, and

(iii) there is an involution of $N_{\text{Sym}(\Omega)}(G)$ that interchanges $M_1$ and $M_2$, and which centralises a point stabiliser of $G$.

---

1That is, $\rho_{\tau g} = \rho_g \tau$ for all $\tau \in \text{Aut}(G)$ and $\rho_g \in G_R$. 

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3.3. SUBDIRECT SUBGROUPS

PROOF. Suppose \( G \) has three distinct transitive minimal normal subgroups \( M_1, M_2, \) and \( M_3 \). Then each pair of the \( M_i \) intersect trivially and hence any two of them are contained in the centraliser of the third. For example \( M_2, M_3 \subseteq C_G(M_1) \). However, by Lemma 3.2.14, \( C_G(M_1) \) is semiregular and it follows that \( M_2, M_3, \) and \( C_G(M_1) \) are regular and hence equal (by the Orbit-Stabiliser Theorem), which is a contradiction. Therefore, there are at most two distinct transitive minimal normal subgroups of \( G \).

Suppose now that \( M_1 \) and \( M_2 \) are distinct transitive minimal normal subgroups of \( G \). As was noted above, \( M_1 = C_G(M_2) \) and \( M_2 = C_G(M_1) \), and they are both regular on \( \Omega \). Let \( M = M_1 \). By Lemma 3.2.4, \( N_{\text{Sym}(\Omega)}(M) \) on \( \Omega \) is permutationally isomorphic to \( \text{Hol}(M) \) on \( M \) via an isomorphism \( \theta \) such that \( \text{Im}(M) = \text{M}_R \). Now \( M_2 = C_G(M) \) is mapped by \( \theta \) to \( C_{\text{Hol}(M)}(M_2) = M_2 \). Let \( \gamma \) be the involution \( x \mapsto x^{-1} \) \((x \in M) \) in \( \text{Sym}(M) \). Then a simple calculation shows that \( \gamma \) centralises \( \text{Aut}(M) \), and \( \lambda\gamma = \lambda_2 \) for all \( y \in M \). Thus, \( \gamma \) interchanges \( M_L \) and \( M_R \). Therefore, if \( \text{Im}(G) \) is the point stabiliser of \( \text{Im}(G) \) in its action on \( M \), then \( \text{Im}(G) \) is a subgroup of \( \text{Aut}(M) \) and hence is centralised by \( \gamma \). So \( \text{Im}(G) = (M_R \text{Im}(G)) \). Now \( \text{Im}(G) = C_{\text{Sym}(M)}(\text{Im}(G)) \). The element \( \gamma \) has the required properties.

Let \( L \) be a subgroup of \( G \) containing \( G_1 \). For the rest of this proof, we will identify \( G \) with its corresponding subgroup of \( \text{Hol}(M) \). Since \( L \leq \text{Hol}(M) \), we must have that \( L = N_L \times H \) for some \( N_L \) and \( H \leq \text{Aut}(M) \).

So the orbit \( 1^L \) consists of elements of the form \( \gamma \tau \) where \( \gamma \in N \) and \( \tau \in H \). It is not difficult to show that \( 1^L \) is in fact a subgroup of \( M \). Recall that in the Holomorph of \( M \), we have the point stabilisers \( (M_1 \times M_R) = \text{Inn}(M) \) and \( \text{Hol}(M) = \text{Aut}(M) \). So since \( M_1 \times M_R \leq G \leq \text{Hol}(M) \), we have \( \text{Inn}(M) \leq G_1 \leq \text{Aut}(M) \). Note that \( 1^L \) is \( G_1 \)-invariant as \( 1^L \) is a block for \( G \) containing \( 1 \) (by Theorem 2.2.11). So \( 1^L \) is \( \text{Inn}(M) \)-invariant and hence \( 1^L \) is normal in \( M \). Hence \( 1^L \) is a normal subgroup of \( G \) (as \( G = G_1 M \) contained in \( M \) and therefore \( 1^L = 1 \) or \( 1^L = M \).

It follows that \( L = G \) or \( L = G_1 \). Therefore, by Corollary 2.2.12, \( G \) is primitive.

Note that the third part of the above lemma says that there is a permutational isomorphism of \( G \) to itself which interchanges its two transitive minimal normal subgroups.

3.3. Subdirect subgroups

When studying topological spaces, modules, or permutation groups, the sub-objects and product-objects are of special interest. In particular, we may be interested in the sub-objects of a product-object and so the embedding and projection maps of a product-object are of central importance. In this section, we revise some of the basic notions concerning certain embeddings of finite groups into finite direct products.

Recall, that for a subset \( H \) of \( G \), the normaliser of \( H \) (in \( G \)) is \( N_G(H) = \{ g \in G : gH = Hg \} \). Note that for all \( H \leq G \), we have \( H \leq N_G(H) \leq G \). We say that an element \( g \in G \) normalises \( H \) if \( g \in N_G(H) \). Similarly, a subgroup \( J \) of \( G \) normalises \( H \) if every element of \( J \) normalises \( H \). Let \( P = H_1 \times H_2 \times \cdots \times H_n \) be a direct product of groups. Given a subset \( J \) of \( \{1, \ldots, n\} \), we denote the natural projection map from \( P \) to \( \prod_{j \in J} H_j \) by \( \pi_j \). If \( J = \{j\} \), then for brevity we write \( \pi_j \) to mean \( \pi_j \).

**Definition 3.3.1** (Subdirect product). A group \( G \) is a subdirect product of \( P = H_1 \times H_2 \times \cdots \times H_n \) if there is an embedding \( \phi : G \to P \) such that \( \phi \circ \pi_j : G \to H_j \) is an epimorphism for each \( j \in I \).

(i) In the case that \( G \) is a subgroup of \( P \) and \( \phi \) is the inclusion map, we say that \( G \) is a subdirect subgroup of \( P \).

(ii) If \( G \) is a subgroup of \( P \), we say that \( G \) is a diagonal subgroup of \( P \) if the restriction of \( \pi_j \) to \( G \) is injective for each \( j \in I \).

(iii) We say that \( G \) is a full diagonal subgroup of \( P \) if it is both a subdirect and diagonal subgroup.

In the latter case, when \( G \) is a full diagonal subgroup of \( H_1 \times \cdots \times H_n \), the direct factors \( H_i \) are each isomorphic to \( G \).

**Lemma 3.3.2.** Let \( G \) be a full diagonal subgroup of the direct product \( H_1 \times \cdots \times H_n \). Then there exist isomorphisms \( \{ \gamma_i : G \to H_i \mid i \in \{1, \ldots, n\}\} \), such that

\[
G = \{(\gamma_1(g), \gamma_2(g), \ldots, \gamma_n(g)) : g \in G\}.
\]

**Proof.** Since \( G \) is a full diagonal subgroup of \( H := H_1 \times \cdots \times H_n \), each projection map \( \pi_i \) is an isomorphism. Now note that each element \( g \in G \) can be written as \( g = g_1 \cdots g_n \) where each \( g_i \) is equal to \((1, \ldots, 1, \pi_i(g), 1, \ldots, 1) \) (where \( \pi_i(g) \) is in the \( i \)-th coordinate). Hence it follows that \( G = \{ (\pi_1(g), \pi_2(g), \ldots, \pi_n(g)) : g \in G \} \).

For every group \( H \) and integer \( n \), the set \( \{ h, h, \ldots, h : h \in H \} \) is a full diagonal subgroup of \( H^n \) called the the straight diagonal subgroup of \( H^n \), which we will denote by \( \text{Diag}(H^n) \). Below is a simple result which played a key role in Scott’s proof of Lemma 3.3.4(i).
Lemma 3.3.3. A full diagonal subgroup of a direct product of nonabelian simple groups is self-normalising.

Proof. Exercise. □

The first part of the following lemma appears in Scott’s paper (see [15, Lemma, p. 328]), and the second result can be found in [8, Proposition 5.2.5(i)]. This lemma is well-known to universal algebraists.

Lemma 3.3.4. Let $P$ be a finite direct product $\prod_{i \in I} T_i$ of finite nonabelian simple groups, and let $Q$ be a subgroup of $P$.

(i) If $Q$ is a subdirect subgroup of $P$, then $Q$ is the direct product $\prod_{i \in I} Q_i$ of full diagonal subgroups of subproducts $\prod_{i \in I} T_i$ where the $I_j$ form a partition of $I$.

(ii) If $Q$ is a normal subgroup of $P$, then $Q = \prod_{i \in J} T_i$ where $J$ is a nonempty subset of $I$.

Proof. We prove (i) first. For all $i \in I$, let $\pi_i : P \to T_i$ be the natural projection map. Now there is a minimal subset $S \subseteq I$ such that $Q$ intersects non-trivially with $\prod_{i \in S} T_i$. Let $D$ be this intersection. Note that $D$ is normal in $Q$ (since $\prod_{i \in S} T_i$ is normal in $\prod_{i \in I} T_i$), and so for all $j \in I$, we have that $\pi_j(Q)$ is normal in $\pi_j(Q)$. By minimality of $S$, the $\pi_j(D)$ are nontrivial, and thus $\pi_j(D) = T_j$ for all $j \in S$ (as each $T_j$ is simple). Therefore, $D$ is a subdirect subgroup of $\prod_{i \in S} T_i$. But minimality of $S$ also ensures that each $\pi_j$ is injective on $D$. Therefore $D$ is in fact a full diagonal subgroup of $\prod_{i \in S} T_i$.

Now let $E$ be the projection of $P$ on $\prod_{i \in S} T_i$. Then $D$ is a normal subgroup of $E$, as noted above, since $D$ is its own projection on this subproduct. But a full diagonal subgroup of a product of nonabelian simple groups is self-normalising by Lemma 3.3.3. Thus $D = E$ and $Q$ is the direct product $D \times D'$ where $D' = Q \cap \prod_{i \in I \setminus S} T_i$. The result follows by repeating the argument inductively to $I \setminus S$ and noting that $D'$ is a subdirect product of $\prod_{i \in S} T_i$.

Now we prove (ii). Since $Q$ is normal, $\pi_i(Q)$ is normal in $T_i$ for all $i \in I$. Because $T_i$ is simple, there exists a subset $J$ of $I$ such that $Q$ is a subdirect product of $\prod_{i \in J} T_i$. Note that $J$ is nonempty since $Q$ is nontrivial. Applying (i), we obtain that $Q$ is a direct product $\prod_{i \in I} Q_i$ of full diagonal subgroups of subproducts of the $T_i$. Now going back to the argument used to prove (i), we see that $D$ is (under the assumption that $Q$ is normal) a normal subgroup of the subproduct $\prod_{i \in J} T_i$, and hence equal to it, as $D$ is self-normalising. Therefore, it follows that $Q = \prod_{i \in J} T_i$. □

We now return to our theory of minimal normal subgroups, and in particular, we present some properties that will provide insight in a later section.

Lemma 3.3.5. Let $G$ be a group and $N$ be a normal subgroup of $G$. Suppose that $N$ is a direct product of nonabelian simple groups $T_1, T_2, \ldots, T_k$.

(i) $G$ acts on $\{T_1, T_2, \ldots, T_k\}$ by conjugation;

(ii) $G$ is transitive on $\{T_1, T_2, \ldots, T_k\}$ if and only if $N$ is a minimal normal subgroup of $G$.

Proof.

(i) Let $g \in G$. Then for all $i \in \{1, \ldots, k\}$, $T_i^g$ is a normal subgroup of $N^g = N$ and hence by Lemma 3.3.4, $T_i^g$ is equal to a direct product of some of the $T_1, \ldots, T_k$. Since $T_i^g$ is simple, it follows that $T_i^g$ is equal to one of the $T_1, \ldots, T_k$. So indeed, conjugation by $G$ is a well-defined group action on $\{T_1, \ldots, T_k\}$.

(ii) Suppose $N$ is a minimal normal subgroup of $G$ and suppose $G$ is intransitive on $\{T_1, \ldots, T_k\}$. So there exists an orbit of some $T_i$ that is not equal to $\{T_1, \ldots, T_k\}$. Let $M$ be the direct product of the elements in this orbit. Without loss of generality, let’s assume that $M = T_1 \times T_2 \times \cdots \times T_j$ where $j < k$. Then $M$ is a proper subgroup of $N$. Let $g \in G$. Then for all $i < j$, $T_i^g \subseteq \{T_1, \ldots, T_j\}$ so $T_i^g \leq M$ and thus $M^g = M$. So $M$ is a non-trivial normal subgroup of $G$, which contradicts the minimality of $N$. Hence $G$ is transitive.

Conversely, suppose $G$ is transitive on $\{T_1, \ldots, T_k\}$ and let $M$ be a normal subgroup of $G$ contained in $N$. By Lemma 3.3.4, $M = T_1^g \times T_2^g \times \cdots \times T_j^g$ where $j \leq k$ and $\{T_1^g, T_2^g, \ldots, T_j^g\} \subseteq \{T_1, T_2, \ldots, T_k\}$. Without loss of generality, we may assume that $T_i^g = T_i$ for all $i \in \{1, \ldots, j\}$. Now since $G$ is transitive on $\{T_1, \ldots, T_k\}$, there exists $g \in G$ such that $T_k^g = T_i$. But $M$ is a normal subgroup of $G$, and hence $T_k^g \leq M$. Therefore $j = k$ and $M = N$, giving that $N$ is a minimal normal subgroup of $G$. □

Exercises 3.3.6.

(i) Let $G$ be a group and let $N$ be a normal subgroup of $G$. A supplement for $N$ in $G$ is a subgroup $H$ of $G$ such that $G = NH$. Suppose that $N$ is a direct product of nonabelian simple groups $T_1, T_2, \ldots, T_k$. Prove that if $G_0$ is a supplement of $N$ in $G$, $N_0 = N \cap G_0$, and $N$ is a minimal normal subgroup of $G$, then $\pi_i(N_0)$ ($1 \leq i \leq k$) are pairwise isomorphic.

(ii) Let $T$ be a nonabelian simple group, let $k$ be a positive integer and consider the straight diagonal subgroup $D$ of $M = T^k$. Consider the wreath product $\text{Out}(T) \wr S_k$ and its natural action on $T^k$:

$$(t_1, \ldots, t_k)^{\text{Im}(T)} = (t_1^{r_1}, \ldots, t_k^{r_k})$$
Let $W$ be the semidirect product of $T^k$ with the subgroup $\text{Diag}(\text{Out}(T)^k) \times S_k$ of $\text{Out}(T) \wr S_k$. Define an action of $W$ on the right cosets $M/D$ as follows:

- $D(t_1, t_2, \ldots, t_k)^{(x_1, x_2, \ldots, x_k)} := D(t_1 x_1, t_2 x_2, \ldots, t_k x_k)$ for all $(x_1, x_2, \ldots, x_k) \in T^k$;
- $D(t_1, t_2, \ldots, t_k)^{(\text{Inn}(T) \tau \ldots \text{Inn}(T) \tau)} := D(t_1^\tau, t_2^\tau, \ldots, t_k^\tau)$ for all $\tau \in \text{Aut}(T)$;
- $D(t_1, t_2, \ldots, t_k)^{\pi} := D(t_1^{\pi}, t_2^{\pi}, \ldots, t_k^{\pi})$ for all $\pi \in S_k$.

(a) Show that the action of $W$ on $M/D$ is faithful.

(b) Prove that the normal subgroup $T^k$ of $W$ is transitive.

(c) Prove that $W$ is quasiprimitive in its action on $M/D$.

Remarks 3.3.7. In the last exercise, we have a generic prototypical example of a quasiprimitive group of Simple Diagonal type. In the later chapters of these notes, we will see that they are important, but we will not investigate them rigorously nor deeply.
CHAPTER 4

Wreath Products

4.1. The definition

Consider the full symmetric group $S_n$ acting on $n$ letters. Every group acting faithfully on a set of size $n$ can be realised as a subgroup of $S_n$, and so we can view such group actions as subgroups of $S_n$. We have seen then that the large intransitive subgroups of $S_n$ are disjoint unions of smaller actions. To get the largest such intransitive subgroups, we consider disjoint unions of symmetric groups. Hence the subgroups of $S_n$ of the form $S_a \times S_b$, where $a + b = n$ are the large intransitive subgroups of $S_n$. Following this line of thought, we may ask what the largest imprimitive subgroups of $S_n$ look like. Suppose we partition $\{1, \ldots, n\}$ into $b$ blocks of size $a$, and we want to determine the largest subgroup of $S_n$ preserving this partition. Such considerations lead naturally to the definition of a wreath product.

**DEFINITION 4.1.1 (Wreath Product).** Let $A$ and $B$ be two finite groups and suppose $B$ acts on the set $\{1, 2, \ldots, n\}$. We define the *wreath product* of $A$ and $B$ with respect to this action, to be

$$A \wr B := A^n \rtimes B$$

where the *top group* $B$ acts on the *base group* $A^n$ by

$$(a_1, a_2, \ldots, a_n)b = (a_{1b}, a_{2b}, \ldots, a_{nb})$$

for all $(a_1, a_2, \ldots, a_n) \in A^n$ and $b \in B$.

**EXERCISES 4.1.2.**

(i) Show that the action of the top group on the base group given in the definition above is well-defined.

(ii) Let $B$ be a partition of $\{1, \ldots, n\}$ into $b$ blocks of size $a$. Show that the full stabiliser of $B$ in $S_n$ is isomorphic to $S_n \wr S_b$.

4.2. Imprimitive action

Now suppose that $A$ acts on a set $\Omega$. There are two important actions of $A \wr B$ that we should recognise. The first is called the *imprimitive action* of $A \wr B$ on $\Omega \times N$, defined by

$$(\omega, i)^{(a_1, \ldots, a_n)b} = (\omega^{a_i}, i^b)$$

for all $(\omega, i) \in \Omega \times N$ and all $(a_1, \ldots, a_n)b \in A \wr B$. Note that the fibre $\{ (\omega, 1) : \omega \in \Omega \}$ of $\Omega \times N$ is a block of imprimitivity for $A \wr B$ on $\Omega \times N$. Note that this is the action referred to in the discussion above about large imprimitive subgroups of $S_n$. The imprimitive action of the wreath product is very important in understanding imprimitive groups. According to Bhattacharjee, Macpherson, Möller, and Neumann (see [4, pp.72]),

“one of the most important properties of wreath products is their universality as embedding groups for imprimitive groups.”

They then prove an embedding theorem [4, Theorem 8.5] for an imprimitive permutation group (see also [11]) in a wreath product. We recast this well-known result below.

**LEMMA 4.2.1.** *Let $G$ be any transitive imprimitive permutation group on a set $\Omega$ and let $\mathcal{P}$ be a $G$-invariant partition of $\Omega$. Let $\Gamma$ be an element of $\mathcal{P}$ and let $C$ be the permutation group induced by the action of $G\Gamma$ on $\Gamma$. Let $D$ be the group of permutations induced by $G$ on $\mathcal{P}$. Then $\Omega$ may be identified with $\Gamma \times \mathcal{P}$ in such a way that $G$ can be embedded into the wreath product $C \wr D$ in imprimitive action.*

If $G$ is a transitive but imprimitive group on a finite set $\Omega$, then by the above lemma, we can embed $G$ into the wreath product $G^\Gamma \wr S_k$ acting in imprimitive action, where $\Gamma$ is a block for $G$, $G^\Gamma$ is the group induced by the action of the setwise stabiliser $G\Gamma$ on $\Gamma$, and $k$ is the size of the orbit of $\Gamma$ under $G$ (see also [6, Theorem 1.8]). If $G^\Gamma$ is also imprimitive, then we can embed $G^\Gamma$ into a wreath product in a similar manner to that above. Since $\Omega$ is
finite, we can continue this process until we have an embedding of $G$ into an iterated wreath product of primitive groups. So we can think of the primitive groups as the building blocks for transitive permutation groups.

4.3. Product action

The second action to recognise, is the so-called product action of $A \wr B$ on $\Omega^n$, defined by

$$(\omega_1, \ldots, \omega_n)(a_1, \ldots, a_n)b^{-1} = (\omega_{a_1b}, \ldots, \omega_{a_nb})$$

for all $a_1, \ldots, a_n \in A$, $\omega_1, \ldots, \omega_n \in \Omega$, and $b \in B$. One can consider this action dual to that of the imprimitive action. The imprimitive action arises as the stabiliser of a decomposition of set into a disjoint union of sets, whereas the product action arises as the stabiliser of a decomposition of a set into a Cartesian product of sets. For $1 \leq i \leq n$, let $p_i : \Omega^n \to \Omega$ be the natural projection map $(\omega_1, \ldots, \omega_n) \mapsto \omega_i$. Then $\Omega_i = \{p_i^{-1}(\omega) : \omega \in \Omega\}$ is a partition of $\Omega^n$, and it is well-known (see for example [6, pp. 103]) that $\text{Sym}(\Omega) \wr S_n$ in product action, is the full stabiliser in $\text{Sym}(\Omega^n)$ of the set $\{\Omega_i : 1 \leq i \leq n\}$. Below is a classical result concerning the product action of a wreath product (see [7, Lemma 2.7A]).

**Theorem 4.3.1.** Suppose $A$ and $B$ are finite groups where $A$ acts on a set $\Omega$ and $B$ acts on a set of $n$ elements. The wreath product $A \wr B$ acts primitively in product action on $\Omega^n$ if and only if $B$ is transitive and $A$ is primitive and not regular on $\Omega$.

Dr. Csaba Schneider, in personal communication with the author, showed that Theorem 4.3.1 also holds if we replace “primitive” with “quasiprimitive”. We will see later that this result generalises to a larger class of permutation groups – innately transitive groups.

**Exercises 4.3.2.**

(i) Let $T$ be a nonabelian simple group and $k$ be a positive integer. Show that $\text{Aut}(T^k) \cong \text{Aut}(T) \wr S_k$. 
CHAPTER 5

Innately Transitive Groups

5.1. Some fundamental theory

Recall that a finite group $G$ is quasiprimitive if all of its minimal normal subgroups are transitive. We have the following weaker notion:

**Definition 5.1.1 (Innately Transitive).** Let $G$ be a finite permutation group. If there exists a transitive minimal normal subgroup $M$ of $G$, then we say that $G$ is innately transitive and that $M$ is a plinth for $G$.

Recall from 3.2.5, that $G$ has at most two transitive minimal normal subgroups, and if it has two, then there is a permutational isomorphism of $G$ which interchanges the minimal normal subgroups. So it does not matter what we choose to be the plinth of $G$. The following proposition gives us necessary and sufficient conditions for an innately transitive group to be quasiprimitive.

**Theorem 5.1.2.** Let $G$ be a finite innately transitive permutation group on a set $\Omega$ with plinth $M$. Then $G$ is quasiprimitive if and only if $C_G(M)$ is transitive or $C_G(M) = 1$.

**Proof.** First suppose that $M$ is abelian. Then $M \leq C_G(M)$. Since $M$ is transitive and $C_G(M)$ is semiregular by Lemma 2.3.14, $M$ and $C_G(M)$ are both regular and hence equal. Recall from Lemma 3.1.5 that each minimal normal subgroup of $G$ distinct from $M$ intersects $M$ trivially and hence is contained in $C_G(M)$. Thus, $M$ is the unique minimal normal subgroup of $G$ and hence $G$ is quasiprimitive. Suppose now that $M$ is nonabelian. Since $M$ is a minimal normal subgroup of $G$, we have $C_G(M) \cap M = 1$. Thus, if $C_G(M) = 1$, then $M$ is the unique minimal normal subgroup of $G$ and hence $G$ is quasiprimitive. Suppose $C_G(M)$ is transitive. By Lemma 2.3.14(i), $M$ is semiregular and hence regular as $M$ is transitive. By Lemma 2.3.14(ii), $C_{\text{Sym}(\Omega)}(M)$ is semiregular, implying that $C_G(M)$ is also semiregular and consequently regular. Therefore, $C_G(M)$ is a minimal normal subgroup of $G$. By Lemma 3.2.5, it follows that $G$ is primitive. Finally, if $C_G(M)$ is non-trivial and intransitive, we have that $G$ is not quasiprimitive.

Let $G$ be a finite innately transitive permutation group on a set $\Omega$ with plinth $M$. Suppose that $G$ is not quasiprimitive. Then there exists an intransitive normal subgroup $N$ of $G$. So $M \cap N = 1$ and hence $N$ centralises $M$. So $C_G(M)$ is the “largest” intransitive normal subgroup of $G$ and so it stands to reason that the orbits of $C_G(M)$ are of interest when studying non-quasiprimitive innately transitive groups. The following lemma shows that the point stabiliser $M_\alpha$ is a normal subgroup of the set-wise stabiliser of $\alpha^{C_G(M)}$ in $M$. We will need this lemma to prove the succeeding proposition.

**Lemma 5.1.3.** Let $G$ be a finite innately transitive permutation group on a set $\Omega$ with plinth $M$, let $\alpha \in \Omega$, and let $\Delta = \alpha^{C_G(M)}$. Then $M_\alpha$ is a normal subgroup of $M_\Delta$, and $|M_\Delta : M_\alpha| = |C_G(M)|$.

**Proof.** Exercise. □

The next result shows that the members of a significant family of innately transitive groups are quasiprimitive.

**Lemma 5.1.4.** Let $G$ be a finite innately transitive permutation group on $\Omega$ with nonabelian and nonsimple plinth $M$, and let $\alpha \in \Omega$. If $M_\alpha$ is a subdirect subgroup of $M$, then $G$ is quasiprimitive.

**Proof.** Let $M = T_1 \times \cdots \times T_k$ where each $T_i$ is a nonabelian simple group and $k \geq 2$. Suppose first that $M_\alpha$ is a subdirect subgroup of $M$ and let $\Delta = \alpha^{C_G(M)}$. By Lemma 3.3.4, $M_\alpha = D_1 \times \cdots \times D_l$ where $l \leq k$ and for all $i$, $D_i$ is a full diagonal subgroup of a subproduct $M_i = \prod_{j \neq i} T_j$ and the $I_i$ form a partition of $\{1, \ldots, k\}$. So clearly, $N_M(M_\alpha) = \prod_{i=1}^l N_{M_i}(D_i)$. By Lemma 3.3.3, $N_M(D_i) = D_I$ for all $i$ and hence $N_M(M_\alpha) = M_\alpha$. Now Lemma 5.1.3 implies that $M_\alpha = M_\Delta$ and hence $C_G(M) = 1$. Thus $G$ is quasiprimitive by Theorem 5.1.2. □

Now we return to wreath products. Recall from Theorem 4.3.1, that the wreath product $A \wr B$ is primitive in product action if and only if $B$ is transitive and $A$ is primitive and not regular. We have an analogous result for innately transitive groups.
Theorem 5.1.5. Suppose $A$ and $B$ are finite groups where $A$ acts on a set $\Omega$ (where $|\Omega| > 1$) and $B$ acts on a set of $n$ elements. The wreath product $A \wr B$ is innately transitive in product action on $\Omega^\nu$ with plinth contained in the base group, if and only if $A$ is innately transitive and not regular on $\Omega$ and $B$ is transitive on $n$ elements.

Proof. Exercise. □

Exercises 5.1.6. Let $G$ be a finite innately transitive permutation group acting on a set $\Omega$, and let $M$ be a plinth of $G$.

(i) Show that if $M$ is abelian, then $G$ is primitive.
(ii) Show that if $M$ is a vector space, then Hol$(M)$ is permutationally isomorphic to AGL$(M)$.
(iii) Let $V$ be a $d$-dimensional vector space over $\mathbb{GF}(p)$ and let $G$ be a transitive subgroup of $\text{AGL}(V)$. Show that $G$ is primitive if and only if the stabiliser $G_0$ of the zero vector of $V$ is irreducible (stabilises no nontrivial proper subspace of $V$).
(iv) Let $M$ be a group and suppose that $G$ is a subgroup of Sym$(M)$ containing $M_R$. Let $\Delta$ be the orbit of 1 under $C_G(M_R)$. Show that $(M_R)_\Delta = \{p_\lambda : \lambda \in C_G(M_R)\}$. 


5.2. The subdivision

With the machinery derived in the previous sections, we will now outline a case subdivision for the analysis and characterisation of innately transitive groups.

![Diagram](image)

**Figure 1.** A diagram representing the case sub-division of innately transitive groups. An “up” arrow represents “true” and a down arrow represents “false”.
We will endeavour to understand each of these types. Throughout, let G be a finite innately transitive permutation group on a set Ω, with plinth M. Since M is characteristically simple, there is a simple group T and a positive integer k such that $M \cong T^k$.

5.3. Abelian plinth type

Suppose that M is abelian. Since M is characteristically simple, we have that M is elementary abelian and hence can be identified with a $d$-dimensional vector space V over $\mathbb{F}(p)$ for some prime $p$. Now M is transitive and abelian, so by a Exercise 2.2.13, M is regular. Therefore, G is permutationally isomorphic to a subgroup of Hol(M). Hence G (see the previous exercises) is primitive and permutationally isomorphic to a subgroup of AGL(V).

Description 5.3.1 (Abelian Plinth type). Let V be a finite vector space and let G be a subgroup of AGL(V), in its natural action. Suppose that G contains the group of translations $M$ of V and suppose that the stabiliser $G_0$ of the zero vector of V does not stabilise a nontrivial proper subspace of V. Then G is primitive with plinth M.

5.4. Simple plinth type

Now suppose that M is nonabelian and simple (so $M \cong T$). We split this case up into three parts:

(HS) $C_G(M)$ is transitive;
(AS) $C_G(M) = 1$;
(ASQ) $C_G(M) \neq 1$ and $C_G(M)$ is intransitive.

Holomorph of a Simple Group type. Recall by Lemma 3.2.5 that if $C_G(M)$ is transitive, then G is primitive and has two regular minimal normal subgroups. Moreover, G is permutationally isomorphic to a subgroup of Hol(M), which is in turn isomorphic to $T \rtimes \text{Aut}(T)$.

Description 5.4.1 (Holomorph of a Simple Group type). Let T be a nonabelian simple group and let G be a subgroup of Hol(T), in its natural action, and suppose that $T_R \rtimes \text{Inn}(T) \leq G$. Then G is primitive with plinth T.

Exercises 5.4.2.
(i) Let T be a nonabelian simple group and let $G = T \rtimes T$. Let $H = \text{Diag}(T \times T)$ and consider the coset space $\Omega := G/H$ and the action of G by right coset multiplication. Show that G is innately transitive of Holomorph of a Simple Group type (i.e., $C_G(T)$ is transitive).
(ii) Let G be an innately transitive group with plinth M such that M is nonabelian and simple, and that $C_G(M)$ is transitive. Show that G is permutationally isomorphic to an innately transitive group constructed as above.

Almost Simple type and Almost Simple Quotient type. The remaining case is when $C_G(M)$ is intransitive, which implies that $C_G(M) = 1$ if G is quasiprimitive. Hence we split this case in two to delineate the quasiprimitive and non-quasiprimitive cases.

Description 5.4.3 (Almost Simple and Almost Simple Quotient type). Let G be a finite permutation group on a set $\Omega$, let $\alpha \in \Omega$, and let M be a nonabelian simple normal subgroup of G such that $G = MG_\alpha$ and $G_\alpha \not\cong M$. We have two subcases:

(AS) Here $C_G(M) = 1$ and G is quasiprimitive with plinth M.
(ASQ) Here $C_G(M) \neq 1$, $G \neq C_G(M)G_\alpha$, and G is innately transitive with plinth M.

Exercises 5.4.4.
(i) Let T be a nonabelian simple group and let $T_0$ be a proper subgroup of T. Show that the action of T on the right cosets of $T_0$ is quasiprimitive of Almost Simple type.
(ii) Show that the full symmetry group of the icosahedron is innately transitive of Almost Simple Quotient type.

Note that it seems almost intractable to know all the innately transitive groups of this type. By the first exercise above, to classify quasiprimitive groups of Almost Simple type requires one to know the subgroup structure (up to conjugacy) of every nonabelian simple group!

5.5. Regular Plinth type

Now suppose that M is nonabelian, not simple, and regular. We split this case up into three parts:

(HC) $C_G(M) \neq 1$ and $C_G(M)$ is transitive;
(TW) $C_G(M) = 1$;
(DQ/PQ) $C_G(M) \neq 1$ and $C_G(M)$ is intransitive.
Holomorph of a Compound Group type. Just as in the Holomorph of a Simple Group type, we have (by Lemma 3.2.5) that $G$ is primitive and has two regular minimal normal subgroups.

**DESCRIPTION 5.5.1 (Holomorph of a Compound Group type).** Let $T$ be a nonabelian simple group, let $M = T^m$ where $m > 1$, and let $G$ be a subgroup of $\text{Hol}(M)$, in its natural action. Suppose that $M_L \times M_R \leq G$. Then $G$ is primitive with plinth $M_R$ (or $M_L$).

Twisted Wreath type. Here we have that $M$ is the unique minimal normal subgroup of $G$. The reason why this is called Twisted Wreath type is that you can describe all of the innately transitive groups with a regular plinth as a subgroup of a twisted wreath product (see Chapter 6). For primitive groups, the only other type to consider is the Holomorph of a Compound group type, which is well described. So, due to history and other matters, this lonely subcase is coined twisted wreath type.

**DESCRIPTION 5.5.2 (Twisted Wreath type).** Let $T$ be a nonabelian simple group, let $M = T^k$ where $k > 1$, and let $G$ be a subgroup of $\text{Hol}(M)$, in its natural action. Suppose that $C_G(M_{R}) = 1$. Then $G$ is quasiprimitive with plinth $M_R$.

Diagonal Quotient and Product Action Quotient type. Here we give some prototypical examples of innately transitive groups of Product Quotient type and Diagonal Quotient type. These groups are strictly not quasiprimitive.

**Example 5.5.3 (An innately transitive group of Product Quotient type).**

Let $H$ be an innately transitive group of Almost Simple Quotient type with regular plinth $T$, and let $k > 1$. Then $G = H \wr S_k$ is innately transitive in product action (see Theorem 5.1.5) with regular plinth $M = T^k$. Moreover, $C_G(M) = (C_H(T))^k$ and $C_H(T) > T_k$, and hence $C_G(M)$ is not a subdirect subgroup of the left representation of $M$. Therefore, $G$ is of Product Quotient type.

**Example 5.5.4 (An innately transitive group of Diagonal Quotient type).**

Let $T$ be a nonabelian simple, let $k > 1$, let $m$ be a divisor of $k$ not equal to $k$, let $A$ be the straight diagonal subgroup of $\text{Aut}(T^{k/m})^m$, and let $G = M \times (A \times S_{k/m}) \wr S_k$ where $M$ acts regularly on itself and $(A \times S_{k/m}) \wr S_k$ acts naturally as a subgroup of $\text{Aut}(M) \cong \text{Aut}(T) \wr S_k$. Then $G$ is innately transitive with regular plinth $M$, and $C_G(M) = C^m$ where $C$ is the straight diagonal subgroup of $T^{k/m}$. Since $C_G(M)$ is a subdirect product of $M$, we have that $G$ is of Diagonal Quotient type.

We will not provide a description here for these types as we have for the other types, but instead claim that the examples above serve the same purpose!

### 5.6. Product Action type

Here we present a large class of groups which, in some sense, preserve the product structure of a set. In the case that $G$ is primitive, we have a far simpler description (see the next exercise).

**DESCRIPTION 5.6.1 (Product Action type).** Suppose we have a $G$-invariant partition $\Psi$ of $\Omega$ and $\Psi$ is the Cartesian product of $k$ copies of a set $\Psi_0$. On $\Psi_0$, we assume there is an innately transitive permutation group $A$ of Almost Simple or Almost Simple Quotient type with non-regular plinth $T$. Next, choose $\psi_0 \in \Psi_0$ and set $U := T_{\psi_0}$. For $\psi = (\psi_0, \ldots, \psi_0) \in \Psi$, we have $M_\psi = U^k$, and for $\alpha \in \psi$, the point stabiliser $M_\alpha$ is a subdirect subgroup of $U^k$ with index the size of a block in $\Psi$. Replacing $G$ by a conjugate in $\text{Sym}(\Omega)$ if necessary, we may assume that $G^\Psi \leq A \wr S_k \leq \text{Sym}(\Psi_0) \wr S_k$. If the point stabiliser $G_\alpha$ projects onto a transitive subgroup of $S_k$, then $G$ is innately transitive of Product Action type.

If we allow the plinth $T$ of $A$ above to be regular, we get the Product Quotient type described in the previous section.

**Exercises 5.6.2.** Let $G$ be a primitive group of Product Action type. Show that $G$ is permutationally isomorphic to $G^\Psi$ (where $\Psi$ was defined above) and hence show that $G$ can be realised as the subgroup of a wreath product acting in product action.

### 5.7. Diagonal Type

In this case, $M$ is non-simple, non-regular, a stabiliser $M_\alpha$ is subdirect in $M$, and $C_G(M)$ is trivial. Thus $G$ is quasiprimitive. It turns out that we have $\Omega = \Delta^k$ and $G \leq B \wr S_1 \leq \text{Sym}(\Delta) \wr S_1$, in product action, for some proper divisor $\ell$ of $k$ where $B$ is a quasiprimitive permutation group on $\Delta$ with plinth $T^{k/\ell}$, and $G$ projects onto a transitive subgroup of $S_1$. We have two subtypes:
5.8. A DESCRIPTION BY QUOTIENTS

- **Simple Diagonal Type**
  Here $\ell = 1$ and $M^\alpha$ is a full diagonal subgroup of $M$. For more details see [13].

- **Compound Diagonal Type**
  In this case, $\ell > 1$ and $B$ is of Simple Diagonal type.

Again, to describe these groups, we present simple examples.

**Example 5.7.1 (A small example...).** Here we give an example of a quasiprimitive group of Simple Diagonal type. Let $T$ be a non-abelian simple group, let $k > 1$, and let $D$ be the straight diagonal subgroup of $M = T^k$. So $D$ is isomorphic to $T$. Now consider the right coset action of $M$ on $D$. Note that we can identify the right cosets $M/D$ with the direct product $T^{k-1}$ (as each coset representative can be assumed to be normalised to have a 1 in the first coordinate). In this action, $M$ is quasiprimitive of Simple Diagonal type.

Note that the above example is a simpler version of a more general diagonal type action which we saw in Exercise 3.3.6. We recast this example below.

**Example 5.7.2 (A bigger example...).** Let $T$ be a nonabelian simple group, let $k$ be a positive integer and consider the straight diagonal subgroup $D$ of $M = T^k$. Recall from exercise 3.3.6 that we can form the wreath product $\text{Out}(T) \wr S_k$ and hence a natural semidirect product $W := M \rtimes (\text{Diag}(\text{Out}(T)^k) \times S_k)$.

Next, there is an action of $W$ on the right cosets $M/D$. Now the representatives of the cosets $M/D$ can be taken to have a 1 in the leading coordinate, and hence we can identify $M/D$ with $T^{k-1}$. This identification induces a faithful action of $W$ on $T^{k-1}$. Any subgroup $G$ of $W$ containing the normal subgroup $M$ acts quasiprimitively on $T^{k-1}$ with plinth $M$. This is a typical example of a quasiprimitive group of Simple Diagonal type.

### 5.8. A description by quotients

We saw briefly in Section 2.3 the role of permutational transformations in describing various group actions. Here, we present a way of describing innately transitive groups via a quasiprimitive quotient action. We will also see why some of the types just listed have been titled as such.

**Lemma 5.8.1.** Let $G$ be an innately transitive permutation group acting on a set $\Omega$, let $M$ be a plinth for $G$, and let $\Sigma$ be the set of orbits of $C_G(M)$ on $\Omega$.

(i) If $G$ is imprimitive and $M$ is nonabelian, then $C_G(M)$ is a maximal intransitive normal subgroup of $G$.

(ii) The group $G^\Sigma$ induced by the quotient action of $G$ on $\Sigma$ is isomorphic to $G/C_G(M)$ and has a unique transitive minimal normal subgroup, namely $M^\Sigma$, which is isomorphic to $M$. Hence $G^\Sigma$ is quasiprimitive.

(iii) If $C_G(M)$ is nontrivial and intransitive, that is, $G$ is not quasiprimitive, then Table 1 lists the possible types for $G$ and $G^\Sigma$ respectively.

<table>
<thead>
<tr>
<th>Type of $G$</th>
<th>Type of $G^\Sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Almost Simple Quotient</td>
<td>Almost Simple Diagonal</td>
</tr>
<tr>
<td>Diagonal Quotient</td>
<td>Diagonal</td>
</tr>
<tr>
<td>Product Quotient</td>
<td>Product Action</td>
</tr>
<tr>
<td>Product Action</td>
<td>Product Action</td>
</tr>
</tbody>
</table>

**Table 1.** The quasiprimitive type of the group $G^\Sigma$ corresponding to the innately transitive type of $G$.

**Proof.** A good exercise! □
5.9. The structure theorem for innately transitive groups

The following is a generalisation of the O’Nan-Scott Theorem that appeared in [2].

THEOREM 5.9.1 (Bamberg and Praeger, 2004). Every finite innately transitive permutation group is permutationally isomorphic to a group described in one of the five types

- Abelian Plinth type
- Simple Plinth type
- Regular Plinth type
- Product Action type
- Diagonal type

Moreover, every group described\(^1\) by these types is innately transitive.

EXERCISES 5.9.2.

(i) Which of the following types above hold for primitive groups?

(ii) Prove that a finite 2-transitive permutation group is innately transitive of Abelian Plinth or Almost Simple type. (This is the first step to proving Burnside’s Theorem!)

\(^1\)In [2], there is more detail for the descriptions of these types.
CHAPTER 6

Miscellanea

6.1. Twisted wreath products

Twisted wreath products were first defined and studied by B. H. Neumann (see [10]) in the early 1960’s. We take the approach given by Suzuki and Baddeley (see [16] and [1] respectively). Our ingredients for the twisted wreath product are a group $T$, a group $P$, a subgroup $Q$ of $P$, and a homomorphism $\phi : Q \to \text{Aut}(T)$. The set $\text{Fun}(P,T)$ of functions from $P$ to $T$ forms a group under point-wise multiplication, i.e., $f_1 f_2 : x \mapsto f_1(x) f_2(x)$. Define the base group $B$ as the subgroup of $\text{Fun}(P,T)$ given by

$$B := \{ f : P \to T \mid f(pq) = f(p)\phi(q) \text{ for all } p, q \in Q \}.$$ 

Now we define an action of $P$ on $B$ by letting $f^p$ be the map defined by

$$f^p : x \mapsto f(px).$$

Then the twisted wreath product of $T$ by $P$ (via $\phi$) is the semi-direct product $T \tw_{\phi} P := B \rtimes P$.

We call $P$ the top group of the twisted wreath product and we get the usual wreath product when $\phi$ is trivial. Any twisted wreath product has an action on its base group in which the base group itself acts by right multiplication and the top group by conjugation. We call this the base group action of the twisted wreath product. Below is a lemma that as far as the author is aware, can be attributed to Bercov and Lafuente ([3] and [9]).

**Lemma 6.1.1** (R. Bercov 1967, and J. Lafuente 1984). Let $G$ be a group with a normal subgroup $M$ complemented by a subgroup $P$. Suppose that $T$ is a subgroup of $M$ such that for some $p_1 = 1, p_2, \ldots, p_k \in P$ we can write $M = T^{p_1} \times \cdots \times T^{p_k}$, where conjugation by $P$ permutes the $T^{p_i}$ amongst themselves, that is, $\{T^{p_1}, \ldots, T^{p_k}\}$ is the set of $P$-conjugates of $T$. Set $Q = N_P(T)$ and let $\phi : Q \to \text{Aut}(T)$ be the map induced by the conjugation action of $Q$ on $T$. Then there exists an isomorphism $G \to T \tw_{\phi} P$ which maps $M$ to the base group and $P$ to the top group of the twisted wreath product.

We prove now that an innately transitive group with a regular plinth is permutationally isomorphic to a twisted wreath product acting in base group action.

**Lemma 6.1.2.** Let $G$ be an innately transitive group on a finite set $\Omega$, let $M$ be the plinth of $G$, and let $\alpha \in \Omega$. Suppose also that $M$ is nonabelian and acts regularly on $\Omega$. Then $G$ is permutationally isomorphic to the twisted wreath product $T \tw_{\phi} G_{\alpha}$ in its base group action, where $\phi : N_G(\alpha) \to \text{Aut}(T)$ is the map induced by the conjugation action of $N_G(\alpha)$ on $T$.

**Proof.** First, note that $G_{\alpha}$ is a complement for $M$ in $G$ as $G = MG_{\alpha}$, $M$ is normal in $G$, and $G_{\alpha} \cap M = 1$ as $M$ is regular on $\Omega$. Let $\phi : N_G(\alpha) \to \text{Aut}(T)$ be the map induced by conjugation of $N_G(\alpha)$ on $T$. So by Lemma 6.1.1, there is an isomorphism from $G$ to $T \tw_{\phi} G_{\alpha}$ which maps $M$ to the base group and $G_{\alpha}$ to the top group. Since the top group $G_{\alpha}$ is a point stabiliser for the base group action, $G$ is permutationally isomorphic to $T \tw_{\phi} G_{\alpha}$ (by Lemma 2.3.9). □

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6.2. The Construction of Innately Transitive Groups

In this section, we define an object called an innate triple which is an abstraction of the “three things” that arise from any given innately transitive group. The following was shown in [2].

**Theorem 6.2.1** (Bamberg and Praeger, 2003). Every finite innately transitive permutation group is permutationally isomorphic to an innately transitive group given by Construction 6.2.6, and every permutation group given by this construction is innately transitive.

**Definition 6.2.2 (Innate Triple).** A triple $(K, \varphi, L)$ satisfying the three conditions below is called an innate triple.

(i) $K \cong T^k$ where $T$ is a simple group (possibly abelian),
(ii) $\varphi$ is an epimorphism with domain a subgroup $K_0$ of $K$, with kernel core-free in $K$, and if $K$ is abelian, then $K_0 = K$.
(iii) $L$ is a subgroup of $\text{Aut}(K)$ such that $K$ is $L$-simple$^1$, $\ker \varphi$ is $L$-invariant, and $L \cap \text{Inn}(K) = \text{Inn}_{K_0}(K)$.

We denote by $\mathcal{D}$ the set of all innate triples.

Note that if $K$ is elementary abelian, then condition (2) implies that $\ker \varphi = 1$ and $\varphi$ has domain $K$ so that $\text{Im} \varphi \cong K$.

**Construction 6.2.3 (**...of innate triples**).** Let $G$ be a finite innately transitive permutation group acting on a set $\Omega$, let $K$ be the plinth of $G$, and let $\alpha \in \Omega$. Let $\Delta = C_G(\alpha)$ and let $\varphi : K_\alpha \to C_G(K)$ be the map where for each $u \in K_\alpha$, $\varphi(u)$ is the unique element of $C_G(K)$ such that $\varphi(u)u \in G_\alpha$ (see Lemma 5.1.3). Let $L$ be the subgroup of $\text{Aut}(K)$ induced by the conjugation action of $G_\alpha$ on $K$. Then $(K, \varphi, L)$ is the innate triple associated to $G$.

**Exercises 6.2.4.** Show that the above construction is well-defined (that is, the output is indeed an innate triple).

**Example 6.2.5.** Let $G$ be the full symmetry group of the icosahedron. Then $G$ is isomorphic to $\mathbb{Z}_2 \times A_5$ where $A_5$ is the subgroup of rotations of the icosahedron, and its centraliser $C = \mathbb{Z}_2$ is generated by the central reflection of the icosahedron which maps a given vertex to its antipodal vertex. The group $G$ is innately transitive on the vertices of the icosahedron with plinth $M = A_5$. For a vertex $v$, $K_v \cong \mathbb{Z}_5$ and hence $N_K(K_v) \cong D_{10}$. There is a section of $K$ isomorphic to $C$, namely $N_K(K_e)/K_e$, and so there is an epimorphism $\varphi : N_K(K_e) \to C$ with kernel $K_e$.

Since $K$ is simple, it is clear that $\ker \varphi$ is core-free and $K$ is $L$-simple for all $L \leq \text{Aut}(K)$. So $\varphi$ satisfies property 2 of Definition 6.2.2. The group induced by the conjugation action of the stabiliser $G_v$ (for some vertex $v$) on the plinth $K_v$ turns out to be the inner automorphisms of $K$ induced by $N_K(K_v) \cong D_{10}$. Also note that $\ker \varphi$ is invariant under $L$ as $\ker \varphi = K_e$ is normal in $N_K(K_v)$. So we have an automorphism group $L = \text{Inn}_{D_{10}}(K) \leq \text{Aut}(K)$ satisfying property 3 of Definition 6.2.2, and hence $(K, \varphi, L)$ is an innately transitive triple.

In some sense, we have cut down the innately transitive group into pieces which are defined in terms of a minimal normal subgroup of the group. This process is reversible, that is, we can construct an innately transitive group from an innate triple. Recall that the graph of a homomorphism $\varphi : B \to A$ is the subgroup of $A \times B$ defined by $\text{Graph}(\varphi) := \{(\varphi(b), b) : b \in B\}$.

**Construction 6.2.6 (**...of innately transitive groups**).** Let $(K, \varphi, L)$ be an innate triple and let $X := (\text{Im} \varphi \times K) \rtimes L$ where $L$ acts on $\text{Im} \varphi \times K$ by $(\varphi(u), y)^L = (\varphi(u^\tau), y^\tau)$ for all $u \in \text{Dom} \varphi$, $y \in K$, and $\tau \in L$. Let $X_0 = \text{Graph}(\varphi) \times L$ and let $X$ act by right coset multiplication on $\Omega := [X : X_0]$. Then the kernel of the action of $X$ on $\Omega$ is $Z := \{\varphi(x)x_1x_1^{-1} : x \in K_0\}$, and $X/Z$ is innately transitive and faithful on $\Omega$ (in its induced action) with plinth $KZ/Z$.

**Exercises 6.2.7.**

(i) Verify that the action of $L$ on $\text{Im} \varphi$ is well-defined.
(ii) Show that the output of Construction 6.2.6 is indeed an innately transitive permutation group.
(iii) Let $K, \varphi, X$, and $Z$ be as in Construction 6.2.6. Show that $Z$ is centralised by $\text{Im} \varphi \times K$, and $C_{X/Z}(KZ/Z) = \text{Im} \varphi Z/Z \cong \text{Im} \varphi$.

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$^1$If $L \leq \text{Aut}(K)$, then we say that $K$ is $L$-simple if it has no nontrivial proper normal subgroups invariant under $L$. 
Bibliography


