Appendix A

Classification of Finite Simple Groups

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Every finite group can be built up of simple groups through successive extensions. The abelian simple groups are the groups of prime order. The finite nonabelian simple groups are broadly classified as: (i) the alternating groups A_n ($n \geq 5$); (ii) the simple groups of *Lie type*; and (iii) the *sporadic* simple groups. The *Classification of Finite Simple* ***Groups*** is the claim (based on many thousands of pages of research papers by dozens of mathematicians) that the only finite nonabelian simple groups are the presently known groups in classes (i)–(iii). The *Classification* was formally announced in Gorenstein (1979). Gorenstein et al (1994) is the first in a series of volumes from a project, presently under way, to present a coherent and accessible proof of the *Classification*.

The alternating groups are, of course, well understood. We briefly describe below the groups of Lie type and the 26 sporadic groups.

The Simple Groups of Lie Type

There are five families of *classical* finite simple groups of Lie type, each of which is obtained by factoring a suitable linear group by its centre (a group of scalars). These are the families of (projective) special linear groups, unitary groups, symplectic groups, and two families of orthogonal groups.

In Table A.1, q denotes a order of finite field and so is a prime power, and d represents the order of the centre which has been factored out. The groups are denoted by a common abbreviated notation where $PSL_n(q)$ is denoted by $L_n(q)$, $PSp_{2m}(q)$ by $S_{2m}(q)$, etc. (see Appendix B). We have $L_2(2) \cong S_3$, $L_2(3) \cong A_4$, $L_2(4) \cong L_2(5) \cong A_5$, $L_2(7) \cong L_3(2)$, $L_2(9) \cong A_6$, $L_4(2) \cong A_8$, $S_4(2) \cong S_6$, $U_4(2) \cong S_4(3)$, and $U_3(2)$ is solvable. With these exceptions the groups listed in Table A.1 are nonisomorphic nonabelian simple groups which are not isomorphic to alternating groups.

In addition to these families of classical groups, there are nine further families of groups of Lie type, parameterized by the prime power q, each of which is derived from a Lie algebra of specific dimension. With the notation introduced by C. Chevalley and R. Steinberg these are denoted: $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, ${}^2B_2(q)$ ($q = 2^{2m+3}$), ${}^3D_4(q)$, ${}^2G_2(q)$ ($q = 3^{2m+3}$), ${}^2F_4(q)$ ($q = 2^{2m+3}$), and ${}^2E_6(q)$. The groups in the family ${}^2B_2$ are known as Suzuki groups, and the groups in the families ${}^2G_2$ and ${}^2F_4$ are known as Ree groups after their discoverers. Finally, there is the single exceptional group ${}^2F_4(2)'$ which is known as Tits' group.

Work of Chevalley in 1955 and of Steinberg in 1959 showed that all groups of Lie type can be defined and analyzed, more or less uniformly, by using the underlying Lie algebra structure. In particular, their groups of outer automorphisms can be constructed; these are all solvable and quite small [see Conway et al (1985)].

TABLE A.1. The Simple Groups of Lie Type

Name	Symbol	Order
Linear	$L_n(q)$	$q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - 1)/d$ where $d = \mathrm{GCD}(n, q-1)$, $n \geq 2$
Unitary	$U_n(q)$	$q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - (-1)^i)/d$ where $d = \mathrm{GCD}(n, q+1)$, $n \geq 2$
Symplectic	$S_{2m}(q)$	$q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1)/d$ where $d = \mathrm{GCD}(2, q-1)$, $m \geq 3$
Orthogonal	$O_{2m+1}(q)$	$q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1)/d$ where $d = \mathrm{GCD}(2, q-1)$, $m \geq 2$
Orthogonal	$O_{2m}(q)$	$q^{m(m-1)}(q^m - \epsilon) \prod_{i=1}^{m-1} (q^{2i} - 1)/d$ where $d = \mathrm{GCD}(4, q^m - \epsilon)$, $m \geq 4, \epsilon = \pm 1$

The Sporadic Simple Groups

These are the finite simple groups which do not fall into infinite families. Twenty six of them are known, and according to the *Classification* these are the only finite nonabelian simple groups which are not alternating or of Lie type. The five Mathieu groups were discovered in the middle of the last century, but the other sporadic simple groups where all discovered between 1964 and 1975. Table A.2 lists these groups with their orders and the date when the group was discovered (or predicted to exist). There are many interesting relations between these groups. In particular, the Mathieu group M_{24} contains all of the smaller Mathieu groups, and the Monster M contains (as sections) many of the other sporadic groups. The group of outer

TABLE A.2. The Sporadic Simple Groups

Name	Symbol	Order	Date
Mathieu	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1861
Mathieu	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	1861
Mathieu	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	1873
Mathieu	M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1873
Mathieu	M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1873
Janko	J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1964
Hall–Janko	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	1967
Suzuki	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	1967
Higman-Sims	HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	1967
McLaughlin	McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	1967
Conway	Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1968
Conway	Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1968
Conway	Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	1968
Janko	J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	1968
Fischer	Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	1968
Held	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	1969
Fischer	Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	1969
Fischer	Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	1969
Lyons	Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	1971
Rudvalis	Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	1972
O'Nan	$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	1973
Harada–Norton	HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	1974
Thompson	Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	1974
Baby Monster	B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot$ $17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	1975
Monster	M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot$ $19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	1975
Janko	J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	1975

automorphisms for each of the groups on this list has order at most 2, thus completing the verification of the Schreier Conjecture.

For further information about the classification, see Gorenstein (1979) and (1982), and Gorenstein et al (1994). Conway and Sloane (1988) and Conway et al (1985) give details about the sporadic groups. Thompson (1983) describes some of the history of the discovery of the sporadic groups. An interesting account of the 19th century search for finite simple groups can be found in Silvestri (1979).

Appendix B

The Primitive Permutation Groups of Degree Less than 1000

This appendix gives a list of all proper primitive permutation groups of degree less than 1000. Such a list is of interest in illustrating in concrete form the kinds of primitive groups which arise, in suggesting conjectures about primitive groups, and settling small exceptional cases which often occur in proofs. Earlier lists (of varying completeness and accuracy) of primitive groups of degree n have been published by Jordan (1872) for $n \leq 17$, by Burnside (1897) for $n \leq 8$, by Manning (in a long series of papers that appeared between 1906 and 1929) for $n \leq 15$, by Sims (1970) for $n \leq 20$ and by Pogorelov (1980) for $n \leq 50$. At about the same time as the list presented here originally appearing in Dixon and Mortimer (1988), a list covering the same range was published by Il'in and Takmakov (1986).

The permutation groups in the list are collected into *cohorts* where all groups in a cohort have the same socle and this socle has the same action in each group of the cohort. Thus an item in the list consists of a transitive action for a group H, the socle, on a set Ω and the normalizer N of H in $Sym(\Omega)$ where N acts primitively on Ω. It may happen that H itself is not primitive or that $\mathrm{soc}(N) \neq H$.

Consider, as an example, the entry for the simple group $T = PSL_2(7) \cong PSL_3(2)$ listed under type B in Table B.2. There are four cohorts. There are two primitive actions with socle T. Considering T as $PSL_2(7)$ there is a natural 2-transitive action of degree 8 with stabilizers isomorphic to 7:3. The image of T in S_8 has index 2 in its normalizer (which is isomorphic to $PGL_2(7)$), as indicated by the entry $H.2$ in this row. Taking T as $PSL_3(2)$ there is a natural 2-transitive action of degree 7 on the Fano plane $PG_2(2)$ with stabilizers isomorphic to S_4. The image of T in S_7 is self normalizing, indicated by the entry H in this row. As described in Example 4.6.1, T also has imprimitive actions of degrees 21 and 28 (on the flags and antiflags of $PG_2(2)$) and there is a group $T.2 \cong PGL_2(7)$ which is primitive in both cases. These actions are recorded with first entry $PSL_2(7).2$, the smallest primitive group containing the socle with this action.

The socle H is a direct power of some simple group T. The various possiblities are that the socle is simple, composite with product action,