

Modeling Parallel Wiener-Hammerstein Systems Using Tensor Decomposition of Volterra Kernels

Philippe Dreesen¹, David T. Westwick², Johan Schoukens¹, Mariya Ishteva¹

¹ Vrije Universiteit Brussel (VUB), Dept VUB-ELEC, Brussels, Belgium.

² University of Calgary, Dept Electrical and Computer Engineering, Calgary, Canada.

`philippe.dreesen@gmail.com`

Abstract. Providing flexibility and user-interpretability in nonlinear system identification can be achieved by means of block-oriented methods. One of such block-oriented system structures is the parallel Wiener-Hammerstein system, which is a sum of Wiener-Hammerstein branches, consisting of static nonlinearities sandwiched between linear dynamical blocks. Parallel Wiener-Hammerstein models have more descriptive power than their single-branch counterparts, but their identification is a non-trivial task that requires tailored system identification methods. In this work, we will tackle the identification problem by performing a tensor decomposition of the Volterra kernels obtained from the nonlinear system. We illustrate how the parallel Wiener-Hammerstein block-structure gives rise to a joint tensor decomposition of the Volterra kernels with block-circulant structured factors. The combination of Volterra kernels and tensor methods is a fruitful way to tackle the parallel Wiener-Hammerstein system identification task. In simulation experiments, we were able to reconstruct very accurately the underlying blocks under noisy conditions.

Keywords: tensor decomposition, system identification, Volterra model, parallel Wiener-Hammerstein system, block-oriented system identification, canonical polyadic decomposition, structured data fusion

1 Introduction

System identification is the art of building dynamical models from noisy measurements of input and output data. Linear system identification is a well-established discipline [11, 13, 20] and has yielded successful applications in a wide variety of fields. In the last decades, the use of nonlinear models has become more important in order to capture the nonlinear effects of the real world. Many different nonlinear identification methods have been proposed [6, 17], but very often these solutions are either tailored to a specific application, or are too complex to understand or study.

In this paper, we will draw ideas from two nonlinear system identification approaches and try to combine the benefits of both. The first approach tackles the disadvantage of increased complexity of nonlinear models by considering

block-oriented models [7], which combine flexibility with user-interpretation by interconnecting linear dynamical blocks and static nonlinear functions. Unfortunately, even simple block-oriented models, such as Wiener (cascade of linear-nonlinear), Hammerstein (cascade nonlinear-linear) or Wiener-Hammerstein (cascade linear-nonlinear-linear) require an iterative optimization on a non-convex objective function, and identification procedures that are tailored towards a specific block-structure [16]. The second approach that we will use is the Volterra model [1, 14], which is an extension of the well-known impulse response model for linear dynamical systems. Volterra models take into account higher-order polynomial nonlinearities and can thus be seen as a generalization of the Taylor series expansion for nonlinear dynamical systems. The advantages of Volterra models are that any fading-memory system can be approximated to an arbitrary degree of accuracy [1, 15] and the parameter estimation task is a linear problem. The disadvantages are that the resulting models contain a very high number of parameters and thus cannot be given physical interpretation.

We generalize the earlier results of [4, 5] on Wiener-Hammerstein system identification using tensor decompositions in two ways: First, we show that the case of parallel branches for a fixed degree d gives rise to a canonical polyadic decomposition with block-structured factors. The study of parallel Wiener-Hammerstein systems is useful, as they are universal approximators, whereas single-branch Wiener-Hammerstein systems are not [12, 14]. Second, we jointly consider Volterra kernels of several degrees by means of the structured data fusion framework of [18] and solve the problem as a joint structured tensor decomposition. By simultaneously decomposing several Volterra kernels, the available information is used maximally. The presented method is implemented by means of structured data fusion [18] using Tensorlab 3.0 [21], and is validated on simulation experiments.

The paper is organized as follows. In Section 2 we illustrate the link between the Volterra kernels and tensors and introduce the canonical polyadic decomposition for tensors. Section 3 illustrates how the Volterra kernels of a parallel Wiener-Hammerstein system have a natural connection to the canonical polyadic decomposition with block-circulant factors. This ultimately leads to a joint structured canonical polyadic decomposition of the Volterra kernels of several orders that solve the parallel Wiener-Hammerstein identification task. We validate the method on numerical simulation examples in Section 4. In Section 5 we draw the conclusions.

2 Volterra kernels, tensors and tensor decomposition

2.1 The Volterra model for nonlinear systems

We consider single-input-single-output systems that map an input u at time instance k onto an output y at time instance k . The Volterra series expansion generalizes the well-known convolution operator for linear dynamical systems to the nonlinear case. Essentially the Volterra model of a nonlinear system expresses the output $y(k)$ as a polynomial function of time-shifted input variables

$u(k), u(k-1), \dots, u(k-m)$, with m denoting the memory length of the model. Formally we can write the Volterra model as

$$y(k) = \sum_{d=1}^D \left(\sum_{s_1, \dots, s_d=0}^m H_d(s_1, \dots, s_d) u(k-s_1) \cdots u(k-s_d) \right), \quad (1)$$

where $H_d(\cdot, \dots, \cdot)$ denotes Volterra kernel of degree d . The Volterra series expansion allows for representing a large class of nonlinear systems up to an arbitrary degree of accuracy [1].

2.2 From polynomials to tensors

It is well-known that multivariate homogeneous polynomials can be identified to symmetric higher-order tensors [3]. For instance, we may represent the quadratic polynomial $p(x_1, x_2) = 5x_1^2 - 8x_1x_2 + x_2^2$ as a matrix multiplied from both sides by a vector containing x_1 and x_2 as

$$\begin{aligned} p(x_1, x_2) &= 5x_1^2 - 8x_1x_2 + x_2^2 \\ &= [x_1 \ x_2] \begin{bmatrix} 5 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

In general, we may thus write a (nonhomogeneous) polynomial as

$$p(x_1, \dots, x_n) = p_0 + \mathbf{x}^T \mathbf{p}_1 + \mathbf{x}^T \mathbf{P}_2 \mathbf{x} + \mathcal{P}_3 \times_1 \mathbf{x}^T \times_2 \mathbf{x}^T \times_3 \mathbf{x}^T + \dots, \quad (2)$$

where $\mathbf{x} = [x_1 \ \dots \ x_n]^T$ and \times_n is the n -mode product defined as follows. Let \mathcal{X} be a $I_1 \times I_2 \times \dots \times I_N$ tensor, and let \mathbf{u}^T be an $1 \times I_n$ row vector, then we have

$$(\mathcal{X} \times_n \mathbf{u}^T)_{i_1 \dots i_{n-1} i_{n+1} \dots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 i_2 \dots i_N} u_{i_n}.$$

Notice that the result is a tensor of order $N-1$, as mode n is summed out.

2.3 Canonical Polyadic Decomposition

It is often useful to decompose a tensor into simpler components, and for the proposed method we will use the canonical polyadic decomposition. The canonical polyadic decomposition [2, 8, 10] (also called CanDecomp or PARAFAC) expresses the tensor \mathcal{T} as a sum of rank-one terms as

$$\mathcal{T} = \sum_{r=1}^R \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r,$$

with \circ denoting the outer product and the number of components R denoting the CP rank of the tensor \mathcal{T} . Often a short-hand notation is used as

$$\mathcal{T} = \llbracket \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket,$$

where $\mathbf{A} = [\mathbf{a}_1 \ \dots \ \mathbf{a}_R]$ and \mathbf{B} and \mathbf{C} are defined likewise.

with the circulant matrix \mathbf{P} of size $m_P + m_Q + 1 \times m_Q + 1$. Notice that we fixed the first coefficient $p_0 = 1$ for uniqueness purposes. The matrix \mathbf{P} will turn out to play a central role in the canonical polyadic decomposition of the Volterra kernels of a Wiener-Hammerstein system.

By fixing both $q_0 = 1$ and $p_0 = 1$, we are excluding the possibility that there is a pure delay present in the system. The presence of a delay in the system can be accounted for by setting $p_0 = 1$ and then performing a scaling on the nonlinearity, rather than on q_0 . In case the system has a delay this will lead to an estimated $q_0 \approx 0$. However, for notational convenience we have chosen $p_0 = q_0 = 1$ in the remainder of this paper, but a more general scaling strategy to ensure uniqueness is possible and compatible with the presented method.

For the current Wiener-Hammerstein system we have $f(x) = x^3$, and hence $y(k) = \mathcal{H}_3 \times_1 \mathbf{u}^T \times_2 \mathbf{u}^T \times_3 \mathbf{u}^T$. In [5] it is shown that the Volterra kernel can be written as the canonical polyadic decomposition $\mathcal{H} = \llbracket \mathbf{P}, \mathbf{P}, \mathbf{P} \text{diag}(\mathbf{q}) \rrbracket$, which we can also write in a more symmetrical expression by extracting \mathbf{q}^T into an extra mode as

$$\mathcal{H} = \llbracket \mathbf{P}, \mathbf{P}, \mathbf{P}, \mathbf{q}^T \rrbracket. \quad (4)$$

This fact can be appreciated by considering output $y(k)$ as

$$\begin{aligned} y(k) &= \mathcal{H} \times_1 \mathbf{u}^T \times_2 \mathbf{u}^T \times_3 \mathbf{u}^T \\ &= \llbracket \mathbf{u}^T \mathbf{P}, \mathbf{u}^T \mathbf{P}, \mathbf{u}^T \mathbf{P}, \mathbf{q}^T \rrbracket \\ &= \llbracket \mathbf{v}^T, \mathbf{v}^T, \mathbf{v}^T, \mathbf{q}^T \rrbracket \\ &= \sum_{i=0}^{m_Q} q(i) v^3(k-i), \end{aligned}$$

in which we recognize the convolution of the impulse response of $Q(z)$ with the time-shifted samples $v^3(k)$ as in (3).³

In case of a general polynomial function $f(x)$, the same reasoning can be developed for each degree d , which will lead to a structured canonical polyadic decomposition of the degree- d Volterra kernel as in (4). For instance, if $f(x) = ax^2 + bx^3$, we find the following expressions

$$\mathbf{H}_2 = a \llbracket \mathbf{P}, \mathbf{P}, \mathbf{q}^T \rrbracket,$$

$$\mathcal{H}_3 = b \llbracket \mathbf{P}, \mathbf{P}, \mathbf{P}, \mathbf{q}^T \rrbracket.$$

In Section 3.3 we will discuss how this leads to a joint tensor decomposition.

³ Remark that the introduction of the extra mode \mathbf{q}^T is similar to the extraction of the weights λ_i in the notation $\llbracket \lambda; \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket$ of [10] where the columns of the factor matrices \mathbf{A} , \mathbf{B} and \mathbf{C} are scaled to have unit norm. Our notation is intentionally different in the sense that we have normalized the first elements of the columns of \mathbf{P} and \mathbf{q} equal to one, for practical purposes.

3.2 Parallel Wiener-Hammerstein structure

To understand how we can extend these results to the parallel case, let us consider a two-branch parallel Wiener-Hammerstein system where both branches have an identical nonlinearity $f_1(x) = f_2(x) = x^3$, as in Figure 2. To avoid a

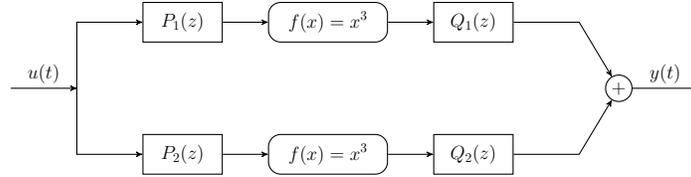


Fig. 2. An example of a two-branch parallel Wiener-Hammerstein system having an identical nonlinear function $f_1(x) = f_2(x) = x^3$.

notational overload, we will assume for the remainder of this paper that the memory lengths of all filters $P_i(z)$ are m_P , and likewise for the filters $Q_i(z)$ the lengths are m_Q . The summation of the two branches leads to

$$\begin{aligned} \mathcal{H}_3 &= \llbracket \mathbf{P}_1, \mathbf{P}_1, \mathbf{P}_1, \mathbf{q}_1^T \rrbracket + \llbracket \mathbf{P}_2, \mathbf{P}_2, \mathbf{P}_2, \mathbf{q}_2^T \rrbracket \\ &= \llbracket [\mathbf{P}_1 \ \mathbf{P}_2], [\mathbf{P}_1 \ \mathbf{P}_2], [\mathbf{P}_1 \ \mathbf{P}_2], [\mathbf{q}_1^T \ \mathbf{q}_2^T] \rrbracket, \end{aligned}$$

with \mathbf{P}_i and \mathbf{q}_i defined similar as in the single-branch case. We may include a scaling of branch one by a scalar c_1 (i.e., $f_1(x) = c_1 x^3$ and branch two by a scalar c_2 (i.e., $f_2(x) = c_2 x^3$ by introducing an additional mode as

$$\mathcal{H}_3 = \llbracket [\mathbf{P}_1 \ \mathbf{P}_2], [\mathbf{P}_1 \ \mathbf{P}_2], [\mathbf{P}_1 \ \mathbf{P}_2], [\mathbf{q}_1^T \ \mathbf{q}_2^T], [c_1 \mathbf{1}_{m_Q+1}^T \ c_2 \mathbf{1}_{m_Q+1}^T] \rrbracket. \quad (5)$$

Introducing the extra factor $[\mathbf{1}_{m_Q+1}^T c_1 \ \mathbf{1}_{m_Q+1}^T c_2]$ does not change the size of the tensor, since it introduces a mode with dimension one. Formally, if we let $m = m_P + m_Q + 1$ denote the memory length of the Volterra model, the tensor in (5) has size $m \times m \times m \times 1 \times 1$ which is equivalent to $m \times m \times m$.

3.3 Coupled tensor and matrix decompositions

The Volterra kernels of the parallel Wiener-Hammerstein model for a particular order d can be decomposed as a structured canonical polyadic decomposition. Hence, if the Volterra kernels of multiple orders are available, a joint decomposition of multiple Volterra kernels should be performed.

Ultimately, we find that the R -branch parallel Wiener-Hammerstein identification task is solved by minimizing the cost criterion

$$\begin{aligned} \underset{\mathbf{P}, \mathbf{q}, \mathbf{c}}{\text{minimize}} \quad & \|\mathbf{h}_1 - \llbracket \mathbf{P}, \mathbf{q}^T, \mathbf{c}_1^T \rrbracket\|_2^2 + \|\mathbf{H}_2 - \llbracket \mathbf{P}, \mathbf{P}, \mathbf{q}^T, \mathbf{c}_2^T \rrbracket\|_F^2 \\ & + \|\mathcal{H}_3 - \llbracket \mathbf{P}, \mathbf{P}, \mathbf{P}, \mathbf{q}^T, \mathbf{c}_3^T \rrbracket\|_F^2 + \dots, \end{aligned} \quad (6)$$

where

$$\begin{aligned} \mathbf{P} &= [\mathbf{P}_1 \cdots \mathbf{P}_R], \\ \mathbf{q}^T &= [\mathbf{q}_1^T \cdots \mathbf{q}_R^T], \\ \mathbf{c}_d^T &= [c_{1d} \mathbf{1}_{m_Q+1}^T \cdots c_{Rd} \mathbf{1}_{m_Q+1}^T]. \end{aligned}$$

The factor matrices \mathbf{P} and \mathbf{q}^T are shared among all joint decompositions while the constants \mathbf{c}_d^T depend on the order d of the considered Volterra kernel. Joint and structured factorizations like (6) can be solved in the framework of structured data fusion [18]. Remark that it is possible to add weighting factors to the different terms in (6), for instance if prior knowledge is available on the accuracy of the estimation of the kernel coefficients of different orders.

4 Numerical results

In this section we validate the proposed identification method on a simulation example. Numerical experiments were performed using MATLAB and structured data fusion [18] in Tensorlab 3.0 [21] (code available on request).

We consider a parallel Wiener-Hammerstein system having two branches with second and third-degree polynomial nonlinearities (Figure 3). The finite

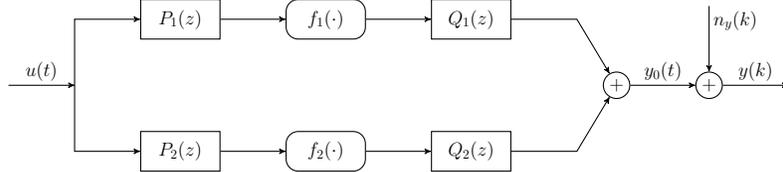


Fig. 3. A two-branch parallel Wiener-Hammerstein system with output noise.

impulse response coefficients of the filters $P_i(z)$ and $Q_i(z)$ are chosen as sums of decreasing exponentials with lengths $m_P = m_Q = 10$, such that the k th impulse response coefficient is given as $\sum_{i=1}^S \alpha_i^{k-1} / \sum_{i=1}^S \alpha_i$, with α_i drawn from a uniform distribution $U[-0.8, 0.8]$. For P_1 we have a sum of $S = 3$ exponentials,

while P_2 , Q_1 and Q_2 consist of a single decreasing exponential ($S = 1$). The coefficients c_{i2} and c_{i3} are drawn from a normal distribution $N(0, 0.1^2)$, and $c_{i0} = c_{i1} = 0$. The input signal is a Gaussian white noise sequence $u(k) \sim N(0, 0.7^2)$ and is applied without noise to the system, for $k = 1, \dots, 10,000$. The outputs $y(k) = y_0(k) + n_y(k)$ are disturbed by additive Gaussian noise n_y with a signal-to-noise ratio of 10 dB.

The Volterra kernel coefficients $H_d(s_1, \dots, s_2)$ with $s_i = 1, \dots, m_P + m_Q + 1$ of degrees two and three are estimated using a standard linear least-squares method on the basis of the second and third degree time-shifted inputs $u(k - s_1) \cdots u(k - s_3)$ and the noisy outputs $y(k)$, for $k = 1, \dots, 10,000$. The memory lengths $m_P = m_Q = 10$ give rise to 21×21 second-order Volterra kernel and a $21 \times 21 \times 21$ third-order Volterra kernel, having in total $231 + 1771 = 2002$ unique kernel elements.

The joint matrix and tensor decomposition with structured factors is then performed using the `sdf_minf` routine of Tensorlab 3.0 [21], returning the parameters of the parallel Wiener-Hammerstein system. We have performed a Monte Carlo experiment with 100 re-initializations of the optimization routine and were able to retrieve the true underlying system parameters in about 10 % of the cases. In Figure 4 (a) we show a typical result of a successful identification that was able to retrieve the underlying system parameters accurately. A zoom of the reconstructed outputs of the identified model together with the true outputs and the noisy measurements from which the Volterra kernels were estimated is shown in Figure 4 (b). It is worth mentioning that in some cases, a relatively small output error was obtained, while the computed system parameters were very different from the underlying parameters. In other experiments we have ob-

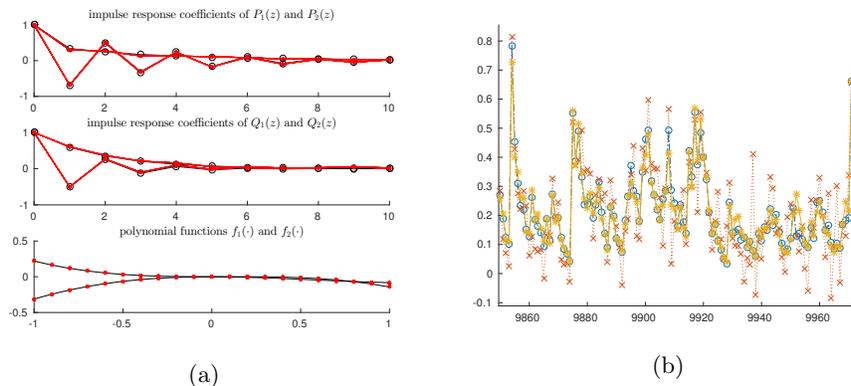


Fig. 4. (a) A typical result of the successful completion, where the method succeeds in retrieving the underlying parallel Wiener-Hammerstein system parameters. (b) The output of the identified parallel Wiener-Hammerstein system ($*$) reconstructs very accurately the noiseless output signal y_0 (\circ), starting from the Volterra kernel coefficients that were computed from the noisy data y with 10 dB SNR (\times).

served that the success rate of the algorithm improves for lower noise levels and shorter filter lengths. If the method failed consistently on a system, this could almost always be understood from problem-specific system properties, such as similar impulse responses in the filters, dominance of one of the branches (i.e., unbalanced values of the coefficients of f_i), etc.

Nevertheless, for extending the method to larger-scale problems, the issue of having good initial estimates becomes more relevant. Possibly the work of [19] which focuses on block-circulant structured canonical polyadic decomposition may provide good starting points for initializing our method.

Finally we would like to remark that in simulations, one can easily compare the retrieved system parameters (or simulated output data) with the underlying true system parameters or signals, which is obviously not possible in real experiments. However, it is worth mentioning that the tensor approximation error (6) was strongly correlated with the error between the simulated output and noiseless output data, which provides a proxy to select the best model among a set of candidates.

5 Conclusions

The joint decomposition of Volterra kernels with structured factors is able to retrieve up to a high degree of accuracy the underlying system parameters. The success rate of the method decreases as the noise level and number of parameters in the system grows, but even up to moderately long impulse responses (memory length of ten samples), the method was successful in about 10 % of the re-initializations of the optimization routine. Ongoing work is concerned with obtaining good initializations [22], as this becomes an important issue when considering filters with longer memories and/or higher noise levels. In future work, the link should be investigated between rank properties and identifiability of the coupled and structured canonical polyadic decomposition and the identifiability of the parallel Wiener-Hammerstein structure, as was done for other block-structures in [9].

Acknowledgments

This work was supported in part by the Fund for Scientific Research (FWO-Vlaanderen), by the Flemish Government (Methusalem), the Belgian Government through the Inter-university Poles of Attraction (IAP VII) Program, by the ERC Advanced Grant SNL-SID under contract 320378 and by FWO projects G.0280.15N and G.0901.17N. The authors want to thank Otto Debals and Nico Vervliet for help with the use of Tensorlab/SDF and the suggestion to extract the vector \mathbf{q} into an additional tensor mode.

References

1. Boyd, S., Chua, L.: Fading memory and the problem of approximating nonlinear operators with volterra series. *IEEE Trans. Circuits and Systems* 32(11), 1150–1171 (November 1985)

2. Carroll, J., Chang, J.: Analysis of individual differences in multidimensional scaling via an N-way generalization of “Eckart-Young” decomposition. *Psychometrika* 35(3), 283–319 (1970)
3. Comon, P., Golub, G., Lim, L.H., Mourrain, B.: Symmetric tensors and symmetric tensor rank. *SIAM J. Matrix Anal. Appl.* 30(3), 1254–1279 (2008)
4. de M. Goulart, J.H., Boizard, M., Boyer, R., Favier, G., Comon, P.: Tensor CP decomposition with structured factor matrices: Algorithms and performance. *IEEE J. Sel. Top. Signal Process.* 10(4), 757–769 (2016)
5. Favier, G., Kibangou, A.Y.: Tensor-based methods for system identification — part 2: Three examples of tensor-based system identification methods. *Int. J. Sci. Techn. Autom. Control & Comp. Eng. (IJ-STA)* 3(1), 870–889 (2006)
6. Giannakis, G.B., Serpedin, E.: A bibliography on nonlinear system identification. *IEEE Trans. Signal Process.* 81(3), 533–580 (2001)
7. Giri, F., Bai, E.W.: Block-oriented Nonlinear System Identification. *Lecture Notes in Control and Information Sciences*, Springer (2010)
8. Harshman, R.A.: Foundations of the PARAFAC procedure: Model and conditions for an “explanatory” multi-mode factor analysis. *UCLA Working Papers in Phonetics* 16(1), 1–84 (1970)
9. Kibangou, A.Y., Favier, G.: Tensor analysis-based model structure determination and parameter estimation for block-oriented nonlinear systems. *IEEE J. Sel. Top. Signal Process.* 4(3), 514–525
10. Kolda, T.G., Bader, B.W.: Tensor decompositions and applications. *SIAM Rev.* 51(3), 455–500 (September 2009)
11. Ljung, L.: *System Identification: theory for the user*. Prentice Hall PTR, Upper Saddle River, NJ (1999)
12. Palm, G.: On representation and approximation of nonlinear systems. *Biol. Cybern.* 34(1), 49–52 (1979)
13. Pintelon, R., Schoukens, J.: *System Identification: A Frequency Domain Approach*. Wiley-IEEE Press, 2nd edn. (2012)
14. Schetzen, M.: *The Volterra and Wiener Theories of Nonlinear Systems*. Wiley, New York (1980)
15. Schoukens, J., Pintelon, R., T. Dobrowiecki, Y.R.: Identification of linear systems with nonlinear distortions. *Automatica* 41, 491–504 (2005)
16. Schoukens, M., Tiels, K.: Identification of nonlinear block-oriented systems starting from linear approximations: a survey (2016), (preprint arXiv:1607.01217)
17. Sjöberg, J., Zhang, Q., Ljung, L., Benveniste, A., Delyon, B., Glorennec, P.Y., Hjalmarsson, H., Juditsky, A.: Nonlinear black-box modeling in system identification: a unified overview. *Automatica* 31(12), 1691–1724 (1995)
18. Sorber, L., Van Barel, M., De Lathauwer, L.: Structured data fusion. *IEEE J. Sel. Top. Signal Process.* 9 (2015)
19. Van Eeghem, F., De Lathauwer, L.: Algorithms for CPD with block-circulant factors. *Tech. Rep. 16-100*, KU Leuven ESAT/STADIUS (2016)
20. Van Overschee, P., De Moor, B.: *Subspace identification for linear systems: theory, implementation, applications*. Kluwer Academic Publishers, Dordrecht, Netherlands (1996)
21. Vervliet, N., Debals, O., Sorber, L., Van Barel, M., De Lathauwer, L.: Tensorlab 3.0 (2016), available online, Mar. 2016. URL: <http://www.tensorlab.net/>
22. Westwick, D.T., Ishteva, M., Dreesen, P., Schoukens, J.: Tensor factorization based estimates of parallel Wiener-Hammerstein models. *Tech. rep.*, University of Calgary, Calgary, Canada and Vrije Universiteit Brussel, Brussels, Belgium (2016), submitted to IFAC World Congress 2017