Comparing Classical and Relativistic Kinematics in First-Order Logic

and introducing work-in-progress: Conceptual Distances between Formal Theories

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Setting the stage:

Studying axiomatic relativity theory in the "Andréka–Németi School", Algebraic Logic department, Alfréd Rényi Institute of Mathematics

- See "On the logical structure of relativity theories" by H. Andréka, J. X. Madarász, and I. Németi; with contributions from: A. Andai, G. Sági, I. Sain, and Cs. Tőke. Research Report, Budapest 2002. https://old.renyi.hu//pub/algebraic-logic/Contents.html
- Inspired by Alfred Tarski's initiative *Logic, Methodology and Philosophy of Science*, by David Hilbert's Sixth problem *Mathematical Treatment of the Axioms of Physics* and by the Wiener Kreis.

Introduction:

Comparing scientific theories which are axiomatized in "First Order Logic with equality":

- **Qualitative:**
  - Which concepts are similar or different between theories?
- **Quantitative:**
  - How many concepts are different?
  - How far are theories away from each other?

Equivalence between formal theories:

There are several ways in which theories can be equivalent to each other:

Logical Equivalence

$\Downarrow \not\Uparrow$

Definitional Equivalence

$\Downarrow \not\Uparrow$

Conceptual Equivalence (Morita Equivalence, definitional equivalence with different sorts)

$\Downarrow \not\Uparrow$

Categorical Equivalence

See "*Morita Equivalence*" by Thomas William Barrett and Hans Halvorson, The Review of Symbolic Logic 9 (3): 556-582 (2016).

A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

An *interpretation* of theory $\mathbf{Th_1}$ in theory $\mathbf{Th_2}$ is a translation Tr which translates all tautologies and all axioms of $\mathbf{Th_1}$ into theorems of $\mathbf{Th_2}$.

A *definitional equivalence* exists between two theories if those theories can be interpreted in each other and if all formulas from both theories translated into the other theory and back are logical equivalent to the original formulas.

See Definition 4.3.42 on p. 167 in "Cylindric Algebras Part II" by L. Henkin, J. D. Monk, and A. Tarski 1985.

For a comparison between the different definitions of definitional equivalence, see "On Generalization of Definitional Equivalence to Languages with Non-Disjoint Signatures" by Koen Lefever & Gergely Székely, submitted 2018, arXiv:1802.06844

Metric distance:

Let X be a set of theories and let $\equiv$ be an equivalence relation on X. We call a map $d : X \times X \to [0, \infty)$ the conceptual distance on $(X, \equiv)$ if it satisfies the following properties:

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ iff $x \equiv y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, z) \leq d(x, y) + d(y, z)$.

Conceptual distance: three approaches

1. A trivial example: discrete distance
2. An ad-hoc example: adding and removing concepts to make classical kinematics and special relativity equivalent
3. A definition using Cylindric Algebras or Concept Algebras

Trivial example: discrete distance

Let X-be any set of theories and $\equiv$ any equivalence relation on X. The discrete conceptual distance on $(X, \equiv)$ is the following:

$$d(x,y) = \begin{cases} 0 & \text{if } x \equiv y, \\ 1 & \text{if } x \not\equiv y. \end{cases}$$

We define the *Conceptual Distance* between two theories as the minimum number of concepts (i.e. logical predicates) which need to be added to or substracted from one theory to make it definitionally (or logically, categorically, etc...) equivalent to the other theory.

Two theories which are equivalent have a Conceptual Distance of zero.

Adding a concept:

- introduce new primitive predicate R
- add axiom $\exists\bar{x}(\varphi_R(\bar{x}))$

Removing concept $\varphi(\bar{x})$:

- add axiom $\neg\exists\bar{x}(\varphi(\bar{x}))$
- change theory T to a weaker theory T^- to avoid contradictions

Example: Using definitional equivalence to calculate the conceptual distance between classical kinematics and relativistic kinematics

See:

"*Using Logical Interpretation and Definitional Equivalence to Compare Classical Kinematics and Special Relativity Theory*" by Koen Lefever
PhD Dissertation, Vrije Universiteit Brussel, 2017-05-26

"*Comparing Classical and Relativistic Kinematics in First-Order Logic*"
by Koen Lefever and Gergely Székely

Logique et Analyse, Vol. 61 Nr. 241 p. 57-117, 2018

There are translations Tr, Tr_+, and Tr'_+ between the languages of Classical Kinematics and Special Relativity Theory such that:

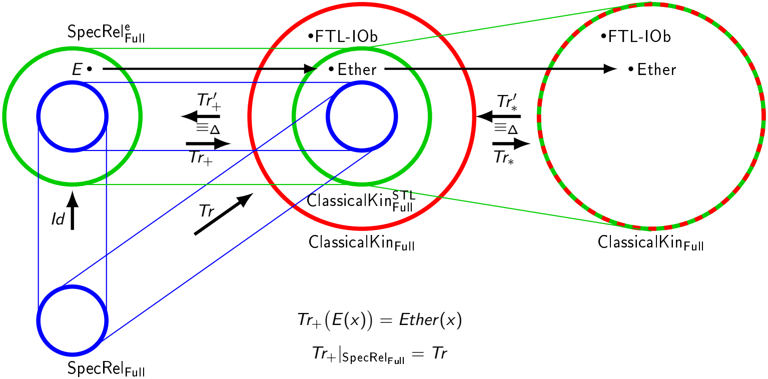
- $ClassicalKin_{Full} \vdash Tr(SpecRel_{Full})$
- $ClassicalKin^{STL}_{Full} \vdash Tr_+(SpecRel^e_{Full})$
- $SpecRel^e_{Full} \vdash Tr'_+(ClassicalKin^{STL}_{Full})$
- Definitional equivalence: $SpecRel^e_{Full} \equiv_\Delta ClassicalKin^{STL}_{Full}$, i.e., Tr_+ and Tr'_+ are inverses of each other up to logical equivalence in $SpecRel^e_{Full}$ and $ClassicalKin^{STL}_{Full}$.

There are translations Tr_*, and Tr'_* between the languages of Classical Kinematics and Classical Kinematics restricted to Slower-Than-Light oberservers such that:

- $ClassicalKin^{STL}_{Full} \vdash Tr_*(ClassicalKin_{Full})$
- $ClassicalKin_{Full} \vdash Tr'_*(ClassicalKin^{STL}_{Full})$
- Definitional equivalence: $ClassicalKin_{Full} \equiv_\Delta ClassicalKin^{STL}_{Full}$,

and hence by transitivity of definitional equivalence:

- $SpecRel^{e}_{Full} \equiv_\Delta ClassicalKin_{Full}$.



$$Tr_{+}(E(x)) = Ether(x)$$

$$Tr_{+}|_{\mathsf{SpecRel}_{\mathsf{Full}}} = Tr$$

$\textbf{Kin} := \{\text{AxEField}, \text{AxEv}, \text{AxSelf}, \text{AxSymD}, \text{AxLine}, \text{AxTriv}, \text{AxNoAcc}\}$

$\textbf{ClassicalKin}_{\textit{Full}} := \textbf{Kin} \cup \{\text{AxEther}, \text{AbsTime}, \text{AxThExp}_{+}\}$

$\textbf{SpecRel}_{\textit{Full}} := \textbf{Kin} \cup \{\text{AxPh}_{c}, \text{AxThExp}\}$

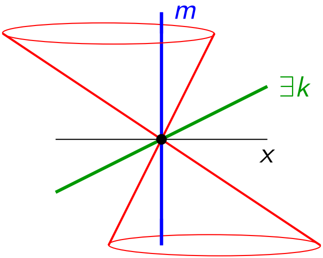
Kin:

- AxEField: The structure of quantities $\langle Q, +, \cdot, \leq \rangle$ is an Euclidean field,
- AxEv: Inertial observers coordinatize the same events (meetings of bodies).
- AxSelf: Every inertial observer is stationary according to himself.
- AxSymD: Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them.
- AxLine: The worldlines of inertial observers are straight lines according to inertial observers.
- AxTriv: Any trivial transformation of an inertial observer is also an inertial observer.
- AxNoAcc: All observers are inertial observers.

ClassicalKin$_{Full}$:

- AxAbsTime: The time difference between two events is the same for all inertial observers.
- AxEther: There exists an inertial observer in which the light cones are right.
- AxThExp$_+$: Inertial observers can move along any non-horizontal straight line:

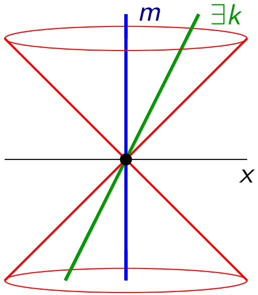
$$(\exists h \in B)\big[IOb(h)\big] \wedge$$
$$(\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)\ \big(x_0 \neq y_0 \rightarrow \big(\exists k' \in IOb\big)\big[\bar{x}, \bar{y} \in wl_k(k')\big]\big)\ .$$



SpecRel_{Full}:

- AxPh: For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.
- AxThExp: Inertial observers can move with any speed slower than that of light:

$$(\exists h \in B)\big[IOb(h)\big] \land (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in \mathrm{Q}^4\big)$$
$$\big(space(\bar{x}, \bar{y}) < \mathfrak{c} \cdot time(\bar{x}, \bar{y}) \rightarrow (\exists k' \in IOb)\big[\bar{x}, \bar{y} \in wl_k(k')\big]\big).$$



$\mathbf{Kin}$:={AxEField, AxEv, AxSelf, AxSymD, AxLine, AxTriv, AxNoAcc}

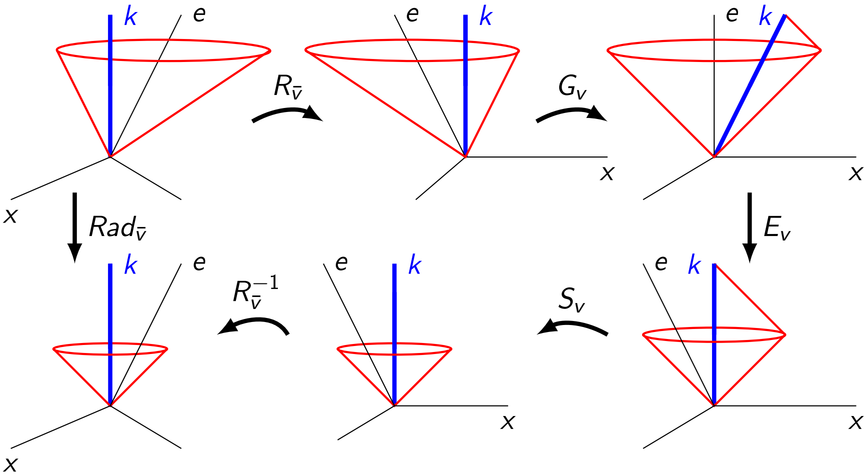
$\mathbf{ClassicalKin}_{\mathit{Full}}$:=$\mathbf{Kin} \cup$ {AxEther, AbsTime, $\mathrm{AxThExp}_{+}$}

$\mathbf{SpecRel}_{\mathit{Full}}$:= $\mathbf{Kin} \cup$ {AxPh_{c}, AxThExp}

Justification Theorems:

$\mathbf{ClassicalKin}_{\mathit{Full}} \vdash$ Worldview transformations are Galilean transformations.

$\mathbf{SpecRel}_{\mathit{Full}} \vdash$ Worldview transformations are Poincaré transformations
(see Theorem 2.1, p.639 in "A logic road from special relativity to general relativity" by H. Andréka, J. X. Madarász, I. Németi, and G. Székely, in *Synthese*, Vol. 186, Nr. 3, pages 633–649, 2012).



$$Rad_{\bar{v}} = R_{\bar{v}}^{-1} \circ S_v \circ E_v \circ G_v \circ R_{\bar{v}}$$

$$Tr(a+b=c) \stackrel{\text{def}}{\equiv} a+b=c$$

$$Tr(a \cdot b=c) \stackrel{\text{def}}{\equiv} a \cdot b=c$$

$$Tr(a<b) \stackrel{\text{def}}{\equiv} a<b$$

$$Tr\big(\mathsf{W}^{SR}(k,b,t,x,y,z)\big) \stackrel{\text{def}}{\equiv}$$
$$\exists t'x'y'z'[\mathsf{W}^{CK}(k,b,t',x',y',z') \wedge Rad_k(t',x',y',z')=(t,x,y,z)]$$

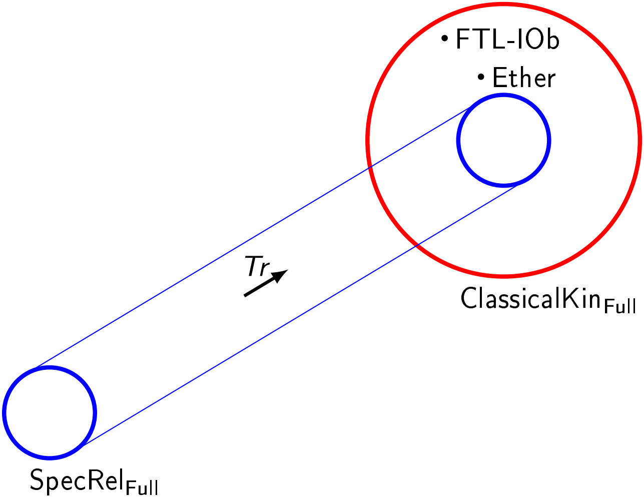
$$Tr\big(\mathsf{IOb}^{SR}(k)\big) \stackrel{\text{def}}{\equiv} \mathsf{IOb}^{CK}(k) \wedge \forall e[Ether(e) \rightarrow speed_e^{CK}(k) < \mathfrak{c}_{\mathfrak{e}}]$$

where

$$Ether(e) \stackrel{def}{\iff} \mathsf{IOb}^{CK}(e) \wedge \forall p[Ph(p) \rightarrow speed_e^{CK}(p) = \mathfrak{c}_{\mathfrak{e}}]$$

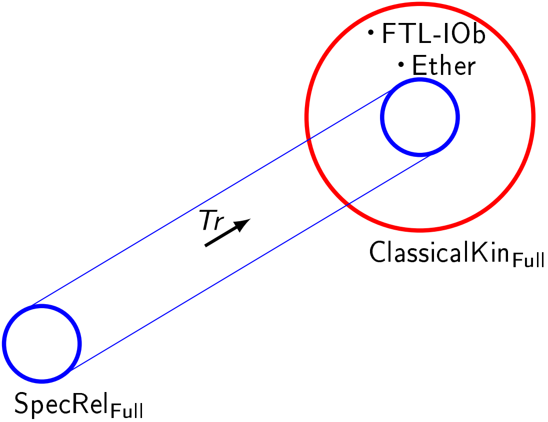
Theorem:

There is an interpretation Tr of **SpecRel**$_{\textit{Full}}$ *in* **ClassicalKin**$_{\textit{Full}}$.



Theorem:

There is no definitional equivalence between **SpecRel**$_{Full}$ *and* **ClassicalKin**$_{Full}$.



$Conceptual\_distance(\mathbf{SpecRel}_{Full}, \mathbf{ClassicalKin}_{Full}) > 0$

Adding and removing concepts to make both theories equivalent:

- Removing FTL observers from $\mathbf{ClassicalKin}_{\mathit{Full}}$
- Adding a "primitive ether" to $\mathbf{SpecRel}_{\mathit{Full}}$

$\mathsf{ClassicalKin}^{STL}_{Full} :=$
$\mathsf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\mathsf{STL}}, \mathsf{AxNoFTL}\}$

AxNoFTL :

All inertial observers move slower than light with respect to the ether frames.

$$\neg\exists m\big(IOb(m) \wedge \exists e[Ether(e) \wedge Speed^{CK}_{e}(m) \geq \mathfrak{c}_{\mathfrak{e}}]\big).$$

$\mathsf{AxThExp}^{\mathsf{STL}}$:

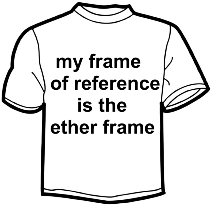
Inertial observers can move with any speed which is in the ether frame slower than that of light.

$$\exists h\ \big(IOb(h)\big) \wedge \forall e\bar{x}\bar{y}\big(Ether(e) \wedge \mathsf{space}(\bar{x},\bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot \mathsf{time}(\bar{x},\bar{y}) \\ \rightarrow \exists k\ \ IOb(k) \wedge W(e,k,\bar{x}) \wedge W(e,k,\bar{y})\big).$$

$$\textbf{SpecRel}^{e}_{Full} := \textbf{SpecRel}_{Full} \cup \{\text{AxPrimitiveEther}\}$$

AxPrimitiveEther :

There is a non-empty class of ether observers, stationary with respect to each other, which is closed under trivial transformations.

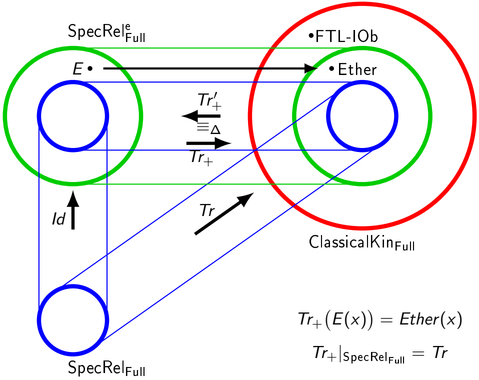


$$\exists e\Big(E(e) \wedge \forall k\big[[IOb(k) \wedge (\exists T \in Triv) w^{SR}_{ek} = T] \leftrightarrow E(k)\big]\Big)$$

Theorem:

There is an interpretation Tr'_+ *of* **ClassicalKin**$^{STL}_{Full}$ *in* **SpecRel**$^{e}_{Full}$.

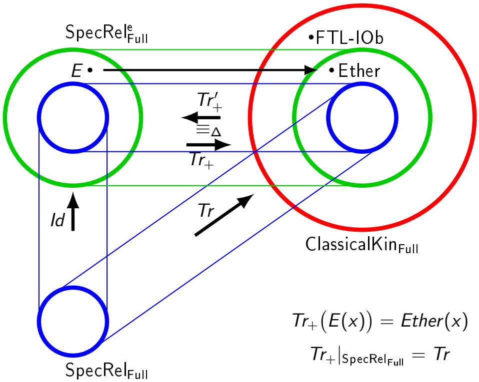
$$\textbf{SpecRel}^{e}_{Full} \vdash Tr'_+(\textbf{ClassicalKin}^{STL}_{Full})$$



Theorem:

Tr_+ and Tr'_+ are a definitional equivalence between $\mathsf{SpecRel}^e_{Full}$ *and* $\mathsf{ClassicalKin}^{STL}_{Full}$.

$Tr'_+ \circ Tr_+(\mathsf{SpecRel}^e_{Full}) \Leftrightarrow \mathsf{SpecRel}^e_{Full}$ and $Tr_+ \circ Tr'_+(\mathsf{ClassicalKin}^{STL}_{Full}) \Leftrightarrow \mathsf{ClassicalKin}^{STL}_{Full}$



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\mathsf{SpecRel}_{Full}} = Tr$$

$Conceptual\_distance(\mathsf{SpecRel}_{Full}, \mathsf{ClassicalKin}_{Full}) \leq 2$

Theorem:

There is an interpretation Tr_* of $\mathsf{ClassicalKin}^{STL}_{Full}$ *in* $\mathsf{ClassicalKin}_{Full}$.

$$\mathit{ClassicalKin}_{Full} \vdash Tr_*(\mathit{ClassicalKin}^{STL}_{Full})$$

Theorem:

There is an interpretation Tr'_* of $\mathsf{ClassicalKin}_{Full}$ *in* $\mathsf{ClassicalKin}^{STL}_{Full}$.

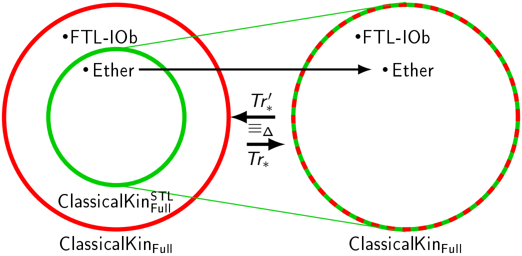
$$\mathit{ClassicalKin}^{STL}_{Full} \vdash Tr'_*(\mathit{ClassicalKin}_{Full})$$

Theorem:

Tr_* and Tr'_* are a definitional equivalence between $\mathbf{ClassicalKin}^{STL}_{Full}$ *and* $\mathbf{ClassicalKin}_{Full}$.

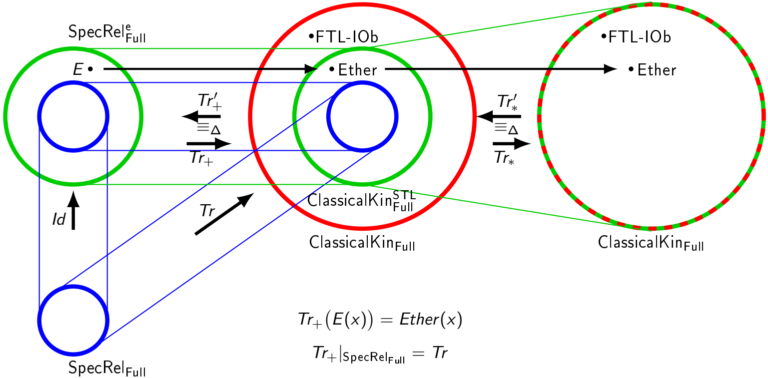
$Tr'_* \circ Tr_*(\mathbf{ClassicalKin}^{STL}_{Full}) \Leftrightarrow \mathbf{ClassicalKin}^{STL}_{Full}$ and

$Tr_* \circ Tr'_*(\mathbf{ClassicalKin}_{Full}) \Leftrightarrow \mathbf{ClassicalKin}_{Full}$



Theorem:

$Tr_* \circ Tr_+$ and $Tr'_+ \circ Tr'_*$ are a definitional equivalence between **$\mathsf{SpecRel}^e_{Full}$** *and* **$\mathsf{ClassicalKin}_{Full}$**.



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\mathsf{SpecRel}_{\mathsf{Full}}} = Tr$$

$Conceptual\_distance(\mathbf{SpecRel}_{Full}, \mathbf{ClassicalKin}_{Full}) = 1$

Definition by Mohamed Khaled using Cylindric Algebras / Concept Algebras:

If $\mathfrak{A} \subset \mathfrak{B}$, then

$$d(\mathfrak{A}, \mathfrak{B}) = min\Big\{n : \exists b_1 \ldots b_n \text{ such that } \langle \mathfrak{A}, b_1 \ldots b_n \rangle = \mathfrak{B}\Big\}$$

where

- $\mathfrak{A} \subset \mathfrak{B}$ means that $\mathfrak{A}$ is a embedable in $\mathfrak{B}$,
- $b_1 \ldots b_n$ are the minimal elements to be added to $\mathfrak{A}$ to generate $\mathfrak{B}$.

For arbitrary algebras $\mathfrak{A}$ and $\mathfrak{B}$:

$$d(\mathfrak{A}, \mathfrak{B}) = min\left\{\sum_{i=0}^{N} d(\mathfrak{B}_i, \mathfrak{B}_{i+1}) : \mathfrak{B}_0 = \mathfrak{A}, \mathfrak{B}_i \sim \mathfrak{B}_{i+1}, \mathfrak{B}_N = \mathfrak{B}\right\},$$

where $\sim$ means $\subset$ or $\supset$.

Upcoming paper: generalization to calculate distances between formal theories, groups, vector spaces, ordered fields, . . .

"Distances Between Formal Theories"
by Michèle Friend, Mohamed Khaled, Koen Lefever & Gergely Székely

Proposed further research:

- What is the conceptual distance between
  - Classical dynamics,
  - Newton-Cartan Theory, and
  - General Relativity Theory?
- Jean Paul Van Bendegem: What is the conceptual distancte between
  - Classical thermodynamics and
  - Statistical thermodynamics?
    - see “Untersuchungen über die Grundlagen der Thermodynamik” by C. Carathéodory, Math. Ann. 64, p. 355–386, 1909