Latin Squares vs. Permutation Arrays

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Latin squares of order \(n\)

An \(n \times n\) matrix in which \(n\) distinct symbols from a symbol set \(S\) are arranged, such that each symbol occurs exactly once in each row and in each column.

Reduced? First row and first column is in the natural order of the symbol set we choose.

Mutually orthogonal latin squares of order \(n\)

- Take two latin squares of order \(n\) \(L_1\) and \(L_2\).
- Superimpose \(L_1\) and \(L_2\).
- All of the \(n^2\) ordered pairs \((i,j)\).
- Mutually orthogonal!

Notation: MOLS.

- Take a set of latin squares \(L_1, L_2, \ldots, L_k\).
- \(L_i\) and \(L_j\) mutually orthogonal for all \(1 \leq i < j \leq k\).
- Set of MOLS?

Reduced set of MOLS?

One latin square is reduced.

First row in every other latin square in this set is in the natural order of the symbol set we chose.

Generalized Room Square Packing

\(T(n, k, l)\) of size \(n\) over a set \(X\) of \(n\) elements and index \(l\).

\(X\) is an \(n \times n\) array satisfying the following properties:

- every cell in the array is a subset of \(X\).
- each symbol of \(X\) occurs exactly once in each row and in each column.

Any two distinct symbols of \(X\) occur together in at most \(t\) cells of the array.

Hamming distance

The Hamming distance \(d_H\) between two permutations \(\alpha, \beta \in S_n\) is defined as,

\[d_H(\alpha, \beta) = |\{k \in \{0, 1, \ldots, n-1\} | \alpha_k \neq \beta_k\}| \]

Permutation array

\(PA(n,d)\) of size \(n\) over \(n\) symbols and of minimum distance \(d\), is an \(n \times n\) array satisfying the following properties:

- each row is a permutation of the symbols of \(S\).
- the distance between two rows is at least equal to \(d\).

Orthogonal array

\(OA(s, k, n)\), with \(n\) levels, \(s\) \(k\)-ary and index \(l\) is an \(s \times k\) array \(A\) with entries from a symbol set \(S\) of cardinality \(n\) satisfying the following properties:

- every \(s \times 1\) subarray of \(A\) contains each \(k\)-tuple based on \(S\) exactly \(l\) times as a row.

Isometry

An isometry on the metric space \((S_n, d)\) is a distance-preserving map, \(f: S_n \rightarrow S_n\). This means, \(\forall \alpha, \beta \in S_n\)

\[d(f(\alpha), f(\beta)) = d(\alpha, \beta)\]

\(ISO(S_n, d):\) set of all isometries on the metric space \((S_n, d)\).

Subgroups of \(ISO(S_n, d)\)

- \(Z = (\{\alpha\} \subseteq S_n)\), the set of left multiplications,
- \(R = (\{s \subseteq S_n\})\), the set of right multiplications,
- \(S\), the group generated by the inverse \(s\),
- \(L = (\{\sigma \subseteq S_n\})\), the group generated by the conjugates.

Bi-invariant distance

Is \(d\) a bi-invariant distance?

- \(ISO(S_n, d) = (\mathbb{C} \times \mathbb{R}) = \mathbb{C}\)

Every isometry \(f \in ISO(S_n, d)\) can be uniquely written as \(\lambda f \circ g\), with \(\lambda \in \mathbb{C}\) and \(g, \beta \in S_n\).

Isomorphic permutation arrays

Consider two permutation arrays \(P_1\) and \(P_2\) of length \(n\).

\(IS\) isomorphic\(\iff IS(L_1) = IS(L_2) = (\mathbb{C} \times \mathbb{C}) = \mathbb{C}\)

Every isometry \(f \in IS(S_n, d)\) can be uniquely written as \(\lambda f \circ g\), with \(\lambda \in \mathbb{C}\) and \(g, \beta \in S_n\).

MOLS vs. permutation arrays

Suppose we have a set of \(q = 1\) MOLS of order \(q\) and an associated permutation array \(PA(q, q-1)\),

- Permutation of the rows in the set of MOLS \(\rightarrow\) permutation on the alphabet in \(P\).
- Permutation of the columns in the set of MOLS \(\rightarrow\) permutation of the columns in \(P\).
- Permutation on the alphabet in the set of MOLS \(\rightarrow\) permutation of the rows in \(P\).

We can easily see that this permutation array is isometric to the previous permutation array.

Edit distances

Hamming distance: most commonly used edit distance.

Other examples? Cayley distance, Levenshtein distance, Ulam-Levenshtein distance, longest common subsequence, …

General case: edit operations \(\rightarrow\) edit distance.

Edit operations: addition \(a\), deletion \(d\), substitution \(s\), transposition of two adjacent characters \(p\) …

\(O(\cdot, \cdot)\): set of edit operations.

Edit distance \(\rightarrow\) subset of \(O\).

Example: Hamming distance \(\rightarrow\) subset \([s, d, p]\) of \(O\).

Levenshtein distance

For any \(a, b \subseteq S_n\), the Levenshtein distance \((LD)\) is the minimum cost of a sequence of editing steps required to convert \(a\) into \(b\). These editing steps are operations from the subset \([s, d, p]\) of \(O\).

Example: \(LD(\{0, 1, 2, 3\} \rightarrow \{1, 2, 0, 3\}) = 5\).

Equivalent edit distances

Size of a permutation array for given length and (Hamming) distance \(\rightarrow\) a lot of bounds found by linear programming and semidefinite programming.

Results for other edit distances?

Good lower and upper bounds?

Exact size of permutation array for given length and distance?

Contact

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