Detecting and Quantifying the Nonlinear and Time-Variant Effects in FRF Measurements Using Periodic Excitations

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Abstract—Frequency response function (FRF) measurements are very often used to get quickly insight into the dynamic behavior of complex systems; even if it is known that these systems are only approximately linear and time-invariant. Therefore, it is important to detect and quantify the deviation from the ideal linear time-invariant framework that is inherent to the concept of an FRF. This paper presents a method to detect and quantify the nonlinear and time-variant effects in FRF measurements using periodic excitations. The proposed method can handle noisy input, noisy output data and nonlinear time-variant systems operating in feedback.

Index Terms—frequency response function, best linear time-invariant approximation, slowly time-varying, nonlinear distortions, feedback, errors-in-variables

I. INTRODUCTION

Frequency response function (FRF) measurements are used in all kinds of engineering applications to get quickly insight in the dynamic behavior of complex systems [1]–[3]. However, a lot of real life systems do not satisfy the linearity and time-invariance assumptions inherent to the concept of an FRF. Think of, for example, pit corrosion of aluminum [4], myocardial tissue impedance measurements [5], flight flutter prediction [6], crane dynamics [7]. . . . In all these applications the true dynamics are nonlinear and time-variant. Therefore, there is a real need for methods that can detect and quantify the deviation from the ideal linear time-invariant (LTI) dynamic behavior. Indeed, from the FRF measurement one should be able to decide whether the LTI framework is accurate enough or not for describing the true system dynamics in a given application.

While the impact of nonlinear distortions on FRF measurements is well understood and is described in detail in the literature [8]–[19], the effect of time-variation on FRF measurements has only recently been studied [20]–[22]. Moreover, the combined effect of nonlinear distortions and time-variation in FRF measurements has never been analyzed [23] and [24] handle the parametric estimation of LTI approximations of nonlinear time-variant systems without detecting and quantifying the nonlinear distortions and the time-variation). The goal of this paper is to fill this important gap.

To achieve this goal, the following contributions are made:

1) A relevant class of nonlinear time-variant systems is proposed. The basic assumption made is that the system response can be split into two contributions: a nonlinear time-invariant part, and a linear time-variant part. Using this assumption it is shown that the properties of the nonlinear distortion and the time-variation are the same as for, respectively, nonlinear time-invariant systems and linear time-variant systems.

2) It is proven that the nonlinear distortion is uncorrelated with the time-variation, which is a non-trivial result because both depend on the input. This key property allows one to distinguish the nonlinear distortion from the time-variation.

3) A method is presented to detect and quantify the nonlinear and time-variant effects in FRF measurements using random phase multisines (sum of harmonically related sinewaves with user specified amplitudes and random phases [3]). It is shown that two periods of the (transient) response to one random phase multisine excitation is enough to measure the FRF, and to quantify the disturbing noise, the time-variation, the non-linear distortion, and the leakage error. The proposed method is a non-trivial combination and generalization of the algorithms in [12], [19], [21], [22] and [25].

The paper is organized as follows. For notational simplicity, it is first assumed that the input is known and that the system operates in open loop (Sections II–V). Next, the proposed method is generalized to noisy input, noisy output observations of systems operating in closed loop (Section VI). The class of nonlinear time-variant systems for which the method applies is defined in Section II. Next, the concept of the best linear time-invariant (BLTI) approximation of a nonlinear slowly time-varying system is introduced (Section III). Further, an LTI equivalent model of a nonlinear slowly time-varying system is constructed (Section IV). This LTI equivalent model is then used to estimate the BLTI approximation, and the levels of the noise, the time-variation, and the nonlinear distortion (Section V). The proposed measurement procedure imposes some constraints on the experimental setup (signal generation and data acquisition). This issues are discussed in detail in Section VII. Section VIII illustrates the theory on two different time-variant electronic circuits operating in open and closed loop respectively. Finally, some conclusions are drawn (Section IX).

II. CLASS OF NONLINEAR SLOWLY TIME-VARYING SYSTEMS

This section defines the class of nonlinear slowly time-varying systems considered. First, the concept of a linear slowly time-
A linear slowly time-varying system is introduced. Next, it is combined with a specific class of nonlinear systems.

A. Linear Slowly Time-Varying

A linear time-variant system is uniquely characterized by its response $g(t, \tau)$ to a Dirac impulse at time instant $\tau$. The time-variant transfer function $G(s, t)$, defined as

$$G(s, t) = L_{\tau}\{g(t, t - \tau)\}$$

with $L_{\tau}\{}$ the Laplace transform w.r.t. $\tau$, has similar properties as the transfer function of a linear time-invariant system. For example, the steady state response to $\sin(\omega_0 t)$ equals $\{G(j\omega_0, t)[\sin(\omega_0 t + ZG(j\omega_0, t))\}$, and the transient response $y(t)$ to an excitation $u(t)$ is obtained as

$$y(t) = L^{-1}\{G(s, t)U(s)\}$$

with $L^{-1}\{}$ the inverse Laplace transform, and $U(s)$ the Laplace transform of $u(t)$ (see [26]–[28]).

Over a finite time interval $[0, T]$, the time-variant transfer function $G(s, t)$ can be expanded in series as

$$G(s, t) = \sum_{p=0}^{\infty} G_p(s) f_p(t) \quad t \in [0, T]$$

with $f_p(t)$, $p = 0, 1, \ldots$, a complete set of basis functions. Without any loss of generality, the basis functions $f_p(t)$ can always be chosen such that

$$f_0(t) = 1 \text{ and } \frac{1}{T} \int_0^T f_p(t) dt = 0 \text{ for } p > 0$$

In this paper, the system is called slowly time-varying if the infinite sum in (3) can be replaced by a finite sum.

**Definition 1.** The time-variant transfer function $G(s, t)$ of a linear slowly time-varying system can be written as

$$G(s, t) = \sum_{p=0}^{N_b} G_p(s) f_p(t) \quad t \in [0, T]$$

where $N_b < \infty$, and where $f_p(t)$ is a polynomial of degree $p$ satisfying (4).

Combining (2) and (5) gives the block diagram shown in Fig. 1. It follows that the slowly time-varying system (5) can be decomposed into a time-invariant branch with transfer function $G_0(s)$, and $N_b$ time-variant branches with time-invariant gains $f_p(s)$ and time-variant output gains $f_p(t)$ ($p = 1, 2, \ldots, N_b$).

B. Nonlinear Slowly Time-Varying

The basic assumption made is that only the time-invariant branch of the slowly time-varying system shown in Fig. 1 behaves nonlinearly. In addition, it is assumed that the output of the nonlinear branch can be approximated arbitrarily well in mean square sense by a Volterra series. Such nonlinear systems have the property that the steady state response to a periodic input is periodic with the same period as the input [29]. Therefore, this class of nonlinear systems is denoted by PISPO (period in, same period out). The PISPO class includes hard nonlinearities such as saturation, clipping, and dead zones; but excludes phenomena such as chaos and subharmonics.

**Assumption 2.** The nonlinear (NL) slowly time-varying system consists of an NL PISPO time-invariant branch, and $N_b$ linear time-variant branches (see Fig. 2, top block diagram).

Note that Assumption 2 excludes any interaction between the nonlinear and the time-varying behavior. This rather severe assumption is needed to separate the nonlinear distortions from the time-variation in frequency response function measurements using periodic excitations (see Section V). In practice, this assumption is fulfilled if the time-variation is small enough in order not to excite the nonlinearities in the time-variant branches (see Section VIII).
III. THE BEST LINEAR TIME-INVARIANT APPROXIMATION

First, we recall the definition and the properties of the best linear time-invariant (BLTI) approximation of a linear time-variant (LTV) system. Next, the definition is applied to the class of nonlinear time-variant systems satisfying Assumption 2. Finally, the properties of the difference between the actual output \( y(t) \) of the nonlinear time-variant system and the output \( y_{\text{BLTI}}(t) \) predicted by the BLTI model are discussed. It is shown that the output residual \( y(t) - y_{\text{BLTI}}(t) \) can be split into the sum of two mutually uncorrelated contributions: the nonlinear distortions and the time-variation.

A. Linear Slowly Time-Varying

The best linear time-invariant (BLTI) approximation of a linear time-variant system over the time interval \([0, T]\) is defined as

\[
G_{\text{BLTI}}(s) = \frac{1}{T} \int_0^T G(s, t) dt
\]

Combining (3), (4), and (6) it follows that the BLTI approximation equals the dynamics of the time-invariant branch: \( G_{\text{BLTI}}(s) = G_0(s) \). Under the following assumption the relationship between the BLTI approximation (6) and the spectral analysis estimate [1] using cross- and auto-power spectra can be established.

**Assumption 3.** The input \( u(t) \) is a random phase multisine

\[
u(t) = \sum_{k=1}^F A_k \sin(k\omega_0 t + \phi_k)
\]

with \( A_k \geq 0 \) user defined deterministic amplitudes, \( \phi_k \) random phases chosen such that \( \mathbb{E}[e^{j\phi_k}] = 0 \) (e.g. \( \phi_k \) is uniformly distributed over \([0, 2\pi]\)), \( \omega_0 = 2\pi f_0 \), and \( f_0 T \in \mathbb{N}_0 \) (an integer number of excitation periods are measured). The linear time-invariant systems \( G_p(s) \), \( p = 0, 1, \ldots, N_b \), operate in steady state.

Under Assumption 3, it has been shown in [20] that the BLTI approximation (6) equals the spectral analysis estimate

\[
G_{\text{BLTI}}(j\omega_k) = \frac{\mathbb{E}\{Y(k)\overline{U}(k)\}}{\mathbb{E}\{|U(k)|^2\}}
\]

where the expected values are taken w.r.t. the random realization of \( u(t) \), and with \( \overline{U}(k) \) the complex conjugate of \( U(k) \). \( U(k) \) and \( Y(k) \) are the discrete Fourier transform (DFT) spectra of \( u(t) \) and \( y(t) \), respectively

\[
X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n + k) e^{-j2\pi n k / N}
\]

with \( T_s \) the sampling period chosen such that \( f_{f_0} < 1/(2T_s) \) (Nyquist condition); \( N \) the number of samples \( (NT_s = T) \); \( X = U, Y \); and \( x = u, y \). The difference \( y_{\text{TV}}(t) \) between the actual output \( y(t) \) of the linear time-variant system, and the output \( y_{\text{BLTI}}(t) \) of the BLTI approximation (6) is given by

![Figure 3. Best linear time-invariant (BLTI) approximation of a nonlinear time-variant (NLTV) system satisfying Assumption 2.](image)

\[
y_{\text{TV}}(t) = y(t) - y_{\text{BLTI}}(t) = \sum_{p=1}^{N_b} y_p(t)
\]

where

\[
y_p(t) = f_p(t) \sum_{k=1}^F (A_k [G_p(j\omega_k) \times \sin(\omega_k t + \phi_k + \angle G_p(j\omega_k))])
\]

with \( \omega_k = k\omega_0 \). Since the DFT \( Y_{\text{TV}}(k) \) of \( y_{\text{TV}}(t) \) has zero mean value and is uncorrelated with – but not independent of – the input DFT \( U(k) \) (proof: see [20]), (6) is the best – in least squares sense – LTI approximation of the linear time-variant system.

B. Nonlinear Slowly Time-Varying

First note that the best linear approximation (BLA) of a nonlinear period in, same period out (NL PISO) system, excited by a random phase multisine in periodic steady state is defined as

\[
G_{\text{BLA}}(j\omega_k) = \frac{\mathbb{E}\{Y(k)\overline{U}(k)\}}{\mathbb{E}\{|U(k)|^2\}}
\]

(see [3], [11], [19]). The difference \( y_s(t) \) between the actual output \( y_0(t) \) of the NL PISO system (see Fig. 2, top branch of the top block diagram), and the output \( y_{\text{BLA}}(t) \) of the BLA (12)

\[
y_s(t) = y_0(t) - y_{\text{BLA}}(t)
\]

is uncorrelated with – but not independent of – the input \( u(t) \) (proof: see [3], [8], [12]). It shows that (12) is the best – in least squares sense – linear approximation of the NL PISO system.

Combining (8) and (12) proves that the BLTI approximation of a nonlinear slowly time-varying system (Assumption 2) excited by a random phase multisine (Assumption 3), is equal to the BLA of the time-invariant nonlinear branch (see Fig. 2, top block diagram). This results in the bottom block diagram of Fig. 2, where \( G_0(s) \) stands for the BLA of the NL PISO branch, and with \( y_0(t) \) the stochastic nonlinear distortion (13).
C. Properties of the Output Residual

Shifting the output nonlinear distortion \( y_s(t) \) in the top branch of the bottom block diagram in Fig. 2 to the output, gives the block diagram shown in Fig. 3, where \( G_{\text{BLTI}}(s) = G_0(s) \) (combine (4), (5), (6), (8), and (12)), and where \( y_{TV}(t) \) and \( y_s(t) \) are defined in, respectively, (10) and (13).

Under Assumption 3, the output time-variation \( y_{TV}(t) \) and its DFT \( Y_{TV}(k) \) have the following properties:

1. \( Y_{TV}(k) \) has zero mean value: \( \mathbb{E}\{Y_{TV}(k)\} = 0 \).
2. \( Y_{TV}(k) \) is uncorrelated with – but not independent of – \( U(k) \): \( \mathbb{E}\{Y_{TV}(k)\overline{U}(k)\} = 0 \).
3. \( \text{var}(Y_{TV}(k)) \) is a continuous function of the frequency with continuous (higher order) derivatives.
4. \( Y_{TV}(k) \) is correlated over the frequency.
5. \( Y_{TV}(k) \) is not asymptotically \( (F = O(N) \to \infty) \) normally distributed.
6. \( y_{TV}(t) \) is non-periodic and depends linearly on the input \( u(t) \) (linear time-variant relationship; see (7), (10), and (11)).

(proof: see [20] for Properties 1–4, and 6; see Appendix A for Property 5). Properties 1–4 show that \( y_s(t) \) acts as frequency correlated noise on frequency response function measurements, that might wrongly be attributed to filtered white noise. However, since \( y_s(t) \) is harmonically related to the input \( u(t) \) (Property 6), while filtered white noise is independent of the input, it is possible to distinguish both error sources (see Section V).

If the number of excited frequencies in the random phase multisine (7) with fixed rms value increases to infinity in a user possible to distinguish both error sources (see Section IV).

Under Assumption 3, the output time-variation \( y_{TV}(t) \) and its DFT \( Y_{TV}(k) \) have the following properties:

1. \( Y_{S}(k) \) has zero mean value: \( \mathbb{E}\{Y_{S}(k)\} = 0 \) for \( k \neq 0 \).
2. \( Y_{S}(k) \) is uncorrelated with – but not independent of – \( U(k) \): \( \mathbb{E}\{Y_{S}(k)\overline{U}(k)\} = 0 \).
3. \( \text{var}(Y_{S}(k)) \) is a continuous function of the frequency with continuous (higher order) derivatives.
4. \( Y_{S}(k) \) is asymptotically \( (F = O(N) \to \infty) \) uncorrelated over the frequency.
5. \( Y_{S}(k) \) is asymptotically \( (F = O(N) \to \infty) \) normally distributed.
6. \( y_{S}(t) \) is harmonically related to the input \( u(t) \) (contains no subharmonics).

(proof: see [3], [8], [12]). Properties 1–5 show that \( y_{S}(t) \) acts as noise on frequency response function measurements, that might wrongly be attributed to filtered white noise. However, since \( y_{S}(t) \) is harmonically related to \( u(t) \) (Property 6), while filtered white noise is independent of the input, it is possible to distinguish both error sources (see Section V).

Finally, since \( y_{TV}(t) \) depends linearly on the input \( u(t) \), and since the cross-correlation between \( Y_{S}(k) \) and \( Y_{TV}(k) \) is zero

\[
\mathbb{E}\{Y_{S}(k)\overline{Y}_{TV}(k')\} = 0 \text{ for any } k, k' \neq 0, N/2
\]

(14)

\[
\begin{align*}
G_{\text{BLTI}}(s) = H_0(s) + \frac{2}{T} \sum_{p=0}^{\left\lfloor \frac{N_0}{2} \right\rfloor} H_2^{(1)}(s) + \\
4 \frac{T^2}{\beta_{2p}} \sum_{p=1}^{\left\lfloor \frac{N_0}{2} \right\rfloor} H_2^{(2)}(s) + O(T^{-3})
\end{align*}
\]

(15)

IV. LINEAR TIME-IN Variant Equivalent of A NONLINEAR SLOWLY TIME-VARYING SYSTEM

First, we recall that a single-input, single-output linear slowly time-varying system satisfying Definition 1 can be modeled exactly by a multiple-input, single-output time-invariant model. Further, the relationship between the input-output DFT spectra of such a system operating in periodic steady state (Assumption 3) is established. Next, this relationship is generalized to nonlinear slowly time-varying systems satisfying Assumption 2.

A. Linear Slowly Time-Varying

In [21] it has been shown that the slowly time-varying system (5) (see also Fig. 1) can be modeled exactly by the multiple-input, single-output (MISO) linear time-invariant (LTI) model shown in Fig. 4. Using Legendre polynomials \( P_p(2t/T - 1), \ t \in [0, T], \) satisfying (4) (see [30]) as basis functions \( (f_p(t) = P_p(2t/T - 1)), \) the best linear time-invariant approximation \( G_{\text{BLTI}}(s) = G_0(s) \) of the time-variant system (5) is related to the transfer functions \( H_p(s), \ p = 0, 1, \ldots, N_b, \) of the MISO time-invariant equivalent model as
Assuming that the linear time-variant (LTV) system operates in periodic steady state (Assumption 3), the relationship between the input-output DFT spectra of the LTV system is given by

\[ Y(k) = G_0(j\omega_k)U(k) + Y_{TV}(k) \]  
(16)

with \( \omega_k = 2\pi k f_s / N \) (\( f_s = 1/T_s \)), \( Y_{TV}(k) \) the DFT (9) of \( y_{TV}(t) \) (10)

\[ Y_{TV}(k) = \sum_{p=1}^{N_b} Y_p(k) \]
(17)

\[ Y_p(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} F_p(k-l) U(l) G_p(j\omega_l) \]
(18)

and \( Y_p(k) \) and \( F_p(k) \) the DFT of \( y_p(t) \) (11) and \( f_p(t) \), respectively. Under the same periodic steady state assumption, the input-output DFT spectra of the MISO LTI model shown in Fig. 4 are related as

\[ Y(k) = H_0(j\omega_k)U(k) + \sum_{p=1}^{N_b} H_p(j\omega_k) U_p(k) + T_H(j\omega_k) \]
(19)

with \( U_p(k) \) the DFT of \( u_p(t) \) = \( \{u(t), f_p(t)\} \), and \( T_H(j\omega_k) \) the transient (leakage) term due to the non-periodicity of \( u_p(t) \), \( p = 1, \ldots, N_b \) [3], [31]. \( T_H(j\omega_k) \) decreases to zero as an \( O(T^{-1/2}) \). Subtracting (19) from (16) gives an explicit expression for \( Y_{TV}(k) \) as a function of the frequency response functions \( H_p(j\omega) \) of the MISO LTI model

\[ Y_{TV}(k) = (H_0(j\omega_k) - G_0(j\omega_k))U(k) + \sum_{p=1}^{N_b} H_p(j\omega_k) U_p(k) + T_H(j\omega_k) \]
(20)

where \( G_0(j\omega_k) = G_{BLTI}(j\omega_k) \) is calculated as in (15).

**Discussion**

- The sum in (20) is the dominant term; the first and the last term being an \( O(T^{-1/2}) \) and \( O(T^{-1/2}) \), respectively.
- Eq. (20) shows the duality between transient (leakage) and time-variation: under periodic steady state, the transient term \( T_H(j\omega_k) \) of the MISO LTI model (19) is a part of the output time-variation \( Y_{TV}(k) \). Under transient conditions the situation is more complicated (see Section VI-A).

**B. Nonlinear Slowly Time-Varying**

Shifting the nonlinear distortions \( y_N(t) \) in the bottom block diagram of Fig. 2 to the output, it can easily be seen that the reasoning of Section IV-A also applies to nonlinear time-variant systems satisfying Assumption 2 (compare Fig. 1 to the bottom block diagram of Fig. 2). The only difference being that \( Y_s(k) \) is added to the right hand side of Eqs. (16) and (19)

\[ Y(k) = G_0(j\omega_k)U(k) + Y_{TV}(k) + Y_S(k) \]
(21)

\[ \begin{array}{c}
\text{Input} \quad u(t) \\
\text{NLTV} \\
\text{Output} \quad y(t) \\
\end{array} \]
\[ \begin{array}{c}
\text{y_{TV}(t)} + y_s(t) + n_y(t) \\
\text{Output} \\
\end{array} \]
\[ \begin{array}{c}
\text{G_{BLTI}(s)} \\
\text{Output} \\
\end{array} \]
\[ \begin{array}{c}
\text{y(t)} \\
\text{Output} \\
\end{array} \]

Figure 5. Known input \( u(t) \) – noisy output \( y(t) \) observations of a nonlinear time-variant (NLTV) system satisfying Assumption 2. \( y_{TV}(t) \), \( y_s(t) \), and \( n_y(t) \) are, respectively, the output time-variation, the output nonlinear distortions, and the output noise. \( y_{TV}(t) \) and \( y_s(t) \) are mutually uncorrelated and uncorrelated with – but not independent of – the input \( u(t) \); while \( n_y(t) \) is independent of \( y_{TV}(t) \), \( y_s(t) \), and \( u(t) \).

\[ Y(k) = H_0(j\omega_k)U(k) + Y_S(k) + \sum_{p=1}^{N_b} H_p(j\omega_k) U_p(k) + T_H(j\omega_k) \]
(22)

Subtracting (22) from (21) shows that (20) remains valid.

**V. MEASURING THE BLTI APPROXIMATION**

Consider the setup of Fig. 5 where known input – noisy output measurements of a nonlinear time-variant system (NLTV) satisfying Assumption 2 are available. The goal is to estimate the best linear time-invariant (BLTI) approximation and the levels of the noise \( n_y(t) \), the nonlinear distortion \( y_s(t) \), and the time-variation \( y_{TV}(t) \), from a single experiment with a random phase multisine (7). First, we explain the basic idea and, next, give a detailed description of the estimation procedure.

**A. Basic Idea**

The DFT spectrum of 2 consecutive periods of the noiseless steady state response of a linear time-invariant (LTI) system to a random phase multisine (7) is shown in the left plot of Fig. 6. It can be seen that the output DFT spectrum is non-zero at the even DFT bins (frequencies) \( 2k \) only. In general, the DFT of \( P \) consecutive periods has signal energy at DFT frequencies (bins) \( Pk \) only and, hence, at all the other DFT frequencies the DFT spectrum is exactly zero. This basic property of a periodic signal is the key to separate the nonlinear distortion (periodic signal) from the disturbing noise (non-periodic signal).

Consider now noisy observations of \( P \) consecutive periods of the periodic steady state response (Assumption 3) of a nonlinear time-variant system satisfying Assumption 2. Since \( y_N(t) \) is harmonically related to \( u(t) \) (see Section III-C), while \( u_p(t) = f_p(t)u(t), \quad p = 1, 2, \ldots, N_b \), and \( n_y(t) \) are non-periodic signals, the output DFT \( Y(k) \) of the MISO LTI model (22) can be split into contributions at non-excited DFT frequencies \( k \neq rP, \quad r = 0, 1, \ldots, N/(2P) - 1 \)

\[ Y(k) = \sum_{p=1}^{N_b} H_p(j\omega_k) U_p(k) + T_H(j\omega_k) + N_Y(k) \]
(23)
and excited DFT frequencies \( k = rP, r = 1, 2, \ldots, N/(2P) - 1 \)

\[
Y(k) = H_0(j\omega_k)U(k) + \sum_{p=1}^{N_b} H_p(j\omega_k)U_p(k) + 
\]

\[
T_H(j\omega_k) + Y_S(k) + N_Y(k) \tag{24}
\]

(see the right plot of Fig. 6 for the case \( P = 2 \)). Note that \( T_H(j\omega_k) \) in Eqs. (23) and (24) also contains an \( O(T^{-1/2}) \) contribution of the noise leakage (transient) error. Here, it is assumed that the noise leakage can be neglected w.r.t. original \( T_H(j\omega_k) \) term in Eq. (22) (see also Section VI-A).

From (23) and (24) it follows that it is easier to estimate the time-variation at the non-excited DFT bins than at the excited ones. Indeed, the former are only disturbed by noise while the latter are disturbed by noise and nonlinear distortion. In addition, if the time-variation is below the noise level at the non-excited DFT bins, it will be below the sum of the noise and nonlinear distortion levels at the excited DFT bins. This motivates the following three step procedure: first, the MISO LTI model (22) is estimated via a two step procedure and, next, the best linear distortions are estimated. Subtracting the noise variance from the total variance gives an estimate of the variance \( \hat{Y}_{\text{LTI}}(k) \) is larger than that of \( Y(k) \), viz.,

\[
\text{var}(\hat{Y}_{\text{LTI}}(k)) = (1 + \lambda^2) \text{var}(N_Y(k)) \tag{26}
\]

with \( \lambda \) defined in Eq. (44) (proof: see Appendix C).

Next, following the same lines of the fast method in [32], nonparametric estimates of the frequency response function \( H_0(j\omega_k) \) and the total variance \( (1 + \lambda^2) \text{var}(N_Y(k)) \) are obtained from \( \hat{Y}_{\text{LTI}}(k) \) (25) and \( U(k) \) at the excited DFT frequencies \( k = rP, r = 1, 2, \ldots \). Due to the noise on the estimates \( \hat{H}_p(j\omega_k) \) and \( \hat{T}_H(j\omega_k) \), the noise variance of \( \hat{Y}_{\text{LTI}}(k) \) is larger than that of \( Y(k) \), viz.,

\[
\text{var}(\hat{Y}_{\text{LTI}}(k)) = (1 + \lambda^2) \text{var}(N_Y(k)) \tag{26}
\]

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\]

with \( \lambda \) defined in Eq. (44) (proof: see Appendix C).

Next, following the same lines of the fast method in [32], nonparametric estimates of the frequency response function \( H_0(j\omega_k) \) and the total variance \( (1 + \lambda^2) \text{var}(N_Y(k)) \) are obtained from \( \hat{Y}_{\text{LTI}}(k) \) (25) and \( U(k) \) at the excited DFT frequencies \( k = rP, r = 1, 2, \ldots \). Due to the noise on the estimates \( \hat{H}_p(j\omega_k) \) and \( \hat{T}_H(j\omega_k) \), the noise variance of \( \hat{Y}_{\text{LTI}}(k) \) is larger than that of \( Y(k) \), viz.,

\[
\text{var}(\hat{Y}_{\text{LTI}}(k)) = (1 + \lambda^2) \text{var}(N_Y(k)) \tag{26}
\]

with \( \lambda \) defined in Eq. (44) (proof: see Appendix C).

Next, following the same lines of the fast method in [32], nonparametric estimates of the frequency response function \( H_0(j\omega_k) \) and the total variance \( (1 + \lambda^2) \text{var}(N_Y(k)) \) are obtained from \( \hat{Y}_{\text{LTI}}(k) \) (25) and \( U(k) \) at the excited DFT frequencies \( k = rP, r = 1, 2, \ldots \). Due to the noise on the estimates \( \hat{H}_p(j\omega_k) \) and \( \hat{T}_H(j\omega_k) \), the noise variance of \( \hat{Y}_{\text{LTI}}(k) \) is larger than that of \( Y(k) \), viz.,

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with \( \lambda \) defined in Eq. (44) (proof: see Appendix C).

Next, following the same lines of the fast method in [32], nonparametric estimates of the frequency response function \( H_0(j\omega_k) \) and the total variance \( (1 + \lambda^2) \text{var}(N_Y(k)) \) are obtained from \( \hat{Y}_{\text{LTI}}(k) \) (25) and \( U(k) \) at the excited DFT frequencies \( k = rP, r = 1, 2, \ldots \). Due to the noise on the estimates \( \hat{H}_p(j\omega_k) \) and \( \hat{T}_H(j\omega_k) \), the noise variance of \( \hat{Y}_{\text{LTI}}(k) \) is larger than that of \( Y(k) \), viz.,

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\text{var}(\hat{Y}_{\text{LTI}}(k)) = (1 + \lambda^2) \text{var}(N_Y(k)) \tag{26}
\]

with \( \lambda \) defined in Eq. (44) (proof: see Appendix C).

Next, following the same lines of the fast method in [32], nonparametric estimates of the frequency response function \( H_0(j\omega_k) \) and the total variance \( (1 + \lambda^2) \text{var}(N_Y(k)) \) are obtained from \( \hat{Y}_{\text{LTI}}(k) \) (25) and \( U(k) \) at the excited DFT frequencies \( k = rP, r = 1, 2, \ldots \). Due to the noise on the estimates \( \hat{H}_p(j\omega_k) \) and \( \hat{T}_H(j\omega_k) \), the noise variance of \( \hat{Y}_{\text{LTI}}(k) \) is larger than that of \( Y(k) \), viz.,

\[
\text{var}(\hat{Y}_{\text{LTI}}(k)) = (1 + \lambda^2) \text{var}(N_Y(k)) \tag{26}
\]

with \( \lambda \) defined in Eq. (44) (proof: see Appendix C).

Next, following the same lines of the fast method in [32], nonparametric estimates of the frequency response function \( H_0(j\omega_k) \) and the total variance \( (1 + \lambda^2) \text{var}(N_Y(k)) \) are obtained from \( \hat{Y}_{\text{LTI}}(k) \) (25) and \( U(k) \) at the excited DFT frequencies \( k = rP, r = 1, 2, \ldots \). Due to the noise on the estimates \( \hat{H}_p(j\omega_k) \) and \( \hat{T}_H(j\omega_k) \), the noise variance of \( \hat{Y}_{\text{LTI}}(k) \) is larger than that of \( Y(k) \), viz.,

\[
\text{var}(\hat{Y}_{\text{LTI}}(k)) = (1 + \lambda^2) \text{var}(N_Y(k)) \tag{26}
\]
(19), the first and second order derivatives are approximated by central differences [33], viz.,

\[
X^{(1)}(j\omega_k) = \frac{X(j\omega_{k+1}) - X(j\omega_{k-1})}{j\omega_{k+1} - j\omega_{k-1}} + O(T^{-2})
\]

\[
X^{(2)}(j\omega_k) = \frac{X(j\omega_{k+1}) - 2X(j\omega_k) + X(j\omega_{k-1})}{(j\omega_{k+1} - j\omega_{k-1})/2^2} + O(T^{-2})
\]

where \( X = H_p, \ p = 1, 2, \ldots, N_p, \) and with \( \omega_k = 2\pi k f_0 \) the excited harmonics. Note that these formulas are valid for equispaced excited harmonics only (Assumption 4).

Next, the noise and total variances of \( G_{BLTI}(j\omega_k) \) (15) are calculated following the same lines of [32] and taking into account the following special properties of the estimated MISO LTI model: (i) \( H_0(j\omega_k) \) depends on the noise \( N_Y(k) \) at the excited and non-excited frequencies (see Eq. (26)), and on the stochastic nonlinear distortions \( Y_S(k) \); (ii) \( H_p(j\omega_k), \ p > 0, \) depend on the noise \( N_Y(k) \) at the non-excited frequencies, and are independent of the nonlinear distortions \( Y_S(k) \); and (iii) \( H_0(j\omega_k) \) is uncorrelated with \( H_p(j\omega_k), \ p > 0 \) (proof: see Appendix D).

Finally, the output time-variation \( Y_{TV}(k) \) is obtained via (20).

C. Uncertainty Evaluation

In the first step of the estimation procedure the noise variance \( \text{var}(N_Y(k)) \) is quantified, while in the second step an estimate of the variance of the nonlinear distortions \( \text{var}(Y_S(k)) \) is obtained. The uncertainty of the BLTI approximation only depends on \( N_Y(k) \) and \( Y_S(k) \) because the time-variant effects have been removed in the estimates. Since \( N_Y(k) \) is asymptotically \( (N \to \infty) \) circular complex normally distributed for filtered white noise disturbances \( n_y(t) \) [1], [3], and since \( Y_S(k) \) is asymptotically \( (F = O(N) \to \infty) \) circular complex normally distributed for random phase multisine excitations \( u(t) \) [3], the most compact \( p \times 100\% \ (p \in [0,1]) \) confidence region of the BLTI approximation estimate is a circle with center \( \hat{G}_{BLTI}(j\omega_k) \) and radius

\[
-\log(1-p)\sigma_{\text{cov}}^2(k)
\]

with \( \log(x) \) the natural logarithm of \( x \) (proof: see [3], [34]) where \( \sigma_{\text{cov}}^2(k) \) stands for the variance due to either the noise, or the nonlinear distortions, or the sum of both contributions (total variance). The variance \( \sigma_{\text{cov}}^2(k) \) is calculated from \( \text{var}(N_Y(k)) \) and \( \text{var}(Y_S(k)) \) following the same lines as in [25], [32].

Notes

1) Because the noise and the nonlinear distortions have a random contribution to the BLTI approximation estimate, a type A evaluation of the uncertainty (see Section 4.3 of [35]) is made in Eq. (27).

2) The bias of the BLTI approximation estimate due to the local polynomial approximation and the numerical differentiation can be kept below the variance error by an appropriate choice of the frequency resolution \( f_0 \), the order \( R \) of the local polynomial approximation, and the number \( N_k \) of time-variant parallel branches (follow the same lines of [21], [38]).

VI. Extensions

A. Non-Steady State Conditions

If Assumption 3 is violated, then a rational form \( T_G(j\omega_k) \) modeling the transient behavior of the LTI systems \( G_p(s), \ p = 0, 1, \ldots, N_p, \) must be added to Eqs. (16) and (21) (proof: see [21]). Eqs. (19) and (22) remain valid but now \( T_H(j\omega_k) \) models the sum of the leakage error due to the non-periodicity of \( u_p(t) \) and the transient term \( T_G(j\omega_k) \). Therefore, in case of non-steady state conditions, the time-variation \( Y_{TV}(k) \) can only be estimated within an \( O(T^{-1/2}) \) bias error

\[
Y_{TV}(k) = \sum_{p=1}^{N_k} H_p(j\omega_k) U_p(k) + O \left( T^{-1/2} \right)
\]

Combining (20) with \( T_H = O(T^{-1/2}) \) and \( H_0 - G_0 = O(T^{-1}) \) (use (15) with \( G_{BLTI} = G_{BLO} \) proves (28).

Note. In case of noisy output observations (23) or (24), the \( O(T^{-1/2}) \) bias error in (28) also includes the noise transient (leakage) term.

B. Noisy Input Observations – Feedback

In case of noisy input-output observations and/or operation in feedback, the spectral analysis methods (8) and (12) are biased and should be replaced by the indirect method for measuring a FRF (see [19], [22], [36], [37])

\[
G_{BLTI}(j\omega_k) = G_{BLA}(j\omega_k) = \mathbb{E} \left\{ Y(k) R(k) \right\} / \mathbb{E} \left\{ U(k) R(k) \right\}
\]

where \( R(k) \) is the DFT of a known reference signal \( r(t) \). This is typically the digital signal stored in the arbitrary waveform generator, and the unknown digital-to-analog characteristics of the waveform generator are included in the unknown actuator dynamics.

First, following the same lines of [19], [22], a multiple-input, two-output LTI equivalent model of the nonlinear time-varying system is constructed from reference \( r(t) \) to input-output \( z(t) = [y(t) u(t)]^T \) simultaneously. Eqs. (20) and (22) are then replaced by, respectively,

\[
\tilde{Z}_{TV}(k) = (H_{rz,0}(j\omega_k) - G_{rz,0}(j\omega_k)) R(k) + \sum_{p=1}^{N_k} H_{rz,p}(j\omega_k) R_p(k) + T_{Hz}(j\omega_k)
\]

\[
Z(k) = H_{rz,0}(j\omega_k) R(k) + \sum_{p=1}^{N_k} H_{rz,p}(j\omega_k) R_p(k) + T_{Hz}(j\omega_k) + \tilde{Z}_S(k) + N_Z(k)
\]

with \( Z(k) \) the input-output DFT spectra; \( H_{rz,p}(j\omega_k) \) the 2 by 1 FRFs from reference \( r_p(t) = r(t)f_p(t) \) to input-output...
$z(t)$; $R_h(k)$ the DFT of $r_h(t)$; $T_{H_s}(j\omega_k)$ the 2 by 1 input-output transient (leakage) term; $\tilde{Z}_S(k) = [\tilde{Y}_S(k) \ U_S(k)]^T$ the observed input-output nonlinear distortions; $N_S(k) = [N_Y(k) \ N_U(k)]^T$ the disturbing input-output noise; $\tilde{Z}_{TV}(k) = [\tilde{Y}_{TV}(k) \ U_{TV}(k)]^T$ the observed input-output time-variation; and $G_{rz,0}(j\omega_k) = [G_{ry,0}(j\omega_k) \ G_{ru,0}(j\omega_k)]^T$ the 2 by 1 best linear approximation from reference $r(t)$ to input-output $z(t)$.

Next, the 3 step estimation procedure of Section V-B is applied to (31) and (30).

Finally, the BLTI approximation $G_{BLTI}(j\omega_k)$ and the output time-variation $Y_{TV}(k)$ are obtained from $G_{rz,0}(j\omega_k)$ and $\tilde{Z}_{TV}(k)$ as

$$G_{BLTI}(j\omega_k) = G_{ry,0}(j\omega_k)/G_{ru,0}(j\omega_k)$$

$$Y_{TV}(k) = \tilde{Y}_{TV}(k) - G_{BLTI}(j\omega_k) \ U_{TV}(k)$$

(proof: see [22]); and the variance of $Y_S(k)$ is calculated from the covariance of $\tilde{Z}_S(k)$ as

$$\text{var}(Y_S(k)) = \text{var}(\tilde{Y}_S(k)) + |G_{BLTI}(j\omega_k)|^2 \text{var}(U_S(k)) - 2\text{Re}(\text{covar}(\tilde{Y}_S(k), U_S(k))G_{BLTI}(j\omega_k))$$

with $\text{Re}(z)$ the real part of the complex number $z$ (proof: use $Y_S(k) = \tilde{Y}_S(k) - G_{BLTI}(j\omega_k) \ U_S(k)$ shown in [19]).

VII. REQUIREMENTS ON THE EXPERIMENTAL SETUP

In this section we discuss in detail the requirements on the generator/actuator and the data acquisition channels.

A. Synchronization

Assumption 4 implies coherent sampling of the multisine excitation signal. This condition is easily fulfilled in practice by synchronizing the arbitrary waveform generator with the data acquisition units: either the generator clock is used as external clock for the acquisition unit, or the generator and acquisition clocks originate from the same mother clock (e.g., VXI and PXI systems).

B. Calibration

Consider first the case where the input is observed without errors (see Fig. 5 and Eq. (21)). Due to the dynamics $G_u(j\omega)$ and $G_y(j\omega)$ of, respectively, the input and output data acquisition channels, one measures $Y_m(k) = G_y(j\omega_k) Y(k)$ and $U_m(k) = G_u(j\omega_k) U(k)$ instead of $Y(k)$ and $U(k)$. Replacing $U$ and $Y$ in (21) by, respectively, $G_u^{-1} U_m$ and $G_y^{-1} Y_m$, gives

$$Y_m = G_y^{-1} G_u^{-1} U_m + G_y Y_{TV} + G_y Y_S + G_y N_Y$$

where the arguments have been dropped for notational simplicity. From (35) it can be seen that the true BLTI approximation is recovered via a relative data calibration of the acquisition channels (only the ratio $G_y/G_u$ should be known at the excited frequencies), while the true variances of the time-variation, the nonlinear distortion and the disturbing noise are recovered via an absolute amplitude calibration of the output acquisition channel (only $|G_y|$ should be known to correct the variances). While

the absolute amplitude calibration requires a calibrated power meter, the relative calibration needs no calibrated measurement device (apply a multisine excitation to both acquisition channels simultaneously and measure the FRF).

If the input observations are noisy and/or the device under test operates in closed loop, then $|G_u|$ is needed to calculate the correct variances of the input noise and of the observed input time-variation and nonlinear distortion. It is readily obtained from the previous relative ($G_y/G_u$) and absolute ($|G_y|$) calibrations: $|G_u| = |G_y|/|G_u/G_u|$.

C. Linearity

The data acquisition channels should be linear – an assumption that is close to be met by a well designed measurement device – otherwise the nonlinear distortions of the acquisition channels would incorrectly be assigned to the device under test (DUT). In practice the nonlinear detection threshold of the DUT is set by the (low) distortion level of the acquisition channels used.

The situation is somewhat more involved for the generator/actuator in the setup. If the input of the DUT is observed without errors (in practice a very high input signal-to-noise ratio), then the detection and quantification of the nonlinear behavior of the DUT is not influenced by the possible nonlinear behavior (spectral impurity) of the generator/actuator. However, in case of noisy input, noisy output measurements and/or systems operating in closed loop, the periodic nonlinear distortions produced by the generator/actuator result in non-periodic output nonlinear distortions that will increase the estimated output noise level. Hence, in this case the nonlinear detection threshold of the DUT is also set by the (low) distortion level of the generator/actuator used.

VIII. EXPERIMENTAL VERIFICATION

Although the 3 step estimation procedure of Section V-B has been validated extensively on simulation examples, we only report here the experimental results on two time-variant electronic circuits. The first circuit operates in open loop (see Fig. 7), while the second circuit operates in closed loop (see Fig. 8).

For all experiments the reference signal $r(t)$ is a random phase multisine (7) consisting of the sum of 522 sinewaves with uniformly distributed phases $\phi_k$, and equal amplitudes $A_k$ in the

Figure 7. Time-variant second order bandpass filter operating in open loop. It consists of the following components: a high gain operational amplifier (CA7414E); a JFET transistor (BF245B); three resistors ($R_1 = R_3 = 10 \, \text{k}\Omega$ and $R_2 = 470 \, \text{k}\Omega$); and two capacitors ($C_1 = C_2 = 10 \, \text{nF}$), $u(t)$, $y(t)$, and $p(t)$ are, respectively, the input, output, and scheduling signals. $r(t)$ is the reference signal stored in the arbitrary waveform generator and $R_0 = 50 \, \Omega$ is the output impedance of the generator.
frequency band \([228.9\,\text{Hz}, 39.98\,\text{kHz}]\) chosen such that the rms value of the input \(u(t)\) (open and closed loop setups) is about 100 mV \((f_0 = 2f_s/N, f_s = 625\,\text{kHz}, N = 16384, F = 524, A_1 = A_2 = 0,\) and \(A_3 = A_4 = \cdots = A_F)\). \(P = 2\) consecutive periods of the steady state response of the input \(u(t)\) and output \(y(t)\) are acquired using a band-limited measurement setup (all signals are lowpass filtered before sampling).

Four experiments are performed for each electronic circuit: one time-variant experiment where a periodic ramp \(p(t)\) with period \(2/f_0\) is applied to either the JFET transistor in Fig. 7 (open loop setup), or the resistors made by electro-optical components in Fig. 8 (closed loop setup); and three time-invariant experiments where the scheduling signal \(p(t)\) is kept constant and equal to, respectively, the minimum, the maximum, and the mean value of \(p(t)\) in the time-variant experiment (see Figures 9 and 10).

**A. Open Loop Example - Known Input, Noisy Output**

In the open loop time-variant experiment the scheduling signal \(p(t)\) decreases linearly -803 mV to -830 mV with a mean value of -818 mV (see Fig. 9). Starting from \(N\) known input, noisy output samples \(u(nT_s)\) and \(y(nT_s), n = 0, 1, \ldots, N - 1\), the MISO model (22), with \(f_p(t)\) Legendre polynomials satisfying (4), is estimated following the procedure of Section V-B (local polynomial method [38] with a fourth-order local polynomial approximation of the FRFs and the leakage term, and six degrees of freedom for the variance estimates). Fig. 11 shows the results for \(N_b = 2\): \(H_1(j\omega_k), H_2(j\omega_k)\) and their noise variance (the estimates do not depend on the nonlinear distortions \(Y_S(k)\)); and \(H_0(j\omega_k)\) and its noise (light gray) and total (dark gray) variances. Increasing \(N_b\) further makes no sense because \(|H_3(j\omega_k)|\) is of the order of its noise standard deviation for all values of \(k\).

Fig. 12 shows the nonparametric estimates of the BLTI approximation \(G_{\text{BLTI}}(j\omega_k)\) and the output time-variation \(Y_{\text{TV}}(k)\) obtained via (15) and (20), respectively. From the top and bottom right plots it can be seen that the total variance (sum of the noise variance and the variance of the stochastic nonlinear distortions) is significantly larger than the noise variance in the band \([300\,\text{Hz}, 20\,\text{kHz}]\). From the bottom right plot it also follows that the time-variation is above the noise level in the band \([300\,\text{Hz}, 15\,\text{kHz}]\). Above 15 kHz no time-variation can be detected within the measurement uncertainty. Note also that, in
The band [300 Hz, 15 kHz], the time-variation has the same order of magnitude as the nonlinear distortions. It illustrates that the time-variation can more easily be detected at the non-excited DFT frequencies (disturbing noise only) than at the excited DFT frequencies (disturbing noise and nonlinear distortions).

Finally, the results of the linear time-variant (LTV) experiment are compared in Fig. 13 to those of the three linear time-invariant (LTI) experiments. From the right plot it can be seen that the noise and total variances estimates of the LTV experiment (black lines) coincide with those of the three LTI experiments (gray lines). So, the time variations were sufficiently slow such that the noise remained stationary, and that the nonlinear distortions remained time-invariant. Comparing the gray lines with the black dashed line in the left plot of Fig. 13, it follows that the BLTI approximation of the LTV experiment is significantly different from the three FRFs of the LTI experiments in the passband only. This is exactly the frequency band where the time-variation is most apparent (see Fig. 12, bottom right plot).

In the closed loop time-variant experiment the scheduling signal \( p(t) \) increases linearly from 820 mV to 831 mV with a mean value of 826 mV (see Fig. 10). Starting from \( N \) known reference, and noisy input-output samples \( r(nT_s), u(nT_s) \) and \( y(nT_s), n = 0, 1, \ldots, N - 1 \), the MITO model (31), with \( f_p(t) \) Legendre polynomials satisfying (4), is estimated following the procedure of Sections V-B and VI-B (local polynomial method [38] with a fourth-order local polynomial approximation of the FRFs and the leakage term, and six degrees of freedom for the variance estimates). Fig. 14 shows the estimated FRFs and their noise variance for the case \( N_0 = 2 \). Increasing \( N_0 \) further makes no sense because \( |H_{ru,3}(j\omega_k)| \) and \( |H_{ry,3}(j\omega_k)| \) are of the order of their noise standard deviation for all values of \( k \).

Fig. 15 shows the nonparametric estimates of the BLTI approximation \( G_{BLTI}(j\omega_k) \), the output time-variation \( Y_{TV}(k) \), and the variance of the output nonlinear distortions \( Y_{n,k}(k) \) obtained via (32), (33), and (34) respectively. From the top and bottom right plots it can be seen that the total variance (sum of the noise variance and the variance of the stochastic nonlinear distortions) is significantly larger than the noise variance over the whole frequency band. From the bottom right plot it also follows that the time-variation is the dominant output error source over the whole frequency band.

Finally, the results of the linear time-variant (LTV) experiment are compared in Fig. 16 to those of the three linear time-invariant (LTI) experiments. From the right plot is can be seen that the noise and total variances estimates of the LTV experiment (black lines) coincide with those of the three LTI experiments (gray lines). Comparing the gray lines with the black dashed line in the left plot of Fig. 16, it follows that the BLTI approximation of the LTV experiment is significantly different from the three FRFs of the LTI experiments over the whole frequency band. This is consistent with the observation that the time-variation is the dominant error source over the whole frequency band (see Fig. 15, bottom right plot).
randomly selected non-excited harmonics, is then needed to confirm or reject the linearity hypothesis.

IX. CONCLUSIONS

For a certain class of nonlinear time-variant systems, the best linear time-invariant approximation and the levels of the noise, the nonlinear distortion, and the time-variation can be estimated using nonparametric methods developed for multiple-input, multiple-output time-invariant systems. Measuring two consecutive periods of the (transient) response to one random phase multisine excitation is sufficient. As such, the approximation error of the LTI framework is completely quantified from a single experiment.

To separate the noise from the nonlinear distortions, it is assumed that the time-variant part of the system is not affected by the nonlinear distortions. Although this assumption seems to be quite restrictive, the experiments on the electronic circuits show that it is practically valid for slow (small) time-variations. Experiments on a vibrating robot arm with time-varying length (results not reported here) confirm this observation.

ACKNOWLEDGMENT

This work is sponsored by the Research Council of the Vrije Universiteit Brussel, the Research Foundation Flanders (FWO-Vlaanderen), the Flemish Government (Methusalem Fund, METH1), and the Belgian Federal Government (Interuniversity Attraction Poles programmeVII, Dynamical Systems, Control, and Optimization).

APPENDIX A

PROOF OF PROPERTY 5 OF $Y_{TV}(k)$

Under Assumption 3, the DFT of (10) is given by Eqs. (17) and (18). Since $F_p(k)$ in (18) has a skirt-like shape [40], only a finite number of frequencies contributes to the sum (18), even if $F = O(N) \rightarrow \infty$. Therefore, the central limit theorem cannot be applied to each term $Y_p(k)$ in the sum (17), and $Y_{TV}(k)$ is not asymptotically ($F = O(N) \rightarrow \infty$) normally distributed.

APPENDIX B

PROOF OF EQ. (14)

Using (17) and (18), the expected value in (14) can be elaborated as

$$
\mathbb{E} \left\{ Y_S(k) \overline{Y}_{TV}(k') \right\} = \frac{1}{\sqrt{N}} \sum_{p=1}^{N_0} \sum_{l=0}^{N-1} F_p(k'-l) G_p(j\omega_l) \times \mathbb{E} \left\{ Y_S(k) \overline{U}(l) \right\}
$$

In the sequel of this appendix we will study (36) for the case $k, k' \neq 0, N/2$. To analyze $\mathbb{E} \left\{ Y_S(k) \overline{U}(l) \right\}$, an explicit expression of $Y_S(k)$ for NL PISPO systems is used.

For an NL PISPO system, the DFT of the output nonlinear distortions (13) can be written as

$$
Y_S(k) = \sum_{\alpha=2}^{\infty} Y_S^\alpha(k) \tag{37}
$$

$$
Y_S^\alpha(k) = N^{-\frac{\alpha}{2}} \sum_{K \in \mathbb{R}s_{k_1} \cdot \ldots \cdot k_{\alpha}} G^n_{k_1, \ldots, k_{\alpha}} U(k_1) \ldots U(k_{\alpha}) \tag{38}
$$

C. Discussion

The experimental results validate the three step estimation procedure proposed in Section V-B: the noise level and the level of the nonlinear distortions predicted by the time-variant experiments coincide with those of the time-invariant experiments (see Figs. 13 and 16).

Although Assumption 2, stating that only the time-variant part of the system is subject to nonlinear distortions, is not fulfilled for the two electronic circuits in Figs. 7 and 8, it turns out from the experiments that Assumption 2 is valid for slow (small) time-variations. For fast (large) time-variations, it has been observed that the estimated noise variance is too large, and that it even might coincide with the total variance. This can be explained by the presence of nonlinear distortions in the time-varying branches of Fig. 2. Hence, if the estimated noise and total variances coincide, one may not immediately conclude that the time-variant system behaves linearly. An additional experiment using, for example, a random phase multisine with

Figure 15. Best linear time-invariant (BLTI) approximation of the time-variant closed loop circuit (top), and its input-output DFT spectra at the excited harmonics (bottom). Top: estimated BLTI approximation (black), its noise variance (light gray), and its total variance (dark gray). Bottom left: input DFT spectrum $E\{U(k)\}$ (black), and input noise variance $\text{var}(N_Y(k))$ (light gray). Bottom right: output DFT spectrum $Y_{BLTI}(k)$ (black) of the BLTI model (see Fig. 3); output noise variance $\text{var}(N_Y(k))$ (light gray); variance of the stochastic nonlinear distortions $\text{var}(Y_S(k))$ (dark gray); and time-variation $Y_{TV}(k)$ (medium gray).

Figure 16. Comparison between the estimated best linear time-invariant (BLTI) approximation of the LTV experiment and the estimated frequency response functions (FRFs) of three LTI experiments - closed loop circuit. Left: BLTI approximation (black line) and its total variance (black dashed line), and magnitude of the complex difference between the BLTI approximation and the FRFs (3 gray lines). Right: noise (3 almost coinciding light gray lines) and total (3 almost coinciding dark gray lines) variances FRFs; and noise (1 black line) and total (1 dashed black line) variance BLTI approximation.
where $G^\alpha_{k_1\ldots k_\alpha}$ is the symmetrized frequency domain representation of the Volterra kernel of degree $\alpha$ [29], $K = (k_1, k_2, \ldots, k_{\alpha-1})$, and $\mathbb{K}_{S_0, \alpha}$ is the following set of indexes

$$\mathbb{K}_{S_0, \alpha} = \left\{ K \left| \begin{array}{l}
\sum_{i=1}^{\alpha} k_i = k \\
\text{and } \sum_{i=1}^{\alpha} \phi_{k_i} \neq \phi_k
\end{array} \right. \right\}$$

(39)

with $\phi_r = LU(r)$ (see [3], Appendix 3.A). Collecting (36) to (39) shows that we should analyze

$$\mathbb{E} \left\{ U(k_1) \ldots U(k_\alpha) \mathcal{U}(l) \right\} | K \in \mathbb{K}_{S_0, \alpha}$$

(40)

Since $\mathbb{E}\{e^{j\phi_r}\} = 0$, the expected value (40) is different from zero only if the indexes $k_1, k_2, \ldots, k_\alpha$, satisfying (39), can be chosen such that

$$\sum_{i=1}^{\alpha} \phi_{k_i} = \phi_l$$

(41)

(the sum of the phases of all factors in the expected value is then zero). We will show in the sequel that this is impossible.

First note that condition (41) can be satisfied only if (i) all indexes $k_i \ (i = 1, 2, \ldots, \alpha)$ but one can be grouped in pairs $(k_i, k_j)$ with $k_i = -k_j$ (=> $\phi_{k_i} + \phi_{k_j} = 0$), and (ii) there exists an index such that $k_i = l$ (=> $\phi_{k_i} = \phi_l$). If $\alpha$ is even, then the pairing procedure gives $\sum_{i=1}^{\alpha} \phi_{k_i} = 0$ and (41) cannot be satisfied. If $\alpha$ is odd, then (41) can be satisfied via the pairing procedure. We show now that (41) can never be fulfilled for indexes belonging to the set $\mathbb{K}_{S_0, \alpha}$ (39). If $k = l$, then (41) is in contradiction with the second constraint in (39). If $k \neq l$, then (41) implies that $\sum_{i=1}^{\alpha} k_i = l$, which is in contradiction with the first constraint in (39).

Since it is impossible to fulfill condition (41), the expected value (40) is zero, which proves that $\mathbb{E} \{ Y_S(k) \mathcal{U}(l) \} = 0$. Combining the latter with (36) shows (14).

**APPENDIX C**

**PROOF OF Eq. (26).**

Since the DFT of filtered white noise is asymptotically (for $N \to \infty$) uncorrelated over the frequency, the noise on $Y(k)$ at the excited DFT frequencies is asymptotically uncorrelated with the estimates $\hat{H}_p(j\omega_k)$ and $\hat{T}_H(j\omega_k)$ that only depend on the noise at the non-excited DFT frequencies. Therefore, the noise variance of $\hat{Y}_{\text{LTI}}(k)$ (25) is given by the sum of the noise variance of $Y(k)$ and the noise variance of $\tilde{Y}(k) = \sum_{p=1}^{N_0} \hat{H}_p(j\omega_k) U_p(k) + \hat{T}_H(j\omega_k)$

$$\text{var} \left( \hat{Y}_{\text{LTI}}(k) \right) = \text{var} \left( N_Y(k) \right) + \text{var} \left( \tilde{Y}(k) \right)$$

(42)

Following the same lines as in [25], [32], the variance of $\tilde{Y}(k)$ is found to be equal to

$$\text{var}(\tilde{Y}(k)) = \lambda^2 \text{var}(N_Y(k))$$

(43)

where

$$\lambda = \left\| S \left[ \begin{array}{c} \hat{U}(k) \\
1 \end{array} \right] \right\|_2$$

(44)

with $\hat{U}(k) = \left[ U_1(k) \ldots U_{N_0}(k) \right]^T$. The matrix $S$ is derived from the singular value decomposition $U \Sigma V^H$ of the hermitian transpose of the regression matrix $K_n$ of the local polynomial approximation of degree $R$ ($K_n^H = U \Sigma V^H$), viz.,

$$S = U \Sigma^{-1} V_{\{\text{set:}\}}^H$$

(45)

where $V_{\{\text{set:}\}}$ selects rows $1, 2, \ldots, N_0$ and $(R+1)N_0 + 1$ and all columns of $V$. Finally, the regression matrix $K_n$ is defined as

$$K_n = \left[ K(k - r_1) \ldots K(k - r_1) K(k + r_1) \ldots K(k + r_n) \right]$$

(46)

with $k \pm r_i, i = 1, 2, \ldots, n$, the first $2n$ non-excited DFT frequencies (left $n$ values) and right (n values) from the excited DFT frequency $k = tP$, and where each column $K (k \pm r_i)$ equals

$$K (k \pm r_i) = \left[ K_1(\pm r_i) \hat{U}(k \pm r_i) \right]$$

(47)

with $K_1(\pm r_i) = [1(\pm r_i) \ldots (\pm r_i)]^T \otimes$ the Kronecker matrix product [39]. Collecting (42) to (47) proves (26).

**APPENDIX D**

**THE ESTIMATES $\hat{H}_0$ AND $\hat{H}_p$, $p > 0$, ARE UNCORRELATED.**

Using (24), the estimated output DFT $\hat{Y}_{\text{LTI}}(k)$ (25) at the excited frequencies can be rewritten as

$$\hat{Y}_{\text{LTI}}(k) = H_0(j\omega_k) U(k) + T_H(j\omega_k) - \hat{T}_H(j\omega_k) + \sum_{p=1}^{N_0} (H_p(j\omega_k) - \hat{H}_p(j\omega_k)) U_p(k) + N_Y(k) + Y_S(k)$$

(48)

where $H_p(j\omega_k) - \hat{H}_p(j\omega_k)$ and $T_H(j\omega_k) - \hat{T}_H(j\omega_k)$ depend on the noise $N_Y(k)$ at the non-excited frequencies only (step 1 of the identification procedure). Since $H_0(j\omega_k)$ is estimated using $\hat{Y}_{\text{LTI}}(k)$ and $U(k)$, it follows from (48) that $H_0(j\omega_k)$ depends on the noise $N_Y(k)$ at the excited and non-excited frequencies, and on the nonlinear distortions $Y_S(k)$.

We show now that $H_0(j\omega_k)$ is asymptotically ($T \to \infty$) uncorrelated with $\hat{H}_p(j\omega_k)$, $p > 0$. Since the noise is independently distributed of the input, the estimates $\hat{H}_p(j\omega_k)$, $p > 0$, are uncorrelated with the terms $(H_p(j\omega_k) - \hat{H}_p(j\omega_k)) U_p(k)$ in (48). However, $\hat{H}_p(j\omega_k)$, $p > 0$, are correlated with $\hat{H}_0(j\omega_k)$ via $T_H(j\omega_k) - \hat{T}_H(j\omega_k)$ in (48). The latter decreases to zero as an $O(T^{-1/2})$ and, therefore, the correlation can asymptotically ($T \to \infty$) be neglected.

**REFERENCES**


