Detection and Quantification of the Influence of Time-Variation in Frequency Response Function Measurements Using Arbitrary Excitations

R. Pintelon, E. Louarroudi, and J. Lataire,
Vrije Universiteit Brussel, dept. ELEC, Pleinlaan 2, 1050 Brussel, Belgium
Tel. (+32) 2.629.29.44, Fax. (+32) 2.629.28.50, E-mail: Rik.Pintelon@vub.ac.be

Abstract—This paper presents a nonparametric method for detecting and quantifying the influence of time-variation in frequency response function (FRF) measurements. The method is based on the estimation of the best linear time-invariant (BLTI) approximation of a linear time-variant (LTV) system from known input, noisy output data. The key idea consists in reformulating the single-input, single-output time-variant problem as a multiple-input, single-output time-invariant problem. Beside the BLTI approximation of the LTV system, the contribution of the disturbing noise, the leakage error, and the time-varying effects at the output is also quantified. As such the approximation error of the time-invariant framework is known.

Index Terms—frequency response function, best linear time-invariant approximation, slowly time-varying, continuous-time, nonparametric modeling

I. INTRODUCTION

The linear time-invariant (LTI) framework has proven to be very successful for describing real life systems in all kinds of applications such as prediction/control in the process industry, structural health monitoring in civil engineering structures, modal analysis in mechanical engineering, monitoring of the corrosion of metals, … [1], [2], [3]. Therefore, nonparametric frequency response function (FRF) estimates are very useful to get quickly insight in the dynamic behavior of complex systems that are only approximately linear and time-invariant. While the impact of nonlinear distortions on classical FRF measurements is well understood [4], [5], [6], [7], much less is known concerning the influence of time-variation on FRF estimates. The goal of this paper is to fill this gap.

Recently it has been shown that time-variant effects act as frequency-correlated noise in classical FRF measurements using random excitations [8]. Therefore, it is difficult to distinguish them from the leakage errors: in [8] the presence of time-variant effects and leakage errors in classical FRF measurements can be detected, but they cannot be quantified. In this paper we present a method for quantifying the noise level, the leakage errors, and the level of the time-variant effects in FRF estimates from one experiment with a random excitation. It is based on the equivalence between a single-input, single-output time-variant system and a multiple-input, single-output (MISO) linear time-invariant system. Therefore, classical nonparametric methods for estimating the FRF of MISO LTI systems can be used to quantify the time-variant effects. A similar idea has been used in [9] to model periodically time-varying systems excited by periodic excitations. In this paper we estimate the best linear time-invariant approximation of arbitrary time-varying systems excited by arbitrary excitations. Compared with the classical FRF estimate via cross- and autopower spectra, the proposed method (i) estimates the best linear approximation with a (significant) smaller uncertainty; (ii) separates the leakage errors from the time-variant effects; and (iii) quantifies the approximation error of the time-invariant framework.

The paper is organized as follows. First, we give a formal definition of the best linear time-invariant approximation and discuss its basic properties (Section II). Next, it is shown that a single-input, single-output linear slowly time-varying system can be modeled exactly by a multiple-input, single output linear time-invariant system (Section III). Based on this LTI equivalence, a procedure for estimating the BLTI is proposed (Section IV). The approach is illustrated on simulation and measurement examples (Sections V and VI) and, finally, some conclusions are drawn (Section VII).

II. BEST LINEAR TIME-INARIANT APPROXIMATION OF A LINEAR TIME-VARIANT SYSTEM

In this section we recall the definition of the best linear time-invariant (BLTI) approximation of a linear time-variant (LTV) system, and discuss the basic properties that are needed in the sequel of this paper.

A. Definition BLTI

A linear time-variant (LTV) system is uniquely characterised by its response \( g(\tau,t) \) to a Dirac impulse at time instant \( \tau \) [10]. The Fourier transform of \( g(\tau,t) \) w.r.t. \( \tau \) defines the time-varying frequency response function, called system function in [10],

\[
G(j\omega,t) = \int_{-\infty}^{+\infty} g(\tau,t - \tau) e^{-j\omega \tau} d\tau.
\]  

(1)

It can be shown that the time-varying FRF (1) of an LTV system has similar properties as the FRF of an LTI system [11]:

obtained via cross- and autopower spectra equals the BLTI system and the output of the BLTI model.

![Diagram](image)

Figure 1. Best linear time-invariant (BLTI) approximation of a linear time-variant (LTV) system excited by periodic noise or random phase multisines \( U(k) \). \( Y_{TV}(k) \) is the difference between the actual output \( Y(k) \) of the LTV system and the output of the BLTI model.

\[ U(k) \xrightarrow{LTV} Y(k) \equiv U(k) \xrightarrow{BLTI} Y_{TV}(k) \xrightarrow{} Y(k) \]

1) The steady state response of a stable LTV system to \( \sin(\omega_0 t) \) is given by \( G(j\omega_0, t) \sin(\omega_0 t + \angle G(j\omega_0, t)) \).

2) The transient response \( y(t) \) to an input \( u(t) \) can be calculated as \( y(t) = L^{-1} \{ G(s, t) U(s) \} \), with \( U(s) \) the Laplace transform of \( u(t) \) and \( L^{-1} \{ \} \) the inverse Laplace transform operator.

(see [12], [13] for determining the stability of LTV systems). The second property will be used in the sequel of this paper.

The best linear time-invariant (BLTI) approximation over the time interval \([0, T]\) of a linear time-variant system is defined as

\[ G_{BLTI}(j\omega) = \frac{1}{T} \int_0^T G(j\omega, t) \, dt. \]  

(2)

Note that \( G_{BLTI}(j\omega) \) depends on \( T \) and, without additional assumptions on the time variation, nothing can be said outside the interval \([0, T]\). However, assuming – for example – that the time variation is periodic (see, for example, [14]) or an ergodic random process (see, for example, [13]), the integral in (2) can be replaced by

\[ G_{BLTI}(j\omega) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} G(j\omega, t) \, dt \]  

(3)

and \( G_{BLTI}(j\omega) \) is valid for \( t \in (-\infty, +\infty) \). Under some assumptions it can be shown that (2) is the best (in mean square sense) LTI approximation of the LTV system (see Section II-B).

B. Properties BLTI

Under the following assumptions

1) The time-varying transfer function \( G(s, t) \) can be expanded in series as

\[ G(s, t) = \sum_{p=0}^{\infty} G_p(s) f_p(t) \quad t \in (0, T) \]  

(4)

with \( f_p(t) \) a complete set of basis functions over the interval \((0, T)\).

2) The excitation \( u(t) \) is either periodic noise or a random phase multisine with period length \( T \).

3) The LTI systems \( G_p(s) \) in (4) operate in steady state, it is shown in [8] that the frequency response function (FRF) obtained via cross- and autopower spectra equals the BLTI approximation (2)

\[ G_{BLTI}(j\omega_k) = \mathbb{E} \left\{ \frac{Y(k) U(k)}{|U(k)|^2} \right\} \]  

(5)

where the expected value \( \mathbb{E} \{ \} \) is taken w.r.t. the random realization of the input \( u(t) \), \( Y(k) \) and \( U(k) \) are the discrete Fourier transform (DFT) spectra of, respectively, \( y(nT_s) \) and \( u(nT_s) \),

\[ X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_s) e^{-j2\pi kn/N} \]  

(6)

with \( T_s \) the sample period, \( T = NT_s \), and where \( x = u, y \) and \( X = U, Y \). Hence, the actual output \( Y(k) \) of the LTV system can be written as (see also Figure 1)

\[ Y(k) = G_{BLTI}(j\omega_k) U(k) + Y_{TV}(k) \]  

(7)

where the residual \( Y_{TV}(k) \) is uncorrelated with – but not independent of – the input \( U(k) \).

Discussion

- The key property for proving result (5) is that \( \mathbb{E} \left\{ \{ U(k) T_l(t) \} \right\} = 0 \) for \( k \neq l \) (see [8]). Hence, (5) is valid independent of the choice of the power spectrum of the periodic noise and random phase multisine excitation. Since FRF measurements over a finite time window \([0, T]\) are subject to leakage errors for nonperiodic band-limited excitations, (5) is true within an \( O(N^{-1}) \) bias error for filtered band-limited white noise inputs.

- The basis functions \( f_p(t) \) in (4) can always be chosen such that

\[ f_0(t) = 1 \quad \text{and} \quad \frac{1}{T} \int_0^T f_p(t) \, dt = 0 \quad \text{for} \quad p > 0. \]  

(8)

With this choice we have that \( G_{BLTI}(j\omega) = G_0(j\omega) \), the zeroth order term in (4).

- For periodic or ergodic random time-variations (5) becomes for \( T \to \infty \)

\[ G_{BLTI}(j\omega) = \frac{S_{uu}(j\omega)}{S_u(j\omega)} \]  

(9)

where \( S_{uu}(j\omega) \) is the Fourier transform w.r.t. \( \tau \) of the crosscorrelation function \( \mathbb{E} \{ y(t) u(t - \tau) \} \), and with \( G_{BLTI}(j\omega) \) the BLTI (3). For systems operating in open loop, the definition in [16] of the second-order equivalent LTI model of a time-varying nonlinear system boils down to Eq. (9).

III. TIME-INVARIANT EQUIVALENCE OF A SLOWLY TIME-VARYING SYSTEM

In the sequel of this paper we will assume that the time-variation is slow. Therefore, we first define what is meant by “slow”. Next, it is shown that the response of a slowly time-varying system can be calculated exactly by a multiple-input, single-output (MISO) linear time-invariant (LTI) model.
Finally, the relationship between the best linear time-invariant (BLTI) approximation and the equivalent MISO LTI model is established.

A. Slowly Time-Varying Systems

Since polynomials can approximate arbitrary well continuous functions with continuous derivative over a finite interval [17], it makes sense to define “slow” as follows.

**Definition 1.** The time-varying transfer function (1) of a slowly time-varying system can be written as

\[ G_s(t) = \sum_{p=0}^{N_b} G_p(s) f_p(t) \quad t \in (0, T) \]  

(10)

where \( N_b < \infty \), and where \( f_p(t) \) are polynomials of order \( p \) satisfying (8).

B. Linear Time-Invariant Equivalence

The following theorem shows that the actual output \( y(t) \) of the single-input, single-output linear slowly time-varying system (10) can be calculated as the response of an \( (N_b + 1) \)-input, single-output LTI system.

**Theorem 2.** For LTV system (10) there exist \( N_b + 1 \) LTI systems \( H_p(j\omega) \), \( p = 0, 1, \ldots, N_b \), such that

\[ y(t) = L^{-1} \left\{ \sum_{p=0}^{N_b} G_p(s) U(s) f_p(t) \right\} \]  

(11)

\[ = L^{-1} \left\{ \sum_{p=0}^{N_b} H_p(s) L{u(t) f_p(t)} \right\} \]  

(12)

where \( X(s) (X = Y, U) \) is the Laplace transform of \( x(t) (x = y, u) \), and with \( L^{-1} \{ \} \) and \( L \{ \} \), respectively, the inverse Laplace and the Laplace transform operators.

**Proof:** See Appendix A.

Eqs. (11) and (12) are visualized in Figure 2. It follows that Eq. (12) can be interpreted as the response of a multiple-input \( u_p(t) = u(t) f_p(t) \), \( p = 0, 1, \ldots, N_b \), single-output \( y(t) \) LTI system.

C. Relationship BLTI Approximation and MISO LTI Model

In the following theorem we establish the relationship between \( G_0(s) (= G_{BLTI}(s)) \) and \( H_p(s) \), \( p = 0, 1, \ldots, N_b \), for a particular choice of the polynomial basis \( f_p(t) \).

**Theorem 3.** Let \( P_p(2t/T - 1), t \in [0, T] \), be Legendre polynomials of order \( p \) (see [18]) satisfying (8). The best linear time-invariant approximation \( G_{BLTI}(s) \) of the linear slowly time-varying system (10), where \( f_p(t) = P_p(2t/T - 1) \), is then related to the transfer functions \( H_p(s) \), \( p = 0, 1, \ldots, N_b \), in (12) as

\[ G_{BLTI}(s) = H_0(s) + 2 \frac{N_b}{T} \sum_{p=1}^{N_b} \beta_{2p} H_p^{(2)}(s) + O(T^{-3}) \]  

(13)

with \( \beta_{2p} = 1.5 + 2.5(p - 1) + (p - 1)^2, [x] \) the largest integer smaller than or equal to \( x \), \( H_p^{(r)}(s) \) the \( r \)-th order derivative of \( H_p(s) \) w.r.t. \( s \), and where \( O(T^{-m}) \) with \( m > 0 \) means that \( \lim_{T \to \infty} T^m O(T^{-m}) < \infty \). The \( O(T^{-3}) \) bias term depends on the higher order derivatives \( H_p^{(r)}(s) \), with \( r, p \geq 3 \) and \( r \leq p \).

**Proof:** See Appendix B.

**Discussion**

- The orthogonal Legendre polynomials are used as basis functions because their mean value over \([-1, 1]\) is zero. To use this property, the interval \([0, T]\) must be mapped on \([-1, 1]\). This time normalization reduces the dynamic range of the powers of \( t \) and, thus, improves the numerical stability of the nonparametric estimates in Section IV.

- Due to the time normalisation, the basis functions \( f_p(t) = P_p(2t/T - 1) \) are dimensionless and independent of \( T (0, T) \) is mapped on \([-1, 1] \) via the transformation \( 2t/T - 1 \). Hence, increasing \( T \) in (10) results in a slower time-variation because the absolute time-variation over \([0, T]\) remains unchanged.

- If the time-variation is sufficiently slow \((T \to \infty)\), then \( G_{BLTI}(s) \approx H_0(s) \) and the first and second sum in (13) are, respectively, an \( O(T^{-1}) \) and \( O(T^{-2}) \) bias correction.
IV. NONPARAMETRIC ESTIMATION OF THE BEST LINEAR
TIME-INVARIANT APPROXIMATION

The estimation starts from sampled input-output data and,
therefore, we must first establish the relationship between
the input-output DFT spectra of a linear slowly time-varying
system. Next, the nonparametric estimation procedure is discussed in detail.

A. Relationship Between the Input-Output DFT Spectra

The relationship between the input and output DFT spectra is established for slowly time-varying systems excited by
band-limited inputs.

**Assumption 4.** The excitation \( u(t) \) is band-limited: the input power spectral density \( S_{uu}(j\omega) \) is zero for \( f > f_B \),
where \( f_B < f_s/2 \), with \( f_s = 1/T_s \) the sampling frequency.

**Theorem 5.** Under Assumption 4, the input-output DFT spectra \( U(k) \) and \( Y(k) \) of a slowly time-varying system (see Definition 1) are related by

\[
Y(k) = \sum_{p=0}^{N_b} H_p(j\omega_k) U_p(k) + T_H(j\omega_k) \tag{14}
\]

with \( U_p(k) = DFT\{u(nT_s) f_p(nT_s)\} \), and where \( T_H(j\omega) \) is a rational form of \( j\omega \) modeling the sum of the spectral leakage and the residual alias errors.

**Proof:** See Appendix C.

B. Estimation Procedure

The goal is (i) to obtain nonparametric estimates of the best linear time invariant (BLTI) approximation \( \hat{G}_{\text{BLTI}}(j\omega_k) \) of the linear time-varying system from known input, noisy output samples \( u(nT_s) \) and \( y(nT_s) \), \( n = 0, 1, \cdots, N-1 \) (see Figure 3); and (ii) to quantify the approximation error \( Y_{TV}(k) \) of the BLTI framework (see Figure 1). In Figure 3, \( v(t) \) is a zero mean, filtered (band-limited) white noise disturbance. The input \( u(t) \) satisfies Assumption 4, and the output \( y(t) \) is lowpass filtered before sampling.

The estimation procedure consists of the following four steps (see Figure 2 where \( G_0 = G_{\text{BLTI}} \)):

1) **Nonparametric estimation \( \hat{H}_p(j\omega_k) \):** From Theorem 5 it follows that the \( N_b + 1 \) FRFs \( \hat{H}_p(j\omega_k) \) can be estimated using nonparametric multiple-input, single-output (MISO) methods. Examples of such estimators are the spectral analysis [19], [20] and the local polynomial [21], [22] methods. We use here the local polynomial method (LPM) because, (i) it suppresses better the sum of the leakage and the residual alias errors \( T_H(j\omega_k) \) in Eq. (14)), and (ii) it estimates the FRFs at the full frequency resolution \( 1/T \) of the experiment duration \( T \).

2) **Estimation \( \hat{N}_b \):** The number \( N_b \) of parallel branches modeling the time-variation in (14) should also be identified from the data. Since the LPM method estimates the FRFs and their covariance, the following approach is used to determine \( N_b \): start with \( N_b = 1 \) and increase \( N_b \) till the estimated FRF \( \hat{H}_{N_b}(j\omega_k) \) is zero within its uncertainty \( (\|\hat{H}_{N_b}\| \approx \text{std}(\hat{H}_{N_b})) \). The number of significant parallel branches is then \( N_b - 1 \).

3) **Nonparametric estimation \( \hat{G}_{\text{BLTI}}(j\omega_k) \):** Using Eq. (13), a nonparametric estimate of the BLTI is obtained as

\[
\hat{G}_{\text{BLTI}}(j\omega_k) = \hat{H}_0(j\omega_k) + \frac{2}{T} \sum_{p=0}^{N_b-1} \hat{H}_{2p+1}^{(1)}(j\omega_k) \tag{15}
\]

where the first order derivatives are replaced by central differences

\[
\hat{H}_{m}^{(1)}(j\omega_k) = \frac{\hat{H}_m(j(\omega_k+1)) - \hat{H}_m(j(\omega_k-1))}{j}\omega_k+1 - j\omega_k-1 \tag{16}
\]

Since the central difference (16) equals the true derivative within an \( O(T^{-2}) \) bias error [23]; and since the bias error of the LPM estimate is an \( O(T^{-\infty}) \) contribution [3], [21]; the bias on \( \hat{G}_{\text{BLTI}}(j\omega_k) \) is given by

\[
E\left\{\hat{G}_{\text{BLTI}}(j\omega_k)\right\} = G_{\text{BLTI}}(j\omega_k) = -4 \frac{N_b}{T^2} \sum_{p=1}^{\lfloor N_b/2 \rfloor} \beta_{2p} H_{2p}^{(2)}(s) + O(T^{-3}) \tag{17}
\]

for \( R \geq 2 \).

4) **Nonparametric estimation \( \hat{Y}_{TV}(k) \):** Using Theorems 2 and 5, the difference between the actual output of the time-variant system and the output of the BLTI model can be estimated as

\[
\hat{Y}_{TV}(k) = \sum_{p=1}^{N_b} \hat{H}_p(j\omega_k) U_p(k) \cdots \hat{G}_{\text{BLTI}}(j\omega_k) - \hat{H}_0(j\omega_k) U(k) \tag{18}
\]

where the second term in the difference has an \( O(T^{-1}) \) contribution (see Eq. (15)).

**Notes**

- The variance estimate of \( \hat{G}_{\text{BLTI}}(j\omega_k) \) (15) requires the knowledge of the covariance of the FRF estimates \( \hat{H}_m(j\omega_k) \) over the neighboring frequencies \( k \) and \( k-1 \).
This information can easily be obtained from the residuals of the local polynomial approximation in the LPM estimates (follow the same lines of [21]).

- Using (17), the $O(T^{-2})$ bias of the BLTI estimate (15) can be reduced to an $O(T^{-3})$ as

$$G_{\text{BLTI}}(j\omega_k) = \tilde{G}_{\text{BLTI}}(j\omega_k) + \cdots$$

$$\frac{4}{T^2} \sum_{p=1}^{\lfloor \frac{N_T}{2} \rfloor} \beta_{2p} \hat{H}_{2p}^{(2)}(j\omega_k)$$  \hspace{1cm} (19)

where the second order derivatives are replaced by second order central differences

$$\hat{H}_{m}^{(2)}(j\omega_k) = \frac{\hat{H}_m(j\omega_k+2) - 2\hat{H}_m(j\omega_k) + \hat{H}_m(j\omega_k-2)}{(j\omega_k+1-j\omega_k-1)^2}$$

(use $\omega_k - \omega_l = 2\pi (k-l)/T$).

- Replacing $G_{\text{BLTI}}(j\omega_k)$ (15) by $\tilde{G}_{\text{BLTI}}(j\omega_k)$ (19) in Eq. (18) reduces the $O(T^{-2})$ bias of $\hat{Y}_{\text{TV}}(k)$ (18) to an $O(T^{-3})$.

V. SIMULATION EXAMPLES

The goal of the simulation examples is to illustrate (i) the relationship (13) between the best linear time-invariant (BLTI) approximation $G_{\text{BLTI}}(s)$ and the transfer functions $H_p(s)$ of the MISO LTI model (12); and (ii) the selection of the number of time-varying branches $N_b$ in (14). Therefore, two different structures of time-varying models are considered: (i) a parallel structure with time-varying gains (see Fig. 2, top block diagram), and (ii) a differential equation with time-varying coefficients.

A. Parallel Structure with Time-Varying Gains

The simulated LTV system is a parallel structure with time-varying gains (see Fig. 2, top block diagram) consisting of $N_b = 6$ time-varying branches, with $f_p(t)$ Legendre polynomials satisfying (8), and $G_p(s)$ second order Chebyshev filters with resonance frequency increasing from 60 Hz ($p = 0$) to 90 Hz ($p = N_b$) in steps of 5 Hz. The passband ripples of $G_{2p}(s)$ and $G_{2p+1}(s)$ are, respectively, 20 dB and 15 dB. From (2) and (10) it follows that $G_{\text{BLTI}}(s) = G_0(s)$.

As input $u(t)$ we take the sum of $F$ harmonically related sinewaves

$$u(t) = \sum_{k=1}^{F} A_k \sin(2\pi f_0 t + \phi_k)$$  \hspace{1cm} (20)

with equal amplitudes $A_k$, uniformly distributed phases $\phi_k$, and $f_0 = f_s/N$. The sampling frequency $f_s$, the number of samples per period $N$, and the number of sinewaves $F$ are chosen such that the frequency band ($0\text{Hz}, 100\text{Hz}$] is excited: $f_s = 1\text{kHz}$, $N = 1024 \times 64$, and $F = 6553$. The response $y(t)$ to one period of the random phase multisine (20) is calculated. Starting from $N$ input-output samples $u(nT_s)$, $y(nT_s)$, $n = 0, 1, \ldots, N-1$, the best linear-time invariant (BLTI) approximation is estimated following the procedure of Section IV-B with $N_b = 6$. Since the goal of the simulation is to check the bias error of the BLTI estimates (15) and (19), no noise is added to the output and the bias error of the local polynomial method [21] is kept small by choosing a 12th order local polynomial approximation of the FRF and the sum of the leakage and residual alias errors in (14).

Figure 4 shows the true BLTI approximation $G_{\text{BLTI}} = G_0$; the bias error of the BLTI estimates $\hat{H}_0$, $G_{\text{BLTI}}$ (15), and $\tilde{G}_{\text{BLTI}}$ (19); and the mean square error (mse) of the local polynomial method [21] is kept small everywhere at the MATLAB® precision. The latter is calculated via the residuals of the local polynomial approximation (see [21] for the details). Note the significant decrease in bias error of $G_{\text{BLTI}}(O(T^{-2}))$ compared with $\hat{H}_0$ ($O(T^{-1})$), and $\tilde{G}_{\text{BLTI}}$ ($O(T^{-3})$) compared with $\tilde{G}_{\text{BLTI}}$ ($O(T^{-2})$). The true bias errors of $\hat{H}_0$ and $\tilde{G}_{\text{BLTI}}$ are compared in Figure 5 to the estimated bias errors obtained via, respectively, the first order derivatives of $H_{2p+1}$ (see Eq. (15)) and the second order derivatives of $H_{2p}$ (see Eq. (19)). Note the nice agreement between the true and the estimated values.

B. Differential Equation with Time-Varying Coefficients

The simulated LTV system is a third order differential equation with time-varying coefficients

$$\sum_{p=0}^{3} a_p(t) y_0^{(p)}(t) = \sum_{p=0}^{3} b_p(t) u^{(p)}(t)$$  \hspace{1cm} (21)

where $x^{(p)}(t) = d^p x(t) / dt^p$, $x = u$, $y_0$. The poles and zeroes of the corresponding frozen transfer function
The results are shown in Figure 9. It can be seen that the bias of the BLTI estimate $\hat{G}_{\text{BLTI}} (j\omega_k)$ is smaller than its uncertainty. Note also that $\tilde{H}_p (j\omega_k)$ only differs from $\tilde{G}_{\text{BLTI}} (j\omega_k)$ in the neighborhood of the resonance frequency where the time-variation is most important.

Finally, the contribution of the time-variation to the output $\hat{Y}_{\text{TV}} (k)$ is estimated via (18). The result is shown in Fig. 7. It can be seen that the time-variation (top light grey line) is well above the estimated noise level (bottom medium grey line), and is of the same order of magnitude as the leakage errors (middle dark grey line). Note also that $\hat{Y}_{\text{TV}} (k)$ is about 20 dB below $Y (k)$, showing that the BLTI (see Fig. 1) can predict the actual output of the LTV system within an error of 10%.

To emphasize the difference of the proposed approach with classical FRF measurements [8], we also estimated (14) with $N_0 = 0$, which coincides with the spectral analysis estimate (5) (proof: see [8]). The result is shown in Figure 10. It can be seen that the standard deviation of the classical FRF measurement (dark grey line) is well above that of the BLTI estimate (black dashed line). It follows that $Y_{\text{TV}} (k)$ is the dominant error term in the classical FRF measurement. Since $Y_{\text{TV}} (k)$ depends linearly on the input (see (18)), increasing
the input rms value will not lower the standard deviation of the classical FRF estimate. This is a major difference with the behavior of the filtered white noise contribution $V(k)$. Note, however, that increasing the input rms value decreases the standard deviation of the BLTI estimates (15) and (19) with $N_b = 3$.

VI. MEASUREMENT EXAMPLE

A. Measurement Setup

The device under test is a time-varying electronic circuit (see Fig. 11) that behaves as a second order bandpass filter with time-varying resonance frequency and damping ratio. The time-variation is controlled via the gate voltage $p(t)$ (scheduling parameter) of the JFET transistor. The input signal $u(t)$ is a random phase multisine (20) with an rms value of 100 mV. The multisine parameters are chosen such that $u(t)$ (20) consists of 522 harmonics with equal amplitudes in the frequency band $[228.9\ \text{Hz}, 39.98\ \text{kHz}]$ ($f_0 = f_s/P, f_s = 62.5\ \text{kHz}, P = 8192, F = 524, A_1 = A_2 = 0, \text{ and } A_3 = A_4 = \cdots = A_F$). A periodic ramp $p(t)$ is applied to the gate of the JFET transistor (period $1/f_0$ and voltage decreasing linearly from $-1.66\ \text{V}$ to $-1.86\ \text{V}$). $N = 15P/16 = 7680$ input-output samples of the steady state response are acquired using a band-limited measurement setup ($u(t)$ and $y(t)$ are lowpass filtered before sampling). Figure 12 shows the corresponding input-output DFT spectra (6). Note that during the acquisition time the gate voltage $p(t)$ decreased linearly from $-1.66\ \text{V}$ to $-1.85\ \text{V}$.

B. Results

Starting from the $N$ known input, noisy output samples $u((nT_s)\quad \text{and}\quad y((nT_s)\quad n = 0, 1, \cdots, N - 1$, the MISO model (14), with $f_p(t)$ Legendre polynomials satisfying (8), is estimated following the procedure of Section IV-B (local polynomial method [21] with a 6th order local polynomial approximation of the FRFs and the leakage term, and six degrees of freedom for the variance estimates). Figure 13 shows the results for $N_b = 2$. Increasing $N_b$ further makes no sense because $|H_3(j\omega_k)| \sim \text{std}(H_3(j\omega_k))$ for all values of $k$.

The nonparametric estimates of the best linear time-invariant (BLTI) approximation are obtained via Eqs. (15) and (16). From Figure 14 it can be seen that the bias of the BLTI estimate $G_{\text{BLTI}}(j\omega_k)$ is smaller than its uncertainty. Note also that $H_0(j\omega_k)$ only differs from $G_{\text{BLTI}}(j\omega_k)$ in the frequency band where the time-variation is the largest.

The contribution of the time-variation to the output $\hat{Y}_{TV}(k)$ is estimated via (18). From Figure 12 it can be seen that the time-variation (middle light grey line) is well above the
accurately the time variation at the output. The latter sets the BLTI approximation, also the noise level, the leakage errors, single-output linear time-variant system. In addition to the

\[ H_{BLTI}(j\omega_k) \]

(left grey: difference \( G_{BLTI}(j\omega_k) \) (black), its variance \( \text{var}(G_{BLTI}(j\omega_k)) \) (dark grey), and its bias (19) \( G_{BLTI}(j\omega_k) - G_{BLTI}(j\omega_k) \) (light grey).

Compared with the classical FRF estimates, the variance of the proposed procedure is (much) smaller in those frequency bands where the time-variation dominates the noise error. Outside these bands the variances of both nonparametric estimates are equal.

**ACKNOWLEDGEMENT**

This work is sponsored by the Research Council of the Vrije Universiteit Brussel, the Research Foundation Flanders (FWO-Vlaanderen), the Flemish Government (Methusalem Fund METH1), and the Belgian Federal Government (Inter-university Attraction Poles programme IAP VII).

**APPENDIX**

A. Proof of Theorem 2

First, we prove Theorem 2 for a particular choice of the polynomial basis \( f_p(t) = t^p \). Next, we generalize the result to any polynomial basis satisfying (8).

Direct application of property 2 of Section II-A to the slowly time-varying system (10) gives Eq. (11). Elaborating (11) we find

\[
y(t) = \sum_{p=0}^{N_b} y_p(t) f_p(t)
\]

(23)

where \( y_p(t) = \mathcal{L}^{-1}\left\{G_p(s) U(s)\right\} \). Using \( f_p(t) = t^p \) and \( L\{t^p x(t)\} = (-1)^p X^{(p)}(s) \), where \( X^{(p)}(s) \) is the \( p \)th order derivative of \( X(s) \) w.r.t. \( s \), the Laplace transform of each term in the sum (23) can be written as

\[
L\{y_p(t) f_p(t)\} = (-1)^p \frac{d^p}{ds^p} \left(G_p(s) U(s)\right)
\]

(24)

with \( C_p^q = p! / (q! (p-q)!) \). Combining the Laplace transform of (23) with (24) gives

\[
Y(s) = \sum_{p=0}^{N_b} \sum_{q=0}^{p} (-1)^p C_p^q G_p^{(p-q)}(s) U^{(q)}(s)
\]

(25)

with \( \hat{H}_q(s) = \sum_{p=0}^{N_b} (-1)^{p-q} C_p^q G_p^{(p-q)}(s) \) and \( L\{t^p u(t)\} = (-1)^p U^{(q)}(s) \). By making appropriate linear combinations of \( t^p \), (25) can be written as (12), where \( f_q(t) \) is a polynomial of order \( q \) satisfying (8), and with \( \hat{H}_q(s) \) an appropriate linear combination of the \( \hat{H}_p(s) \), \( p = 0, 1, \ldots, N_b \).
B. Proof of Theorem 3

Since the calculations are quite tedious, the theorem will only be proven here for \( N_0 = 3 \). The cases \( N_0 > 3 \) follow exactly the same lines.

From (12) it follows that the Laplace transform of the output is given by

\[
Y(s) = H_0(s)U(s) + H_1(s)U_1(s) + H_2(s)U_2(s) + H_3(s)U(s) \tag{26}
\]

where \( U_p(s) = L\{u(t)f_p(t)\} \). To simplify the calculations we replace without any loss in generality the interval \([0, T]\) by \([-T/2, T/2]\). Hence, the basis functions \( f_p(t) \) are related to the Legendre polynomials as \( f_1(t) = \alpha_t, f_2(t) = (3\alpha_t^2 - 1)/2, \) and \( f_3(t) = (5\alpha_t^3 - 3\alpha_t)/2, \) where \( \alpha = 2/T \). In the sequel of this appendix we elaborate the second, the third, and the fourth term in (26).

Using \( L\{y(t)\} = -U'(s) \) and \( f(s)g'(s) = (f(s)g(s))' - f'(s)g(s) \), with \( t = d/ds \), the second term in (26) can be rewritten as

\[
H_1(s)U_1(s) = -\alpha(H_1(s)U(s))' + \alpha H_1'(s)U(s)
= L\{f_1(t)z_1(t)\} + \alpha H_1'(s)U(s) \tag{27}
\]

where \( z_1(t) = L^{-1}\{H_1(s)U(s)\} \). Using \( L\{t^2u(t)\} = U''(s) \) and \( f(s)g''(s) = (f(s)g(s))'' - 2(f'(s)g(s))' + f''(s)g(s) \) we find in a similar way an expression for the third term in (26)

\[
H_2(s)U_2(s) = L\{f_2(t)z_2(t)\} + 3\alpha L\{f_1(t)w_2(t)\}
+ 1.5\alpha^2 H''_2(s) \tag{28}
\]

where \( z_2(t) = L^{-1}\{H_2(s)U(s)\} \) and \( w_2(t) = L^{-1}\{H'_2(s)U(s)\} \). Following the same lines the fourth term becomes

\[
H_3(s)U(s) = L\{f_3(t)z_3(t)\} + 5\alpha L\{f_2(t)w_3(t)\}
+ 7.5\alpha^2 L\{f_1(t)v_3(t)\} + \alpha H_3'(s)U(s)
+ 2.5\alpha^3 H'''_3(s)U(s) \tag{29}
\]

where \( z_3(t) = L^{-1}\{H_3(s)U(s)\} \), \( w_3(t) = L^{-1}\{H'_3(s)U(s)\} \), and \( v_3(t) = L^{-1}\{H''_3(s)U(s)\} \).

Collecting (11), (12), (26), (27), (28), and (29) gives

\[
G_0(s) = H_0(s) + \alpha H_1'(s) + \alpha H_1'(s)
+ 1.5\alpha^2 H''_2(s) + 2.5\alpha^3 H'''_3(s)
\]

where \( 2.5\alpha^3 H'''_3(s) = O(T^{-3}) \). This concludes the proof since \( G_0(s) = G_{BLTI}(s) \).

C. Proof of Theorem 5

Under Assumption 4, the Fourier spectrum of the windowed signals \( w(t)tpu(t) \), with \( w(t) = 1 \) for \( t \in (0, T) \) and zero elsewhere, is no longer band-limited. However, it is a smooth function of the frequency for \( f > f_B \). Indeed,

\[
F\{w(t)tpu(t)\} = F\{w(t)tp\} * F\{w(t)u(t)\}
\]

with \( F\{\} \) the Fourier transform and \( \ast \) the convolution product;

and where \( F\{w(t)tp\} \) and \( F\{w(t)u(t)\} \) are smooth functions of the frequency for \( f > 0 \). Therefore, at the DFT frequencies \( \omega_k = 2\pi k f_B/N \), the alias errors in the band \((0, f_B)\) can be approximated arbitrarily well by a polynomial in \( j\omega \). Hence, following the same lines as in [3], [24], [25], we find at the DFT frequencies \( \omega_k \)

\[
F\{w(t)u_p(t)\}\big|_{\omega_k} = \frac{\sqrt{N}}{fs}\text{DFT}\{u_p(nT_s)\} + I(j\omega_k) \tag{30}
\]

with \( u_p(t) = tpu(t) \) and \( I(j\omega) \) a polynomial in \( j\omega \). Eq. (30) shows that the results of [3], [25] also apply to the MISO LTI system (12), which proves (14).

REFERENCES


