Nonparametric regression estimation :

an empirical process approach to uniform in bandwidth consistency of kernel–type estimators and conditional U–statistics

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I. A short introduction

Nonparametric regression estimation

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Nonparametric regression estimation

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Examples :

- weight $\longleftrightarrow$ length
- length $\longleftrightarrow$ nationality
- life expectancy $\longleftrightarrow$ age of grandparents
- type of car $\longleftrightarrow$ city in which you live
- salary $\longleftrightarrow$ education level
- etc...

Nonparametric regression estimation

In mathematical words :

Let (X, Y) be a random vector where Y is assumed to be related to X. Then we denote the regression function of Y w.r.t. X as

$$m(t) = \mathbb{E}[Y|X = t].$$

read it as : *"the expected value of* Y *given that* $X = t$*".*

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- let X = number of years of study

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Example : expected salary in function of the study time

- let $Y =$ salary
- let $X =$ number of years of study

$$\Longrightarrow \quad m(4) = \mathbb{E}[Y|X = 4] = \text{expected salary if you have studied for 4 years}$$

Nonparametric regression estimation

Estimation :

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  - something more sophisticated
- we will focus on the so–called "kernel–type estimators".

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nonparametric = the idea of making no assumption about underlying distribution of the variables

- in contrast with parametric models described only by parameters
- much more generally applicable than parametric estimation
- many many methods available, e.g. : histogram, averages, splines, kernel–type estimators, ...

II. Consistency

Consistency

What is consistency ?

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Example : estimators for the mean length of coffee beans

- take an average of the lengths of the beans in the sample
  $\hookrightarrow$ consistent
- take the longest bean in the sample and divide it by 2
  $\hookrightarrow$ (generally) not consistent

Consistency

In mathematical words :

a consistent estimator = an estimator that converges to the estimated quantity as the sample size $n \to \infty$.

More precisely, let $\hat{m}_n(t)$ be an estimator for $m(t)$ based upon i.i.d. random variables $(X_1, Y_1), \ldots, (X_n, Y_n)$. Then $\hat{m}_n(t)$ is consistent if

$$\hat{m}_n(t) \longrightarrow m(t), \quad \text{when } n \to \infty.$$

III. Kernel–type estimators

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- $\int K(u)du = 1$,
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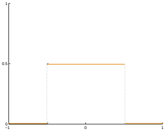
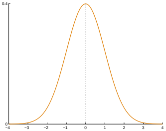
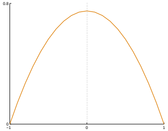


Uniform kernel



Gaussian kernel



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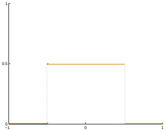

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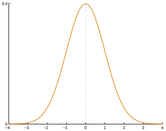

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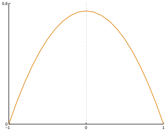

Epanechnikov kernel

(2) a bandwidth :

- a number $0 < h < 1$,
- usually (and necessarily for consistency) $h_n \to 0$.

Kernel–type estimators

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be i.i.d. random variables in $\mathbb{R}^d \times \mathbb{R}^r$, and $\varphi : \mathbb{R}^r \to \mathbb{R}$ a measurable function. A typical kernel–type estimator looks like

$$\hat{\varphi}_{n,h}(t) = \frac{1}{nh^d} \sum_{i=1}^{n} \varphi(Y_i) K\Big(\frac{t - X_i}{h}\Big), \quad t \in \mathbb{R}^d.$$

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- **role of the kernel** : determines the shape of the weights that are assigned in a neighborhood of the observation X_i
- **role of the bandwith** : controls the size of the neighborhood

Examples of particular interest

1. the kernel density estimator

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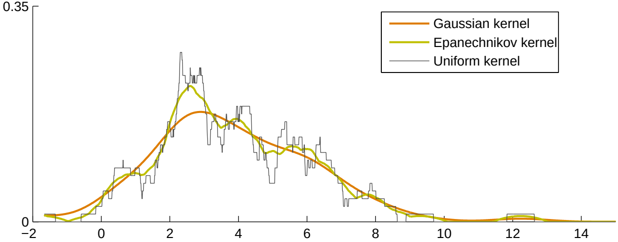

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2. the Nadaraya–Watson–type estimator

$$\hat{m}_{n,h,\varphi}(t) = \frac{\sum_{i=1}^{n} \varphi(Y_i) K\left(\frac{t-X_i}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{t-X_i}{h}\right)}.$$

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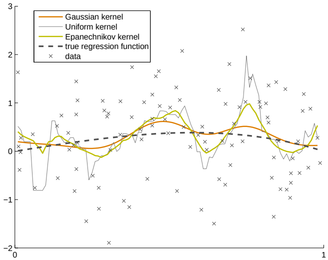

Three NW estimators for

$$m_\varphi(t) = t(1 - t^2)$$

based upon three different kernel functions.

Examples of particular interest

3. the conditional U–statistic of order $m \leq n$

$$\hat{m}_{n,h,\varphi}(\mathbf{t}) := \frac{\sum_{\mathbf{i}\in I_n^m} \varphi(Y_{i_1},\ldots,Y_{i_m}) K\left(\frac{t_1 - X_{i_1}}{h}\right)\cdots K\left(\frac{t_m - X_{i_m}}{h}\right)}{\sum_{\mathbf{i}\in I_n^m} K\left(\frac{t_1 - X_{i_1}}{h}\right)\cdots K\left(\frac{t_m - X_{i_m}}{h}\right)},$$

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- generalization of the Nadaraya–Watson–type estimators $(m = 1)$
- useful to estimate "multivariate" regression functions

$$m_\varphi(\mathbf{t}) = \mathbb{E}[\varphi(Y_1,\ldots,Y_m)|(X_1,\ldots,X_m) = \mathbf{t}], \mathbf{t} \in \mathbb{R}^m.$$

Examples of particular interest

Examples :

- conditional moments : take $\varphi(y_1) = y_1^k$, then for $\mathbf{t} = t$,

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- conditional variance : take $\varphi(y_1, y_2) = (y_1 - y_2)^2/2$, then for $\mathbf{t} = (t, t)$,

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- conditional probabilities : take $\varphi(y_1, y_2) = \mathbb{I}\{y_1 > y_2\}$, then for $\mathbf{t} = (t_1, t_2)$,

$$m_\varphi(\mathbf{t}) = \mathbb{P}\{Y_1 > Y_2|X_1 = t_1, X_2 = t_2\}.$$

Examples of particular interest

Known results when $h_n \to 0$:

- uniform consistency : under some appropriate assumptions, $\hat{f}_{n,,h_n}(t)$, $\hat{m}_{n,,h_n,\varphi}(t)$ and $\hat{m}_{n,,h_n,\varphi}(\mathbf{t})$ are uniformly consistent estimators, i.e.,

$$\sup_{t\in\mathbb{R}^d} |\hat{f}_{n,h_n}(t) - f_X(t)| \longrightarrow 0, \quad \text{a.s.}$$

$$\sup_{t\in I}\sup_{\varphi\in\mathcal{F}} |\hat{m}_{n,h_n,\varphi}(t) - m_\varphi(t)| \longrightarrow 0, \quad \text{a.s.}$$

$$\sup_{\mathbf{t}\in I^m} |\hat{m}_{n,h_n,\varphi}(\mathbf{t}) - m_\varphi(\mathbf{t})| \longrightarrow 0, \quad \text{a.s.}$$

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Known results when $h_n \to 0$:

- convergence rates : under some appropriate assumptions, it holds when in particular $h_n \to 0$ and $nh_n^d / \log n \to \infty$ that

$$\sup_{t \in \mathbb{R}^d} |\hat{f}_{n,h_n}(t) - \mathbb{E}\hat{f}_{n,h_n}(t)| = O\left(\sqrt{\frac{|\log h_n|}{nh_n^d}}\right), \quad \text{a.s.}$$

$$\limsup_{n \to \infty} \sup_{\varphi \in \mathcal{F}} \sup_{t \in I} \frac{\sqrt{nh_n^d}|\hat{m}_{n,h_n,\varphi}(t) - \widehat{\mathbb{E}}\hat{m}_{n,h_n,\varphi}(t)|}{\sqrt{|\log h_n|}} < \infty, \quad \text{a.s.}$$

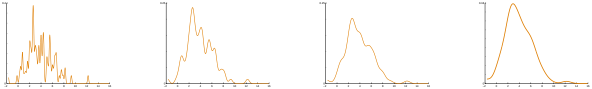
$$\sup_{\mathbf{t} \in I^m} |\hat{m}_{n,h_n,\varphi}(\mathbf{t}) - m_\varphi(\mathbf{t})| = O\left(\sqrt{\frac{\log n}{nh_n^{2m}}}\right), \quad \text{a.s.}$$

IV. A brief motivation for UiB results

Motivation of UiB

Issue : choice of the bandwidth

- kernel density estimator



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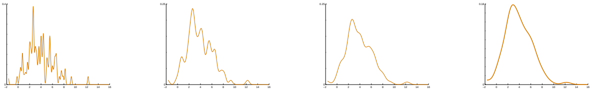

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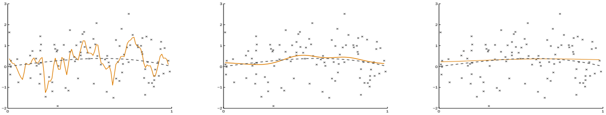

$h = 0.01$ $h = 0.1$ $h = 0.3$

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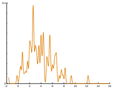
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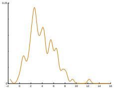
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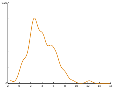
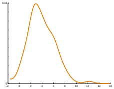

$h = 0.5$


$h = 0.8$

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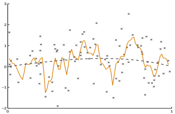
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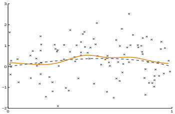
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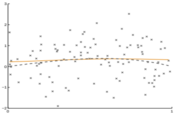
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$\Rightarrow$ **bias–variance trade–off**

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Question : what is the optimal bandwidth ?

- a huge amount of literature exists about how to find the "optimal bandwidth" for $\hat{m}_{n,h,\varphi}(t)$, e.g.,
  - cross validation : Härdle and Marron (1985)
  - plug–in methods : Tsybakov (1987)
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$\Rightarrow$ classical consistency results with fixed $h_n \to 0$ do not apply

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- **Application to random bandwidths :**

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$\hookrightarrow$ we do not lose the consistency as long as $\hat{h}_n \in [\frac{c_n \log n}{n}, b_n]$.

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Motivation of UiB

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Suppose that : you go for a coffee break, so

- you bring your favorite coffee ,
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$\Longrightarrow$  is useless, and thus no coffee.

Motivation of UiB

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Imagine : a universal wondermachine able to make coffee with any sort of coffee system :

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Then this wondermachine would be called a

"uniform–in–coffee–system" coffee machine...

Motivation of UiB

UiCS coffee machines $\longleftrightarrow$ UiB results :

Aim :	make your favorite coffee 	analyze estimators based upon optimal bandwidths
existing tool : (useless)	 for 	consistency results for $h_n \rightarrow 0$
solution :	 , , , 	UiB consistency results $a_n \leq h \leq b_n$

V. A general UiB result

General Theorem [GT]

The first UiB result for general processes of the form

$$W_{n,h,\varphi}(t) := \sum_{i=1}^{n} \{c_\varphi(t)\varphi(Y_i) + d_\varphi(t)\} K\left(\frac{t - X_i}{h}\right)$$
$$- n\mathbb{E}\{c_\varphi(t)\varphi(Y) + d_\varphi(t)\} K\left(\frac{t - X}{h}\right)$$

was proved by Einmahl and Mason (2005).

General Theorem [GT]

Under the "Usual Assumptions [UA]", it holds that with probability 1,

$$\limsup_{n\to\infty} \sup_{c(\log n/n)^{\gamma} \le h \le b_0} \sup_{\varphi\in\mathcal{F}} \sup_{t\in I} \frac{|W_{n,h,\varphi}(t)|}{\sqrt{nh^d(|\log h| \vee \log\log n)}} = O(1).$$

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The General Theorem implies the UiB consistency of

- kernel density estimators $\hat{f}_{n,h}(t)$,
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- kernel–type estimators at a fixed point $\hat{\varphi}_{n,h}(t)$,
- the kernel–based Hill estimator $\hat{\tau}_{n,h}$.

Main steps of the proof :

Standard Methodology – [SM]

Standard Methodology [SM]

STEP 1 : empirical process representation

$$\begin{aligned} W_{n,h,\varphi}(t) &= \sum_{i=1}^{n}\{c_\varphi(t)\varphi(Y_i)+d_\varphi(t)\}K\left(\frac{t-X_i}{h}\right) \\ &\qquad -n\mathbb{E}\{c_\varphi(t)\varphi(Y)+d_\varphi(t)\}K\left(\frac{t-X}{h}\right) \\ &= \sqrt{n}\alpha_n(g), \end{aligned}$$

where α_n is the empirical process indexed by a suitable subclass $\mathcal{G}_n$ of the class of functions

$$\mathcal{G}=\{(x,y)\mapsto c_\varphi(t)\varphi(y)K\left(\frac{t-x}{h}\right):\varphi\in\mathcal{F},t\in I,0<h<1\}.$$

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STEP 2 : blocking argument and moment bound

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$$\mathbb{E} G_n^2(X,Y) \equiv \beta_{k,j}^2 \quad \text{and} \quad \sup_{g \in \mathcal{G}_{n_k,j}} \mathbb{E} g^2(X,Y) \leq 2^j C_1 \beta_{k,j}^2 a_{n_k}^d =: \sigma_{k,j}^2.$$

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- Moment bound of Einmahl and Mason (2005) :

$$\mathbb{E} \| \sum_{i=1}^{n_k} \varepsilon_i g(X_i, Y_i) \|_{\mathcal{G}_{n_k,j}}^2 \leq n_k a_{n_k}^d (|\log a_{n_k}| \vee \log\log n_k) =: \lambda_{k,j}^2.$$

Standard Methodology [SM]

STEP 3 : exponential inequality

- Talagrand's exponential inequality (1994) with $t = \rho\lambda_{k,j}, \rho > 1$:

$$\mathbb{P}\left\{ \max_{n_{k-1}<n\leq n_k} \|\sqrt{n}\alpha_n(g)\|_{\mathcal{G}_{n_k,j}} \geq A_1(1+\rho)\lambda_{k,j} \right\} \leq 4(\log n_k)^{-A_2\rho}.$$

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- Set $L(k) = \max\{j \geq 0 : 2^j a_{n_k} \leq 2b_0\}$, then

$$[a_{n_k}, b_0] \subseteq [a_{n_k}, 2^{L(k)}a_{n_k}] \quad \text{and} \quad \mathcal{G}_{n_k} \subseteq \bigcup_{j=0}^{L(k)} \mathcal{G}_{n_k,j},$$

and

$$L(k) \leq 2\log n_k.$$

Standard Methodology [SM]

STEP 4 : Borel–Cantelli

- a little calculation yields that

$$\mathbb{P}\left\{ \max_{n_{k-1}<n\le n_k} \sup_{a_n\le h\le b_0} \frac{\|\sqrt{n}\alpha_n(g)\|_{\mathcal{G}_{n_k}}}{\sqrt{nh^d(|\log h| \vee \log\log n)}} \ge A_1(1+\rho) \right\}$$
$$\le \sum_{j=0}^{L(k)} 4(\log n_k)^{-A_2\rho} \le 8(\log n_k)^{1-A_2\rho}.$$

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$$\le\sum_{j=0}^{L(k)}4(\log n_k)^{-A_2\rho}\ \le\ 8(\log n_k)^{1-A_2\rho}.$$

- Borel–Cantelli implies for a large enough $\rho>1$ that wp.1,

$$\limsup_{k\to\infty}\max_{n_{k-1}<n\le n_k}\sup_{a_n\le h\le b_0}\sup_{\varphi\in\mathcal{F}}\sup_{t\in I}\frac{|W_{n,h,\varphi}(t)|}{\sqrt{nh^d(|\log h|\vee\log\log n)}}<\infty.$$