On $i$-tight sets of the Hermitian polar space with small parameter $i$

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Finite classical polar spaces

- point-line geometry
- one or all axiom
- classical examples: associated to a sesquilinear or quadratic form on a vector space.
Let $\mathbb{F}_q$ be the finite field of order $q$.
Let $V(d, q)$ be the $d$-dimensional vector space over $\mathbb{F}_q$.
Let $f$ be a non-degenerate sesquilinear or non-singular quadratic form on $V(d, q)$.

Definition

The finite classical polar space $\mathcal{P}$ associated to $f$ is the geometry of totally isotropic/totally singular subspaces with respect to $f$. The Witt index of $f$ is the rank of the polar space.

Finite classical polar spaces are naturally embedded in the projective space $\text{PG}(d, q)$. 
Definition

A *generator* is a subspace of maximal dimension.

A polar space of rank $r > 1$ is a geometry with points, lines, \ldots, $r - 1$-dimensional projective spaces.
Finite classical polar spaces

<table>
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<th>form</th>
<th>polar space</th>
<th>notation</th>
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<tr>
<td>quadratic</td>
<td>orthogonal</td>
<td>$Q(2n, q)$, $Q^-(2n+1, q)$, $Q^+(2n+1, q)$</td>
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<tr>
<td>alternating</td>
<td>symplectic</td>
<td>$W(2n+1, q)$</td>
</tr>
<tr>
<td>hermitian</td>
<td>hermitian</td>
<td>$H(n, q^2)$</td>
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- **Orthogonal forms:** quadratic when $q$ is even, both quadratic and bilinear when $q$ is odd.
- **Symplectic polar space:** is isomorphic with parabolic quadric when $q$ is even.
### Finite classical polar spaces

<table>
<thead>
<tr>
<th>Polar Space</th>
<th>Rank</th>
<th>Points</th>
<th>Generators</th>
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<tr>
<td>$Q(2n, q)$</td>
<td>$n$</td>
<td>$(q^n + 1) \frac{q^n - 1}{q - 1}$</td>
<td>$\prod_{i=1}^{n} (q^i + 1)$</td>
</tr>
<tr>
<td>$W(2n - 1, q)$</td>
<td>$n$</td>
<td>$(q^n + 1) \frac{q^n - 1}{q - 1}$</td>
<td>$\prod_{i=1}^{n} (q^i + 1)$</td>
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<tr>
<td>$H(2n + 1, q^2)$</td>
<td>$n$</td>
<td>$(q^{2n+1} + 1) \frac{q^{2n+2} - 1}{q^2 - 1}$</td>
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Strongly regular graphs

Definition

A graph $\Gamma$ on $v$ vertices is a strongly regular $(v, k, \lambda, \mu)$-graph if

- the valency is constant $k$ for every vertex,
- every two adjacent vertices have exactly $\lambda$ common adjacent vertices,
- every two non-adjacent vertices have exactly $\mu$ adjacent vertices.
Definition

Let $\mathcal{P}$ be a polar space with $\nu$ points. Define the graph $\Gamma$ with vertices the points of $\mathcal{P}$ and two different vertices being adjacent if and only if they are collinear as points in $\mathcal{P}$.

Theorem (proof: see e.g. Brouwer, Cohen, Neumaier)

The 1-adjacency matrix has exactly three eigenvalues, $\epsilon^-$, $k$, $\epsilon^+$, and $\mathbb{R}^\nu$ is the sum of the eigenspaces.
Definition (S.E. Payne, 1987)

A point set $A$ of a finite generalized quadrangle is *tight* if on average, each point of $A$ is collinear with the maximum number of points of $A$. 
Tight sets

Definition (S.E. Payne, 1987)
A point set $A$ of a finite generalized quadrangle is tight if on average, each point of $A$ is collinear with the maximum number of points of $A$.

Definition (not a formal definition!)
An $i$-tight set is a set of points that behaves as if it is the disjoint union of $i$ generators.
Tight sets

Definition

Let \( P \) be a polar space of rank \( r \) over \( \mathbb{F}_q \). A set \( T \) of points is an \( i \)-tight set of \( P \) if the following holds:

\[
|P^\perp \cap T| = \begin{cases} 
    i \frac{q^{r-1} - 1}{q-1} + q^{r-1} & \text{if } P \in T \\
    i \frac{q^{r-1} - 1}{q-1} & \text{if } P \notin T
\end{cases}
\]
Tight sets

Definition

Let $\mathcal{P}$ be a polar space of rank $r$ over $\mathbb{F}_q$. A set $\mathcal{T}$ of points is an $i$-tight set of $\mathcal{P}$ if the following holds:

$$|\mathcal{P} \perp \cap \mathcal{T}| = \begin{cases} \frac{iq^{r-1} - 1}{q-1} + q^{r-1} & \text{if } P \in \mathcal{T} \\ \frac{iq^{r-1} - 1}{q-1} & \text{if } P \notin \mathcal{T} \end{cases}$$

Theorem (Bamberg et al., after Delsarte et al.)

The characteristic vector of a tight set is orthogonal to one of the eigenspaces.
Tight sets: examples in hermitian polar spaces

Lemma (many references)

- The set of points of $\mathbb{W}(2n + 1, q)$ embedded in $\mathbb{H}(2n + 1, q^2)$ is a $(q + 1)$-tight set.
- The set of points of $\mathbb{H}(2n - 1, q^2)$ embedded in $\mathbb{H}(2n, q^2)$ is a $(q^{2n-1} + 1)$-tight set.
Tight sets: examples in hermitian polar spaces

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- The set of points of $W(2n + 1, q)$ embedded in $H(2n + 1, q^2)$ is a $(q + 1)$-tight set.
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Lemma (many references as well)

Let $q$ be odd. The set of points of $Q(2n, q)$ embedded in $H(2n, q^2)$ is a $(q + 1)$-tight set.
We consider the polar space $H(4, q^2)$. If $q$ is odd, two examples of $(q + 1)$-tight sets.

A non-degenerate hyperplane section yields a $q^3 + 1$-tight set.

Natural question: what about $i$-tight sets, $i < q + 1$?
Theorem (DB–Metsch, 201x)

An \( i \)-tight set, \( i < q + 1 \) of \( \mathbb{H}(4, q^2) \), is the disjoint union of \( i \) lines of \( \mathbb{H}(4, q^2) \).
Small tight sets

Theorem (DB–Metsch, 201x)

An \( i \)-tight set, \( i < q + 1 \) of \( H(4, q^2) \), is the disjoint union of \( i \) lines of \( H(4, q^2) \).

Conjecture

A \( q + 1 \)-tight set of \( H(4, q^2) \) is the set of points of a sub generalized quadrangle of order \( q \).
Small tight sets

Theorem (DB–Metsch, 201x)

An $i$-tight set, $i < q + 1 - \sqrt{2q}$ of $H(6, q^2)$, is the disjoint union of $i$ planes of $H(6, q^2)$. 
Theorem (DB–Metsch, 201x)

An $i$-tight set, $i < q + 1 - \sqrt{2q}$ of $H(6, q^2)$, is the disjoint union of $i$ planes of $H(6, q^2)$.

Conjecture

An $i$-tight set, $i < q + 1$ of $H(2n, q^2)$, is the disjoint union of $i$ generators of $H(2n, q^2)$. 
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