Do $i$-tight sets and $m$-ovoids hate each other?

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(joint work John Bamberg and Ferdinand Ihringer)

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Galois geometry

- $\text{PG}(d, q)$: projective space of dimension $d$ over finite field $\text{GF}(q)$, elements are subspaces of dimension at least 1 of the $d + 1$ dimensional vector space over $\text{GF}(q)$.

- Sesquilinear and quadratic forms: totally isotropic elements of underlying vector space make a nice geometry: *classical polar space*. 
Galois geometry

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- Sesquilinear and quadratic forms: totally isotropic elements of underlying vector space make a nice geometry: \textit{classical polar space}. 
Polar spaces

- Witt index of underlying form = rank of polar space
- subspaces of maximal dimension: *generators*
Substructures

**Definition**

An $m$-ovoid is a set $O$ of points such that every generator meets $O$ in exactly $m$ points.
Notation: \( \theta_{r-1}(q) := \frac{q^{r-1}}{q-1} \) = number of points in an \( r - 1 \) dimensional projective space.

Definition
An \( i \)-tight set \( T \) of a finite classical polar space \( P \) of rank \( r \geq 2 \), is a set of \( i\theta_{r-1}(q) \) points, such that

\[
|P^\perp \cap T| = \begin{cases} 
  i\theta_{r-2}(q) + q^{r-1} & \text{if } P \in T \\
  i\theta_{r-2}(q) & \text{if } P \notin T.
\end{cases}
\]
Substructures

Notation: $\theta_{r-1}(q) := \frac{q^r - 1}{q - 1} = \text{number of points in an } r - 1 \text{ dimensional projective space.}$

**Definition**

An $i$-tight set $\mathcal{T}$ of a finite classical polar space $\mathcal{P}$ of rank $r \geq 2$, is a set of $i\theta_{r-1}(q)$ points, such that

$$|P^\perp \cap \mathcal{T}| = \begin{cases} 
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\end{cases}$$
Graphs and eigenspaces

- **collinearity graph** of polar space: strongly regular graph: three eigenspaces (two non-trivial).
- \(m\)-ovoid: orthogonal to first non-trivial eigenspace.
- \(i\)-tight set: orthogonal to second non-trivial eigenspace.

Results of several people are relevant: J. Bamberg, M. Law, S. Kelly, T. Penttila.
Ovoids of projective spaces

- Ovoids in projective spaces (general dimension, arbitrary field): introduced by J. Tits
- Ovoids do not exist in projective spaces of rank more than three (J. Tits)
- Ovoids in $\text{PG}(3, q)$: combinatorial definition is possible (and used frequently).

**Definition**

An ovoid of $\text{PG}(3, q)$ is a set of $q^2 + 1$ points no three collinear.
Ovoids of polar spaces

- Original definition: see $m$-ovoid.
- Ovoids in finite classical polar spaces and generalized quadrangles: introduced by J. Thas.
- Ovoids seem to be *rare* in polar spaces of high rank.
- There are some cases where existence can be proven relatively easy.
- There are some cases where non-existence can be proven relatively easy.
Non-existence results

By one proof (Essentially due to J. Thas):
- $Q^{-}(2n+1, q), \, n \geq 2$
- $W(2n+1, q), \, n \geq 2$
- $H(2n, q^2), \, n \geq 2$

With (some) extra work:
- $W(3, q), \, q \text{ odd (J. Thas)}$
- $Q(2n, q) \, n \geq 4 \text{ (A. Gunawardena and E. Moorehouse)}$
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Existence results

By easy observation:
- $Q(4, q)$
- $H(3, q^2)$

By hard work:
- $Q(6, q)$, $q = 3^h$: from polarity of Split Cayley Hexagon
- $Q^+(7, q)$, some particular values of $q$: using spreads and triality
**Difficult cases**

- $Q^+(2n + 1, q)$, $n \geq 4$.
- $H(2n + 1, q^2)$, $n \geq 2$.

**Theorem (A. Blokhuis, G.E. Moorehouse)**

The hyperbolic quadric $Q^+(2n + 1, q)$, $q = p^h$, $n \geq 3$ has no ovoids if

$$p^n > \left( \frac{2n + p}{2n + 1} \right)^2 - \left( \frac{2n + p - 2}{2n + 1} \right)^2.$$

**Theorem (G.E. Moorehouse)**

The hermitian variety $H(2n + 1, q^2)$, $q = p^h$, $n \geq 2$ has no ovoids if

$$p^{2n+1} > \left( \frac{2n + p}{2n + 1} \right)^2 - \left( \frac{2n + p - 1}{2n + 1} \right)^2.$$
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More results on hermitian varieties

Theorem (JDB, K. Metsch)

The hermitian variety $H(5, 4)$ has no ovoid.

Theorem (A. Klein)

The hermitian variety $H(2n − 1, q^2)$ has no ovoid if $n > q^3$. 
Using tight sets

- Construct a weighted tight set
- Use the intersection with an ovoid and a combinatorial argument
- One line non-existence proof for $Q^-(5, q)$, $W(5, q)$ and $H(4, q^2)$.
- Results on $m$-ovoids of $Q^-(5, q)$.

**Theorem**

If $\mathcal{O}$ is an $m$-ovoid of $Q^-(5, q)$, then $m = \frac{q+1}{2}$

This theorem is due to B. Segre (1965), generalized to generalized quadrangles of order $(s, s^2)$ by J.A. Thas (1989), and shown by J. Bamberg, A. Devillers and Schillewaert using the intriguing set approach (2012).
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Approach applied on $\mathbb{H}(5, q^2)$

- Embedded $W(5, q)$ in $\mathbb{H}(5, q^2)$: $(q + 1)$-tight set
- $\pi$: plane meeting $\mathbb{H}(5, q^2)$ in $\mathbb{H}(2, q^2)$: using $\pi$, $\pi^\perp$, and the closure: weighted $(q^2 - 1)^2$-tight set, weights of $\pi$ and $\pi^\perp$ equal $-q^2 + 1$.
- Currently: only an alternative proof for non-existence of ovoids of $\mathbb{H}(5, 4)$. 

Jan De Beule  hating structures