Classification results on weighted minihypers

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Introduction

Consider ϵ_t subspaces $\mathrm{PG}(t,q)$, ϵ_{t-1} subspaces $\mathrm{PG}(t-1,q)$, . . . , ϵ_1 subspaces $\mathrm{PG}(1,q)$ (lines) and ϵ_0 points. The union of this objects is an

$$\{\sum_{i=0}^{t}\epsilon_i v_{i+1}, \sum_{i=1}^{t}\epsilon_i v_i; n, q\}\text{−minihyper}(F,w)$$

$$v_i = |\mathrm{PG}(i,q)| = \frac{q^{i+1}-1}{q-1}$$

Disjoint subspaces give a non-weighted minihyper. By a theorem of Hamada, Helleseth and Maekawa, also the reverse is true when $\sum_{i=0}^{t}\epsilon_i \leqslant \sqrt{q}$.

question: can we prove the same result for weighted minihypers?

The planar case

The first step is the planar case. What exists for non-weighted minihypers in the plane?

Non-weighted $\{f, m; 2, q\}$ minihypers are often called m-fold blocking sets.

Theorem 1. (S. Ball [1], K. Metsch) *An ϵ_1-fold blocking set in $\mathrm{PG}(2, q)$, ϵ_1 small, not containing a line, has at least $\epsilon_1 q + \sqrt{\epsilon_1 q} + 1$ points.*

it follows:

Lemma 1. *An $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 2, q\}$-minihyper (F, w) contains a line if $\epsilon_1 + \epsilon_0 \leqslant \sqrt{q}$.*

Can we remove this line and still have a $(\epsilon_1 - 1)$-fold blocking set?

The planar case (continued)

Suppose that the $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 2, q\}$-minihyper (F, w) contains a line L.

- If $|(F, w) \cap L| \geqslant q + \epsilon_1$, reducing the weight of every point of L with one gives a new $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; 2, q\}$-minihyper (F', w').
- If $q + 1 \leqslant |(F, w) \cap L| \leqslant q + \epsilon_1 - 1$, it is not immediately clear that this procedure works. It is possible that we have to add at most $\epsilon_1 - 1$ points p_i again.

The planar case (continued)

Using polynomial techniques we obtain that L is the only line on p_i containing exactly $\epsilon_1 - 1$ points of the new minihyper. But:

Lemma 2. (A. Blokhuis, L. Storme and T. Szőnyi [2]) *Let* (F, w) *be an* $\{(\epsilon_1 - 1)(q+1) + c, \epsilon_1 - 1; 2, q\}$*-minihyper,* $\epsilon_1 - 1 + c < \frac{q}{2}$*, and let* $p \in F$ *be a point of weight* 1*. Then* p *lies on at least* $q - c$ *lines intersecting* (F, w) *in* $\epsilon_1 - 1$ *points.*

This gives a contradiction, in other words:

Theorem 2. *An* $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; 2, q\}$*-minihyper,* $\epsilon_1 + \epsilon_0 \leqslant \sqrt{q}$*, is a sum of* ϵ_1 *lines and* ϵ_0 *points.*

The situation in 3-space

We consider a $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; 3, q\}$-minihyper (F, w), $\epsilon_1 + \epsilon_0 \leqslant \sqrt{q}$.

Projecting (F, w) from a point $p \notin (F, w)$ gives a $\{(\epsilon_1' - 1)(q+1) + \epsilon_0', \epsilon_1' - 1; 2, q\}$-minihyper (F', w'). Using that (F', w') is a sum of lines and points, we can prove that (F, w) is the sum of ϵ_1 lines and ϵ_0 points. Inductively, we can prove:

Theorem 3. *A* $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; k, q\}$*-minihyper* (F, w)*,* $\epsilon_1 + \epsilon_0 \leqslant \sqrt{q}$*,* $k \geqslant 2$*, is a sum of* ϵ_1 *lines and* ϵ_0 *points.*

More parameters

We consider now $\{\epsilon_2(q^2+q+1)+\epsilon_1(q+1)+\epsilon_0, \epsilon_2(q+1)+\epsilon_1; k, q\}$-minihypers (F, w), $\epsilon_2+\epsilon_1+\epsilon_0 \leqslant \sqrt{q}$, $k \geqslant 3$.

Using the results on $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; k, q\}$-minihypers, and using an induction hypothesis, we prove:

Theorem 4. *An* $\{\epsilon_2(q^2+q+1)+\epsilon_1(q+1)+\epsilon_0, \epsilon_2(q+1)+\epsilon_1; k-1, q\}$*-minihyper* (F, w), $\epsilon_2+\epsilon_1+\epsilon_0 \leqslant \sqrt{q}$, $k \geqslant 4$, *is a sum of* ϵ_2 *planes,* ϵ_1 *lines and* ϵ_0 *points.*

The general case

We consider a

$$\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=1}^{t} \epsilon_i v_i; k, q\}\text{-minihyper}(F, w)$$

$k \geqslant 2, 1 \leqslant t < k, \sum_{i=0}^{t} \epsilon_i \leqslant \sqrt{q}$

Using an induction hypothesis on k, and the obtained characterisation results for smaller t, we can prove that (F, w) is a sum of ϵ_t t-dimensional subspaces $\mathrm{PG}(t, q)$, ϵ_{t-1} $t-1$-dimensional subspaces $\mathrm{PG}(t-1, q)$, . . . , ϵ_1 lines and ϵ_0 points.

References

[1] A. Blokhuis, L. Storme, and T. Szőnyi. Lacunary polynomials, multiple blocking sets and Baer subplanes. *J. London Math. Soc. (2)*, 60(2):321–332, 1999.

[2] A. Blokhuis, L. Storme, and T. Szőnyi. Lacunary polynomials, multiple blocking sets and Baer subplanes. *J. London Math. Soc. (2)*, 60(2):321–332, 1999.