Lower and upper bounds of maximal partial ovoids of orthogonal polar spaces

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Geometric and algebraic combinatorics 4
in $\mathrm{PG}(2n, q)$, $n \geq 2$,
$Q(2n, q) : X_0^2 + X_1 X_2 + \ldots X_{2n-1} X_{2n} = 0$.

in $\mathrm{PG}(2n+1, q)$, $n \geq 2$,
$Q^-(2n+1, q) : f(X_0, X_1) + X_2 X_3 + \ldots X_{2n} X_{2n+1} = 0,
\quad f(X_0, X_1) : \text{irreducible, homogeneous, of degree 2}.$

in $\mathrm{PG}(2n+1, q)$, $n \geq 2$,
$Q^+(2n+1, q) : X_0 X_1 + X_2 X_3 + \ldots X_{2n} X_{2n+1} = 0.$
Quadrics

Rank

- \( \text{Q}(2n, q) \): rank \( n \).
- \( \text{Q}^- (2n + 1, q) \): rank \( n \).
- \( \text{Q}^+ (2n + 1, q) \): rank \( n + 1 \).
Let $\mathcal{P}$ be a finite classical polar space.

**Definition**

**ovoid**: every generator meets $\mathcal{O}$ in exactly one point.

First question: existence?
### Existence of ovoids (low rank)

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Existence of ovoids (high rank)

$Q^+(2n + 1, q)$, $n \geq 3$, $q = p^h$: no, if

$$p^n > \binom{2n + p}{2n + 1} - \binom{2n + p - 2}{2n + 1}$$

$Q^+(7, q)$: yes if $q$ is odd prime or $q \equiv 0$ or $2 \mod 3$
Questions

- If there is no ovoid, what is the (size of) the *largest* partial ovoid?
- If there are ovoids, what is the (size of) the *largest* maximal partial ovoid different from an ovoid?
- What is the (size of) the *smallest* maximal partial ovoids?
Questions

- If there is no ovoid, what is the (size of) the largest partial ovoid?
- If there are ovoids, what is the (size of) the largest maximal partial ovoid different from an ovoid?
- What is the (size of) the smallest maximal partial ovoids?
Let \( \mathcal{P} \) be a finite classical polar space.

**Definition**

**Partial ovoid:** every generator meets \( \mathcal{O} \) in at most one point.

**Definition**

\( \mathcal{O} \) is **maximal:** \( \mathcal{O} \) cannot be extended.

**Definition**

\( \mathcal{O} \) is **maximal:** \( P^\perp \) meets \( \mathcal{O} \) in at least one point.
Let $\mathcal{P}$ be a finite classical polar space.

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**Definition**

$\mathcal{O}$ is **maximal:** $P^\bot$ meets $\mathcal{O}$ in at least one point.
How to obtain a lower bound

\( n_i := \) number of points of \( Q(d, q) \) collinear with \( i \) points of \( \mathcal{O} \).

\[
\sum n_i = |Q^{\pm}(2n + 1, q)| - w
\]

\[
\sum in_i = wq|Q^{\pm}(2n - 1, q)|
\]

\[
\sum i(i - 1)n_i = w(w - 1)|Q^{\pm}(2n - 1, q)|
\]

\[
\sum i(i - 1)(i - 2)n_i = w(w - 1)(w - 2)|Q(2n - 2, q)|
\]
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Counting
Using the equations . . .

\[
0 \leq \sum_{i} n_i(i - 1)(i - a)(i - a - 1)
= \sum_{i} n_i i(i - 1)(i - 2) - (2a - 1) \sum_{i} n_i i(i - 1) + (a^2 + a) \sum_{i} n_i(i - 1)
\]
Resulting bounds for $Q^{-}(5, q)$ and $Q^{+}(5, q)$

**Theorem**

$Q^{-}(5, q) : q \geq 4 \Rightarrow w \geq 2q + 2$, $q < 4 \Rightarrow w \geq 2q + 1$

$Q^{-}(2n+1, q) : n \geq 3 \Rightarrow w \geq 2q + 1$

$Q^{+}(5, q) : w \geq 2q$

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Resulting bounds for $Q^-(5, q)$ and $Q^+(5, q)$

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Orthogonal polar spaces
Ovoids and partial ovoids
Lower bounds
Upper bounds

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How to obtain a lower bound

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Resulting bounds for $Q(2n, q)$

**Theorem**

$Q(4, q) : q_{odd} \Rightarrow w \geq 1.419q$

$Q(6, q) : q \in \{3, 5, 7\} \Rightarrow w \geq 2q, q \geq 9 \Rightarrow w \geq 2q - 1$

$Q(8, 3) : w \geq 2q$

$Q(2n, q) : n \geq 4 : \geq 2q + 1$
A lower bound for $Q(2n, q)$, $q$ even

- $Q(2n, q)$, $q$ even has a nucleus
- Projecting from this nucleus yields the symplectic polar space $W(2n − 1, q)$.

**Theorem**

The smallest maximal partial ovoids of $W(2n − 1, q)$ are the hyperbolic lines.

**Theorem**

The smallest maximal partial ovoids of $Q(2n, q)$ are conics whose nucleus coincides with the nucleus of $Q(2n, q)$. 
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**Theorem**

*The smallest maximal partial ovoids of $Q(2n, q)$ are conics whose nucleus coincides with the nucleus of $Q(2n, q)$.*
Upper bounds

An upper bound for $Q^+(2n+1, q)$

- $|\mathcal{O}| = q^n + 1 - \delta$.
- $|P^\perp \cap \mathcal{O}| \geq q^{n-1} + 1 - \delta$
- $n_i = 0$ for $i < q^{n-1} + 1 - \delta$ and $i > q^{n-1} + 1$
- $0 \leq \sum_i n_i(i - q^{n-1} - 1)(i - q^{n-1})(i - q^{n-1} - 1 + \delta)$

Theorem

A maximal partial ovoid $\mathcal{O}$ of $Q^+(2n+1, q)$, that is not an ovoid, has at most $q^n - q^{(n-1)/2}$ points.
An upper bound for $Q^+(2n + 1, q)$

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- $|O| = q^n + 1 - \delta$.
- $|P^\perp \cap O| \geq q^{n-1} + 1 - \delta$
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An upper bound for $Q(2n, q)$, $q$ odd non prime

**Lemma**

Consider $Q(2n, q) \subseteq Q^+(2n+1, q)$, $n \geq 3$, $q$ not a prime, and suppose that $Q^+(2n+1, q)$ has an ovoid with $q^n + 1 - \delta$, $\delta > 0$, points in $Q(2n, q)$. Then $\delta \geq 2(q^{n-2} + q^{n-3} + \ldots + q + 1) + 1$.

**Theorem (A. Gács and JDB)**

$Q(4, q)$ has no maximal partial ovoids when $q$ is odd and non prime.

**Corollary**

$Q(6, q)$, $q$ not a prime, does not have a maximal partial ovoid of size $q^3 + 1 - \delta$ with $0 < \delta < q + 1$. 
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Consider $Q(2n, q) \subseteq Q^+(2n + 1, q)$, $n \geq 3$, $q$ not a prime, and suppose that $Q^+(2n + 1, q)$ has an ovoid with $q^n + 1 - \delta$, $\delta > 0$, points in $Q(2n, q)$. Then $\delta \geq 2(q^{n-2} + q^{n-3} + \ldots + q + 1) + 1$.

**Theorem (A. Gács and JDB)**

$Q(4, q)$ has no maximal partial ovoids when $q$ is odd and non prime.

**Corollary**

$Q(6, q)$, $q$ not a prime, does not have a maximal partial ovoid of size $q^3 + 1 - \delta$ with $0 < \delta < q + 1$. 
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Let $S$ be a partial spread of $\mathbb{H}(3, q^2)$. Then $|S| \leq \frac{1}{2}(q^3 + q + 2)$.

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An upper bound for $Q^-(5, q)$

**Theorem**

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This dualizes to an upper bound for partial ovoids of $Q^-(5, q)$. 
$\mathcal{P}_n$ denotes a finite classical polar space of rank $n$.

**Theorem**

If partial ovoids of $\mathcal{P}_r$ have deficiency $\epsilon_r$, then partial ovoids of $\mathcal{P}_{r+1}$ have deficiency at least $q\epsilon_r$. 