The smallest sets of points meeting all generators of $\mathrm{H}(2n, q^2)$

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Classical Polar Spaces

- $\mathrm{W}(2n+1,q)$, the symplectic polar space, rank $n+1$
- $\mathrm{Q}^-(2n+1,q)$, the non singular elliptic quadric, rank n
- $\mathrm{Q}(2n,q)$, the non singular parabolic quadric, rank n
- $\mathrm{Q}^+(2n+1,q)$, the non singular hyperbolic quadric, rank $n+1$
- $\mathrm{H}(2n+1,q^2)$ and $\mathrm{H}(2n,q^2)$, the non singular hermitian variety, rank $n+1$ and rank n resp.

Polar Spaces - Definitions

An ovoid $\mathcal{O}$ of a classical polar space $\mathcal{P}$ is a set of points such that every generator meets $\mathcal{O}$ in exactly one point.

A blocking set $\mathcal{K}$ of $\mathcal{P}$ is a set of points such that every generator meets $\mathcal{K}$ in at least one point. $\mathcal{K}$ is minimal iff $\mathcal{K} \setminus \{p\}$ is not a blocking set $\forall p \in \mathcal{K}$.

Ovoids - existence

J.A. Thas (1981) proved non-existence in the following cases:

- $\mathrm{Q}^-(2n+1,q)$, $n \geq 2$
- $\mathrm{W}(2n+1,q) \cong \mathrm{Q}(2n+2,q)$, q even, $n \geq 2$
- $\mathrm{W}(2n+1,q)$, q odd, $n \geq 1$
- $\mathrm{H}(2n,q^2)$, $n \geq 2$.

some open cases:

- $\mathrm{Q}^+(2n+1,q)$, $q > 3$, $n > 3$
- $\mathrm{H}(2n+1,q^2)$, $n > 1$

partial results for $\mathrm{Q}^+(7,q)$ and $\mathrm{Q}(6,q)$, q odd and $\mathrm{H}(2n+1,q^2)$, $n > 1$

Blocking all generators

- K. Metsch charaterised the smallest minimal blocking sets of $\mathrm{Q}^-(2n+1,q)$, $n>1$ and of $\mathrm{W}(2n+1,q)$, $n>1$, q even. (no ovoids exist in this cases).

- open: $\mathrm{W}(2n+1,q)$, $n \geq 1$, q odd, $\mathrm{Q}^+(2n+1,q)$, $\mathrm{Q}(2n,q)$, q odd, $\mathrm{H}(2n,q^2)$ and $\mathrm{H}(2n+1,q^2)$.

- with L. Storme $\mathrm{Q}(6,q)$, q even

- with L. Storme and K. Metsch, $\mathrm{Q}(2n,q)$, q odd prime

Starting with $\mathrm{H}(4, q^2)$: Combinatorial facts

Suppose $\mathcal{K}$ is a minimal blocking set of $\mathrm{H}(4, q^2)$, $|\mathcal{K}| = q^5 + \delta$, $1 \leqslant \delta \leqslant q^2$.

Lemma 1. *If $p \in \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \leqslant \delta$.*

Lemma 2. *If $r \in \mathrm{PG}(4, q) \setminus \mathcal{K}$, then $|r^\perp \cap \mathcal{K}| \geqslant q^3 + 1$.*

Lemma 3. *If $p \in \mathcal{K}$, then $|p^\perp \cap \mathcal{K}| \geqslant q^2 - q + 1$.*

Part 2: finding lines with a lot of points

Suppose $r \in \mathrm{H}(4, q^2)$. Define $w_r + 1$ as the smallest number of points of $\mathcal{K}$ that lie on a line of $\mathrm{H}(4, q^2)$ on r.

Lemma 4. *If L is a generator of $\mathrm{H}(4, q^2)$ meeting $\mathcal{K}$ in more than one point, then L contains a point $r \notin \mathcal{K}$, with $w_r > 0$.*

Lemma 5. *If $r \in \mathrm{H}(4, q^2) \setminus \mathcal{K}$ and $w_r > 0$, then $w_r > q^2 - q$.*

Lemma 6. *There exists a point $r \in \mathrm{H}(4, q^2)$ such that $\mathcal{K}$ is the truncated cone $(r^\perp \cap \mathrm{H}(4, q^2)) \setminus \{r\}$.*

Part 3: $\mathrm{H}(2n, q^2)$, $n \geqslant 3$

Suppose $\mathcal{K}$ is a minimal blocking set of $\mathrm{H}(2n, q^2)$, $|\mathcal{K}| = q^{2n+1} + \delta$, $1 \leqslant \delta \leqslant q^{2n-2}$.

Lemma 7. *There exists a point* $r \in \mathrm{H}(2n, q^2) \setminus \mathcal{K}$*, such that* $|r^{\perp} \cap \mathcal{K}| = q^{2n-2} + q^{2n-5}$

Lemma 8. *Suppose that* $r \in \mathrm{H}(2n, q^2) \setminus \mathcal{K}$ *with* $|r^{\perp} \cap \mathcal{K}| = q^{2n-2} + q^{2n-5}$*. Then* $r^{\perp} \cap \mathcal{K} = \pi_{n-3}\mathrm{H}(2, q^2) \setminus \pi_{n-3}$, $r \notin \pi_{n-3}$, $\pi_{n-3} \subset \mathrm{H}(2n, q^2)$

Theorem 1. *The smallest minimal blocking sets of* $\mathrm{H}(2n, q^2)$, $n \geq 2$, *are truncated cones* $\pi_{n-2}\mathrm{H}(2, q^2) \setminus \pi_{n-2}$, $\pi_{n-2} \subset \mathrm{H}(2n, q^2)$.