Point sets in $\mathrm{AG}(n, q)$ (not) determining certain directions

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Directions in $\mathrm{AG}(n, q)$

Definition

Consider $\mathrm{AG}(n, q)$ with plane at infinity π. Given a point set $U \subseteq \mathrm{AG}(n, q)$, then a point $p \in \pi$ is a *determined direction* of U if and only if there exists a line of $\mathrm{AG}(n, q)$ through p, meeting U in at least two points. Denote the set of all determined directions of U by D_U.

Corollary

If $|U| > q^n$, then D_U contains all points of π.

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Blocking sets of $\mathrm{PG}(2, q)$

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A point set $B \subseteq \mathrm{PG}(2, q)$ is called a *blocking set* if every line of $\mathrm{PG}(2, q)$ contains at least one point of B.

A line of $\mathrm{PG}(2, q)$ is an example of a blocking set, but such a blocking set is called *trivial*

Definition

A blocking set B is called *minimal* if $B \setminus \{p\}$ is not a blocking set for any $p \in B$.

Theorem (Bruen, 1971)

If B is a minimal blocking set of a projective plane of order n, then $|B| \geq n + \sqrt{n} + 1$.

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Let p be prime. Let

$$f = \prod_{i=1}^{p+k} (X + a_i Y + b_i) \, ,$$

and suppose that there are at least $(p+1)/2 + k \leq p - 1$ elements s of $\mathbb{F}_p$ with the property that $X^p - X \mid f(X, s)$.

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Lemma

Suppose that $f(X) = g(X)X^q + h(X)$ is a polynomial in $\mathbb{F}_q[X]$ factorising completely into linear factors in $\mathbb{F}_q[X]$. If $max(deg(g), deg(h)) \leq (q-1)/2$ then $f(X) = g(X)(X^q - X)$ or $f(X) = gcd(f, g)e(X^p)$ for $e \in \mathbb{F}_q[X]$, where $q = p^h$.

Theorem

Let p be prime. Let

$$f = \prod_{i=1}^{p+k} (X + a_i Y + b_i)\,,$$

and suppose that there are at least $(p+1)/2 + k \leq p - 1$ *elements s of* $\mathbb{F}_p$ *with the property that* $X^p - X \mid f(X, s)$*. Then f contains a factor*

$$\prod_{x_i \in \mathbb{F}_q} (X + x_i Y + m x_i + c)$$

blocking sets

Corollary

Let U be a set of points of $\mathrm{AG}(2, p)$. If there are at least $|U| - (p-1)/2$ and at most $p - 1$ parallel classes for which the lines of these parallel classes are all incident with at least one point of U, then U contains all points of a line.

Corollary (Blokhuis, 1994)

Let B be a blocking set of $\mathrm{PG}(2, p)$. If $|B| \leq (3p + 1)/2$, then B contains all the points of a line.

one of the original theorems

Theorem (Rédei, 1973)

A function $\phi : \mathbb{F}_q \to \mathbb{F}_q$ determining less than $(q+3)/2$ directions is linear over a subfield of $\mathbb{F}_q$.

Theorem (Szőnyi, 1996)

A set U of $q - k > q - \sqrt{q}/2$ points of $\mathrm{AG}(2, q)$ which does not determine a set E of more than $(q + 1)/2$ directions, can be extended to a set of q points not determining the set E.

particular point sets of $\mathrm{AG}(3, q)$

Theorem

Let U be a point set of $\mathrm{AG}(3, q)$, $= p^h$, $|U| = q^2$, and suppose that U does not determine the directions on a conic at infinity. Then every hyperplane of $\mathrm{AG}(3, q)$ intersects U in $0 \pmod p$ points.

Corollary (Ball, 2004; Ball, Govaerts, Storme, 2006)

Consider $Q(4, q)$. When $q = p$ prime, any ovoid of $Q(4, q)$ is contained in a hyperplane section, and so it is necessarily an elliptic quadric $Q^-(3, q)$.

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a generalization of the direction result

Theorem (Ball)

Let U be a set of q^{n-1} points of $\mathrm{AG}(n, q)$, $q = p^h$. Suppose that for $0 \leq e \leq (n-2)h-1$, more than $p^e(q-1)$ directions are not determined by U. Then every hyperplane of $\mathrm{AG}(3, q)$ is incident with a multiple of p^{e+1} points.

Theorem (DB, Gács, 2005)

Let U be a set of $q^2 - 2$ points of $\mathrm{AG}(3, q)$, $q = p^h$, $h > 1$. If U does not determine a set E of $p + 2$ directions at infinity, then U can be extended to a set of size q^2, not determining the directions of E.

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application for $Q(4, q)$

Corollary (DB, Gács, 2005)

A partial ovoid of $Q(4, q)$, $q = p^h$, $h > 1$, of size $q^2 - 1$ can be extended to an ovoid.

sets of size $q^2 - \epsilon$

Lemma (DB, Tákats, Sziklai, 20XX)

Let U be a point set of $\mathrm{AG}(3, q)$, of size $q^2 - \epsilon$, such that E is the set of non-determined directions. If U cannot be extended without determining directions of E, then E is contained in a planar algebraic curve of degree $\epsilon^4 - 4\epsilon^3 + \epsilon$.

sets of size $q^2 + \epsilon$

Can we characterise such a set for *small* ϵ? (motivated by an application?)