On the smallest minimal blocking sets of $Q(2n, q)$, for $q$ an odd prime

J. De Beule
Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, 9000 Gent, Belgium (http://cage.ugent.be/~jdebeule, jdebeule@cage.ugent.be)

L. Storme
Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, 9000 Gent, Belgium (http://cage.ugent.be/~ls, ls@cage.ugent.be)

Abstract
We characterize the smallest minimal blocking sets of $Q(2n, q)$, $q$ an odd prime, in terms of ovoids of $Q(4, q)$ and $Q(6, q)$. The proofs of these results are written for $q = 3, 5, 7$ since for these values it was known that every ovoid of $Q(4, q)$ is an elliptic quadric. Recently, in [2], it has been proven that for all $q$ prime, every ovoid of $Q(4, q)$ is an elliptic quadric. Since as many proofs as possible were written for general $q$, using the classification result of De Beule and Metsch [9] on the smallest blocking sets of $Q(6, q)$, $q > 3$ prime, the results for $Q(2n, q)$, $n \geq 4$, $q = 5, 7$, are also valid for $q$ prime, $q > 7$. The case $q = 3$ is treated separately since this is the only value for $q$ an odd prime for which $Q(6, q)$ has an ovoid. We end the article by discussing the possibilities and remaining problems to obtain the characterization for general $q$ odd.

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1 Introduction and definitions

Let $Q(2n, q)$, $n \geq 2$, be the non-singular parabolic quadric in $PG(2n, q)$. An ovoid of the polar space $Q(2n, q)$ is a set of points $O$ such that every maximal singular subspace (or generator) of $Q(2n, q)$ intersects $O$ in exactly one point. For $Q(2n, q)$, the generators are spaces of dimension $n-1$. A blocking set of the polar space $Q(2n, q)$ is a set of points $K$ such that every generator intersects
\(\mathcal{K}\) in at least one point. If \(\mathcal{O}\) is an ovoid of \(Q(2n,q)\), then \(\mathcal{O}\) has size \(q^n + 1\). So if \(\mathcal{K}\) is a blocking set of \(Q(2n,q)\) different from an ovoid, then \(\mathcal{K}\) has size \(q^n + 1 + r\), with \(r > 0\). A blocking set \(\mathcal{K}\) is called \textit{minimal} if for every point \(p \in \mathcal{K}, \mathcal{K} \setminus \{p\}\) is not a blocking set, or equivalently, if for every point \(p \in \mathcal{K}\), there is a generator \(\alpha\) such that \(\alpha \cap \mathcal{K} = \{p\}\).

It is known that if \(p\) is a point of \(Q(2n,q)\) and \(\alpha\) is the tangent hyperplane of \(Q(2n,q)\) at \(p\), then \(\alpha \cap Q(2n,q) = pQ(2n - 2,q)\), a singular quadric with vertex \(p\) and base \(Q(2n - 2,q)\). This tangent hyperplane is often denoted by \(T_p(Q(2n,q))\). If \(\mathcal{K}\) is an ovoid or blocking set of \(Q(2n,q)\) and \(p \notin \mathcal{K}\), then \(p\) projects \(\mathcal{K} \cap \alpha\) on an ovoid or blocking set of the base \(Q(2n - 2,q)\) of \(\alpha \cap Q(2n,q)\). So if the base \(Q(2n - 2,q)\) has no ovoid, then \(Q(2n,q)\) has no ovoid.

The problem on the existence of ovoids of \(Q(2n,q)\), \(q\) odd, is not solved for general odd \(q\). It will become clear that to characterize the smallest minimal blocking sets of \(Q(2n,q)\), one must know whether \(Q(2n_0,q)\) has an ovoid for \(n_0 < n\). Since the non-existence of an ovoid of \(Q(2n,q)\) implies the non-existence of an ovoid of \(Q(2n + 2,q)\), it is sufficient to know if \(Q(2n_0,q)\) has no ovoid for a certain \(n_0\). It is known that \(Q(4,q)\) always has an ovoid [21]. In [10], we could characterize the smallest minimal blocking sets of \(Q(6,q)\), \(q\) even, knowing that \(Q(6,q), q\) even, has no ovoid [24]. This was also independently done by K. Metsch who, using a different method, characterized the smallest minimal blocking sets of \(W(2n + 1,q)\), \(n \geq 2, q\) even, completely (see [17]). Since \(W(2n + 1,q), q\) even, is isomorphic to \(Q(2n + 2,q), q\) even, this result also gives the complete characterization of the smallest minimal blocking sets of \(Q(2n + 2,q), q\) even, \(n \geq 2\).

For \(q\) odd, the situation is different. For some values of \(q\), there are different classes of ovoids known for \(Q(6,q)\), see [15], [23], [24] and [25]. Up to projectivity, the quadric \(Q(6,3)\) has just one ovoid, see [20], [22]. We also know that \(Q(6,q), q = 5, 7\), has no ovoid, see [19]. From [13], we know that \(Q(8,q), q\) odd, and hence \(Q(2n,q), q\) odd and \(n \geq 4\), has no ovoids.

If \(Q(6,q)\) has no ovoid, the smallest known examples of blocking sets of \(Q(2n + 2,q), n \geq 2\), are constructed as follows. Consider the tangent space on an \((n - 2)\)-dimensional space \(\pi\) of \(Q(2n + 2,q)\), then \(T_\pi(Q(2n + 2,q)) \cap Q(2n + 2,q) = \pi Q(4,q)\), a cone with vertex \(\pi\) and base a 4-dimensional parabolic quadric \(Q(4,q)\). Define \(\mathcal{K}\) as the cone with vertex \(\pi\) and base \(\mathcal{O}\), an ovoid of \(Q(4,q)\), minus the points of the vertex \(\pi\). Then \(\mathcal{K}\) is a minimal blocking set of \(Q(2n+2,q)\) of size \(q^{n+1} + q^{n-1}\). This construction yields also the smallest known minimal blocking sets of \(Q(6,q)\) different from an ovoid of \(Q(6,q)\) if \(Q(6,q)\) has an ovoid, but we can find smaller examples in higher dimensions if \(Q(6,q)\) has ovoids, with actually the same construction using now the ovoids of \(Q(6,q)\). Suppose that \(Q(6,q)\) has ovoids. Consider now the tangent space of an \((n - 3)\)-
dimensional space $\pi$ of $Q(2n + 2, q)$, then $T_\pi(Q(2n + 2, q)) \cap Q(2n + 2, q) = \pi Q(6, q)$. Define now $\mathcal{K}$ as the cone with vertex $\pi$ and base $\mathcal{O}$, an ovoid of $Q(6, q)$, minus the points of the vertex $\pi$. Then $\mathcal{K}$ is a minimal blocking set of $Q(2n + 2, q)$ of size $q^{n+1} + q^{n-2}$.

Using the known fact that every ovoid of $Q(4, q)$, $q = 3, 5, 7$, is an elliptic quadric [4–7,16,18], and that $Q(6, 3)$ has a unique type of ovoid [20,22], we proceed as follows. In Section 4, we prove the characterization for $Q(6, 3)$ of the smallest minimal blocking sets different from an ovoid. As in [10], we use projection arguments and the characterization of the smallest minimal blocking sets of $Q(4, q)$, $q > 3$ prime, is an elliptic quadric, the following theorem is also valid for $q > 7$, since we have written as many proofs as possible for general $q$. The case $n = 2$ for $q > 7$ prime is of De Beule and Metsch [9].

**Theorem 1** Let $\mathcal{K}$ be a minimal blocking set different from an ovoid of $Q(2n + 2, q)$, $q$ odd prime, $q > 3$, $n \geq 2$, $|\mathcal{K}| \leq q^{n+1} + q^{n-1}$, or $n = 2$, $q = 3$, and $|\mathcal{K}| \leq 3^3 + 3$. Then there is an $(n - 2)$-dimensional space $\pi_{n-2}$, $\pi_{n-2} \subset Q(2n + 2, q)$, $\pi_{n-2} \cap \mathcal{K} = \emptyset$, with the following property: $T_{\pi_{n-2}}(Q(2n + 2, q)) \cap Q(2n + 2, q) = \pi_{n-2} Q(4, q)$ and $\mathcal{K}$ is a cone with vertex $\pi_{n-2}$ and base $\mathcal{O}$, where $\mathcal{O}$ is a 3-dimensional elliptic quadric of $Q(4, q)$, minus the points of the vertex $\pi_{n-2}$, and $|\mathcal{K}| = q^{n+1} + q^{n-1}$.

Furthermore, since it is now known from the results of Ball et al. [2], and O’Keefe and Thas [19] that the quadrics $Q(2n + 2, q)$, $n \geq 2$, $q > 3$ prime, do not have ovoids, the preceding theorem classifies the smallest blocking sets of $Q(2n + 2, q)$, $n \geq 2$, $q > 3$ prime.

In the fourth section, we prove the following result.

**Theorem 2** Let $\mathcal{K}$ be a minimal blocking set of $Q(2n + 2, q = 3)$, $n \geq 3$, $|\mathcal{K}| \leq 3^{n+1} + 3^{n-2}$. Then there is an $(n - 3)$-dimensional space $\pi_{n-3}$, $\pi_{n-3} \subset Q(2n + 2, 3)$, $\pi_{n-3} \cap \mathcal{K} = \emptyset$, with the following property: $T_{\pi_{n-3}}(Q(2n + 2, 3)) \cap Q(2n + 2, 3) = \pi_{n-3} Q(6, 3)$ and $\mathcal{K}$ is a cone with vertex $\pi_{n-3}$ and base $\mathcal{O}$, where $\mathcal{O}$ is an ovoid of $Q(6, 3)$, minus the points of the vertex $\pi_{n-3}$, and $|\mathcal{K}| = 3^{n+1} + 3^{n-2}$.

In the next section, we present some lemmas which will be used and which are valid in general situations. Furthermore, we present some technical lemmas.
about minimal blocking sets of $Q(4, q)$. To end the introduction, we define the concept \textit{truncated cone} $\alpha\mathcal{O} \setminus \alpha$, for $\alpha$ a subspace and $\mathcal{O}$ an arbitrary geometric object lying in some subspace $\pi$ such that $\alpha \cap \pi = \emptyset$, as the set of points of the cone with vertex $\alpha$ and base $\mathcal{O}$, without the points of the space $\alpha$. If the dimension of $\alpha$ is -1, we define $\alpha\mathcal{O} \setminus \alpha$ as $\mathcal{O}$ itself. We will often denote an $n$-dimensional space by $\pi_n$.

2 Important lemmas

The first two lemmas are the key lemmas for our projection arguments. For both lemmas, we suppose $p$ to be a point of $Q(2n+2, q)$ and $\mathcal{K}$ to be a minimal blocking set of $Q(2n+2, q)$ different from an ovoid, $|\mathcal{K}| \leq q^{n+1} + 1 + r$, $r < q^{n-1}$.

Lemma 3 If $p \in \mathcal{K}$, then $|T_p(Q(2n+2, q)) \cap \mathcal{K}| \leq 1 + r$.

\textbf{PROOF.} Since $\mathcal{K}$ is minimal, there exists a generator $\pi_n$ such that $\pi_n \cap \mathcal{K} = \{p\}$. Consider the $q^n$ hyperplanes of $\pi_n$ not on $p$. On each hyperplane there are exactly $q$ generators different from $\pi_n$ which must be blocked by at least one point of $\mathcal{K}$ and which cannot share any point of $\mathcal{K}$. Hence at least $q \cdot q^n$ points of $\mathcal{K}$ lie outside $T_p(Q(2n+2, q))$ and so $|T_p(Q(2n+2, q)) \cap \mathcal{K}| \leq 1 + r$.

Lemma 4 If $p \in Q(2n+2, q) \setminus \mathcal{K}$, then $p$ projects $\mathcal{K} \cap T_p(Q(2n+2, q))$ onto $\mathcal{K}_p$, which is a minimal blocking set of $Q(2n, q)$, with $Q(2n, q)$ a $2n$-dimensional parabolic quadric in $T_p(Q(2n+2, q)) \cap Q(2n+2, q)$.

\textbf{PROOF.} Let $T_p(Q(2n+2, q)) \cap Q(2n+2, q) = pQ(2n, q)$. Choose $Q$ as fixed base for this tangent cone. Let $\mathcal{K}_p$ be the projection of $\mathcal{K} \cap T_p(Q(2n+2, q))$ from $p$ onto $Q$. Suppose that $\mathcal{K}_p$ is not minimal, then there exists a point $p' \in \mathcal{K}_p$ such that every generator $\pi_{n-1}$ of $Q$ contains at least one other point of $\mathcal{K}_p$. There are $(q+1)(q^2+1) \ldots (q^{n-1}+1)$ generators of $Q$ on $p'$, and every point of $\mathcal{K}_p$ that lies in $T_{p'}(Q)$ can block $(q+1)(q^2+1) \ldots (q^{n-2}+1)$ of these generators. So if $\mathcal{K}_p$ is not minimal, then there are at least $q^{n-1} + 1$ points of $\mathcal{K}_p$ different from $p'$ necessary to block all generators on $p'$. Hence for some point $r \in \mathcal{K}$ on the line $pp'$, $|T_r(Q(2n+2, q)) \cap \mathcal{K}| > q^{n-1}$, a contradiction. We conclude $\mathcal{K}_p$ to be minimal.

The following lemmas are about minimal blocking sets of $Q(4, q)$, $q = 3, 5, 7$. From [11], we have the following important theorem if $q$ is even.

\textbf{Theorem 5} A minimal blocking set on $Q(4, q)$, $q$ even, $q \geq 32$, of size $q^2 + 1 + r$, with $0 < r \leq \sqrt{q}$, consists of an ovoid and $r$ extra points.
The next three lemmas are the extension of this theorem for \( q = 3, 5, 7 \). In the proof of these lemmas, another theorem from [11] when \( q \) is odd is extremely useful. To formulate the theorem, we need the following definitions.

Let \( \mathcal{B} \) be a minimal blocking set of \( Q(4, q) \) of size \( q^2 + 1 + r \). We call \( r \) the excess of the minimal blocking set. A line of \( Q(4, q) \) is called a multiple line when it contains at least two points of \( \mathcal{B} \). The excess of a line is the number of points of \( \mathcal{B} \) it contains, minus one. The weight of a point of \( Q(4, q) \) with respect to \( \mathcal{B} \) is the minimum of the excesses of the lines of \( Q(4, q) \) passing through this point.

A blocking set \( \mathcal{D} \) of the projective plane \( PG(2, q) \) is a set of points such that each line of \( PG(2, q) \) contains at least one point of \( \mathcal{D} \). A blocking set \( \mathcal{D} \) containing a line of \( PG(2, q) \) is called a trivial blocking set. When \( \mathcal{D} \) does not contain a line, then it is called a non-trivial blocking set. It is known that the smallest non-trivial blocking sets in \( PG(2, q) \), \( q \) an odd prime, have size \( q + (q + 3)/2 \) ([3]).

A sum of lines \( \mathcal{S} \) of \( PG(d, q) \) is a collection of lines of \( PG(d, q) \), where each line is accorded a positive integer, called its weight. Furthermore, the weight of a point with respect to \( \mathcal{S} \) is the sum of the weights of the lines of \( \mathcal{S} \) through this point.

A pencil of \( Q(4, q) \) is the set of \( q + 1 \) generators on a point of \( Q(4, q) \). Let \( \mathcal{P} \) be a collection of pencils of \( Q(4, q) \), where each pencil is accorded a positive integer, called its weight. The set of lines on the elements of \( \mathcal{P} \) is called the sum of the pencils \( \mathcal{P} \). Furthermore, the weight of a line \( L \) in the sum of pencils \( \mathcal{P} \) is the sum of the weights of the pencils containing \( L \).

A cover \( \mathcal{C} \) of a generalized quadrangle \( S \) is a set of lines of \( S \) such that each point of \( S \) belongs to at least one line of \( \mathcal{C} \). A multiple point of a cover \( \mathcal{C} \) is a point of \( S \) belonging to at least two lines of \( \mathcal{C} \). The excess of a point with respect to \( \mathcal{C} \) is the number of lines of \( \mathcal{C} \) passing through this point, minus one. The weight of a line with respect to \( \mathcal{C} \) is the minimum of the excesses of the points belonging to this line.

**Theorem 6 (Eisfeld et al. [11])** Let \( \mathcal{C} \) be a cover of a classical generalised quadrangle \( S \) of order \((q, t)\) embedded in \( PG(d, q) \). Let \( |\mathcal{C}| = qt + 1 + r \), with \( q + r \) smaller than the cardinality of the smallest non-trivial blocking sets in \( PG(2, q) \). Then the multiple points of \( \mathcal{C} \) form a sum of lines of \( PG(d, q) \), where the weight of a line in this sum is equal to the weight of this line with respect to the cover, and with the sum of the weights of the lines equal to \( r \).\(^1\)

\(^1\) This is exactly the formulation found in [11], except for the notation \( S \) for the GQ, and where we replaced “contained in \( Q \)” by “of \( PG(d, q) \)”. 

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However, one has to be extremely careful in the case $S = W(3, q)$ concerning the interpretation of this theorem. It is clear that all the multiple points are points of the GQ $S$. This does not, however, imply in general that the lines of the sum are lines of the GQ $S$. If $S$ is $W(3, q)$, which is the situation we need, then every point of $\text{PG}(3, q)$ is a point of $W(3, q)$, hence it is possible that projective lines which are not lines of the GQ $S$ are lines of the sum of lines describing the set of multiple points. Hence the interpretation is that the sum of lines is a sum of lines of $\text{PG}(3, q)$.

Consider a cover $C$ of the GQ $S = W(3, q)$, satisfying the conditions of Theorem 6. This cover $C$ dualises to a blocking set $B$ of the GQ $S' = Q(4, q)$. The sum of multiple lines can now be described by pencils, i.e. $q + 1$ lines of $S'$ on a point, the dual of a line of $\text{PG}(3, q)$ which is also a line of $S$, and reguli, lying in 3-dimensional spaces intersecting $S' = Q(4, q)$ in a hyperbolic quadric $Q^+(3, q)$, corresponding to the $q + 1$ points on a line of $\text{PG}(3, q)$ which is not a line of $S$. We will be interested in the situation where only pencils occur in this sum.

The following lemma is based on similar results from [12].

**Lemma 7** Suppose that $C$ is a cover of $S = W(3, q)$, of size $q^2 + 1 + r$, with $q + r$ smaller than the cardinality of the smallest non-trivial blocking sets in $\text{PG}(2, q)$, such that the multiple points of $C$ are a sum $\mathcal{A}$ of lines of $\text{PG}(3, q)$. If $L$ is a line of $\mathcal{A}$, $L$ not a line of $W(3, q)$, then $L^\perp \in \mathcal{A}$, with $\perp$ the symplectic polarity corresponding to $W(3, q)$.

**PROOF.** Suppose that $L$ is a line of $\mathcal{A}$, but not a line of $S$. Since $L \not\in S$, $L \not\in C$, so $L$ is intersected by at least $2q + 2$ lines of $C$. Then also $L^\perp$ is intersected by these at least $2q + 2$ lines of $C$.

If $L^\perp \not\in \mathcal{A}$, then $L^\perp$ intersects at most $r$ lines of $\mathcal{A}$, so the sum of the excesses of the points of $L^\perp$ is at most $r$. But it is at least $2q + 2 - (q + 1) = q + 1$, so also $L^\perp \in \mathcal{A}$.

Consider now a blocking set $B$ of $Q(4, q)$, $q$ odd, of size $|B| = q^2 + 1 + r$. This corresponds to a cover $C$ of $W(3, q)$. If $r = 1$, then Theorem 6 implies that the multiple points of $C$ lie on a unique line $L$ of $\text{PG}(3, q)$, with weight 1. This implies that $L^\perp = L$, since otherwise the sum of lines would contain two lines with weight 1, a contradiction since the sum of the weights of the lines equals $r = 1$. If $r = 2$, then either the multiple points of $C$ lie on a unique line of $\text{PG}(3, q)$ with weight 2, or on two lines of $\text{PG}(3, q)$ with weight one. If we suppose that all lines of the sum have weight 2, then, by the same arguments as for $r = 1$, the sum of lines consists of a unique line of weight 2 belonging to $W(3, q)$. We can formulate the following corollary.
Corollary 8 Let \( B \) be a minimal blocking set of \( Q(4,q) \), \( q \) odd, \( |B| = q^2 + 1 + r \), \( q + r \) smaller than the cardinality of the smallest non-trivial blocking sets in \( PG(2,q) \). If \( r = 1 \), then the multiple lines pass through a common point \( p \in Q(4,q) \setminus B \). If \( r = 2 \), and all multiple lines have excess 2, then all multiple lines pass through a common point \( p \in Q(4,q) \setminus B \).

This corollary is not a complete replacement of Theorem 5, since it only gives information on how the multiple lines are structured. Still it enables to prove for \( q = 3 \), and for \( q = 5, 7 \) with the aid of the computer [8], the following results.

Lemma 9 If \( B \) is a minimal blocking set of \( Q(4,3) \) different from an ovoid, then \( |B| > 11 \).

PROOF. Suppose that \( B \) is a minimal blocking set of \( Q(4,3) \), \( |B| = 11 \). Note that an ovoid \( \mathcal{O} \) of \( Q(4,3) \) has size 10. Then there exists a point \( p \notin B \) such that all 4 lines on \( p \) contain 2 points of \( B \) (Corollary 8), i.e. \( T_p(Q(4,3)) \cap B = \{p_1, \ldots, p_8\} \). There remain 3 points \( r_1, r_2, r_3 \) in \( B \). Those points cannot be collinear with the points \( p_1, \ldots, p_8 \), since the only 2-secants to \( B \) pass through \( p \). Hence \( \cap_{i=1}^{3} T_{r_i}(Q(4,3)) \cap T_p(Q(4,3)) \cap Q(4,3) = \mathbb{p}^+ \setminus (B \cup \{p\}) = \{s_1, \ldots, s_4\} \). But the polar space of \( \langle s_1, s_2, s_3, s_4 \rangle \) is at most a line, so this would be a line containing \( r_1, r_2, r_3 \), a contradiction.

The following two lemmas are computer results [8].

Lemma 10 If \( B \) is a minimal blocking set of \( Q(4,q) \), \( q = 5, 7 \), different from an ovoid of \( Q(4,q) \), then \( |B| > q^2 + 2 \).

We want to exclude a particular minimal blocking set \( B \) of \( Q(4,7) \) of size 52 = \( q^2 + 3 \). From Corollary 8, one of the possibilities is that there is one point \( p \in Q(4,7) \setminus B \) with \( q + 1 \) lines on it being blocked by exactly three points of \( B \). Excluding this situation (one pencil of weight two) will be sufficient for our proofs. This is what the next lemma does.

Lemma 11 There is no minimal blocking set \( B \) of size 52 on \( Q(4,7) \) such that there is one point \( p \in Q(4,7) \setminus B \) with \( q + 1 \) lines on it being blocked by exactly three points of \( B \).

Finally, we prove in this section the first step to the characterization. In the next lemma, we prove that a secant line to a minimal blocking set must contain "a lot" of points. It seems that this result is always an immediate corollary of the results about minimal blocking sets of \( Q(4,q) \). Namely, this result is also proven in [10] in the case \( q = 3, 5, 7 \).
Lemma 12. Suppose that $K$ is a minimal blocking set of $Q(6,q)$, not an ovoid, and $|K| \leq q^3 + q$. If $L$ is a line of $Q(6,q)$, then $|L \cap K| = 0, 1$ or $|L \cap K| \geq 3$.

PROOF. Suppose that $|L \cap K| = 2$. Consider a generator $\pi \subseteq Q(6,q)$ such that $L \subseteq \pi$ and $L \cap K = \pi \cap K$. Such a generator $\pi$ exists. Suppose that every generator on $L$ contains a point of $K$ not on $L \cap K$. This would imply $|T_p(Q(6,q)) \cap K| > 1 + q$, for every $p \in L \cap K$, a contradiction with Lemma 3. Count the number of pairs $(u,v)$, $u \in \pi \setminus L$, $v \in K \setminus \pi$, $u \in T_v(Q(6,q))$. By Lemmas 4, 9 and 10, $|T_u(Q(6,q)) \cap K| > q^2 + 2 \geq q^2 + |L \cap K|$. We find a lower bound of $q^2(q^2 + 1)$. If $v \in K \setminus \pi$, then $T_v(Q(6,q))$ intersects $\pi$ in a line, hence with $v$ correspond $q$ or 0 points of $\pi \setminus L$, which gives as upper bound $(q^3 + q - |L \cap K|)q$. Necessarily $(q^3 + q - 2q)q \geq (q^2 + 1)q^2$, a contradiction. Hence, $|L \cap K| < 2$ or $|L \cap K| > 2$.

3 The characterization when $Q(6,q)$ has no ovoids

In this section we prove Theorem 1 for $q > 3$ prime. When possible, proofs are given for general $q$. It will be mentioned when we suppose that $q = 5, 7$. We will first prove Theorem 1 for $n = 2$. We suppose that $K$ is a minimal blocking set of $Q(6,q)$, $|K| \leq q^3 + q$.

Lemma 13. If $\pi$ is a generator of $Q(6,q)$, $q = 5, 7$, then $|\pi \cap K| = 1$ or $|\pi \cap K| \geq 3$ and all points of $\pi \cap K$ lie on a line of $Q(6,q)$.

PROOF. If $|\pi \cap K| \geq 2$, then $\pi$ contains at least 3 points on a line $L$ (Lemma 12). Suppose that $\pi$ contains another point $p \in K$ not on $L$. There are at least 3 lines on $p$ containing at least 3 points of $K$ ($p$ included). Hence, $\pi$ would contain at least 7 points of $K$. Lemma 3 states $|T_p(Q(6,q)) \cap K| \leq q$. So $q = 7$. Suppose that $|\pi \cap K| = 7$ and $\pi \cap K = \{p_1, \ldots, p_7\}$. Since every line $p_ip_j (i \neq j)$ must contain a third point $p_k$, the points $\{p_1, \ldots, p_7\}$ must form a PG$(2,2) \subset PG(2,7)$, a contradiction, hence $|\pi \cap K| \geq 8$, contradicting Lemma 3.

Lemma 14. Suppose that $p \in Q(6,q) \setminus K$, with $q = 5, 7$. If there is a generator on $p$ containing exactly one point of $K$, then $|T_p(Q(6,q)) \cap K| \leq q^2 + q$; otherwise $|T_p(Q(6,q)) \cap K| \geq 3(q^2 + 1)$.

PROOF. Suppose that $\pi$ is a generator of $Q(6,q)$ on $p$ containing exactly one point $s \in K$. Consider the $q^2 - q$ lines of $\pi$ not through $s$ or $p$. Every such line lies in $q$ generators of $Q(6,q)$ different from $\pi$. The $q(q^2 - q)$ planes of
Q(6, q) on these lines of \( \pi \) are blocked by at least one point of \( \mathcal{K} \), so at most \( q^2 + q \) points of \( \mathcal{K} \) remain for \( T_p(Q(6, q)) \). If all generators on \( p \) contain at least 3 points of \( \mathcal{K} \), then \( |T_p(Q(6, q)) \cap \mathcal{K}| \geq 3(q^2 + 1) \).

**Lemma 15** Suppose that \( \pi \) is a generator of \( Q(6, q) \), \( q = 5, 7 \), containing at least 3 points of \( \mathcal{K} \) on the line \( L \). Let \( p \in \pi \setminus L \), then \( |T_p(Q(6, q)) \cap \mathcal{K}| \leq q^2 + q \).

**PROOF.** Denote \( \pi \cap \mathcal{K} = \{ s_0, \ldots, s_n \} \) \((n \geq 2)\). If every generator on \( p \) contains more than one point of \( \mathcal{K} \), then also the \( q + 1 \) generators on \( \langle s_0, p \rangle \) contain more than one point of \( \mathcal{K} \) while the only point of \( \mathcal{K} \) they share is the point \( s_0 \). Hence, \( |T_{s_0}(Q(6, q)) \cap \mathcal{K}| > 1 + q \), a contradiction. By the previous lemma, \( |T_p(Q(6, q)) \cap \mathcal{K}| \leq q^2 + q \).

**Lemma 16** If \( L \) is a line of \( Q(6, q) \), \( q = 5, 7 \), and \( |L \cap \mathcal{K}| > 1 \), then there exists a point \( p \in L \setminus \mathcal{K} \) such that \( |T_p(Q(6, q)) \cap \mathcal{K}| \geq 3(q^2 + 1) \); furthermore, exactly \( q^2 + 1 \) lines on \( p \) meet \( \mathcal{K} \) in at least 3 points and \( p \) projects those lines onto an ovoid of \( Q(4, q) \), where \( Q(4, q) \) is the base of the cone \( T_p(Q(6, q)) \cap Q(6, q) \).

**PROOF.** Consider a plane \( \pi \) such that \( |\pi \cap \mathcal{K}| \geq 3 \) and \( L := \langle \pi \cap \mathcal{K} \rangle \). Let \( L = \{ p_0, \ldots, p_q \} \), and suppose that \( p_q \not\in \mathcal{K} \). Such a point exists, since otherwise \( |T_{p_i}(Q(6, q)) \cap \mathcal{K}| \geq q + 1 \) for \( p_i \in \mathcal{K} \), a contradiction with Lemma 3. Suppose that \( |\pi \cap \mathcal{K}| = N, 3 \leq N \leq q \). Let \( \pi \cap \mathcal{K} = \{ p_0, \ldots, p_{N-1} \} \). Then \( |(\cup_{i=0}^{N-1}T_{p_i}(Q(6, q)) \cap \mathcal{K})| := M \leq Nq - (N-1)N \). So \( |(T_{p_N}(Q(6, q)) \cap \mathcal{K}) \cup \ldots \cup (T_{p_0}(Q(6, q)) \cap \mathcal{K})| := M' \geq q^3 + 2 - N(q - (N-1)). \) Suppose that for all \( p_i \in L \setminus \mathcal{K}, |T_{p_i}(Q(6, q)) \cap \mathcal{K}| \leq q^2 + q \), then \( M' \leq ((q+1)-N)(q^2 + q - N) + N \). For \( 3 \leq N \leq 5, q = 5 \), and \( 3 \leq N \leq 7, q = 7 \), \( ((q+1)-N)(q^2 + q - N) + N < q^3 + 2 - N(q - N + 1) \), hence for some \( p_i \in L \setminus \mathcal{K}, |T_{p_i}(Q(6, q)) \cap \mathcal{K}| \geq 3(q^2 + 1) \) and by Lemma 14, \( p_i \) lies on a line \( L \), \(|L \cap \mathcal{K}| \geq 3 \). Since, by Lemma 13, two lines containing 3 points of \( \mathcal{K} \) cannot span a plane of \( Q(6, q) \), Lemma 15 implies that there are \( q^2 + 1 \) lines on \( p_i \) each containing at least 3 points of \( \mathcal{K} \) and \( p_i \) projects these points onto an ovoid of \( Q(4, q) \).

We will call lines of \( Q(6, q) \) intersecting \( \mathcal{K} \) in more than one point secants to \( \mathcal{K} \).

**Lemma 17** If \( L \) is a line of \( Q(6, 7) \), then \(|L \cap \mathcal{K}| = 0, 1 \) or \(|L \cap \mathcal{K}| \geq 4 \).

**PROOF.** Consider a secant to \( \mathcal{K} \). By the previous lemma there is a point \( p \) on that secant on which there are \( q^2 + 1 \) secants to \( \mathcal{K} \). Each secant on \( p \) intersects \( \mathcal{K} \) in at least 3 points. Suppose that \( p \in L \) and \(|L \cap \mathcal{K}| = 3 \). Choose an arbitrary plane \( \pi \) of \( Q(6, 7) \) on \( L \), choose \( u \in \pi \setminus L \) arbitrary. Suppose now that \(|T_u(Q(6, 7)) \cap \mathcal{K}| = q^2 + 3 \). From Lemma 15, we know that
\[ |T_u(Q(6, 7)) \cap K| \leq q^2 + 7. \]

Consider all generators on \( u \). The \( q+1 \) generators on \( \langle u, p \rangle \) are blocked by at least 3 points. Hence \( u \) projects \( T_u(Q(6, 7)) \cap K \) onto a minimal blocking set of \( Q(4, 7) \) (Lemma 4) of size \( q^2 + 3 = 52 \) with the property that all multiple blocked lines have excess exactly 2 and share a common point not in the projected blocking set. Such a minimal blocking set is excluded by Lemma 11. Hence, \( |T_u(Q(6, 7)) \cap K| > q^2 + 3 \) for all \( u \in \pi \setminus L \) if there exists a 3-secant \( L \). We can now repeat the counting arguments of Lemma 12. Count the number of pairs \( (u, v) \), \( u \in \pi \setminus L, v \in K \setminus \pi, u \in T_v(Q(6, 7)) \). Necessarily \( |T_u(Q(6, 7)) \cap K| > q^2 + 3 \geq q^2 + |L \cap K| \). We find a lower bound of \( q^2(q^2 + 1) \). If \( v \in K \setminus \pi \), then \( T_v(Q(6, 7)) \) intersects \( \pi \) in a line, hence with \( v \) correspond \( q \) or 0 points of \( \pi \setminus L \), which gives as upper bound \( (q^2 + q - |L \cap K|)q \). Necessarily \( (q^2 + q - 3)q \geq (q^2 + 1)q^2 \), a contradiction. Hence \( |L \cap K| < 2 \) or \( |L \cap K| > 3 \).

**Corollary 18** If \( L \) is a line of \( Q(6, 7) \) and \( |L \cap K| > 1 \), then there exists a point \( p \in L \setminus K \) such that \( |T_p(Q(6, 7)) \cap K| \geq 4(q^2 + 1) \); furthermore, exactly \( q^2 + 1 \) lines on \( p \) meet \( K \) in at least 4 points and \( p \) projects those lines onto an ovoid of \( Q(4, 7) \), where \( Q(4, 7) \) is the base of the cone \( T_p(Q(6, 7)) \cap Q(6, 7) \).

So far we know that \( K \) contains at least \( q^2 + 1 \) lines on a common point \( p \) intersecting \( K \) in at least 3 \( (q = 5) \) or 4 \( (q = 7) \) points; \( p \notin K \). We now prove that \( K \) cannot contain 2 such secant cones. From now on we will use the fact that all the ovoids of \( Q(4, q) \), \( q = 5, 7 \), are elliptic quadrics \( Q^{-}(3, q) \).

**Lemma 19** Consider a secant cone on \( p \notin K \) as described in Lemma 16 and Corollary 18. There cannot exist such a second secant cone on a point \( p' \notin K \).

**PROOF.** Suppose that there is such a cone on \( p \) and \( p' \). The point \( p \) cannot be collinear with \( p' \) since if \( p' \in T_p(Q(6, q)) \), the cones can only share at most \( q-1 \) points, which implies \( |K| \geq 3(q^2+1)+3(q^2+1)-4 > 5(q^2+1) \) (if \( q = 5 \) or \( |K| \geq 4(q^2+1)+4(q^2+1)-6 > 7(q^2+1) \) (if \( q = 7 \)), in both cases a contradiction to \( |K| \leq q^3 + q. \) Hence \( p' \notin T_p(Q(6, q)) \) and \( |T_p(Q(6, q)) \cap T_p(Q(6, q)) \cap K| = q^2 + 1 \); and this intersection is necessarily an ovoid of \( Q(4, q) \). We will now distinguish between \( q = 5 \) and \( q = 7 \).

**Case 1:** \( q = 5 \). Consider two lines \( L, M, p \in L, p \in M \), of the secant cone on \( p \), with necessarily \( |L \cap K| = |M \cap K| = 3 \). Denote \( L \setminus K = \{ p, r_0, r_1 \}, M \setminus K = \{ p, r_0', r_1' \} \). It is clear that \( r_0, r_1, r_0', r_1' \notin T_p(Q(6,5)) \). For, suppose that e.g. \( r_0 \in T_p(Q(6,5)) \). Since \( |T_p(Q(6,5)) \cap K| \geq 3(q^2+1) \) and \( |T_p(Q(6,5)) \cap K| \geq 3(q^2+1) \), \( r_0 \in T_p(Q(6,5)) \) implies that \( T_p(Q(6,5)) \cap K \) and \( T_p(Q(6,5)) \cap K \) share less than \( q^2 + 1 \) points, hence \( |K| > 5(q^2 + 1) \), a contradiction. Let \( Q_{r_1'}(3, 5) \) be the elliptic quadric \( T_{r_1'}(Q(6,5)) \cap T_p(Q(6,5)) \cap K \), and let \( Q_{r_1'}^{-}(3, 5) \) be the elliptic quadric \( T_{r_1'}(Q(6,5)) \cap T_p(Q(6,5)) \cap K \). Furthermore, if \( Q_{r_1}^{-}(3, 5) \) is the intersection \( T_p(Q(6,5)) \cap T_p(Q(6,5)) \cap K \), then \( Q_{r_1}^{-}(3, 5) \) shares exactly 1
point with $Q_p^-(3,5)$, namely the point $\langle r_0, r_1 \rangle \cap Q_{r_0}^-(3,5)$. The same arguments hold for the points $r_0'$ and $r_1'$. The quadrics $Q_{r_0}^-(3,5)$ share the point $\langle r_0', r_1' \rangle \cap Q_{r_0}^-(3,5)$. But $Q_{r_0}^-(3,5)$ consists of $q^2$ points of $(T_{r_0}(Q(6,5)) \cap K) \setminus T_p(Q(6,5))$ plus one point of $\langle r_0', r_1' \rangle \cap K$, hence $Q_{r_0}^-(3,5)$ has at least 12 points in common with some $Q_{r_0}^-(3,5)$, but 12 points define the solid of $Q_{r_0}^-(3,5)$ completely. So for example, $r_0'$ shares the same elliptic quadric $Q_{r_0}^-(3,5)$ of $K$ with $T_{r_0}(Q(6,5))$ as $r_0$ or $r_1$, which implies that $q^2 + 1$ points of the cone of secants to $K$ on $p$ have the same elliptic quadric in their tangent hyperplane; a contradiction since the polar space of a 3-dimensional space, containing some elliptic quadric of $Q(6,5)$, is a plane, which cannot contain all these $q^2 + 1$ points since they define at least a 3-dimensional space.

**Case 2:** $q = 7$. We again consider two lines $L, M$, $p \in L$, $p \in M$, of the secant cone on $p$ with necessarily $|L \cap K| = |M \cap K| = 4$. Denote now $L \setminus K = \{p, r_0, r_1, r_2\}$, $M \setminus K = \{p, r_0', r_1', r_2'\}$. Again $T_{r_0}(Q(6,7)) \cap T_{r_0'}(Q(6,7)) \cap K = Q_{r_0}^-(3,7)$, an elliptic quadric, and $T_{r_0}(Q(6,7)) \cap T_{r_0'}(Q(6,7)) \cap K = Q_{r_0}^-(3,7)$, an elliptic quadric. Also here the intersection $T_p(Q(6,7)) \cap T_{r_0'}(Q(6,7)) \cap K = Q_{r_0}^-(3,7)$ shares exactly one point with $Q_{r_0}^-(3,7)$, namely the point $\langle r_0, r_1 \rangle \cap Q_{r_0}^-(3,7)$. We again consider the points $r_0', r_1'$ and $r_2'$. The quadrics $Q_{r_0}^-(3,7)$ share the point $\langle r_0', r_1' \rangle \cap Q_{r_0}^-(3,7)$. Also here $Q_{r_0}^-(3,7)$ consists of $q^2$ points of $(T_{r_0'}(Q(6,7)) \cap K) \setminus T_p(Q(6,7))$ plus the point $\langle r_0', r_1' \rangle \cap Q_{r_0}^-(3,7)$, hence $Q_{r_0}^-(3,7)$ has at least 16 points in common with some quadric $Q_{r_0}^-(3,7)$; which determine the solid of $Q_{r_0}^-(3,7)$ completely. So for example $r_0'$ shares the same elliptic quadric $Q^-(3,7)$ of $K$ with $T_{r_0'}(Q(6,7))$ as $r_0$, $r_1$ or $r_2$, which implies that $q^2 + 1$ points of the secant cone on $p$ have the same elliptic quadric in their tangent hyperplane. As in the case $q = 5$, this is a contradiction.

**Lemma 20** If there is a cone of secants to $K$ on $p$ and $L$ is a secant line on $p$ and $r \in L \setminus K$, $r \neq p$, then $|T_r(Q(6,q)) \cap K| = q^2 + |L \cap K|$ and $r$ projects all points of $T_r(Q(6,q)) \cap K$ onto an ovoid of $Q(4,q)$, where $Q(4,q)$ is a base of the cone $T_r(Q(6,q)) \cap Q(6,q)$.

**Proof.** It is known that $|T_r(Q(6,q)) \cap K| \leq q^2 + q$ or $|T_r(Q(6,q)) \cap K| \geq 3(q^2 + 1)$. Since no two cones of secants can exist, $|T_r(Q(6,q)) \cap K| \leq q^2 + q$. Necessarily $|L \cap K| \leq q - 1$, but suppose that $|T_r(Q(6,q)) \cap K| > q^2 + |L \cap K|$. There are at least $q^2$ points of $K \setminus L$ necessary to block all generators on $r$. If $|T_r(Q(6,q)) \cap K| > q^2 + |L \cap K|$, then $|(T_r(Q(6,q)) \cap K) \setminus (T_p(Q(6,q)) \cap K)| > q^2$, hence some generator on $r$ not on $p$ must be blocked by at least 2 points of $K$, which implies the existence of a second cone of secants (Corollary 18), a contradiction. Necessarily $r$ projects all points of $T_r(Q(6,q)) \cap K$ onto an ovoid of $Q(4,q)$.
We know so far that there exists exactly one cone of secants on a point $p$ for which $|T_p(Q(6, q)) \cap K| \geq 3(q^2 + 1)$, $4(q^2 + 1)$, for respectively $q = 5$, $q = 7$. In the next lemmas we will consider this cone of secants and reduce the number of possibilities on the number of points of $K$ secants can have. We will first prove a corollary of Lemma 20.

**Corollary 21** If $L$ is a line of the cone of secants on $p$ and there exists a point $r \in L \setminus K$, $r \neq p$, then no point of $(T_r(Q(6, q)) \cap K) \setminus T_p(Q(6, q))$ is collinear with a point of $T_p(Q(6, q)) \cap K$.

**PROOF.** Put $L = \langle r, p \rangle$. To block all generators on $r$, at least $q^2$ points of $K \setminus T_p(Q(6, q))$ are needed. Put $S = (T_r(Q(6, q)) \cap K) \setminus T_p(Q(6, q))$. Suppose that $s \in S$ is collinear with $t \in T_p(Q(6, q)) \cap K$; which implies $|\langle s, t \rangle \cap K| > 1$. Then there is a point $p' \in \langle s, t \rangle \cap K$ on which there exists a cone of secants (Lemma 16 and Corollary 18), a contradiction with Lemma 19.

**Lemma 22** There are no $(q - 1)$-secants to $K$ on $p$. If there is a $q$-secant on $p$, then $|T_p(Q(6, q)) \cap K| = q^3 + q$ and Theorem 1 is proven for $q = 5, 7, n = 2$.

**PROOF.** Suppose that $L$ and $M$ are secant lines on $p$. Suppose that $|L \cap K| = q$ and $|M \cap K| < q$. Hence we find a point $r \in M \setminus K$, $r \neq p$. Now $s \in (T_r(Q(6, q)) \cap K) \setminus T_p(Q(6, q))$ is collinear with exactly one point of $L \cap K$. This contradicts Corollary 21. So all secants on $p$ are $q$-secands and Theorem 1 is proven. Suppose now that $|L \cap K| = q - 1$. If $M$ is a secant line on $p$ such that $|M \cap K| < q - 1$, we find $r, r' \in M \setminus K$, $r, r' \neq p$, each point having at least $q^2$ points of $K \setminus T_p(Q(6, q))$ in its tangent hyperplane. Since no point of the $2q^2$ different points of $K \setminus T_p(Q(6, q))$ can be collinear with a point of $T_p(Q(6, q)) \cap K$, $2q^2$ points of $K \setminus T_p(Q(6, q))$ must be collinear with the unique point $p' \in L \setminus K$, $p' \neq p$, hence $|T_p(Q(6, q)) \cap K| \geq 2q^2 + (q - 1)$ contradicting Lemma 20. Hence $|M \cap K| = q - 1$ for every secant line $M$ on $p$ and $|T_p(Q(6, q)) \cap K| = (q - 1)(q^2 + 1)$. With the same arguments about collinearity we find that $q^2$ points of $(T_r(Q(6, q)) \cap K) \setminus T_p(Q(6, q))$, $r \in L \setminus K$, $r \neq p$; spanning at least a 3-dimensional space, must be collinear with the $q^2 + 1$ points not in $K$ on the secants to $K$ through $p$, also spanning at least a 3-dimensional space, a contradiction.

**Lemma 23** If $p$ is the point on which there is a secant cone to $K$, not all $q^2 + 1$ secants to $K$ are $(q - 2)$-secants.

**PROOF.** Denote all the points on the lines of the secant cone on $p$ by $C$. Suppose that there are $q^2 + 1$ $(q - 2)$-secants on $p$. On every secant $L$ on $p$, 2 points $r, r' \in L \setminus K$, $r \neq p \neq r'$, remain. Put $S = ((T_r(Q(6, q)) \cup T_{r'}(Q(6, q))) \cap
\( \mathcal{K} \setminus T_p(\mathbb{Q}(6, q)) \). Then \(|S| = 2q^2\) and every point \( s \in S \) is collinear with exactly one point of \( M \setminus \mathcal{K} \), \( M \) a secant on \( p \); and for every point \( s \in S \), \( T_s(\mathbb{Q}(6, q)) \cap T_p(\mathbb{Q}(6, q)) \cap \mathcal{C} \) is an elliptic quadric. Not all \( 2(q^2 + 1) \) points of \( C \setminus (\mathcal{K} \cup \{ p \}) \) can be collinear with the same points of \( S \), since the polar space of a 3-dimensional space is a plane. So we find 2 points \( s, s' \in S \) for which the corresponding elliptic quadrics \( Q_1^1(3, q) \) and \( Q_2^1(3, q) \) on the cone are distinct. Two such quadrics share at most \( q + 1 \) points; hence we have found at least \( 2(q^2 + 1) - (q + 1) \) points of \( C \setminus (\mathcal{K} \cup \{ p \}) \) collinear with \( s \) or \( s' \). The \( 2q^2 \) points of \( S \) each define an elliptic quadric of \( C \setminus (\mathcal{K} \cup \{ p \}) \); each such elliptic quadric contains at least \( \frac{q^2 + 1 - (q + 1)^2}{2} \) points of one of the previous quadrics. Since \( \frac{q^2 + 1 - (q + 1)^2}{2} > q + 1 \), the third elliptic quadric must coincide with one of the previous two. So every point of \( S \) is collinear with all points of either \( Q_1^1(3, q) \) or \( Q_2^1(3, q) \); hence at least \( q^2 \) points of \( S \) are collinear with a quadric \( Q^-(3, q) \) of \( \mathbb{Q}(6, q) \), again a contradiction.

So far we have found the following situation:

- there is exactly one cone of secants to \( \mathcal{K} \).
- If there is a \( q \)-secant, Theorem 1 is proven for \( q = 5, 7, n = 2 \).
- There are no \((q - 1)\)-secants to \( \mathcal{K} \).
- Not all secants of the secant cone can be \((q - 2)\)-secants.

Applied to the \( q = 5 \) case, Theorem 1 is proven for \( n = 2 \), since all secants must be at least \( 3 = (q - 2) \)-secants. We need to exclude for \( q = 7 \) the 4-secants.

**Lemma 24** If \( q = 7 \), then there are no 4-secants to \( \mathcal{K} \).

**PROOF.** Suppose that \( L \) is a 4-secant on \( p \). Suppose that \( M \) is a 5-secant on \( p \). Hence we find 2 points \( r, r' \in M \setminus \mathcal{K} \) different from \( p \), and 3 points \( s, s', s'' \in L \setminus \mathcal{K} \) different from \( p \). From Lemma 20, \( 3q^2 \) points of \( \mathbb{K} \setminus T_p(\mathbb{Q}(6, q)) \) are necessary to block all generators on the points \( s, s' \) and \( s'' \). But all of these \( 3q^2 \) points are collinear with \( r \) or \( r' \) (not both), hence \(|T_r(\mathbb{Q}(6, 7)) \cap \mathcal{K}| \geq 3q^2/2\) or \(|T_{r'}(\mathbb{Q}(6, 7)) \cap \mathcal{K}| \geq 3q^2/2\). A contradiction with Lemma 20, since \( 3q^2/2 > q^2 + |M \cap \mathcal{K}| \). We conclude that there are no 5-secants if there is a 4-secant. We repeat the arguments of the previous lemma. Put \( S = ((T_s(\mathbb{Q}(6, 7)) \cup T_{s'}(\mathbb{Q}(6, 7)) \cup T_{s''}(\mathbb{Q}(6, 7))) \cap \mathcal{K}) \setminus T_p(\mathbb{Q}(6, q)) \). Then \( |S| = 3q^2 \) and every point \( s \in S \) is collinear with exactly one point of \( M \setminus \mathcal{K} \), \( M \) a secant on \( p \). As in Lemma 23, we now find 3 points in the set \( S \) giving rise to three distinct elliptic quadrics \( Q_1^1(3, 7), Q_2^1(3, 7), Q_3^1(3, 7) \), using at least \( 3(q^2 + 1) - 3(q + 1) = 126 \) points, not in \( \mathcal{K} \), of the secant cone \( pQ^-(3, 7) \). A fourth point \( t \in S \) defines an elliptic quadric \( Q^-(3, q) \) containing at least \( \frac{q^2 + 1 - 3(q + 1)}{3} = \frac{26}{3} > 8 \), hence 9 points of one of the elliptic quadrics and so these elliptic quadrics must coincide. So every point of \( S \) is collinear with all points of either \( Q_1^1(3, 7) \),
Q(3, 7) or Q(3, 7); hence at least \( q^2 \) points of \( S \) are collinear with an elliptic quadric of \( Q(6, q) \), a contradiction, since the polar space of a 3-dimensional space is a plane, but these \( q^2 \) latter points do not lie in a plane.

As a corollary we may now also state Theorem 1 for \( q = 7 \). At this point, we have proven the following theorem.

**Theorem 25** Let \( \mathcal{K} \) be a minimal blocking set of \( Q(6, q) \), \( q = 5, 7 \), \( |\mathcal{K}| \leq q^3 + q \). Then there is a point \( p \in Q(6, q) \setminus \mathcal{K} \) with the following property: \( T_p(Q(6, q)) \cap Q(6, q) = pQ(4, q) \), and \( \mathcal{K} \) consists of all the points of the lines \( L \) on \( p \) meeting \( Q(4, q) \) in an ovoid \( \mathcal{O} \), minus the point \( p \) itself, and \( |\mathcal{K}| = q^3 + q \).

We will now do the characterization for \( Q(2n + 2, q) \), \( q = 5, 7, n \geq 3 \). For the following lemmas we suppose that every ovoid of \( Q(4, q) \), \( q \) odd, is an elliptic quadric. Hence \( Q(6, q) \) has no ovoid \([19]\). This hypothesis is true for \( q = 5, 7 \). Furthermore we suppose as induction hypothesis that Theorem 1 is true for \( Q(6, q), \ldots, Q(2n, q) \). Using this hypothesis we will prove the characterization for \( Q(2n + 2, q) \).

For all lemmas we suppose that \( \mathcal{K} \) is a minimal blocking set of \( Q(2n + 2, q) \) of size at most \( q^{n+1} + q^{n-1} \).

**Lemma 26** There exists a point \( p \in Q(2n + 2, q) \setminus \mathcal{K} \) such that \( |T_p(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-2} \).

**PROOF.** Consider \( s \in Q(2n + 2, q) \setminus \mathcal{K} \). Since minimality is preserved under projection (Lemma 4), \( |T_s(Q(2n + 2, q)) \cap \mathcal{K}| \geq q^n + q^{n-2} \) if we assume the induction hypothesis. Count the number of incidences \( (p, H), p \in \mathcal{K}, H \) a tangent hyperplane to \( Q(2n + 2, q) \) in a point not in \( \mathcal{K} \). Denote this number by \( N \); \( N \) is an upper bound since \( |\mathcal{K}| \leq q^{n+1} + q^{n-1} \). With every point of \( \mathcal{K} \) corresponds exactly 1 tangent hyperplane. The total number of incidences \( (p, H), p \in \mathcal{K} \) and \( H \) a hyperplane \( T_r(Q(2n + 2, q)), r \notin \mathcal{K} \), is at most \( (q^{n+1} + q^{n-1})q|Q(2n, q)| \) under the assumption that \( |\mathcal{K}| \) equals the upper bound \( q^{n+1} + q^{n-1} \). We find \( N = (q^n + q^{n-2} + 1)(|Q(2n + 2, q) \setminus \mathcal{K}|) + (-q^{2n} + q^{2n-1} - q^{2n-2} - q^{n-1} - 2q^{n-2} - q^{n-3} - \ldots - q - 1) \) (\( n > 4 \), for \( n = 3, 4 \), the remainder is \( -q^6 + q^5 - q^4 - q^3 - 2q - 1 \) and \( -q^5 + q^4 - 2q^3 - 2q^2 - q - 1 \) respectively), where we again used the value \( q^{n+1} + q^{n-1} \) for \( |\mathcal{K}| \). Since the remainder is negative, there must exist a point \( r \in Q(2n + 2, q) \setminus \mathcal{K} \) such that \( |T_r(Q(2n + 2, q)) \cap \mathcal{K}| < q^n + q^{n-2} + 1 \), which implies \( |T_r(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-2} \).

**Lemma 27** If \( L \) is a line of \( Q(2n + 2, q) \), \( L \cap \mathcal{K} = \emptyset \), \( |T_L(Q(2n + 2, q)) \cap \mathcal{K}| = q^{n-1} + q^{n-3} \), then \( |T_{r_i}(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-2} \), for all points \( r_i \in L \).
PROOF. Necessarily \(|T_r,(Q(2n+2,q)) \cap \mathcal{K}| \geq q^n + q^{n-2}\) by the induction hypothesis and Lemma 4. The sets \(T_r,(Q(2n+2,q)) \cap \mathcal{K}\) have exactly \(q^{n-1} + q^{n-3}\) points in common which implies \(\sum_{i=0}^{q-1}|T_r,(Q(2n+2,q)) \cap \mathcal{K}| \geq (q+1)(q^n + q^{n-2} - (q^{n-1} + q^{n-3})) + q^{n-1} + q^{n-3} \geq q^{n+1} + q^{n-1} \geq |\mathcal{K}|\). Hence \(|T_r,(Q(2n+2,q)) \cap \mathcal{K}| = q^n + q^{n-2}\).

Lemma 28 If \(L\) is a line of \(Q(2n+2,q)\), \(L \cap \mathcal{K} = \emptyset\), and \(|T_L,(Q(2n+2,q)) \cap \mathcal{K}| = q^n + q^{n-3}\), then the points of \(T_L,(Q(2n+2,q)) \cap \mathcal{K}\) are the points of a truncated cone \(\pi_{n-4} \mathcal{O} \setminus \pi_{n-3}\); \(\mathcal{O}\) an elliptic quadric of \(Q(4,q)\).

PROOF. Choose \(r_0 \in L\). From Lemma 27, \(|T_{r_0}(Q(2n+2,q)) \cap \mathcal{K}| = q^n + q^{n-2}\) and these points of \(T_{r_0}(Q(2n+2,q)) \cap \mathcal{K}\) are projected from \(r_0\) onto a truncated cone \(\pi_{n-3} \mathcal{O} \setminus \pi_{n-3}\) (induction hypothesis and Theorem 1). Denote by \(\pi_{2n}^0\) a \(2n\)-space such that \(T_{r_0}(Q(2n+2,q)) = r_0 \pi_{2n}^0\) and let \(\pi_{2n}^0\) be the space in which we consider the projection of \(T_{r_0}(Q(2n+2,q)) \cap \mathcal{K}\) from \(r_0\). The points of \(T_{r_1}(Q(2n+2,q)) \cap \mathcal{K}\), projected from \(r_0\), cannot span an \((n+1)\)-dimensional space; or else all projected points of \(T_{r_0}(Q(2n+2,q)) \cap \mathcal{K}\) lie in this space; then all \(q^n + q^{n-2}\) points of \(T_{r_0}(Q(2n+2,q)) \cap \mathcal{K}\) lie in \(\langle r_0, S \rangle \subseteq T_L(Q(2n+2,q))\) \((S = \text{set of projected points of } T_L(Q(2n+2,q)) \cap \mathcal{K})\), a contradiction. So these \(q^n + q^{n-2}\) projected points span an \(n\)-dimensional space. Denote by \(\alpha\) the space spanned by the points of \(T_L(Q(2n+2,q)) \cap \mathcal{K}\) themselves.

We now prove that \(L\) cannot intersect \(\pi_{2n}^0\) in a point of \(T_{\pi_{n-3}}(Q(2n,q))\). Assume the contrary. We have two possibilities. Suppose that \(L\) contains a point \(r_1\) of \(\pi_{n-3}\). Then this point \(r_1\) is collinear with all points of \(\pi_{n-3} \mathcal{O} \setminus \pi_{n-3}\). But then \(T_L(Q(2n+2,q)) = T_{r_0}(Q(2n+2,q)) \cap T_{r_1}(Q(2n+2,q)) \cap \mathcal{K}\) contains all \(q^n + q^{n-2}\) points of \(T_{r_0}(Q(2n+2,q))\), a contradiction. Suppose secondly that \(L\) contains a point \(r_1\) of \(\pi_{n-3} \mathcal{O} \setminus \pi_{n-3}\). Consider \(T_{r_1}(Q(2n,q))\). This contains \(\pi_{n-3}\), but intersects \(Q(4,q)\) in a cone \(r_1 Q(2,q)\), where \(\pi_{n-3} Q(4,q) = T_{\pi_{n-3}}(Q(2n,q)) \cap Q(2n,q)\). Necessarily \(r_1 \notin \mathcal{O}\) since \(L \cap \mathcal{K} = \emptyset\). All generators of \(Q(4,q)\) on \(r_1\) are blocked by exactly \(q+1\) points of \(\mathcal{O}\). This shows that \(T_{r_0}(Q(2n+2,q))\) and \(T_{r_1}(Q(2n+2,q))\) share exactly \((q+1)q^{n-2}\) points. With \(T_L(Q(2n+2,q)) = T_{r_0}(Q(2n+2,q)) \cap T_{r_1}(Q(2n+2,q))\), this is a contradiction, since we supposed that \(|T_L(Q(2n+2,q)) \cap \mathcal{K}| = q^n + q^{n-3}\). So we conclude that \(L\) cannot intersect \(\pi_{2n}^0\) in a point of \(T_{\pi_{n-3}}(Q(2n,q))\). Using this we find that \(r_0\) projects the points of \(T_L(Q(2n+2,q)) \cap \mathcal{K}\) onto a truncated cone \(\pi_{n-4} \mathcal{O} \setminus \pi_{n-4}\). It is clear that all points of \(T_L(Q(2n+2,q)) \cap \mathcal{K}\) lie in \(\langle r_0, S \rangle\). Since \(\langle S \rangle\) is an \(n\)-dimensional space, the space \(\alpha\) is an \(n\)- or \((n+1)\)-dimensional space. Now \(L\) cannot lie in \(\alpha\) since \(L\) does not intersect \(T_{\pi_{n-3}}(Q(2n,q))\).

Repeat the arguments for two points \(r_0\) and \(r_0'\) of \(L\). Then the two spaces \(\alpha\) and \(\alpha'\) intersect in an \(n\)-dimensional space \(\beta\) and \(L\) is skew to \(\beta\) since \(L\) shared two distinct points \(r_0\) and \(r_0'\) with \(\alpha\) and \(\alpha'\). Now \(r_0\) projected the \((q^n + q^{n-3})\) points of \(T_L(Q(2n+2,q)) \cap \mathcal{K}\) onto an \(n\)-dimensional space sharing a truncated cone.
π_{n-4}O \setminus π_{n-4} \) with the projection. So also originally before the projection, the \(q^{n-1} + q^{n-3}\) points of \(T_L(Q(2n+2, q)) \cap \mathcal{K}\) formed a truncated cone \(π_{n-4}O \setminus π_{n-4}\).

In the following lemma, the existence of such lines \(L\) is proven in a constructive way.

**Lemma 29** There exists a line \(L\) of \(Q(2n + 2, q)\), \(L \cap \mathcal{K} = \emptyset\), such that \(|T_L(Q(2n + 2, q)) \cap \mathcal{K}| = q^{n-1} + q^{n-3}\).

**Proof.** Consider a point \(p \in Q(2n + 2, q) \cap \mathcal{K}\) such that \(|T_p(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-2}\) (Lemma 26). Denote by \(\mathcal{K}_p\) the projection of \(\mathcal{K} \cap T_p(Q(2n + 2, q))\) from \(p\) in a fixed \(Q(2n, q)\). Since \(|\mathcal{K}_p| = q^n + q^{n-2}\), every line \(\langle p, r \rangle\), \(r \in \mathcal{K}_p\), contains exactly 1 point of \(\mathcal{K}\). Denote \(\mathcal{K}_p = π_{n-3}O \setminus π_{n-3}\); \(O\) a 3-dimensional elliptic quadric. Choose \(s\) such that \(s \notin T_{π_{n-3}}(Q(2n + 2, q))\), and \(\langle p, s \rangle\) a line of \(Q(2n + 2, q)\). Then \(T_s(Q(2n + 2, q))\) intersects \(\mathcal{K}_p\) in a truncated cone \(π_{n-4}O \setminus π_{n-4}\) which is the projection of \(T_p(Q(2n + 2, q)) \cap T_s(Q(2n + 2, q)) \cap \mathcal{K}\), \(T_p(Q(2n + 2, q)) \cap T_s(Q(2n + 2, q)) = T_{(p, s)}(Q(2n + 2, q))\) contains exactly \(q^{n-3}(q^2 + 1)\) projected points of \(\mathcal{K}\); \(p\) never projected two points of \(\mathcal{K}\) onto the same point; so \(T_{(p, s)}(Q(2n + 2, q))\) contains exactly \(q^{n-1} + q^{n-3}\) points of \(\mathcal{K}\).

**Remark 30** Consider a non-singular parabolic quadric \(Q(2n, q)\). By the notation \(S(m, t, 2n, 1)\), we denote the set of \(m\)-dimensional spaces \(π_m\) such that \(π_m \cap Q(2n, q) = π_{m-t-1}Q(t, q)\), where \(Q(t, q)\) is a non-singular parabolic quadric. We will consider quadrics of type \(π_{n-4}Q(4, q)\) (and \(π_{n-5}Q(6, q)\) in the next section) on \(Q(2n, q)\), \(q\) odd; and we are interested in the polar space of such a cone \(π_{n-4}Q(4, q)\) with respect to \(Q(2n, q)\). In this section, \(t = 4\), \(m = n + 1\), and we use \(T = 2n + t - 2m = 2\). In [14, Theorem 22.7.2], we find the possibilities for the polar space of \(\langle π_{m-1}Q(t, q) \rangle\); it is either a cone \(π_{m-t-1}Q^+(T - 1, q)\) or a cone \(π_{m-t-1}Q^-(T - 1, q)\). So when \(t = 4\), \(m = n + 1\), then the polar space intersects \(Q(2n, q)\) in the union of two distinct \((n - 3)\)-dimensional spaces or one (doubly counted) \((n - 4)\)-dimensional space. We will consider a cone \(π_{n-4}O\) in a cone \(π_{n-3}O\); \(O\) an ovoid of \(Q(4, q)\). So \(π_{n-4}Q(4, q) \subset π_{n-3}Q(4, q)\); so \(π_{n-3} \subseteq \langle π_{n-4}Q(4, q) \rangle^φ\), \(φ\) the polarity of \(Q(2n, q)\). Hence, there is a second \((n - 3)\)-dimensional space \(π_{n-3}'\) of \(Q(2n, q)\) in \(\langle π_{n-4}Q(4, q) \rangle^φ\). It is clear that \(π_{n-3}' \not\subseteq T_{π_{n-3}}(Q(2n, q))\); so there are points \(r \in π_{n-3}' \setminus π_{n-3}\) for which \(T_r(Q(2n, q)) \cap T_{π_{n-3}}(Q(2n, q)) \cap Q(2n, q) = π_{n-4}Q(4, q)\).

In Section 4, we will consider a cone \(π_{n-5}O\) in a cone \(π_{n-4}O\); \(O\) an ovoid of \(Q(6, q)\). Hence \(t = 6\), \(m = n + 2\) and \(T = 2\). So \(π_{n-5}Q(6, q) \subset π_{n-4}Q(6, q)\); so \(π_{n-4} \subseteq \langle π_{n-5}Q(6, q) \rangle^φ\), \(φ\) the polarity of \(Q(2n, q)\). Hence there is a second \((n - 4)\)-dimensional space \(π_{n-4}'\) of \(Q(2n, q)\) in \(\langle π_{n-5}Q(6, q) \rangle^φ\). It is clear that \(π_{n-4}' \not\subseteq T_{π_{n-4}}(Q(2n, q))\); so there are points \(r \in π_{n-4}' \setminus π_{n-4}\) for which \(T_r(Q(2n, q)) \cap T_{π_{n-4}}(Q(2n, q)) \cap Q(2n, q) = π_{n-5}Q(6, q)\).
In the following lemma, we will often work with point sets and their projections. If we work with a point set $S$, we will denote its projection by $(S)^p$. It will be clear from the context from which subspace we are projecting.

**Lemma 31** If $r$ is a point, $r \in Q(2n + 2, q) \setminus \mathcal{K}$, and $|T_r(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-2}$, then the points of $T_r(Q(2n + 2, q)) \cap \mathcal{K}$ form a truncated cone $\pi_{n-3}O \setminus \pi_{n-3}$, $O$ an elliptic quadric $Q^-(3, q)$.

**PROOF.** By Lemma 4 and the induction hypothesis, $r$ projects the points of $T_r(Q(2n + 2, q)) \cap \mathcal{K}$ onto a truncated cone $\pi_{n-3}O \setminus \pi_{n-3}$. Let $T_r(Q(2n + 2, q)) \cap \mathcal{K}$ and let $\pi_{2n}$ be the space in which we consider the base $Q(2n, q)$ of the cone $T_r(Q(2n + 2, q)) \cap \mathcal{K}$. Let $L$ be a line of $Q(2n + 2, q)$ through $r$ intersecting $Q(2n, q)$ in a point $r'$, $r' \not\in T_{\pi_{n-3}}(Q(2n, q))$. Then $T_{r'}(Q(2n, q))$ intersects $T_{\pi_{n-3}}(Q(2n, q))$ in a space sharing $\pi_{n-3}$ with $\pi_{n-3}O$. Also $T_{(r,r')}(Q(2n + 2, q))$ shares a truncated cone $\pi_{n-4}O \setminus \pi_{n-4}$ with $\mathcal{K}$ (see Lemma 28 and Lemma 29). Since we supposed that every ovoid of $Q(4, q)$ is an elliptic quadric, every cone $\pi_{n-4}O \setminus \pi_{n-4}$ has dimension $n$. Assume now that $n \geq 4$. Consider two lines $L_1$ and $L_2$ of $Q(2n + 2, q)$ through $r$ such that the corresponding cones $T_{L_1}(Q(2n + 2, q)) \cap \mathcal{K}$ share the same base $\mathcal{O}$, i.e. $T_{L_1}(Q(2n + 2, q)) \cap \mathcal{K} = \langle \pi_{n-4}O \setminus \pi_{n-4} \rangle$ and $T_{L_2}(Q(2n + 2, q)) \cap \mathcal{K} = \langle \pi_{n-4}O \setminus \pi_{n-4} \rangle$. This is possible if $n \geq 4$ since $\langle \langle \pi_{n-3}O \rangle^p \rangle$ has points of $Q(2n, q)$ outside $\langle \langle \pi_{n-3}O \rangle^p \rangle$ in its polar space with respect to $Q(2n, q)$ (see the previous remark). Hence the projections share the same base and so $\pi_{n-4}O \setminus \pi_{n-4}$ and $\pi_{n-4}O \setminus \pi_{n-4}$ share the same base $\mathcal{O}$. Since we assumed that $n \geq 4$, dim($\pi_{n-4}O \setminus \pi_{n-4}$) = $n - 5 \geq -1$. This also shows that the two cones $\pi_{n-4}O$ and $\pi_{n-4}O$ define an $(n + 1)$-dimensional space. The point $r$ does not lie in this $(n + 1)$-dimensional space, otherwise the projections of $\pi_{n-4}O$ and $\pi_{n-4}O$ would define an $n$-dimensional space, which is false.

Let $\mathcal{O}_3$ be a 3-dimensional elliptic quadric lying on the cone $\pi_{n-4}O$, suppose that $|O_3 \cap \mathcal{O}| > 0$, and consider the cone $\pi_{n-4}O$. This truncated cone lies in some $T_{L_3}(Q(2n + 2, q))$, $L_3$ a line of $Q(2n + 2, q)$ through $r$. We show that $\pi_{n-4}O \setminus \pi_{n-4} \subseteq \mathcal{K}$. For, consider the cone $(\mathcal{O}_3)^p$ which has a non-empty intersection with $(\mathcal{O}_3)^p$. For every $t \in \mathcal{K}$ with $t^p \in (\mathcal{O}_3)^p \cap (\mathcal{O})^p$, we know already where the cone $t \pi_{n-4}$ lies (this is at least a line since $n \geq 4$). The points of the vertex $\pi_{n-4}$ of the cone $\mathcal{O}_3 \pi_{n-4}$, with $\pi_{n-4}O \setminus \pi_{n-4} = T_{L_3}(Q(2n + 2, q)) \cap \mathcal{K}$, are the only points in $T_{L_3}(Q(2n + 2, q))$ lying on $q$-secants to $\mathcal{K}$. The points of $\pi_{n-4}$ lie already on $q$-secants to $\mathcal{O}_3$; so $\pi_{n-4} = \pi_{n-4}$.

Repeating this argument for all elliptic quadrics $\mathcal{O}_3$ on the cone $\pi_{n-4}O$, we are actually proving that all points of the truncated cone $\langle \pi_{n-4}, \pi_{n-4} \rangle \pi_{n-4}O \setminus \pi_{n-4}$ lie in $\mathcal{K}$. This truncated cone has size $q(q^{n-1} + q^{n-3}) = |T_r(Q(2n + 2, q))\cap \mathcal{K}| = q^n + q^{n-2}$.
2, q)) \cap \mathcal{K}, which shows that \( T_r(Q(2n+2, q)) \cap \mathcal{K} = \langle \pi_{n-4}^{(1)}, \pi_{n-4}^{(2)} \rangle \mathcal{O} \setminus \langle \pi_{n-4}^{(1)}, \pi_{n-4}^{(2)} \rangle \).

We now will do the case \( n = 3 \). Reformulating the lemma, we must prove that \( T_r(Q(8, q)) \cap \mathcal{K} \) is a truncated cone \( p\mathcal{O} \setminus \{ p \} \). The point \( r \) projects \( T_r(Q(8, q)) \cap \mathcal{K} \) onto a truncated cone \( r_0 \mathcal{O} \setminus \{ r_0 \} \). For every line \( L = \langle r, r' \rangle \) of \( Q(8, q) \), \( r' \not\in T_{r_0}(Q(8, q)) \), \( T_{L}(Q(8, q)) \) contains \( q^3 + 1 \) points of \( \mathcal{K} \) and \( T_{L}(Q(8, q)) \) intersects \( \mathcal{K} \) in an elliptic quadric (Lemma 28). There are \( q^3 \) such lines through \( r \). Denote the associated elliptic quadrics in \( T_{L}(Q(8, q)) \cap \mathcal{K} \) by \( \mathcal{O}_L \). Consider two such elliptic quadrics \( \mathcal{O}_L \), \( \mathcal{O}_L' \) for which the intersection of \( \mathcal{O}_L \) and \( \mathcal{O}_L' \) is a conic \( C \), and consequently \( \langle \mathcal{O}_L, \mathcal{O}_L' \rangle = \pi_4 \) is a 4-dimensional space. Again, \( r \) does not lie in \( \pi_4 \). We prove that \( \pi_4 \) contains \( T_r(Q(8, q)) \cap \mathcal{K} \) and that \( T_r(Q(8, q)) \cap \mathcal{K} \) coincides with the truncated cone \( p\mathcal{O}_L \setminus \{ p \} \), where \( p = \pi_4 \cap \langle r, r_0 \rangle \). Consider the projections \( (\mathcal{O}_L)_{p} \) and \( (\mathcal{O}_L')_{p} \) in the space \( \pi_6 \). Consider a fixed conic \( C \) in \( (\mathcal{O}_L')_{p} \) disjoint from the intersection \( (\mathcal{O}_L)_{p} \cap (\mathcal{O}_L')_{p} \). Exactly one 3-dimensional space through \( C \) in \( \langle r_0, (\mathcal{O}_L')_{p} \rangle \) is \( \langle C \rangle_{p}, r_0 \rangle \), but the other \( q - 1 \) 3-dimensional spaces in \( \langle r_0, (\mathcal{O}_L)_{p} \rangle \) through \( C \) intersect the cone \( r_0 (\mathcal{O}_L')_{p} \) in an elliptic quadric, one of which is \( (\mathcal{O}_L')_{p} \). Consider the \( q - 1 \) other 3-dimensional spaces. Each one of them intersects the solid of \( (\mathcal{O}_L)_{p} \) in a plane. This plane contains at least one point of \( (\mathcal{O}_L)_{p} \setminus (\mathcal{O}_L')_{p} \), so the elliptic quadric \( (\mathcal{O}_L')_{p} \) in this solid lies completely in \( (\pi_4)_{p} \). And before the projection, the original elliptic quadric \( \mathcal{O} \) lies in \( \pi_4 \). The only points of \( p\mathcal{O}_L \setminus \{ p \} \) not yet discussed are the points of the cone \( pC \). Consider a conic \( C' \) of \( \mathcal{O}_L' \) skew to \( C \) and to \( C \). Then \( (C')_{p} r_0 \cap (C')_{p} r_0 = \{ r_0 \} \); so repeating the previous arguments, every point of \( pC \setminus \{ p \} \) lies in \( \mathcal{K} \). We have proven that \( T_r(Q(8, q)) \cap \mathcal{K} \) is equal to \( p\mathcal{O}_L \setminus \{ p \} \). This concludes the lemma.

**Theorem 32** If every ovoid of \( Q(4, q), q \) odd, is an elliptic quadric and the smallest minimal blocking set of \( Q(6, q) \) is a truncated cone \( r\mathcal{O} \setminus \{ r \} \), where \( \mathcal{O} \) is a 3-dimensional elliptic quadric, then the smallest minimal blocking set of \( Q(2n+2, q), n \geq 3 \), is a truncated cone \( \pi_{n-2} \mathcal{O} \setminus \pi_{n-2} \), \( \mathcal{O} \) an ovoid of \( Q(4, q) \).

**Proof.** Suppose that \( \mathcal{K} \) is a minimal blocking set of \( Q(2n+2, q) \), \( |\mathcal{K}| \leq q^{n+1} + q^{n-1} \). Suppose that \( r \in Q(2n+2, q) \setminus \mathcal{K} \) for which \( |T_r(Q(2n+2, q)) \cap \mathcal{K}| = q^n + q^{n-2} \). Such a point exists (Lemma 26) and the points of \( T_r(Q(2n+2, q)) \cap \mathcal{K} \) are the points of a truncated cone \( \pi_{n-3} \mathcal{O} \setminus \pi_{n-3} \). Consider a line \( L \) of \( Q(2n+2, q) \) on \( r \) such that \( |T_{L}(Q(2n+2, q)) \cap \mathcal{K}| = q^{n-1} + q^{n-3} \) (such a line exists, see the constructive proof of Lemma 29). From Lemma 27, we have \( |T_{r_i}(Q(2n+2, q)) \cap \mathcal{K}| = q^n + q^{n-2} \) for all points \( r_i \in L \). It is clear that the truncated cones \( T_{r_i}(Q(2n+2, q)) \cap \mathcal{K} \) share the truncated cone \( T_{L}(Q(2n+2, q)) \cap \mathcal{K} \), denoted by \( \pi_{n-3} \mathcal{O} \setminus \pi_{n-4} \). Furthermore, \( r \) projects \( \pi_{n-3} \mathcal{O} \setminus \pi_{n-3} \) onto \( \pi_{n-3} \mathcal{O} \setminus \pi_{n-3} \) \( r \) \( \mathcal{K} \) \( r \) the space containing the base \( Q(2n, q) \) of the cone \( T_{r}(Q(2n+2, q)) \cap Q(2n+2, q) \). Consider a line \( L' \) of \( Q(2n+2, q) \) on \( r \) such that \( L' \cap Q(2n, q) \not\in T_{(\pi_{n-3})}(q(2n, q)) \). Considering the projection from the point \( r \) and using the remark preceding Lemma 31, it is possible to
select $L'$ such that $|O \cap O'| \geq 1$, $O'$ being the base of the truncated cone $\pi_{n-4}^{L'} \setminus \pi_{n-4}^{L'} = T_{L'}(Q(2n+2,q)) \cap K$. It is again clear that $\pi_{n-4}^{L'} \setminus \pi_{n-4}^{L'}$ lies on $q+1$ truncated cones $\pi_{n-3}^{L'} \setminus \pi_{n-3}^{L'} = T_s(Q(2n+2,q)) \cap K$, for all points $s \in L'$. Actually with $s$ varying over $L'$, these tangent hyperplanes vary over the hyperplanes through $T_{L'}(Q(2n+2,q))$; so we get every point of the spaces $\pi_{n-3}^{L'}$, $r_i \in L$. These points lie on lines with $q$ points of $K$ to $O \cap O'$; so they belong to the vertices of the cones $\pi_{n-3}^{\{r_i\}} \setminus \pi_{n-3}^{\{r_i\}}$, $s \in L'$. Letting vary $O'$ over the points of $\pi_{n-3}^{L} \setminus \pi_{n-3}^{\{r_i\}}$, we obtain that every point of this truncated cone lies on a line containing $q$ points of $K$ and passing through an arbitrary point of $\pi_{n-3}^{\{r_i\}}$, $r_i \in L \setminus \{r_1\}$. Consider two points $p_1, p_2 \notin \pi_{n-3}^{\{r_1\}} \cap \pi_{n-3}^{\{r_2\}}$, $p_1 \in \pi_{n-3}^{\{r_1\}}$, $p_2 \in \pi_{n-3}^{\{r_2\}}$. Consider an arbitrary line $M$ of $\pi_{n-3}^{\{r_1\}} \setminus \pi_{n-3}^{\{r_1\}}$ passing through $p_1$ and containing $q$ points of $K$. The $q^2$ points of $(M, p_2) \setminus \{p_1, p_2\}$ all lie in $K$; this implies that the truncated cone $(\pi_{n-3}^{\{r_1\}}, \pi_{n-3}^{\{r_2\}}) \setminus (\pi_{n-3}^{\{r_1\}}, \pi_{n-3}^{\{r_2\}})$ lies in $K$. Since $|K| = |(\pi_{n-3}^{\{r_1\}}, \pi_{n-3}^{\{r_2\}}) \setminus (\pi_{n-3}^{\{r_1\}}, \pi_{n-3}^{\{r_2\}})| = (q^2 + 1)q^{n-1} = q^{n+1} + q^{n-1}$, this cone must be equal to $K$.

Theorem 1 is now a corollary of this theorem, since for $q = 5, 7$, the conditions on the ovoids of $Q(4, q)$ are true and the first step in the induction is provided by Theorem 25.

**Remark 33** Recently, using the $1 \pmod{p}$ result on ovoids $O$ of $Q(4, q)$, $q = p^h$, $h \geq 1$, stating that every elliptic quadric of $Q(4, q)$ intersects $O$ in $1 \pmod{p}$ points $[1, 2]$, Ball et al. proved that effectively for $q$ prime, every ovoid of $Q(4, q)$ is an elliptic quadric $[2]$.

De Beule and Metsch [9] used this result to prove Theorem 25 for all $q$ odd prime, $q > 3$.

This implies that the conditions of the preceding theorem are indeed valid for all $q$ odd prime, $q > 3$, so that we can extend our results.

To make these recent improvements clear, we summarize these results in the next theorems.

**Theorem 34** (Ball et al. [2]) Every ovoid of the 4-dimensional parabolic quadric $Q(4, q)$, $q$ prime, is an elliptic quadric.

**Theorem 35** (De Beule and Metsch [9]) The smallest blocking sets of the 6-dimensional parabolic quadric $Q(6, q)$, $q > 3$ prime, are truncated cones $r Q^-(3, q) \setminus \{r\}$, with $Q^-(3, q)$ a 3-dimensional elliptic quadric contained in the tangent hyperplane $T_r(Q(6, q))$ of $Q(6, q)$ in $r$.

**Theorem 36** Let $K$ be a minimal blocking set of $Q(2n + 2, q)$, $n \geq 2$, $q > 3$ prime, $|K| \leq q^{n+1} + q^{n-1}$. Then there is an $(n - 2)$-dimensional space $\pi_{n-2}$, $\pi_{n-2} \subset Q(2n + 2, q)$, $\pi_{n-2} \cap K = \emptyset$, with the following property: $T_{\pi_{n-2}}(Q(2n + 2, q)) \setminus (\pi_{n-2} \setminus K) = \emptyset$. 19
2, q)) \cap Q(2n + 2, q) = \pi_{n-2}Q(4, q) and \mathcal{K} is a cone with vertex \pi_{n-2} and base \mathcal{O}, where \mathcal{O} is a 3-dimensional elliptic quadric of Q(4, q), minus the points of the vertex \pi_{n-2}, and |\mathcal{K}| = q^n + 1 + q^{n-1}.

4 The characterization when Q(6, q) has ovoids

If Q(6, q) has an ovoid, for example when q = 3, then the situation is different from the one in the previous section. We will characterize the smallest minimal blocking sets of Q(2n + 2, q), now using ovoids of Q(6, q). Therefore we must again start with characterizations in low dimensions, now for Q(6, q) and Q(8, q). We will do as much as possible for general q, while the characterization in the lowest dimension will be for q = 3 only, depending on Lemma 9. We start with the following situation. Suppose that \mathcal{K} is a minimal blocking set of Q(6, 3) different from an ovoid, and that |\mathcal{K}| \leq q^3 + q = 30. From Lemmas 3 and 12, we immediately find the following corollary.

**Corollary 37** If \pi is a plane of Q(6, 3), then |\pi \cap \mathcal{K}| = 1 or |\pi \cap \mathcal{K}| = 3, and all points of \pi \cap \mathcal{K} are collinear.

This corollary leads to another corollary.

**Corollary 38** Consider a line L of Q(6, 3) with the property that |L \cap \mathcal{K}| = 3, and that L \setminus \mathcal{K} = \{p\}. Then \mathcal{K} \subseteq T_p(Q(6, 3)).

**PROOF.** Consider a line L such that |L \cap \mathcal{K}| = 3. Let \{p\} = L \setminus \mathcal{K}; then all \(q^3\) points of \mathcal{K} \setminus L must lie in \(T_p(Q(6, 3))\); also \mathcal{K} \setminus L lies in \(T_p(Q(6, 3))\). So \mathcal{K} \subseteq T_p(Q(6, 3)).

This corollary is sufficient to prove the following theorem.

**Theorem 39** Let \mathcal{K} be a minimal blocking set of Q(6, q = 3), different from an ovoid, with |\mathcal{K}| \leq 3^3 + 3. Then there is a point p \in Q(6, 3) \setminus \mathcal{K} with the following property: \(T_p(Q(6, 3)) \cap Q(6, 3) = pQ(4, 3)\) and \mathcal{K} consists of all the points of the lines L on p meeting Q(4, 3) in an ovoid \mathcal{O}, minus the point p itself, and |\mathcal{K}| = 3^3 + 3.

**PROOF.** We know from the corollary that there exists a point p \in Q(6, 3) \setminus \mathcal{K}, such that \mathcal{K} \subseteq T_p(Q(6, 3)). Suppose that some line of Q(6, 3) to \mathcal{K} through p intersecting \mathcal{K} contains a point r, r \neq p, r \notin \mathcal{K}. Then \(T_r(Q(6, 3)) \cap \mathcal{K}\) contains points not in \(T_p(Q(6, 3))\) which is false. So the only possibility is that p lies on
$q^2 + 1$ distinct 3-secants to $\mathcal{K}$ which project from $p$ onto an ovoid of $Q(4, 3)$. We conclude that $\mathcal{K}$ is a truncated cone $p\mathcal{O} \setminus \{p\}$, $\mathcal{O}$ an ovoid of $Q(4, 3)$.

It is known that all ovoids of $Q(4, 3)$ are elliptic quadrics. This yields the complete classification of the smallest minimal blocking sets of $Q(6, 3)$, different from an ovoid.

We now will prove Theorem 2 supposing that $Q(6, q)$ has an ovoid and that the smallest minimal blocking set of $Q(6, q)$, different from an ovoid, is a truncated cone $p\mathcal{O} \setminus \{p\}$, $\mathcal{O}$ an ovoid of $Q(4, q)$. As in the previous section we will start with the lowest dimensional case: $Q(8, q)$. Since $Q(8, q)$, $q$ odd, has no ovoids [13], we suppose that $\mathcal{K}$ is a minimal blocking set of $Q(8, q)$ of size $|\mathcal{K}| = q^4 + 1 + r$, $0 < r < q$.

**Lemma 40** If $L$ is a line of $Q(8, q)$, then $|L \cap \mathcal{K}| = 0, 1$ or $|L \cap \mathcal{K}| = q$.

**PROOF.** Suppose that $q - 1 \geq |L \cap \mathcal{K}| \geq 2$. Consider a generator $\pi$ on $L$ such that $L \cap \mathcal{K} = \pi \cap \mathcal{K}$. For the same reason as in Lemma 12, such a generator exists. Count the number of pairs $(u, v)$, $u \in \pi \setminus L$ and $v \in \mathcal{K} \setminus \pi$, $u \in T_v(Q(8, q))$. Since $u \in \pi \setminus L$ and $|L \cap \mathcal{K}| \geq 2$, $u$ cannot project $T_u(Q(8, q)) \cap \mathcal{K}$ on an ovoid of $Q(6, q)$, so $|T_u(Q(8, q)) \cap \mathcal{K}| \geq q^3 + q \geq q^3 + 1 + |L \cap \mathcal{K}|$ since the projection is a minimal blocking set of $Q(6, q)$ (Lemma 4). We find a lower bound of $(q^3 + q^2)(q^3 + 1)$. The first factor comes from the number of points in $\pi \setminus L$. If $v \in \mathcal{K} \setminus \pi$, then $T_v(Q(8, q))$ intersects $\pi$ in a plane, hence with $v$ correspond $q^2 + q$ or $q^2$ points of $\pi \setminus L$. So we find $(q^4 + 1 + r - |L \cap \mathcal{K}|)(q^2 + q)$ as upper bound for the number of pairs $(u, v)$, and since $2 \leq |L \cap \mathcal{K}|$, we can increase this upper bound to $(q^4 + r - 1)(q^2 + q)$. So necessarily $(q^4 + r - 1)(q^2 + q) \geq (q^3 + q^2)(q^3 + 1)$ or, since $r \leq q - 1$, $(q^4 + q - 2)(q^2 + q) \geq (q^3 + q^2)(q^3 + 1)$, a contradiction.

**Corollary 41** If $\pi$ is a generator of $Q(8, q)$, then $|\pi \cap \mathcal{K}| = 1$ or $|\pi \cap \mathcal{K}| = q$, and all points of $\pi \cap \mathcal{K}$ lie on a line.

**PROOF.** If $|\pi \cap \mathcal{K}| \geq 2$, then the line $L = \langle \pi \cap \mathcal{K} \rangle$ contains already $q$ points of $\mathcal{K}$. Lemma 3 admits no further points of $\mathcal{K}$ in $\pi$.

**Lemma 42** Suppose that $p \notin \mathcal{K}$. If there is a generator $\pi$ on $p$ containing exactly 1 point of $\mathcal{K}$, then $|T_p(Q(8, q)) \cap \mathcal{K}| \leq q^3 + q$; otherwise $|T_p(Q(8, q)) \cap \mathcal{K}| = q(q^3 + 1)$.

**PROOF.** Suppose that $p \in \pi$, $\pi \cap \mathcal{K} = \{s\}$. There are $q^3 - q^2$ planes in $\pi$ not through $s$ or $p$. All generators of $Q(8, q)$ on these planes only share points.
with $K \setminus \pi$. For, a point $r$ of $K \setminus \pi$ has a tangent hyperplane not containing $\pi$; so $\pi$ intersects this tangent hyperplane in a plane $\Omega$; this plane $\Omega$ and $r$ define a unique generator. Hence $q(q^3 - q^2)$ points of $K$ are needed to block them; so at most $q^4 + q - (q^4 - q^3) = q^3 + q$ points of $K$ lie in $T_p(Q(8, q)) \cap K$. If every generator on $p$ contains $q$ points of $K$, then $|T_p(Q(8, q)) \cap K| = q^4 + q$.

**Lemma 43** Suppose that $\pi$ is a generator of $Q(8, q)$, $|\pi \cap K| = q$, $\langle \pi \cap K \rangle = L$. If $p \in \pi \setminus L$, then $|T_p(Q(8, q)) \cap K| = q^3 + q$.

**PROOF.** If all generators on $p$ contain $q$ points of $K$, then in particular also all generators on $(s, p)$, $s \in \pi \cap K$, hence $|T_s(Q(8, q)) \cap K| > q$, contradicting Lemma 3. So $|T_p(Q(8, q)) \cap K| \leq q^3 + q$ (Lemma 42). Since $p$ must project the points of $T_p(Q(8, q)) \cap K$ on $K_p$, a minimal blocking set of $Q(6, q)$ different from an ovoid (Lemma 4), $|K_p| \geq q^3 + q$. Hence $|T_p(Q(8, q)) \cap K| = q^3 + q$.

We now can prove the following theorem.

**Theorem 44** If $Q(6, q)$, $q$ odd, has an ovoid and the smallest minimal blocking set of $Q(6, q)$ different from an ovoid, is a truncated cone $pO \setminus \{p\}$, with $O$ an ovoid of $Q(4, q)$, then the smallest minimal blocking set of $Q(8, q), q$ odd, is a truncated cone $pO' \setminus \{p\}$, with $O'$ an ovoid of $Q(6, q)$.

**PROOF.** Consider a generator $\pi$ of $Q(8, q)$ such that $|\pi \cap K| = q$, $\langle \pi \cap K \rangle = L$. Consider $s \in \pi \setminus L$. By the previous lemma, $|T_s(Q(8, q)) \cap K| = q^3 + q$, and $s$ projects the points of $T_s(Q(8, q)) \cap K$ onto a minimal blocking set $K_s$ of $Q(6, q)$, being a truncated cone $pO \setminus \{p\}$, $O$ an ovoid of $Q(4, q)$. The $q^2 + 1$ lines containing $q$ points of $K_s$ are projections of $q^2 + 1$ lines $M_i$ of $Q(8, q)$, $|M_i \cap K| = q$. Suppose that $M_i \cap \langle s, p \rangle = p_i$. Suppose that $|T_{pi}(Q(8, q)) \cap K| \leq q^3 + q$. For some $i$, for instance $i = 1$, $p' = p_i$ lies on one line $M_1$. The point $p'$ projects $T_{p'}(Q(8, q)) \cap K$ necessarily on a minimal blocking set of $Q(6, q)$ which is an ovoid, hence $|T_{p'}(Q(8, q)) \cap K| = q^3 + q$. Consider now all generators on the line $M_1$. There are $(q^2+1)(q+1)$ such generators and they are all blocked by the points of $M_1 \cap K$. Since there are $q^4(q^2+1)(q+1)$ generators left in $T_{p'}(Q(8, q))$, and every point of $K \cap T_{p'}(Q(8, q))$ blocks $(q^2+1)(q+1)$ of them, every generator on $p'$ not on $M_1$ contains exactly 1 point of $K$. This is a contradiction since every generator on a plane $\langle p', M_i \rangle$ is a generator on $p'$ containing $q$ points of $K$. We conclude that $|T_{p'}(Q(8, q)) \cap K| = q(q^3 + 1) = q^4 + q$, and $|K| = q^4 + q$. Furthermore, $p'$ projects all points of $K$ on an ovoid of $Q(6, q)$, by Lemma 4 and Lemma 43.

Theorem 2 for $n = 3$ is an immediate corollary of this theorem, since for $q = 3$, all conditions about $Q(6, q)$ are satisfied.
We now will do the characterization for $Q(2n + 2, q)$, $q = 3$. For the following lemmas, we suppose that $Q(6, q)$ has ovoids and that the smallest minimal blocking sets of $Q(6, q)$, different from an ovoid, are truncated cones $p\mathcal{O} \setminus \{p\}$, $\mathcal{O}$ an ovoid of $Q(4, q)$. This hypothesis is true for $q = 3$. Furthermore, we suppose as induction hypothesis that Theorem 2 is true for $3 \leq n_0 < n$ for a fixed $n > 3$. Using these hypotheses, we will prove the characterization for $Q(2n + 2, q)$.

For all lemmas, we suppose that $\mathcal{K}$ is a minimal blocking set of $Q(2n + 2, q)$, $n \geq 4$, of size at most $q^{n+1} + q^{n-2}$. Some lemmas will be very analogous to lemmas from previous sections.

**Lemma 45** There exists a point $p \in Q(2n + 2, q) \setminus \mathcal{K}$ such that $|T_p(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-3}$.

**PROOF.** Consider $s \in Q(2n + 2, q) \setminus \mathcal{K}$, then $|T_s(Q(2n + 2, q)) \cap \mathcal{K}| > q^n + q^{n-3}$ if we assume the induction hypothesis. Count the number of incidences $(p, H)$, $p \in \mathcal{K}$, $H$ a tangent hyperplane to $Q(2n + 2, q)$ in a point not in $\mathcal{K}$. Denote this number by $N$; $N$ is an upper bound since $|\mathcal{K}| \leq q^{n+1} + q^{n-2}$. With every point of $\mathcal{K}$ corresponds exactly one tangent hyperplane. Then the total number of incidences $(p, \mathcal{K})$, $p \in \mathcal{K}$ and $H$ a hyperplane $T_r(Q(2n + 2, q))$, $r \notin \mathcal{K}$, is at most $(q^{n+1} + q^{n-2})q|Q(2n, q)|$ under the assumption that $|\mathcal{K}|$ equals the upper bound $q^n + q^{n-2}$. We find $N = (q^n + q^{n-3} + 1)(|Q(2n + 2, q) \setminus \mathcal{K}|) + (-q^{2n} - q^{2n-1} + q^{2n-2} - q^{2n-3} - q^{2n-4} - q^{2n-6} - \ldots - q^{n+1} + 2q^n - q^{n-1} - q^{n-2} - 2q^{n-3} - q^{n-4} - \ldots - q^2 - q - 1)$ (n > 6, for n = 4, 5, 6, the remainder is $-q^8 - q^7 + q^5 - 2q^4 - q^3 - 2q - 1$, $-q^{10} - q^9 + q^7 - q^5 - q^3 - 2q^2 - q - 1$ and $-q^{12} - q^{11} + q^9 - q^8 - 2q^6 - q^5 - q^3 - 2q^2 - q - 1$ respectively) where we again assumed that $|\mathcal{K}| = q^n + q^{n-2}$. Since the remainder is negative, there must exist a point $r \in Q(2n + 2, q) \setminus \mathcal{K}$ such that $|T_r(Q(2n + 2, q)) \cap \mathcal{K}| < q^n + q^{n-3} + 1$, which implies $|T_r(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-3}$.

**Lemma 46** If $L$ is a line of $Q(2n + 2, q)$, $L \cap \mathcal{K} = \emptyset$, $|T_L(Q(2n + 2, q)) \cap \mathcal{K}| = q^{n-1} + q^{n-4}$, then $|T_r(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-3}$, for all points $r_i \in L$.

**PROOF.** Necessarily $|T_r(Q(2n + 2, q)) \cap \mathcal{K}| \geq q^n + q^{n-3}$, by the induction hypothesis and Lemma 4. The sets $T_r(Q(2n + 2, q)) \cap \mathcal{K}$ have exactly $q^{n-1} + q^{n-4}$ points in common. Suppose that for some $r_i$, $|T_{r_i}(Q(2n + 2, q)) \cap \mathcal{K}| > q^n + q^{n-3}$. This implies $\sum_{i=0}^{n} |T_{r_i}(Q(2n + 2, q)) \cap \mathcal{K}| > (q + 1)(q^n + q^{n-3} - (q^{n-1} + q^{n-4})) + q^{n-1} + q^{n-4} \geq q^{n+1} + q^{n-2} \geq |\mathcal{K}|$, a contradiction. Hence $|T_r(Q(2n + 2, q)) \cap \mathcal{K}| = q^n + q^{n-3}$.

In the next lemma, we work with ovoids of $Q(6, q)$. For some values of $q$, different non-isomorphic classes of ovoids are known. Since we want to prove
this lemma independently from \( q \), only using some hypotheses formulated independent of \( q \), we have to deal with some aspects of ovoids. If \( O \) is an ovoid of \( Q(6, q) \), then the dimension of \( \langle O \rangle \) equals 6. For, all quadrics in 3 dimensions don’t contain enough points. Actually, since every 2 points of \( O \) must be non-collinear, we don’t have to consider singular quadrics. Now the dimensions don’t contain enough points. Actually, since every 2 points of \( O \) contains \( O \). Choose \( \pi \). From Lemma 46, \( |O| = q^{n-1} + q^{n-4}, \) then the points of \( T_L(Q(2n + 2, q)) \cap K \) are the points of a truncated cone \( \pi_{n-5}O \setminus \pi_{n-5} \). \( O \) an ovoid of \( Q(6, q) \).

**Proof.** Choose \( r_0 \in L \). From Lemma 46, \( |T_{r_0}(Q(2n + 2, q)) \cap K| = q^n + q^{n-3} \) and the points of \( T_{r_0}(Q(2n + 2, q)) \cap K \) are projected onto a truncated cone \( \pi_{n-4}O \setminus \pi_{n-4} \), \( O \) an ovoid of \( Q(6, q) \). Denote by \( \pi_{2n}^0 \) the \( 2n \)-space such that \( T_{r_0}(Q(2n + 2, q)) = r_0 \pi_{2n}^0 \) and let \( \pi_{2n}^0 \) be the space in which we consider the projection of \( T_{r_0}(Q(2n + 2, q)) \cap K \) from \( r_0 \). The dimension of \( \langle \pi_{n-4}O \setminus \pi_{n-4} \rangle \) equals \( n + 3 \). The points of \( T_L(Q(2n + 2, q)) \cap K \), projected from \( r_0 \), cannot span an \((n+3)\)-dimensional space; or else all projected points of \( T_{r_0}(Q(2n + 2, q)) \cap K \) lie in this space; then all \( q^n + q^{n-3} \) points lie in \( \langle r_0, S \rangle \subseteq T_L(Q(2n + 2, q)) \) (\( S \) = set of projected points of \( T_L(Q(2n + 2, q)) \cap K \), a contradiction. So these \( q^n + q^{n-3} \) projected points span at most an \((n+2)\)-dimensional space. Denote by \( \alpha \) the space spanned by the points of \( T_L(Q(2n + 2, q)) \cap K \) themselves.

We now prove that \( L \) cannot intersect \( \pi_{2n}^0 \) in a point of \( T_{\pi_{n-4}}(Q(2n, q)) \). Assume the contrary. We have two possibilities. Suppose that \( L \) contains a point \( r_1 \) of \( \pi_{n-4} \). Then this point \( r_1 \) is collinear with all points of \( \pi_{n-4}O \setminus \pi_{n-4} \). But then \( T_L(Q(2n + 2, q)) = T_{r_0}(Q(2n + 2, q)) \cap T_{r_1}(Q(2n + 2, q)) \) contains all \( q^n + q^{n-3} \) points of \( T_{r_0}(Q(2n + 2, q)) \), a contradiction. Suppose secondly that \( L \) contains a point \( r_1 \) of \( \pi_{n-4}Q(6, q) \setminus \pi_{n-4} \). Assume that \( r_1 \in Q(6, q) \). Consider \( T_{r_1}(Q(2n, q)) \). This contains \( \pi_{n-4} \), but intersects \( Q(6, q) \) in a cone \( r_1Q(4, q) \). The generators of \( Q(6, q) \) on \( r_1 \) are blocked by exactly \( q^2 + 1 \) points of \( O \). This shows that \( T_{r_0}(Q(2n + 2, q)) \) and \( T_{r_1}(Q(2n + 2, q)) \) share exactly \( (q^2 + 1)q^{n-3} \) points with \( K \). With \( T_L(Q(2n + 2, q)) = T_{r_0}(Q(2n + 2, q)) \cap T_{r_1}(Q(2n + 2, q)) \), this
Consider a point \( \pi \) sharing a truncated cone \( q \) shared two distinct points \( \pi \). This we find that \( L \) is skew to \( T \). Hence \( r \) projects the points of \( T \) onto a truncated cone \( \pi_{n-5} \setminus \pi_{n-5} \), which must be the set \( S \). Necessarily the base ovoid of this set is the same as the base ovoid of the truncated cone \( \pi_{n-4} \setminus \pi_{n-4} \). It is clear that all points of \( T \) lie in \( (r_0, S) \). Since \( (S) \) is an \((n + 2)\)-dimensional space, the space \( \alpha \) is an \((n + 2)\)- or \((n + 3)\)-dimensional space. Now \( L \) cannot lie in \( \alpha \) since \( L \) is skew to \( T_{n-4} \).

Repeat the arguments for two points \( r_0 \) and \( r'_0 \) of \( L \). Then the two spaces \( \alpha \) and \( \alpha' \) intersect in an \((n + 2)\)-dimensional space \( \beta \), and \( L \) is skew to \( \beta \) since \( L \) shared two distinct points \( r_0 \) and \( r'_0 \) with \( \alpha \) and \( \alpha' \). Now \( r_0 \) projected the \( q^{n-1} + q^{n-4} \) points of \( T \) onto a truncated cone \( \pi_{n-5} \setminus \pi_{n-5} \) with the projection. So also originally before the projection, the \( q^{n-1} + q^{n-4} \) points of \( T \) formed a truncated cone \( \pi_{n-5} \setminus \pi_{n-5} \).

**Lemma 48** There exists a line \( L \) of \( Q(2n + 2, q) \), \( L \cap K = \emptyset \), such that \( |T_L(Q(2n + 2, q)) \cap K| = q^{n-1} + q^{n-4} \).

**Proof.** Consider a point \( p \in Q(2n + 2, q) \setminus K \) for which \( |T_p(Q(2n + 2, q)) \cap K| = q^n + q^{n-3} \) (Lemma 45). Denote by \( K_p \) the projection of \( K \cap T_p(Q(2n + 2, q)) \) from \( p \) in a fixed \( Q(2n, q) \). Since \( |K_p| = q^n + q^{n-3} \), every line \( \langle p, r \rangle \), \( r \in K_p \), contains exactly one point of \( K \). Denote \( K_p = \pi_{n-4} \setminus \pi_{n-4} \). Choose \( s \) such that \( s \not\in T_{n-4}(Q(2n + 2, q)) \) and \( \langle r, s \rangle \) a line of \( Q(2n + 2, q) \). Then \( T_s(Q(2n + 2, q)) \) intersects \( K_p \) in a truncated cone \( \pi_{n-5} \setminus \pi_{n-5} \) which is the projection of \( T_p(Q(2n + 2, q)) \cap T_s(Q(2n + 2, q)) \). Then \( T_p(Q(2n + 2, q)) \cap T_s(Q(2n + 2, q)) = T_s(Q(2n + 2, q)) \) contains exactly \( q^{n-4}(q^2 + 1) \) projected points of \( K \); \( p \) never projected two points of \( K \). Denote \( T_{(p,s)}(Q(2n + 2, q)) \) contains exactly \( q^{n-1} + q^{n-4} \) points of \( K \).

**Lemma 49** If \( r \) is a point of \( Q(2n + 2, q) \setminus K \) for which \( |T_r(Q(2n + 2, q)) \cap K| = q^n + q^{n-3} \), then the points of \( T_r(Q(2n + 2, q)) \cap K \) are the points of a truncated cone \( \pi_{n-4} \setminus \pi_{n-4} \) for some ovoid \( O \) of \( Q(6, q) \).

**Proof.** By Lemma 4, \( r \) projects \( T_r(Q(2n + 2, q)) \) onto a truncated cone \( \pi_{n-4} \setminus \pi_{n-4} \). Denote \( T_r(Q(2n + 2, q)) \cap Q(2n + 2, q) = rQ(2n, q) \) and let \( \pi_{2n} \) be the space in which we consider the base \( Q(2n, q) \) of the cone \( T_r(Q(2n + 2, q)) \) \( \cap Q(2n + 2, q) \). Let \( L \) be a line of \( Q(2n + 2, q) \) through \( r \) intersecting \( Q(2n, q) \) in a point \( r' \), \( r' \not\in T_{n-4}(Q(2n, q)) \). Then \( T_r(Q(2n, q)) \) intersects \( T_{n-4}(Q(2n, q)) \) in a space sharing \( \pi_{n-5} \setminus \pi_{n-5} \). Also \( T_{(r', r)}(Q(2n + 2, q)) \) shares a truncated cone \( \pi_{n-5} \setminus \pi_{n-5} \) with \( K \) (see Lemma 47 and Lemma 48). Since we know that every ovoid of \( Q(6, q) \) spans a 6-dimensional space, every truncated cone
$\pi_{n-5}\setminus\pi_{n-5}$ has dimension $n + 2$. Assume now that $n \geq 5$. Consider two lines $L_1$ and $L_2$ of $Q(2n + 2, q)$ through $r$ such that the corresponding cones $T_{L_1}(Q(2n + 2, q)) \cap \mathcal{K}$ share the same base, i.e. $T_{L_1}(Q(2n + 2, q)) \cap \mathcal{K} = \pi_{n-5}\setminus\pi_{n-5}$ and $T_{L_2}(Q(2n + 2, q)) \cap \mathcal{K} = \pi_{n-5}\setminus\pi_{n-5}$. This is possible if $n \geq 5$ since the polar space of $\langle \pi_{n-5} \rangle$ has points of $Q(2n + 2, q)$ outside $\langle \pi_{n-4} \rangle$. (see the remark following Lemma 29). Hence the projections share the same base and also the original point sets share the same base. Since we assumed that $n \geq 5$, the dimension of $\pi_{n-5}\setminus\pi_{n-5}$ equals $n - 6 \geq -1$.

Let $\mathcal{O}_3$ be an ovoid lying on the cone $\pi_{n-5}\setminus\pi_{n-5}$, suppose that $|\mathcal{O}_3 \cap \mathcal{O}| > 0$, and consider the cone $\pi_{n-5}\setminus\mathcal{O}_3$. This cone lies in some $T_{L_3}(Q(2n + 2, q))$, $L_3$ a line of $Q(2n + 2, q)$ through $r$ and $\pi_{n-5}\setminus\mathcal{O}_3 \cap \pi_{n-5}$ lies completely in $\mathcal{K}$. Namely $T_{L_3}(Q(2n + 2, q)) \cap \mathcal{K}$ is a truncated cone $\pi_{n-5}\setminus\mathcal{O}_3 \cap \pi_{n-5}$. We prove that $\pi_{n-5} = \pi_{n-5}$. For, consider the cone $(\mathcal{O}\pi_{n-5})\setminus\pi_{n-5}$ which has a non-empty intersection with $(\mathcal{O}\pi_{n-5})\setminus\pi_{n-5}$. For every $t \in (\mathcal{O}\setminus\pi_{n-5})$, we know already exactly where the cone $(t\pi_{n-5})\setminus\pi_{n-5}$ lies; this is at least a line since $n \geq 5$. So we can locate exactly where the vertex $\pi_{n-5}$ of the truncated cone $T_{L_3}(Q(2n + 2, q)) \cap \mathcal{K}$ lies before the projection, and in fact $\pi_{n-5} = \pi_{n-5}$.

Repeating this argument for all ovoids $\mathcal{O}_3$ on the truncated cone $\pi_{n-5}\setminus\pi_{n-5}$, we are actually proving that all points of the truncated cone $\langle \pi_{n-5}, \pi_{n-5} \rangle\setminus\langle \pi_{n-5}, \pi_{n-5} \rangle$ are lying in $\mathcal{K}$. This truncated cone has size $q(q^{n-1} + q^n) = |T_r(Q(2n + 2, q)) \cap \mathcal{K}|$ which shows that $T_r(Q(2n + 2, q)) \cap \mathcal{K} = \langle \pi_{n-5}, \pi_{n-5} \rangle\setminus\langle \pi_{n-5}, \pi_{n-5} \rangle$.

We now will discuss the case $n = 4$. Reformulating the lemma, we must prove that $T_r(Q(10, q)) \cap \mathcal{K}$ is a truncated cone $\mathcal{O}\setminus\{p\}$. The point $r$ projects $T_r(Q(10, q)) \cap \mathcal{K}$ onto a truncated cone $r_0\mathcal{O}\setminus\{r_0\}$.

We make use of the following computer result. Consider the ovoid $\mathcal{O}$ on $Q(6, 3)$ and consider all the 4-spaces $\pi_4$ of $PG(6, 3)$ for which $\pi_4 = \langle \pi_4 \cap \mathcal{O} \rangle$. Consider the 5-spaces through such a 4-space $\pi_4$. For some 4-spaces $\pi_4$, every 5-space $\pi_5$ through $\pi_4$ contains points of $\mathcal{O} \setminus \pi_4$, but there are also 4-spaces $\pi_4$ such that some 5-space $\pi_5$ through $\pi_4$ contains no points of $\mathcal{O} \setminus \pi_4$.

We will use in the next paragraphs the 4-spaces $\pi_4$ for which $\pi_4 = \langle \pi_4 \cap \mathcal{O} \rangle$ and for which every 5-space $\pi_5$ through $\pi_4$ contains points of $\mathcal{O} \setminus \pi_4$. Considering all these 4-spaces, it is observed that they have an empty intersection, and every point of $\mathcal{O}$ belongs to at least one such 4-space. We now present the proof for $n = 4$.

For every line $L = \langle r, r' \rangle$ of $Q(10, q)$, $r' \notin T_{r_0}(Q(10, q))$, $T_L(Q(10, q))$ contains $q^3 + 1$ points of $\mathcal{K}$ and $T_L(Q(10, q))$ intersects $\mathcal{K}$ in an ovoid of a 6-dimensional
parabolic quadric $Q(6, q)$ (Lemma 47). There are $q^7$ such lines through $r$. Call the associated ovoids $O_L$. Consider two such ovoids $O_{L_1}$ and $O_{L_2}$ for which $O_{L_1} \cap O_{L_2}$ generates a 5-space, and consequently $\langle O_{L_1}, O_{L_2} \rangle = \pi_7$ is a 7-dimensional space. We prove that $\pi_7$ contains $T_r(Q(10, q)) \cap \mathcal{K}$. Note that $r \not\in \pi_7$, or else $(O_{L_1})^p \cup (O_{L_2})^p$ generates a 6-dimensional space; this is false. Note that $O_{L_1}$ and $O_{L_2}$ define a truncated cone $pO_{L_1} \setminus \{p\}$, $\{p\} = \pi_7 \cap \langle r, r_0 \rangle$. Consider the projections $(O_{L_1})^p$ and $(O_{L_2})^p$ in the space $\pi_8^p$. Let $\pi_5 = \langle (O_{L_1})^p \rangle \cap \langle (O_{L_2})^p \rangle$. Select $(O_{L_1})^p$ and $(O_{L_2})^p$ also in such a way that $\pi_5$ contains a 4-space $\pi_4$ for which $\pi_4 = \langle \pi_4 \rangle \cap (O_{L_1})^p$ and for which every 5-space through $\pi_4$ in $\langle (O_{L_1})^p \rangle$ contains points of $(O_{L_1})^p \setminus \pi_4$. Note that since $(O_{L_2})^p$ is projected from $r_0$ onto $(O_{L_1})^p$, also all 5-dimensional spaces in $\langle (O_{L_2})^p \rangle$ through $\pi_4$ contain at least one point of $(O_{L_2})^p \setminus \pi_4$.

Consider a 5-space $\pi_5'$ in $\langle (O_{L_1})^p \rangle$ through $\pi_4$ different from $\langle (O_{L_1})^p \cap (O_{L_2})^p \rangle$. Exactly one 6-dimensional space through $\pi_5'$ is $\langle r_0, \pi_5' \rangle$, but the other $q$ 6-dimensional spaces in $\langle (O_{L_1})^p, (O_{L_2})^p \rangle$ through $\pi_5'$ intersect the cone $r_0(O_{L_2})^p$ in an ovoid, one of which is $(O_{L_1})^p$. Consider the $q - 1$ other 6-dimensional spaces. Each one of them intersects the 6-dimensional space of $(O_{L_2})^p$ in at least one point of $(O_{L_2})^p \setminus \langle (O_{L_1})^p \cap (O_{L_2})^p \rangle$. So this 6-dimensional space shares an ovoid $(O_L)^p$ with the cone $r_0(O_{L_2})^p$ and the points of $(O_{L_1})^p \cap \langle (O_{L_1})^p \cup (O_{L_2})^p \rangle$ generate the 6-space of $(O_L)^p$ completely. As indicated, this ovoid $(O_L)^p$ corresponds with some ovoid $O_L$ in a tangent space $T_{L}(Q(10, q))$, $L$ a line of $Q(10, q)$ through $r$. And the preceding arguments show that before the projection, the original ovoid $O_L$ lies in $\pi_7$.

Letting vary the 5-space $\pi_5'$ through $\pi_4$, we show that all points of the truncated cone $pO \setminus \{p\}$ lie in $\mathcal{K}$, except maybe for the points of the truncated cone $p(O_{L_1} \cap \pi_4) \setminus \{p\}$.

But in order to prove that these points lie in $\mathcal{K}$, we simply repeat the arguments for $O_{L_1}$ and an other ovoid $O_{L_3}$ such that $\dim(O_{L_1} \cap O_{L_3}) = 5$, and such that $\langle O_{L_1}, O_{L_3} \rangle$ contains at least one 4-space $\pi'_4$ such that $\pi'_4 = \langle O_{L_1} \cap \pi'_4 \rangle$ and such that all 5-spaces in $\langle O_{L_1} \rangle$ through $\pi'_4$ contain points of $O_{L_1} \setminus \pi'_4$. Then we can prove that all points of the truncated cone $pO_{L_1} \setminus \{p\}$, not lying in $\langle p, \pi'_4 \rangle$, lie in $\mathcal{K}$.

Note that it is possible to find $O_{L_3}$ since we already know that $pO_{L_1} \setminus \{p\}$ is contained in $\mathcal{K}$, up to maybe $p(O_{L_1} \cap \pi_4) \setminus \{p\}$. Since the computer searches showed that there are a lot of choices for $\pi'_4$, in fact all the possible choices for $\pi'_4$ have an empty intersection, it is possible to prove that all points of $T_r(Q(10, q)) \cap \mathcal{K}$ lie in $\pi_7$ and $T_r(Q(10, q)) \cap \mathcal{K} = pO_{L_1} \setminus \{p\}$.

This concludes the lemma.

**Theorem 50** The smallest minimal blocking sets of $Q(2n + 2, q = 3)$, $n \geq 3,$
are truncated cones \( \pi_{n-3}O \setminus \pi_{n-3} \), \( O \) an ovoid of \( Q(6,q) \).

**PROOF.** Suppose that \( K \) is a minimal blocking set of \( Q(2n+2,q) \), \( |K| \leq q^{n+1}+q^{n-2} \). Suppose that \( r \in Q(2n+2,q) \setminus K \) for which \( |T_r(Q(2n+2,q))\cap K| = q^n+q^{n-3} \). Such a point exists (Lemma 45) and the points of \( T_r(Q(2n+2,q))\cap K \) are the points of a truncated cone \( \pi_{n-4}^rO\setminus\pi_{n-4}^r \), \( O \) an ovoid of \( Q(6,q) \). Consider a line \( L \) on \( r \) such that \( |T_L(Q(2n+2,q))\cap K| = q^{n-1}+q^{n-4} \) (such a line exists, see the constructive proof of Lemma 48). From Lemma 46, we have \( |T_r(Q(2n+2,q))\cap K| = q^n+q^{n-3} \) for all points \( r_j \in L \). It is clear that the truncated cones \( T_r(Q(2n+2,q))\cap K \) are truncated cones \( T_L(Q(2n+2,q))\cap K \), denoted by \( \pi_{n-5}^rO\setminus\pi_{n-5}^r \), with \( \pi_{n-5}^r = \pi_{n-4}^r\cap\pi_{n-4}^{r_j} \), \( r_i \neq r_j \), \( r_i, r_j \in L \). Furthermore, \( r \) projects \( \pi_{n-4}^rO\setminus\pi_{n-4}^r \) onto \( \pi_{n-4}^rO\setminus\pi_{n-4}^r \) for all points \( s \in L' \). Actually with \( s \) varying over \( L' \), these tangent hyperplanes vary over the hyperplanes through \( T_L(Q(2n+2,q)) \); so we get every point of the spaces \( \pi_{n-4}^r \), \( r_i \in L \). These points lie on lines with \( q \) points of \( K \) to \( O \cap O' \); so they belong to the vertices of the cones \( \pi_{n-4}^rO\setminus\pi_{n-4}^r, s \in L' \). Letting vary \( O' \) over the points of \( \pi_{n-4}^1O\setminus\pi_{n-4}^1 \), we obtain that every point of this truncated cone lies on a line containing \( q \) points of \( K \) and passing through \( \pi_{n-4}^r \). Consider two points \( p_1, p_2 \not\in \pi_{n-4}^r \cap \pi_{n-4}^{p_1}, p_1 \in \pi_{n-4}^r, p_2 \in \pi_{n-4}^r \). Consider an arbitrary line \( M \) of \( \pi_{n-4}^1O\setminus\pi_{n-4}^1 \) passing through \( p_1 \) and containing \( q \) points of \( K \). The \( q^2 \) points of \( \langle M, p_2 \rangle \setminus \langle p_1, p_2 \rangle \) all lie in \( K \); this implies that the truncated cone \( \langle \pi_{n-4}^r, \pi_{n-4}^r \rangle O\setminus\langle \pi_{n-4}^r, \pi_{n-4}^r \rangle \) lies in \( K \). Since \( |K| = |\pi_{n-4}^rO\setminus\pi_{n-4}^r| = (q^3 + 1)q^{n-2} = q^{n+1} + q^{n-2} \), this truncated cone must be equal to \( K \).

5 Concluding remarks

Finding the complete characterization of the smallest minimal blocking sets of \( Q(2n+2,q), q \) odd, \( n \geq 3 \), is, using these techniques, still dependent on some open problems. The first one is the existence or non-existence of ovoids of \( Q(6,q) \). However, as we tried to prove as much as possible lemmas for general \( q \) odd, it is clear that formulating the characterization dependent on the existence or non-existence of ovoids of \( Q(6,q) \) is possible. The second important problem is the size of the smallest minimal blocking sets of \( Q(4,q), q \) odd, different from an ovoid. Comparing proofs of Lemma 12 for the \( q \) even and odd case, it is clear that the proof is only depending on this size. Furthermore,
the better this lower bound is, the stronger corollaries the next key lemmas have. Finding an equivalent theorem for the $q$ odd case as Theorem 5 can be very useful to obtain the characterization for $q$ odd in general, at least using these techniques. Finally, we used a computer search to find on the ovoid $O$ of $Q(6,3)$ 4-spaces $\pi_4$ such that $\pi_4 = \langle \pi_4 \cap O \rangle$ and such that every 5-space $\pi_5$ through $\pi_4$ contains points of $O \setminus \pi_4$. Proving this property for arbitrary ovoids of $Q(6,q)$, $q$ odd, will make a lot of proofs valid for arbitrary $q$ odd.

References


