New classification results for a certain class of weighted minihypers

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Abstract

Minihypers were introduced for their relation with linear codes meeting the Griesmer bound. Results on minihypers have therefore important applications in coding theory. On the other hand, minihypers are nice geometrical structures since they are a generalization of blocking sets, which have been studied a lot. Therefore also minihypers deserve to be studied from a purely geometrical point of view. At last, results on minihypers have also important applications in finite projective geometry.

1 Introduction

A linear $[N, k, d; q]$-code is a k-dimensional subspace of the N-dimensional vectorspace $V(N, q)$ over the galois field $\mathrm{GF}(q)$ having minimum Hamming distance d. It is interesting to use linear codes having a minimal length for given k, d and q. Every linear code satisfies $N \geq \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. This inequality is known as the *Griesmer bound*.

Suppose that $d \geq 1$ and $k \geq 2$. Then d can be written in an unique way as $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$, with $\theta \geq 1$ and $0 \leq \zeta_i \leq q-1$, $i = 0, 1, \ldots, k-2$. Using this expression for d, the Griesmer bound for an $[N, k, d; q]$-code can be expressed as $N \geq \theta v_k - \sum_{i=0}^{k-2} \zeta_i v_{i+1}$, defining $v_i := \frac{q^i - 1}{q-1}$ and $v_0 := 0$.

We continue with the definition of a minihyper as found in [8].

Definition 1.1 *An* $\{f, m; N, q\}$-minihyper *is a pair* (F, w)*, where* F *is a subset of the point set of* $\mathrm{PG}(N, q)$ *and where* w *is a weight function* w*:* $\mathrm{PG}(N, q) \to \mathbb{N}$*:* $x \mapsto w(x)$*, satisfying*

1. $w(x) > 0 \iff x \in F$,
2. $\sum_{x \in F} w(x) = f$*, and*
3. $\min\{\sum_{x \in H} w(x) \| H \in \mathcal{H}\} = m$*, where* $\mathcal{H}$ *is the set of hyperplanes of* $\mathrm{PG}(N, q)$*.*

Let C be an $[N,k,d;q]$-code meeting the Griesmer bound and let $G = (g_1^T \cdots g_n^T)$ be a $(k \times N)$-generator matrix of C. Then, for each $i \in \{1,2,\ldots,N\}$, g_i is a nonzero vector in $V(k,q)$; hence it defines a point $p(g_i) \in \mathrm{PG}(k-1,q)$. Now define a weight function $w' : \mathrm{PG}(k-1,q) \to \mathbb{N}$:

$$w'(p) = |\{i \in \{1,2,\ldots,N\} : p = p(g_i)\}$$

If $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$, with θ and ζ_i, $i = 1,2,\ldots,k-2$, as above, then $\max\{w'(p) : p \in \mathrm{PG}(k-1,q)\} = \theta$. Let $w : \mathrm{PG}(k-1,q) \to \mathbb{N} : P \mapsto w(p) = \theta - w'(p)$ be a weight function and let $F = \{p \in \mathrm{PG}(k-1,q) : w(p) > 0\}$. Then (F,w) is a $\{\sum_{i=0}^{k-2} \zeta_i v_{i+1}, \sum_{i=0}^{k-2} \zeta_i v_i; k-1, q\}$-minihyper. This relation is clearly expressed in the following theorem.

Theorem 1.2 (Hamada [?]) *Let q be a prime power and let k, θ and ζ_i, $i = 0,1,\ldots,k-2$, be integers satisfying $k \geq 3$, $\theta \geq 1$, $0 \leq \zeta_i \leq q-1$ and $(\zeta_0, \zeta_1, \ldots, \zeta_{k-2}) \neq (0,0,\ldots,0)$. Let $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$. Then there is a one-to-one correspondence between the set of all nonequivalent $[N,k,d;q]$-codes meeting the Griesmer bound and the set of all*

$$\left\{ \sum_{i=0}^{k-2} \zeta_i v_{i+1}, \sum_{i=0}^{k-2} \zeta_i v_i; k-1, q \right\} - \textit{minihypers } (F,w)$$

satisfying $w(p) \leq \theta$ for each point $p \in \mathrm{PG}(k-1,q)$.

Both the example and the theorem show clearly that there is an interesting link between minihypers and linear codes meeting the Griesmer bound. From now on, we will consider minihypers as purely geometrical objects.

Minihypers can easily be constructed in $\mathrm{PG}(k-1,q)$ as a *sum* of certain geometrical objects. Consider for example an arbitrary set $\mathcal{L}$ of lines of $\mathrm{PG}(k-1,q)$ The sum of lines of $\mathcal{L}$ is defined as the set of all points contained in a line of $\mathcal{L}$, and the weight of a point $P \in \mathrm{PG}(k-1,q)$ is the number of lines of $\mathcal{L}$ on P. If all lines are two by two skew, then all weights of the points in the induced set F are one. Hence the weigth function itself is not necessary to determine the minihyper. We call a minihyper (F,w) a *non-weighted* minihyper if $w(p) = 1$ for all points $p \in F$. If this condition on the weight function is not satisfied, then (F,w) is called a *weighted* minihyper and the minihyper is not completely determined by the set F. We can construct minihypers using different subspaces of $\mathrm{PG}(k-1,q)$, e.g. planes, 3-dimensional susbpaces ..., even subgeometries of $\mathrm{PG}(k-1,q)$ (if they exist) can be used.

2 An application of a known classification result

In this section we describe a known classification result of a certain class of minihypers. We also mention an interesting application of this result in finite geometry. We need the following definition.

Definition 2.1 *A* blocking set *of* $\mathrm{PG}(2,q)$ *is a set B of points such that every line of* $\mathrm{PG}(2,q)$ *meets B in at least one point. A* non-trivial *blocking set B in* $\mathrm{PG}(2,q)$ *is a blocking set containing no line of* $\mathrm{PG}(2,q)$.

On minihypers with a non-trivial weight function, the following result is known:

Theorem 2.2 (Govaerts and Storme [6]) *Let (F, w) be a weighted $\{\delta v_{\mu+1}, \delta v_\mu; k-1, q\}$-minihyper, where $\delta \leq \epsilon$ with $q+\epsilon+1$ the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$*, then (F, w) is a sum of δ μ-dimensional subspaces of* $\mathrm{PG}(k-1,q)$.

A lot of other results on weighted and non-weighted minihypers are known. We refer to [5, 6, 2, 3, 4]

Consider now the quadric $\mathrm{Q}(4,q)$, q odd. This quadric contains points and lines but no planes of $\mathrm{PG}(4,q)$. A *spread* of $\mathrm{Q}(4,q)$ is a set S of lines of $\mathrm{Q}(4,q)$ partitioning the pointset. A *partial spread* is a set P of two by two skew lines of $\mathrm{Q}(4,q)$. A partial spread is *maximal* if and only if it cannot be extended to a larger partial spread. Using the result of theorem 2.2, the following result on partial spreads of $\mathrm{Q}(4,q)$ is proved:

Theorem 2.3 *Let P be a partial spread of size $q^2+1-\delta$. If $\delta < \epsilon$, with $q+\epsilon$ the size of the smallest non-trivial blocking sets in* $\mathrm{PG}(2,q)$*, $q > 2$. Then P can be extended to a spread.*

To prove this theorem, we prove that there is a link between the set of points of $\mathrm{Q}(4,q)$ that do not lie on a line of the partial spread S and a $\{\delta v_{\mu+1}, \delta v_\mu; k-1, q\}$-minihyper. Then the characterisation of such a minihyper from Theorem 2.2 is used to find lines extending the partial spraed P. This result shows clearly that characterisation results on minihypers can have nice applications in projective geometry. For more applications of these results we refer to [1, 7]

The following result concerns non-weighted minihypers.

Theorem 2.4

The aim of this research is to find an analogues characterisation for weighted minihypers

3 New results

In this section we describe a characterisation of

$$\left\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\right\} - \text{minihypers } (F, w),$$

where $k \geq 3$ and $\sum_{i=0}^{k-2} \epsilon_i \leq \sqrt{q}$.

We will use several inductive arguments. Therefore, we start with the case $k=3$. We prove the following theorem.

Theorem 3.1 *Let (F, w) be a $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; k-1, q\}$-minihyper with $\epsilon_0+\epsilon_1 \leq \sqrt{q}+1$. Then (F, w) is the sum of ϵ_0 points and ϵ_1 lines of* $\mathrm{PG}(k-1,q)$.

We start with a minihyper in $\mathrm{PG}(2,q)$, i.e. $k=3$. Such a minihyper (F, w) is a weighted ϵ_1-fold blocking set of $\mathrm{PG}(2,q)$, i.e., every line contains at least ϵ_1 points of (F, w), were a point is counted according to its weight. Since we supposed that $\epsilon_1+\epsilon_0 \leq \sqrt{q}$, a reuslt of S. Ball [?] implies that F contains a complete line L of $\mathrm{PG}(2,q)$, i.e. $w(p) > 0$ for all points $p \in L$. Any line $M \neq L$ of $\mathrm{PG}(2,q)$ intersects L in exactly one point. Define $w'(p) = w(p)$ for all points of $\mathrm{PG}(2,q) \setminus L$ and $w'(p) := w(p) - 1$ for all points $p \in L$. The weight function w' induces a new set of points F'. It is clear that every line $M \neq L$ contains at least $\epsilon_1 - 1$ points of F', but it is unsure if L meets F' in at least $\epsilon_1 - 1$ points.

Using so-called *lacunary polynomials*, we prove the following lemma:

Lemma 3.2 *The set (F', w') is a $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; k-1, q\}$-minihyper with $\epsilon_0 + \epsilon_1 - 1 \leq \sqrt{q}$*

This lemma implies that "removing" the line L from the minihyper (F, w), i.e. substracting 1 from the weight of every point of L, yields a minihyper (F', w') containing $q+1$ points less and satisfying the same conditions of the minihyper (F, w). It is clear that the process of removing a line can be repeated, we may conclude the following theorem:

Theorem 3.3 *Let (F, w) be a $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 2, q\}$-minihyper with $\epsilon_0 + \epsilon_1 \leq \sqrt{q} + 1$. Then (F, w) is the sum of ϵ_0 points and ϵ_1 lines of* $\mathrm{PG}(2, q)$.

We consider now a $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 3, q\}$-minihyper (F, w). If we project this minihyper from an arbitrary point $p \notin F$, then a $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 2, q\}$-minihyper (F', w') is obtained. We know that (F', w') is a sum of ϵ_1 lines and ϵ_0 points of $\mathrm{PG}(2, q)$. Suppose that L' is a line of (F', w'). The plane $\langle p, L' \rangle$ intersects (F, w) in a $\{\mu_1(q+1) + \mu_0, \epsilon_1; 2, q\}$-minihyper (F'', w''), with $\mu_1 + \mu_0 \leq \sqrt{q}$. This minihyper is again a sum of μ_1 lines and μ_0 points. The μ_1 lines are projected on the line L', hence, L' has weight μ_1 in (F', w'). We can consider any line in (F', w'), and we get a total weight of ϵ_1. Hence (F, w) contains ϵ_1 lines, and, necessary, (F, w) is the sum of ϵ_1 lines and ϵ_0 points of $\mathrm{PG}(3, q)$.

We can now repeat this argument for any $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; k-1, q\}$-minihyper with $\epsilon_0 + \epsilon_1 \leq \sqrt{q} + 1$, using an induction on k, and we conclude Theorem 3.1.

In the next step, we consider a $\{\epsilon_2(q^2 + q + 1) + \epsilon_1(q+1) + \epsilon_0, \epsilon_1; k-1, q\}$-minihyper (F, w), $k \geq 4$, $\epsilon_2 + \epsilon_1 + \epsilon_0 \leq \sqrt{q}$.

We start with a lemma that reveals partially some structure of (F, w).

Lemma 3.4 *The minihyper (F, w) contains a sum of ϵ_2 planes.*

To prove this rather technical lemma, we use the following lemma ([5, Lemma 1.1]).

Lemma 3.5 *Consider the minihyper (F, w) and suppose that H is a hyperplane of* $\mathrm{PG}(k-1, q)$ *containing $\epsilon_2(q+1) + \epsilon_1$ points of (F, w). Then (F, w) induces a $\{\epsilon_2(q+1) + \epsilon_1, \epsilon_1; k-2, q\}$-minihyper (F', w') contained in H.*

By Theorem 3.3 we know that (F', w') is the sum of ϵ_2 lines and ϵ_1 points. This enables us, using Theorem 3.3 inductively, and, using another thechnical lemma from [5], to prove the lemma in several steps. We end this part with a new lemma, which finally characterizes the minihyper (F, w).

Lemma 3.6 *If (F, w) is a $\{\epsilon_2(q^2 + q + 1) + \epsilon_1(q+1) + \epsilon_0, \epsilon_1; k-1, q\}$-minihyper, $k \geq 4$, where $\epsilon_2 + \epsilon_1 + \epsilon_0 \leq \sqrt{q}$, containing a sum of ϵ_2 planes, then (F, w) is a sum of ϵ_2 planes, ϵ_1 lines and ϵ_0 points.*

At this point, we have a theorem for arbitrary $k \geq 3$, but with the restriction that $\epsilon_j = 0$, for all $j > 2$. As induction hypothesis we assume now that any

$$\left\{ \sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=0}^{t-1} \epsilon_i v_i; k-1, q \right\} - \text{minihyper} ,$$

where $t-1 < k-2$, $\sum_{i=0}^{t-1} \epsilon_i \leq \sqrt{q}$ is a sum of ϵ_{t-1} $t-1$-dimensional subspaces, ϵ_{t-2} $(t-2)$-dimensional subspaces, ..., ϵ_1 (1)-dimensional subspaces (lines) and ϵ_0 (0)-dimensional subspaces (points). We suppose that (F, w) is a

$$\left\{ \sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=0}^{t} \epsilon_i v_i; k-1, q \right\} - \text{minihyper} ,$$

where $t \leq k-2$, $\sum_{i=0}^{t} \epsilon_i \leq \sqrt{q}$. The induction hypothesis is crucial to prove the next lemma, but quite some technical steps are also needed.

Lemma 3.7 *The minihyper* (F, w) *contains a sum of* ϵ_t t*-dimensional subspaces of* $\mathrm{PG}(k-1, q)$.

Having this lemma, which is very similar to Lemma 3.4, we can now prove the final theorem

Theorem 3.8 *If* (F, w) *is a*

$$\left\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=0}^{t} \epsilon_i v_i; k-1, q\right\} - \textit{minihyper}\ ,$$

where $k \geq 3$, $t \leq k-2$ *and* $\sum_{i=0}^{t} \epsilon_i \leq \sqrt{q}$, *then* (F, w) *is a sum of* ϵ_t (t)*-dimensional subspaces,* ϵ_{t-1} $(t-1)$*-dimensional subspaces,* $\ldots$, ϵ_1 (1)*-dimensional subspaces (lines) and* ϵ_0 (0)*-dimensional subspaces (points) of* $\mathrm{PG}(k-1, q)$.

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